FRACTIONAL INTEGRATION AND IMPULSE RESPONSES: A BIVARIATE APPLICATION TO REAL OUTPUT IN THE US AND THE SCANDINAVIAN COUNTRIES

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ABSTRACT

This paper analyses impulse response functions in the context of vector fractionally integrated time series. We derive analytically the restrictions required to identify the structural-form system. As an illustration of the recommended procedure, we also carry out an empirical application based on a bivariate system including real output in the US and, in turn, in one of four Scandinavian countries (Denmark, Finland, Norway and Sweden). The empirical results appear to be sensitive to some extent to the specification of the stochastic process driving the disturbances, but generally a positive shock to US output has a positive effect on the Scandinavian countries which tends to disappear in the long run.

Keywords: Long memory; Multivariate time series; Impulse response functions

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1. Introduction

This paper analyses impulse response functions in the context of vector fractionally integrated time series. Impulse responses have been studied extensively in the literature, especially in the case of Vector Autoregressions (VAR) with series previously detrended using deterministic polynomials or first differences. In the latter case the series are normally assumed to have a unit root. However, this is a rather restrictive assumption, as the differencing parameter required to achieve I(0) stationarity might in fact be any real number, not necessarily an integer. In such a case, the series are said to be fractionally integrated. Univariate fractional integration has been widely examined in the literature, and many test statistics have been developed for estimating and testing the fractional differencing parameter. Examples are Sowell (1992), Robinson (1994), Tanaka (1999) etc. in parametric contexts and Geweke and Porter-Hudak (1983), Robinson (1995), Shimotsu and Phillips (2004), etc. in semiparametric models (see also Beran, 1994, and Baillie, 1996 for surveys of I(d) univariate models). By contrast, the literature on multivariate fractional integration models is very limited. A few exceptions are Gil-Alana (2003a,b), who extended the univariate frequency domain tests of Robinson (1994) to the multivariate case, and Nielsen (2005), who proposed similar tests in the time domain.

The present paper also adopts a multivariate fractional integration approach to examine system dynamics, first deriving the structural form of the model from the reduced one, and then computing impulse responses. The outline of this paper is as follows: Section 2 deals with the identification of the structural parameters in a vector fractionally integrated model. Section 3 examines in more detail the bivariate case. Section 4 briefly describes a procedure for estimating the parameters in an I(d) system. An empirical application is carried out in Section 5, while Section 6 concludes.

2. Identification of the structural parameters in a fractionally integrated system

The starting point is the following structural model:

$$A D y_t = u_t, \qquad t = 1, 2, \ldots \tag{1}$$

$$u_t = G u_{t-1} + v_t, \quad t = 1, 2, \ldots, \tag{2}$$

where A is a (nxn) matrix of parameters; D is an (nxn) diagonal matrix of the form:

$$\begin{pmatrix} (1-L)^{d_1} & 0 & 0 \\ 0 & --- & 0 \\ 0 & 0 & (1-L)^{d_n} \end{pmatrix},$$

where $d_1, d_2, \ldots, d_n$ can be any real value; y_t is a (nx1) vector of the observable variables; u_t is a (nx1) vector, which is assumed to be I(0); G is another (nxn) matrix of parameters, and w_t is a (nx1) structural error vector with zero mean and diagonal variance-covariance matrix V. Substituting (1) into (2), we obtain

$$A D y_t = G A D y_{t-1} + v_t, \qquad t = 1, 2, \ldots \tag{3}$$

implying that

$$D y_t = A^{-1} G A D y_{t-1} + A^{-1} v_t, \qquad t = 1, 2, \ldots \tag{4}$$

Using now the lag-operator (i.e. $L y_t = y_{t-1}$):

$$\left[I - A^{-1} G A L\right] D y_t = A^{-1} v_t, \qquad t = 1, 2, \ldots,$$

we get

$$y_t = D^{-1} \left[I - A^{-1} G A L\right]^{-1} A^{-1} v_t, \qquad t = 1, 2, \ldots, \tag{5}$$

which is the structural MA(∞) representation of y_t.

Let us consider now the reduced-form model:

$$D y_t = \varepsilon_t, \qquad t = 1, 2, \ldots \tag{6}$$

$$\varepsilon_t = F \varepsilon_{t-1} + w_t, \qquad t = 1, 2, \ldots \tag{7}$$

where ε_t is a (nx1) vector of the d-differenced variables; F is a (nxn) matrix of parameters, and w_t is an I(0) vector with variance-covariance matrix W. Substituting now (6) into (7)

$$D\, y_t \; = \; F\, D\, y_{t-1} \; + \; w_t, \qquad t \; = \; 1, 2, ..., \tag{8}$$

implying that

$$[I \; - \; F\, L] D\, y_t \; = \; w_t, \quad t \; = \; 1, 2, \; ...,$$

and then

$$y_t \; = \; D^{-1} [I \; - \; F\, L]^{-1}\, w_t, \quad t \; = \; 1, 2, ..., \tag{9}$$

which is the reduced-form MA(∞) representation of y_t.

