EXISTENCE AND STABILITY OF MULTIPLE SPOT SOLUTIONS FOR THE
GRAY-SCOTT MODEL IN $R^2$

JUNCHENG WEI AND MATTHIAS WINTER

Abstract. We study the Gray-Scott model in a bounded two dimensional domain and establish the existence and stability of symmetric and asymmetric multiple spotty patterns. The Green’s function and its derivatives together with two nonlocal eigenvalue problems both play a major role in the analysis. For symmetric spots, we establish a threshold behavior for stability: If a certain inequality for the parameters holds then we get stability, otherwise we get instability of multiple spot solutions. For asymmetric spots, we show that they can be stable within a narrow parameter range.

1. Introduction: Self-replicating spots

We study the existence and stability of multiple spotty patterns in the two-dimensional Gray-Scott model. The Gray-Scott system, introduced in [9], [10], models an irreversible reaction involving two reactants in a gel reactor, where the reactor is maintained in contact with a reservoir of one of the two chemicals in the reaction. In nondimensional variables, it can be written as

\begin{align*}
V_t &= D_V \Delta V - (F + k)V + UV^2, \ x \in \Omega, \ t > 0 \\
U_t &= D_U \Delta U - UV^2 + F(1 - U), \ x \in \Omega, \ t > 0 \\
\frac{\partial V}{\partial t} = \frac{\partial U}{\partial t} &= 0 \text{ on } \partial \Omega,
\end{align*}

where $D_U > 0, D_V > 0$ are the two diffusivities, $F$ denotes the feed rate, $k > 0$ is a reaction-time constant, and $\Omega \subset R^N, N \leq 3$ is the container. For various ranges of these parameters, (GS) are known to admit a rich solution structure involving pulse, spots, rings, stripes, traveling waves, pulse-replication pattern, and spatio-temporal chaos. See [21], [22], [23], [14], [15] for numerical simulations and experimental observations.

Some important analytic work is the following, first for the case of 1-D: single and multiple pulse solutions [7], stability [5], [6], stability index [4], slowly modulated two-pulse solutions [2], [3], dynamics of pulses (formal) [22], [23], skeleton structure, spatiotemporal chaos [18], dynamics [8], case of equal diffusivities [11], [12], scattering and separators [19], [20], symmetric and asymmetric patterns, Hopf bifurcation, pulse-splitting in a bounded interval [24].

In higher dimensions there are the following results: formal approach in 2-D and 3-D [16], shadow system in higher dimensions [26], ground state in 2-D [27], bounded domain case in 2-D, symmetric and asymmetric multiple spots, which is the basis for this paper [28], [29].

Let us first rescale the system (GS). Set

\[ \epsilon^2 = \frac{D_V}{F + k}, \ D = \frac{D_U}{F}, \ A = \frac{\sqrt{F}}{F + k}, \ \tau = \frac{F + k}{F}, \]

\[ \bar{x} = \frac{D_U}{F}, \ \bar{t} = \frac{1}{F + k}, \ V(x, t) = \sqrt{F}v(\bar{x}, \bar{t}), \ U(x, t) = u(\bar{x}, \bar{t}). \]

1991 Mathematics Subject Classification. Primary 35B40, 35B45; Secondary 35J40.
Key words and phrases. Pattern formation, Self-replication, Spotty solutions, Reaction-diffusion systems.
Let us drop the bar from now on. Then (GS) is equivalent to

\[
(GS1) \quad \begin{cases}
\frac{\partial v}{\partial t} = \epsilon^2 \Delta v - v + A w^2, & x \in \Omega, t > 0 \\
\tau \frac{\partial u}{\partial t} = D \Delta u - u v^2 + (1 - u), & x \in \Omega, t > 0 \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Throughout this paper, we assume that $\epsilon << 1, D = D(\epsilon) \to \infty$ as $\epsilon \to 0$, $A > 0$ may depend on $\epsilon$, $\tau > 1$ does not depend on $\epsilon$, $\Omega \subset \mathbb{R}^2$ is a bounded and smooth domain.