Note that the structural model in (5) has $2n^2+2n$ parameters to estimate: n corresponding to the fractional differencing parameters in D; $2n^2$ of the two matrices A and G; and the n variances in V. On the other hand, the reduced-form MA(∞) representation in (9) contains $n+n^2+n(n+1)/2$ parameters: the n d-parameters in D; n^2 in F, and $n(n+1)/2$ parameters of the variance-covariance matrix W. Therefore, in order to identify the system we need to impose $(n/2)(n+1)$ restrictions in the structural model. N restrictions can be obtained by imposing a 1-unit variance in the variance-covariance matrix V in (2). However, $(n^2-n)/2$ restrictions will still be required. Here, there are two possibilities: one is to impose triangularity of the A matrix - this would imply that the contemporaneous and the future effects of some of the variables on the others will be zero, which may be a relatively strong assumption in some cases. The second approach uses the Blanchard and Quah (1989) decomposition, which implies that in the long run some variables have no effect on the others. This is illustrated in the following section for the bivariate case.

3. The bivariate model

Let us consider the following structural bivariate model:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} (1-L)^{d_1} & 0 \\ 0 & (1-L)^{d_2} \end{pmatrix} \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}, \quad t = 1, 2, ..., \tag{10}$$

where, initially, u_{1t} and u_{2t} are assumed to be serially uncorrelated, mutually orthogonal structural disturbances, whose variances are normalized to unity. Note that this model can be expressed as:

$$\begin{pmatrix} (1-L)^{d_1}\, y_{1t} \\ (1-L)^{d_2}\, y_{2t} \end{pmatrix} = \begin{pmatrix} \dfrac{d}{ad-bc} u_{1t} - \dfrac{b}{ad-bc} u_{2t} \\ \dfrac{-c}{ad-bc} u_{1t} + \dfrac{a}{ad-bc} u_{2t} \end{pmatrix}, \quad t = 1, 2, ... \tag{11}$$

Considering now the transformed disturbances:

$$u_{1t}^{*} = \frac{1}{ad-bc}\left(d\, u_{1t} - b\, u_{2t}\right), \tag{12}$$

and

$$u_{2t}^{*} = \frac{1}{ad-bc}\left(a\, u_{2t} - c\, u_{1t}\right), \tag{13}$$

and using the Binomial expansions in the fractional differencing polynomials in the left-hand-side of (11), we obtain

$$y_{1t} = \sum_{j=0}^{\infty} \psi_j^{(1)}\, u_{1t-j}^{*}, \tag{14}$$

and

$$y_{2t} = \sum_{j=0}^{\infty} \psi_j^{(2)}\, u_{2t-j}^{*}, \tag{15}$$

where

$$\psi_j^{(i)} = \frac{\Gamma(j+d_i)}{\Gamma(j+1)\,\Gamma(d_i)}, \quad i = 1, 2,$$

where $\Gamma(x)$ stands for the Gamma function and d_i, i = 1, 2 are the orders of integration of the two series. Substituting now (12) into (14) and (13) into (15) leads to:

$$y_{1t} = \sum_{j=0}^{\infty} \phi_j^{(1,1)} u_{1t-j} + \sum_{j=0}^{\infty} \phi_j^{(1,2)} u_{2t-j}, \qquad (16)$$

and

$$y_{2t} = \sum_{j=0}^{\infty} \phi_j^{(2,1)} u_{1t-j} + \sum_{j=0}^{\infty} \phi_j^{(2,2)} u_{2t-j}, \qquad (17)$$

where the impulse response coefficients are:

$$\phi_j^{(1,1)} = \frac{d\,\psi_j^{(1)}}{ad - bc}; \qquad \phi_j^{(1,2)} = \frac{-b\,\psi_j^{(1)}}{ad - bc}; \qquad (18)$$

$$\phi_j^{(2,1)} = \frac{-c\,\psi_j^{(2)}}{ad - bc}; \qquad \phi_j^{(2,2)} = \frac{a\,\psi_j^{(2)}}{ad - bc}. \qquad (19)$$

3.1 Identification in a pure vector fractional model

From the reduced-form system:

$$\begin{pmatrix} (1-L)^{d_1} & 0 \\ 0 & (1-L)^{d_2} \end{pmatrix} \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}, \qquad t = 1, 2, ..., \qquad (20)$$

we can obtain the estimates of d_1 and d_2 under the assumption that ε_t is a white noise vector process. For this purpose we can use, for example, the parametric approach of Gil-Alana (2003a,b) or Nielsen (2005). Note that, once d_1 and d_2 have been estimated in (20), we can directly obtain the coefficients $\psi_j^{(1)}$ and $\psi_j^{(2)}$, j = 0, 1, …, from their Binomial expansions. Using now (11) and (20):

$$\varepsilon_{1t} = \frac{d}{ad - bc} u_{1t} - \frac{b}{ad - bc} u_{2t},$$

and

$$\varepsilon_{2t} = \frac{-c}{ad - bc} u_{1t} + \frac{a}{ad - bc} u_{2t},$$

implying that

$$\sigma_{11}^{\varepsilon} = \frac{1}{(ad - bc)^2}\left(d^2\sigma_{11}^{u} + b^2\sigma_{22}^{u} - 2bd\,\sigma_{12}^{u}\right), \tag{21}$$

$$\sigma_{22}^{\varepsilon} = \frac{1}{(ad - bc)^2}\left(c^2\sigma_{11}^{u} + a^2\sigma_{22}^{u} - 2ac\,\sigma_{12}^{u}\right), \tag{22}$$

and

$$\sigma_{12}^{\varepsilon} = \frac{1}{(ad - bc)^2}\left((ad + bc)\sigma_{12}^{u} - dc\,\sigma_{11}^{u} + ab\,\sigma_{22}^{u}\right). \tag{23}$$

Note that in this context we have three equations for seven unknowns (a, b, c, d, σ_{11}^{u}, σ_{12}^{u} and σ_{22}^{u}), but using the restrictions imposed on the variance-covariance matrix of u_t (i.e., $\sigma_{12}^{u} = 0$ and $\sigma_{11}^{u} = \sigma_{22}^{u} = 1$), the system given by (21) – (23) reduces to:

$$\sigma_{11}^{\varepsilon} = \frac{1}{(ad - bc)^2}\left(d^2 + b^2\right), \tag{24}$$

$$\sigma_{22}^{\varepsilon} = \frac{1}{(ad - bc)^2}\left(a^2 + c^2\right), \tag{25}$$

and

$$\sigma_{12}^{\varepsilon} = \frac{1}{(ad - bc)^2}\left(-dc + ab\right). \tag{26}$$

The new system of equations (24) – (26) is still not identified, as there are only three equations for four unknowns. Here economic theory might play a role. One possibility is to assume that one of the coefficients (a, b, c or d) is equal to 0 but, in doing so, we lose part of the dynamic structure of the system. For example, b = 0 implies, according to (16) and (18), that a structural shock to y_{2t} (u_{2t}) has no effect on y_{1t} neither contemporaneously nor in the long run. Similarly, if c = 0, a shock to y_{1t} will have no effect on y_{2t}. This is a plausible assumption, for instance, in a bivariate case with a single variable for two countries, one of them being a large economy affecting a smaller one. The assumption that one of the variables does not affect the others in the long run might be more realistic in the context of a macroeconomic system. In such a case, an appropriate restriction is the following:

$$\sum_{j=0}^{\infty} \phi_j^{(1,2)} = 0 \qquad (27)$$

or, alternatively:

$$\sum_{j=0}^{\infty} \phi_j^{(2,1)} = 0. \qquad (28)$$

Combining (24) – (26) with (27) or (28) the system is now completely identified and the impulse response functions can easily be estimated.

3.2 A (2x1) vector fractionally autoregressive model

Here, we extend the structural model (10) to the case of weak parametric autocorrelation in u_t. In particular, we consider the case of a VAR(1) system for u_t. Thus, the structural model is now (10) with

$$\begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} u_{1t-1} \\ u_{2t-1} \end{pmatrix} + \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix}, \quad t = 1, 2, ..., \qquad (29)$$

where v_{1t} and v_{2t} are serially uncorrelated and mutually orthogonal with unit variance (i.e., $\sigma_{11}^{v} = \sigma_{22}^{v} = 1$ and $\sigma_{12}^{v} = 0$) and with all the roots lying outside the unit circle. First, we describe the impulse response functions. Assuming that u_t is stationary, (29) can be written as:

$$\begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} = \begin{pmatrix} C_{11}(L) & C_{12}(L) \\ C_{21}(L) & C_{22}(L) \end{pmatrix} \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix}, \quad t = 1, 2, ..., \qquad (30)$$

where $C_{ij}(L)$, i, j = 1, 2 are polynomials of infinite order in L. From (11) and (30):

$$\begin{pmatrix} (1-L)^{d_1}\, y_{1t} \\ (1-L)^{d_2}\, y_{2t} \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} C_{11}(L) v_{1t} + C_{12}(L) v_{2t} \\ C_{21}(L) v_{1t} + C_{22}(L) v_{2t} \end{pmatrix} =$$

$$\frac{1}{ad - bc}\begin{pmatrix} d\,C_{11}(L)v_{1t} + d\,C_{12}(L)v_{2t} - b\,C_{21}(L)v_{1t} - b\,C_{22}(L)v_{2t} \\ -c\,C_{11}(L)v_{1t} - c\,C_{12}(L)v_{2t} + a\,C_{21}(L)v_{1t} + a\,C_{22}(L)v_{2t} \end{pmatrix} = \begin{pmatrix} w_{1t} \\ w_{2t} \end{pmatrix}. \quad (31)$$

Hence, the model becomes:

$$\begin{pmatrix} (1 - L)^{d_1}\, y_{1t} \\ (1 - L)^{d_2}\, y_{2t} \end{pmatrix} = \begin{pmatrix} w_{1t} \\ w_{2t} \end{pmatrix}, \qquad t = 1, 2, ...,$$

implying that

$$y_{1t} = \sum_{j=0}^{\infty} \psi_j^{(1)} w_{1t-j}, \quad (32)$$

and

$$y_{2t} = \sum_{j=0}^{\infty} \psi_j^{(2)} w_{2t-j}. \quad (33)$$

Substituting now w_t from (31) into (32) and (33) we obtain

$$y_{1t} = \sum_{j=0}^{\infty} \rho_j^{(1,1)}\, v_{1t-j} + \sum_{j=0}^{\infty} \rho_j^{(1,2)}\, v_{2t-j}, \quad (34)$$

and

$$y_{2t} = \sum_{j=0}^{\infty} \rho_j^{(2,1)}\, v_{1t-j} + \sum_{j=0}^{\infty} \rho_j^{(2,2)}\, v_{2t-j}, \quad (35)$$

where

$$\rho_j^{(1,1)} = \psi_j^{(1)} \frac{\left(d\,C_{11}(L) - b\,C_{21}(L)\right)}{ad - bc}; \qquad \rho_j^{(1,2)} = \psi_j^{(1)} \frac{\left(d\,C_{12}(L) - b\,C_{22}(L)\right)}{ad - bc}; \quad (36)$$

$$\rho_j^{(2,1)} = \psi_j^{(2)} \frac{\left(-c\,C_{11}(L) + a\,C_{21}(L)\right)}{ad - bc}; \quad \rho_j^{(2,2)} = \psi_j^{(2)} \frac{\left(-c\,C_{12}(L) + a\,C_{22}(L)\right)}{ad - bc}, \quad (37)$$

which are the impulse response functions.