We define two important parameters:

\[
\eta_\epsilon = \frac{|\Omega|}{2\pi D} \log \frac{\sqrt{|\Omega|}}{\epsilon}, \quad L_\epsilon = \frac{\epsilon^2 \int_{\mathbb{R}^2} w^2 \, dy}{A^2 |\Omega|}.
\]

Let us assume that

\[
\lim_{\epsilon \to 0} L_\epsilon = L_0 \in [0, +\infty], \quad \lim_{\epsilon \to 0} \eta_\epsilon = \eta_0 \in [0, +\infty].
\]

All our results will be stated in terms of (the real constants) $L_0, \eta_0, \text{ and } \tau$.

Let $w$ be the unique solution of the following problem (ground state):

\[
(G) \quad \Delta w - w + w^2 = 0, \quad w > 0 \quad \text{in } \mathbb{R}^2, \quad w(0) = \max_{y \in \mathbb{R}^2} w(y), \quad w(y) \to 0 \quad \text{as } |y| \to +\infty.
\]

The uniqueness of the solution to (G) was proved [13]. We shall prove the existence and stability of steady-state solutions to (GS1) of the following shape:

\[
v_\epsilon \sim \sum_{j=1}^K \frac{1}{A \xi_j} w\left(\frac{x - P_j^\epsilon}{\epsilon}\right), \quad u_\epsilon(P_j^\epsilon) \sim \xi_j,
\]

where $K$ is the number of spots, $P_j^\epsilon \in \Omega, j = 1, ..., K$ is the location of the spots, $\frac{1}{A \xi_j}$, $j = 1, ..., K$ is the amplitude of the spots, $w$ is the shape of the spots.

We call the steady state “$K$ - symmetric spots” if the amplitudes of the spots are asymptotically the same in the leading order, i.e.,

\[
\lim_{\epsilon \to 0} \frac{\xi_i^\epsilon}{\xi_1^\epsilon} = 1, \quad \text{for all } i = 2, ..., N.
\]

Otherwise, we call it “$K$ - asymmetric spots”, i.e., if

\[
\lim_{\epsilon \to 0} \frac{\xi_i^\epsilon}{\xi_1^\epsilon} \neq 1, \quad \text{for some } i = 2, ..., N.
\]

2. Existence and Stability of Multiple Symmetric Spots

The result on the existence of multiple symmetric spots is the following

**Theorem 2.1.** Suppose that

\[ (T1) \quad 4(\eta_0 + K)L_0 < 1 \]

and

\[ (T2) \quad \frac{(2\eta_0 + K)^2}{\eta_0} L_0 \neq 1. \]

Then, for $\epsilon$ sufficiently small and $D$ not too large, problem (GS1) has two steady-state solutions $(v_\epsilon^+, u_\epsilon^+)$ with the following properties:

1. $v_\epsilon^+(x) = \sum_{j=1}^K \frac{1}{A \xi_j^+} (w\left(\frac{x - P_j^\epsilon}{\epsilon}\right) + o(1))$ uniformly for $x \in \bar{\Omega}$. Here

\[
\xi_j^\pm = \frac{1 \pm \sqrt{1 - 4(\eta_0 + K)L_0}}{2}.
\]
(2) \( u_\varepsilon^\pm(x) = \xi(x) \) for \( x \in \Omega \).

(3) \( P_j^\varepsilon \to P_j^0 \) as \( \varepsilon \to 0 \) for \( j = 1, ..., K \).

The locations of the spots are determined by using a Green’s function and its derivatives as follows: Define \( G_0(x, y) \) by

\[
\Delta G_0(x, y) - \frac{1}{|\Omega|} + \delta(x - y) = 0, \quad x \in \Omega, \quad \frac{\partial G_0(x, y)}{\partial y_x} = 0, \quad x \in \partial \Omega,
\]

where \( y \in \Omega \). Set \( H_0(x, y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} - G_0(x, y) \).