3.3 Identification in a VAR fractional model

The reduced-form model is now (20) with

$$\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \begin{pmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \end{pmatrix} + \begin{pmatrix} w_{1t} \\ w_{2t} \end{pmatrix}, \qquad t = 1, 2, ..., \tag{38}$$

and using again any of the parametric procedures for vector fractional integration we can obtain estimates of d_1 and d_2, ξ_{11}, ξ_{12}, ξ_{21} and ξ_{22}, along with the coefficients of the variance-covariance matrix of w_t, i.e., σ_{11}^{w}, σ_{12}^{w} and σ_{22}^{w}.

Identification follows here the same lines as in the previous case, noting that

$$\begin{pmatrix} w_{1t} \\ w_{2t} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix},$$

implying three equations of the same form as in (21) – (23), and that

$$\begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus, we add four equations with four unknowns, so the same restrictions as in the previous case apply here.

4. A testing procedure for fractional integration in multivariate contexts

A simple version of the procedure proposed in Gil-Alana (2003a,b) consists in testing the null hypothesis:

$$H_o: \; d \equiv (d_1, d_2, ..., d_n) = (d_{1o}, d_{2o}, ..., d_{no}) \equiv d_o, \tag{39}$$

for any real vector d_o, in the model given by (20), where ε_t is supposed to be an I(0) vector process with spectral density function $F(\lambda)$ that is positive definite. Therefore, ε_t may be white noise, but VAR structures can also be incorporated. To allow for some degree of generality, let us suppose that ε_t in (20) is generated by a parametric model of the form:

$$\varepsilon_t = \sum_{j=0}^{\infty} A_j(\tau) w_{t-j}, \qquad t = 1, 2, ..., \tag{40}$$

where w_t is white noise and W is the unknown variance-covariance matrix of w_t. The spectral density matrix of ε_t is then

$$f_{\varepsilon}(\lambda;\tau) = \frac{1}{2\pi} w(\lambda;\tau) W \, w(\lambda;\tau)^{*} \qquad (41)$$

where $w(\lambda;\tau) = \sum_{j=0}^{\infty} A_j(\tau) e^{i\lambda j}$, and w^* stands for the complex-conjugate transpose of w. A number of conditions are required on A and f_ε when deriving the test statistic, implying that, although ε_t can exhibit a much stronger degree of autocorrelation than multiple ARMA processes, its spectral density matrix must be finite, with eigenvalues bounded away from zero. In Gil-Alana (2003a) it is shown that a Lagrange Multiplier (LM) test of H_o (39) in (20) takes the form:

$$\tilde{S} = T \tilde{b}^T \left[\tilde{C} - \tilde{D}^T \tilde{E}^{-1} \tilde{D} \right]^{-1} \tilde{b}, \qquad (42)$$

where T is the sample size and

$$\tilde{b} = \frac{-1}{T} \sum_{r=1}^{T-1} \psi(\lambda_r) \, tr\left(I_{\varepsilon}(\lambda_r) \, \tilde{f}(\lambda_r;\tilde{\tau}) \right); \quad \tilde{C} = \frac{4}{T} \sum_{r=1}^{T-1} \psi(\lambda_r) \psi(\lambda_r)^T ;$$

$$\psi(\lambda_r) = \log \left| 2 \sin \frac{\lambda_r}{2} \right|, \text{ with } \lambda_r = \frac{2\pi r}{T},$$

$$\tilde{D}^T = \frac{-1}{T} \sum_{r=1}^{T-1} \psi(\lambda_r) \left[tr\left(\tilde{f}^{-1}(\lambda_r;\tilde{\tau}) \frac{\partial \tilde{f}(\lambda_r;\tilde{\tau})}{\partial \tau_1} \right); \ldots; tr\left(\tilde{f}^{-1}(\lambda_r;\tilde{\tau}) \frac{\partial \tilde{f}(\lambda_r;\tilde{\tau})}{\partial \tau_q} \right) \right];$$

$$\tilde{E}_{uv} = \frac{1}{2T} \sum_{r=1}^{T-1} tr\left(\tilde{f}^{-1}(\lambda_r;\tilde{\tau}) \frac{\partial \tilde{f}(\lambda_r;\tilde{\tau})}{\partial \tau_u} \tilde{f}^{-1}(\lambda_r;\tilde{\tau}) \frac{\partial \tilde{f}(\lambda_r;\tilde{\tau})}{\partial \tau_v} \right),$$

where $I_\varepsilon(\lambda_r)$ is a matrix with $(u,v)^{th}$ element:

$$I_{uv}(\lambda_r) = W_u(\lambda_r) \overline{W}_v(\lambda_r); \quad W_u(\lambda_r) = \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{T} \tilde{\varepsilon}_{ut} e^{i\lambda_r t},$$

$$\tilde{\varepsilon} = \begin{pmatrix} \tilde{\varepsilon}_{1t} \\ -- \\ \tilde{\varepsilon}_{nt} \end{pmatrix} = \begin{pmatrix} (1-L)^{d_{1o}} & -- & -- \\ & -- & \\ -- & -- & (1-L)^{d_{no}} \end{pmatrix} \begin{pmatrix} y_{1t} \\ -- \\ y_{nt} \end{pmatrix},$$

where the bar over W denotes complex conjugate, and $\tilde{f}$ is the spectral density matrix of $\tilde{\varepsilon}$:

$$\tilde{f}(\lambda;\tau) = \frac{1}{2\pi} \tilde{w}(\lambda;\tau) \tilde{W}\, \tilde{w}(\lambda;\tau)^{*};$$

with

$$\tilde{w}(\lambda;\tau) = \sum_{j=0}^{\infty} A_j(\tau) \tilde{\varepsilon}^{i\lambda j} \quad \text{and} \quad \tilde{W} = \frac{1}{T} \sum_{t=1}^{T} \tilde{\varepsilon}_t\, \tilde{\varepsilon}_t^{T}.$$

Finally,

$$\tilde{\tau} = \arg\min_{\tau \in T^{*}} \left(\frac{T}{2} \log\det \tilde{f}(\lambda_r;\tau) + \frac{1}{2} \sum_{r=1}^{T-1} tr\left(\tilde{f}^{-1}(\lambda_r;\tau)\, I_{\varepsilon}(\lambda_r) \right) \right),$$

where T* is a compact subset of q-dimensional Euclidean space. Extending the conditions derived by Robinson (1994) for the univariate case, Gil-Alana (2003a) shows that:

$$\tilde{S} \to_d \chi_n^2 \qquad as \quad T \to \infty. \tag{43}$$

Thus, the limit distribution is standard, in contrast to the case of most procedures for testing, for instance, unit roots, in models based on AR (VAR) alternatives, where the null limit distribution is non-standard and critical values have to be calculated in each case using simulation techniques.

5. An empirical application

In this section we apply the techniques outlined above to a bivariate system including real GDP in the US and, in turn, in one of four Scandinavian countries, i.e. Denmark, Finland, Norway and Sweden. The series are annual, for the period 1870-2000, and are taken from the Eurostat website: http:\www.fgn.unisg.ch/eumacro/macrodata/dmtrxneu.htm as well as

Maddison (1995), and have been demeaned prior to estimation to eliminate possible deterministic trends. Note that the choice of one large economy (i.e. the US) and four smaller ones is made to be consistent with the restrictions to be imposed on the model, as a shock to US real output is likely to affect the European countries, including the Scandinavian ones, whilst the opposite should not hold.

First, we apply the procedure described in Section 4, assuming that the disturbances are white noise. Denoting the US real output series by y_{1t}, and each of the Scandinavian countries in turn by y_{2t}, we test H_o: $(d_1, d_2)' = (d_{1o}, d_{2o})'$ in the model given by equation (20) for (d_{1o}, d_{2o})-values ranging from 0 to 2, with 0.01 increments. Table 1 displays the values of d_1 and d_2 for which the null hypothesis cannot be rejected at the 95% level. These values are very similar for the four countries: for the US, the order of integration ranges from 0.57 to 0.62, and it is slightly higher for the Scandinavian countries, being between 0.63 and 0.67. Table 2 reports the values of d_1 and d_2 producing the lowest statistic for each country, along with the values corresponding to the associated variance-covariance matrix of the differenced processes. It can be seen that d_1 (the US order of integration) is 0.60 when the model includes Danish or Finnish real output, and it is slightly smaller (0.58) in the two other systems including Swedish or Norwegian output. On the other hand, the orders of integration for the Scandinavian countries are 0.66 and 0.63 in the case of Denmark and Norway, and Finland and Sweden respectively. Note that these values should be an approximation to the maximum likelihood estimates, as our procedure uses the Whittle function, which is an approximation to the likelihood function.

[Insert Tables 1 and 2 about here]

Next, we allow for autocorrelation in ε_t and assume that it follows a VAR(1) process as in (38). A larger percentage of non-rejection values is then obtained. The lowest statistics for the values of d_1 and d_2 are displayed in Table 3. It can be seen that now d_1 is strictly above

1 in all four cases, with values ranging from 1.08 to 1.31. On the other hand, the values for the Scandinavian countries are all below 1, ranging from 0.64 (Denmark) to 0.97 (Sweden). According to this specification, real output is nonstationary in all cases, though mean-reverting in case of the European countries. Table 4 presents the coefficients of the variance-covariance matrices of the differenced processes and the residuals, both being required for the computation of the impulse response functions.

[Insert Tables 3 and 4 about here]

These are shown in Figures 1 – 4 for the case of white noise disturbances. The observed pattern is very similar in all four countries, namely shocks to real output are mean-reverting in all cases. However, the process of convergence is slower in the US (top-left panels in Figures 1 – 4) compared with the Scandinavian countries (bottom-right panels). This result may appear surprising at first, especially when noting that in Table 2 the order of integration in the US (d_1) is smaller than the corresponding one for the Scandinavian countries (d_2). The reason for the higher persistence in the US case is the interaction between the coefficients of the variance-covariance matrix of the disturbance term and those of the structural-form model given by (10). By construction, shocks to the Scandinavian countries do not affect the US economy (top-right panels), while a 1-unit positive shock to US real output produces a positive effect on the Scandinavian countries, though this disappears in the long run (bottom-left panels).