For \( P \in \Omega^K \) with \( P_j \neq P_l \) for \( j \neq l \) we define

\[
F_0(P_1, ..., P_K) = \sum_{i=1}^{K} H_0(P_i, P_i) - \sum_{j \neq l} G_0(P_j, P_l).
\]

Then, if \( P^0 = (P_1^0, ..., P_K^0) \) is a nondegenerate critical point of \( F_0 \), the solutions exist.

We call \((v_-^\varepsilon, u_-^\varepsilon)\) the small solution and \((v_+^\varepsilon, u_+^\varepsilon)\) the large solution.

Next we consider the stability of such solutions

**Theorem 2.2.** Assume that (T1) and (T2) hold. Let \( P^0 = (P_1^0, ..., P_K^0) \in \Omega^K \) be a nondegenerate local maximum point of \( F_0 \). Let \((v_-^\varepsilon, u_-^\varepsilon)\) be the \( K \)-spot solutions constructed in Theorem 2.1.

Then, for \( \varepsilon \) sufficiently small and \( D \) not too large, the solution \((v_+^\varepsilon, u_+^\varepsilon)\) is linearly unstable for all \( \tau \geq 0 \). For the small solutions the following holds:

**Case 1.** \( \eta_\varepsilon \to 0 \).

If \( K = 1 \), there exists a unique \( \tau_1 > 0 \) such that for \( \tau < \tau_1 \), \((v_-^\varepsilon, u_-^\varepsilon)\) is linearly stable, while for \( \tau > \tau_1 \), \((v_-^\varepsilon, u_-^\varepsilon)\) is linearly unstable.

If \( K > 1 \), \((u_-^\varepsilon, v_-^\varepsilon)\) is linearly unstable for any \( \tau \geq 0 \).

**Case 2.** \( \eta_\varepsilon \to +\infty \).

Then \((v_-^\varepsilon, u_-^\varepsilon)\) is linearly stable for any \( \tau \geq 0 \).

**Case 3.** \( \eta_\varepsilon \to \eta_0 \in (0, +\infty) \), (i.e., \( D \sim \log \frac{1}{\varepsilon} \)).

If \( L_0 < \frac{\eta_0}{(2\eta_0 + K)^2} \), then \((u_-^\varepsilon, v_-^\varepsilon)\) is linearly stable for \( \tau \) small enough or \( \tau \) large enough.

If \( K = 1 \), \( L_0 > \frac{\eta_0}{(2\eta_0 + 1)^2} \), there exist \( \tau_2 > 0, \tau_3 > 0 \) such that \((v_-^\varepsilon, v_-^\varepsilon)\) is linearly stable for \( \tau < \tau_2 \) and linearly unstable for \( \tau > \tau_3 \).

If \( K > 1 \) and \( L_0 > \frac{\eta_0}{(2\eta_0 + K)^2} \), then \((v_-^\varepsilon, u_-^\varepsilon)\) is linearly unstable for any \( \tau \geq 0 \).

3. Existence and Stability of \( K \)-Asymmetric Spots

Our first theorem shows that the asymmetric patterns exist only in a narrow parameter range.

**Theorem 3.1** Asymmetric patterns can exist only if \( \lim_{\varepsilon \to 0} \eta_\varepsilon = \eta_0 \in (0, +\infty) \). In other words, \( D \sim C \log \frac{1}{\varepsilon} \).

Our next theorem shows that the asymmetric patterns are generated by exactly two types of spots.

**Theorem 3.2** The asymmetric solutions is generated by exactly two kinds of spots, called type A and type B, respectively, which differ by their amplitudes.

The following theorem gives the existence of \( K \)-asymmetric spots.
Theorem 3.3. Fix any two integers \( k_1 \geq 1, k_2 \geq 1 \) such that \( k_1 + k_2 = K \geq 2 \). If
\[
L_0 \leq \frac{\eta_0}{4(\eta_0 + k_1)(\eta_0 + k_2)},
\]
there are asymmetric \( K \)-spotty solutions with \( k_1 \) type A spots and \( k_2 \) type B spots. The locations of these \( K \) spots are determined by a Green’s function which depends on the number \( k_1, k_2 \) of the spots.