The results obtained when the disturbances are modelled as VAR(1) processes (not reported here for reasons of space) differ in one important respect, i.e. shocks to US output are found not to be mean-reverting, unlike in the Scandinavian countries, where mean reversion still occurs. However, the cross impulse responses are similar to those computed in the white noise case, with shocks producing a positive effect though disappearing in the long run.

6. Conclusions

In this paper we have analysed impulse response functions in the context of vector fractionally integrated time series. Specifically, we have derived analytically the restrictions required to identify the structural-form system. Our framework improves in two ways upon earlier studies. First, it is much more general compared to standard impulse response analysis (see, e.g., Blanchard and Quah, 1989), as it allows for fractional degrees of integration. Second, it is of a multivariate nature, in contrast to most of the earlier literature on fractional integration which only focuses on the univariate case (as in Robinson, 1994, inter alia). An empirical application based on a bivariate system including real output in the US and in one of four Scandinavian countries in turn (Denmark, Finland, Norway, Sweden) is also carried out as an illustration of the recommended procedure. The empirical results vary depending on how the I(0) vector of disturbances is specified. Specifically, when this is assumed to follow a white-noise process, the series appear to be mean-reverting in all cases. By contrast, when imposing a VAR(1) structure on the differenced series, US real output is found not to exhibit mean-reversion any longer, while output in the Scandinavian countries still does. The cross impulse responses suggest that positive shocks to the US economy affect positively Scandinavian output, but this effect is estimated to be relatively small and tends to disappear in the long run.

The present study could be extended in several ways. For instance, impulse response functions could also be obtained using the Blanchard and Quah (1989) decomposition, i.e., imposing long-run zero restrictions, although this should not affect significantly the empirical findings as the estimates are obtained from the reduced-form model, independently of the restrictions imposed, and consequently the impulse responses should not be much affected. More interestingly, one could consider higher-order systems, examining not simply bilateral country linkages for a given series but rather a full macroeconomic dynamic system. Also, one could allow for structural breaks. We are investigating these issues at present.

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TABLE 1			
Values of d_1 and d_2 for which H_o cannot be rejected			
Country	d_1	d_2	Test statistic
DENMARK	0.57	0.67	4.473
	0.58	0.66	5.128
	0.58	0.67	2.183
	0.59	0.66	1.124
	0.59	0.67	3.025
	0.60	**0.66**	**0.007**
	0.61	0.65	4.320
	0.61	0.66	1.004
	0.62	0.65	3.577
	0.62	0.66	3.401
	0.63	0.67	4.320
FINLAND	0.57	0.64	1.823
	0.58	0.64	2.301
	0.59	0.63	1.641
	0.60	**0.63**	**1.398**
	0.61	0.63	4.659
SWEDEN	0.56	0.64	2.415
	0.57	0.63	3.535
	0.58	**0.63**	**0.925**
NORWAY	0.57	0.66	4.118
	0.58	**0.66**	**0.880**
	0.59	0.65	4.757
	0.59	0.66	2.462
	0.60	0.65	1.629
	0.61	0.65	1.849
	0.62	0.65	4.214

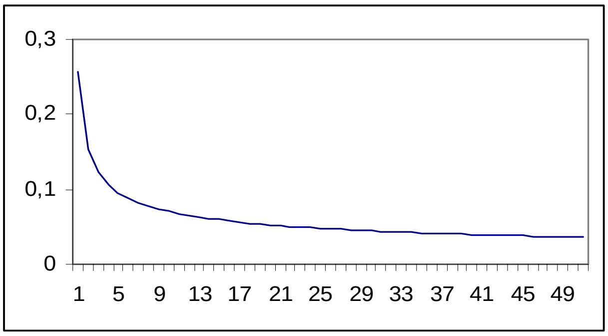
In bold the values producing the lowest statistics for each series

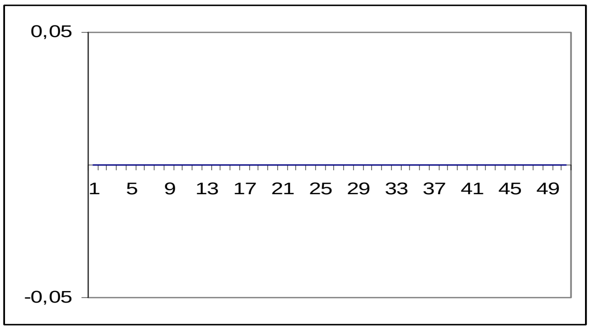
TABLE 2					
d-values corresponding to the lowest statistic and variance-covariance matrix					
	d_1	d_2	σ_{11}	σ_{12}	σ_{22}
DENMARK	0.60	0.66	0.0657	0.0514	0.0480
FINLAND	0.60	0.63	0.0657	0.0519	0.0461
SWEDEN	0.58	0.63	0.0696	0.0450	0.0316
NORWAY	0.58	0.66	0.0696	0.0464	0.035

TABLE 3						
Values of of d_1 and d_2 corresponding to the lowest statistic along with the VAR coefficients						
	d_1	d_2	α_{11}	α_{12}	α_{21}	α_{22}
DENMARK	1.31	0.64	-0.432	-0.878	0.369	1.002
FINLAND	1.22	0.73	-0.610	-0.981	0.682	1.183
SWEDEN	1.08	0.97	-0.799	-0.743	1.155	1.069
NORWAY	1.17	0.77	-0.619	-0.902	0.793	1.207

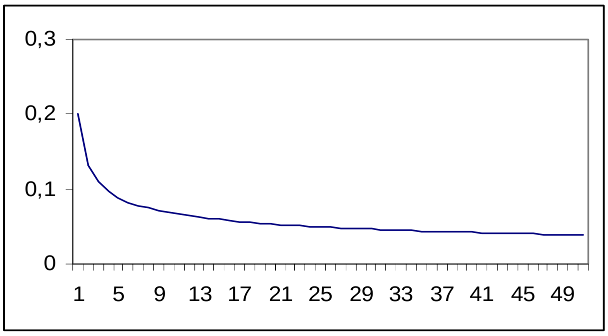
TABLE 4						
Variance-Covariance matrix coefficients						
	Differenced process			Residuals		
	σ_{11}	σ_{12}	σ_{22}	σ_{11}	σ_{12}	σ_{22}
DENMARK	0.0462	0.0260	0.0509	0.1182	-0.0049	0.0476
FINLAND	0.0440	0.0286	0.0358	0.1569	-0.0558	0.0847
SWEDEN	0.0419	0.0268	0.0194	0.176	-0.113	0.145
NORWAY	0.0430	0.0265	0.0272	0.1471	-0.0651	0.0933

FIGURE 1	
Impulse response functions in the case of DENMARK with white noise disturbances	
Effect of a 1-unit shock in u_{1t} on y_{1t}	Effect of a 1-unit shock in u_{2t} on y_{1t}





Effect of a 1-unit shock in u_{1t} on y_{2t}



Effect of a 1-unit shock in u_{2t} on y_{2t}

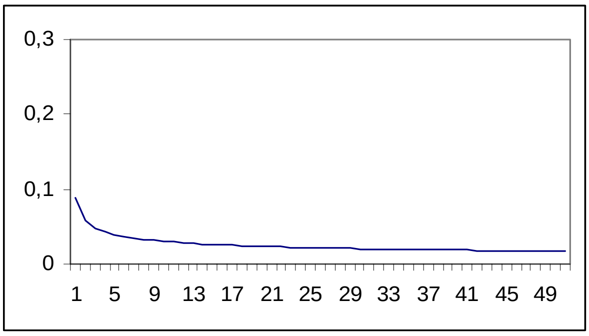
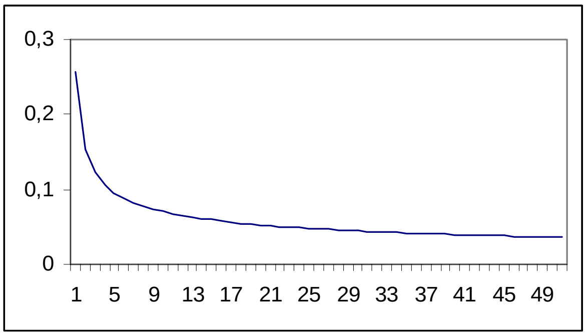
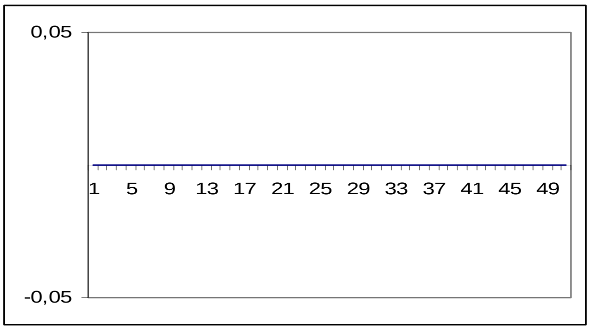
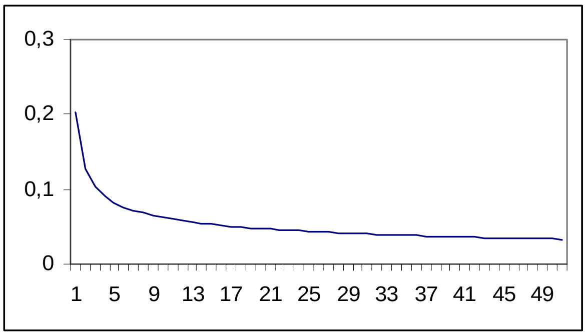


FIGURE 2	
Impulse response functions in the case of FINLAND with white noise disturbances	
Effect of a 1-unit shock in u_{1t} on y_{1t}	Effect of a 1-unit shock in u_{2t} on y_{1t}





Effect of a 1-unit shock in u_{1t} on y_{2t}



Effect of a 1-unit shock in u_{2t} on y_{2t}

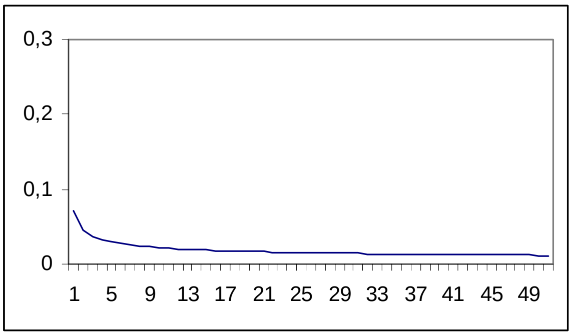
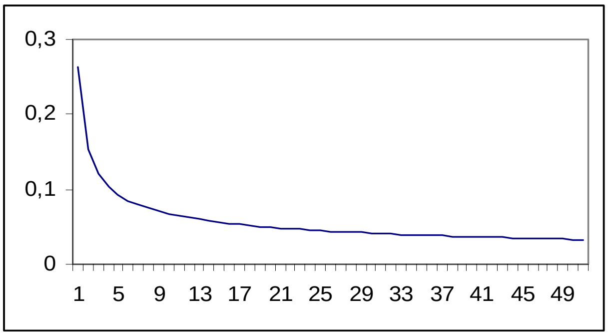


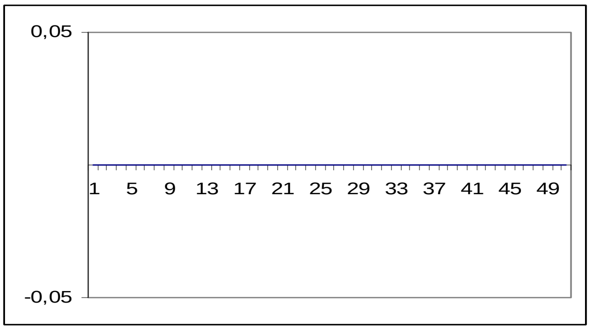
FIGURE 3

Impulse response functions in the case of SWEDEN with white noise disturbances

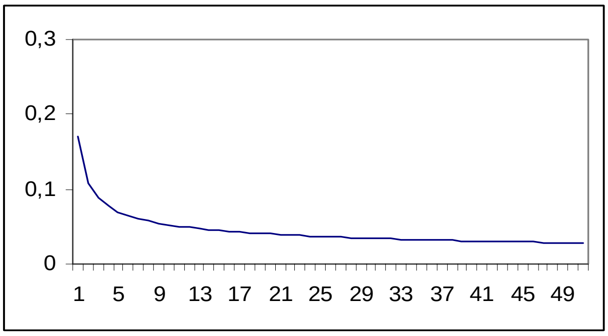
Effect of a 1-unit shock in u_{1t} on y_{1t}



Effect of a 1-unit shock in u_{2t} on y_{1t}



Effect of a 1-unit shock in u_{1t} on y_{2t}



Effect of a 1-unit shock in u_{2t} on y_{2t}

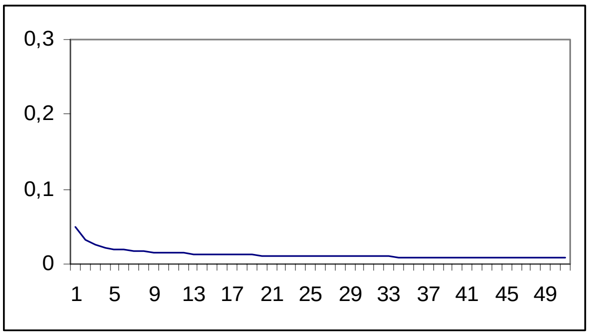
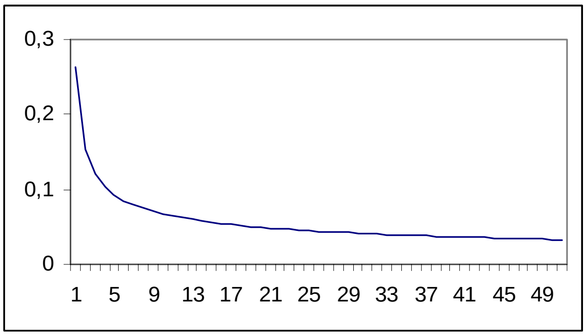


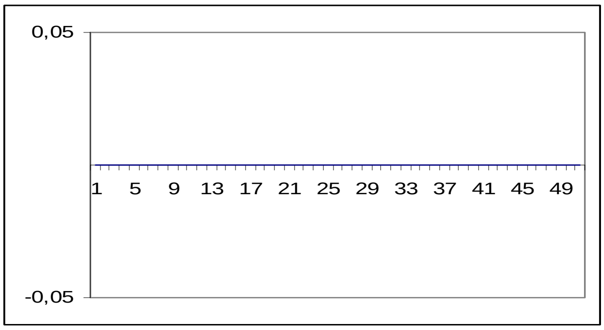
FIGURE 4

Impulse response functions in the case of NORWAY with white noise disturbances

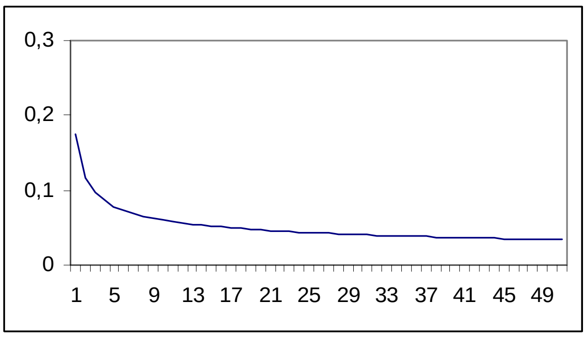
Effect of a 1-unit shock in u_{1t} on y_{1t}



Effect of a 1-unit shock in u_{2t} on y_{1t}



Effect of a 1-unit shock in u_{1t} on y_{2t}



Effect of a 1-unit shock in u_{2t} on y_{2t}