The last theorem classifies the stability of asymmetric patterns

Theorem 3.4.

(1) (stability) The \( K \)-asymmetric spots are stable if
\[
\frac{\eta_0}{(2\eta_0 + K)^2} < L_0 \leq \frac{\eta_0}{4(\eta_0 + k_1)(\eta_0 + k_2)}
\]
and \( \tau \) small.

(2) (Instability) Assume that
\[
L_0 < \frac{\eta_0}{(2\eta_0 + K)^2}
\]
or
\[
\tau \text{ is large enough.}
\]
Then the \( K \)-asymmetric spots are unstable.

4. Main Steps in the Existence Proof

The existence of \( K \) symmetric and asymmetric spots is obtained by the Liapunov-Schmidt reduction method.

We calculate the equations for the amplitudes by solving the second equation of (GS2) with a suitable Green’s function and expanding this Green’s function (here \( G_0 \) is needed). Assuming asymptotically that
\[
\lim_{\epsilon \to 0} \xi_{\epsilon,j} = \xi_j, \quad j = 1, \ldots, K,
\]
we obtain the following system of algebraic equations
\[
1 - \xi_i - \frac{\eta_0 L_0}{\xi_i} = \sum_{j=1}^{K} \frac{L_0}{\xi_j}, \quad i = 1, \ldots, K.
\]
In the symmetric case, i.e., \( \xi_1 = \ldots = \xi_K \), we have
\[
\xi_1 = \ldots = \xi_K = \xi,
\]
where \( \xi \) satisfies the quadratic equation
\[
1 - \xi - \frac{\eta_0 L_0}{\xi} = \frac{KL_0}{\xi}.
\]
This gives two branches of solutions.

In the asymmetric case, it follows by an elementary argument that (assuming w.l.o.g. that \( \xi_2 \neq \xi_1 \)) for \( \xi_j, j = 3, \ldots, K \), we have either \( \xi_j = \xi_1 \) or \( \xi_j = \xi_2 \). This shows that asymmetric patterns are generated by exactly two types of spots.

Let \( k_1 \) be the number of \( \xi_1 \)’s in \( \{\xi_1, \ldots, \xi_K\} \) and \( k_2 \) the number of \( \xi_2 \)’s in \( \{\xi_1, \ldots, \xi_K\} \). Then \( \xi_1 \) must satisfy
\[
1 - \xi_1 = \frac{(k_1 + \eta_0)L_0}{\xi_1} + \frac{k_2}{\eta_0} \xi_1
\]
and therefore
\[
(k_2 + \eta_0)\xi_1^2 - \eta_0 \xi_1 + (k_1 + \eta_0)\eta_0 L_0 = 0.
\]
which has a solution if and only if
\[ \eta_0 \geq 4(k_1 + \eta_0)(k_2 + \eta_0)L_0. \]

Thus we have determined the amplitudes of the spots. Now we have to glue the spots together in \( \Omega \). This is be done by the Liapunov-Schmidt reduction process and a fixed point theorem argument.

5. Stability Proof: Systems of NLEPs

To study the stability of \( K \)-spots, we consider two cases: Large eigenvalues \( \lambda_\epsilon \rightarrow \lambda_0 \) and small eigenvalues \( \lambda_\epsilon \rightarrow 0 \). The small eigenvalues are related to the domain geometry via the Green’s function \( G_0 \).

The analysis of the large eigenvalues gives us the critical threshold.

By some lengthy asymptotic analysis, we arrive at the following system of nonlocal eigenvalue problems (NLEP-system):
\[
\Delta \Phi - \Phi + 2w\Phi - 2B \int_{R^2} \frac{w\Phi}{w^2} w^2 = \lambda_0 \Phi, \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_K \end{pmatrix} \in (H^2(R^2))^K,
\]
where
\[
B = L_0 \left( F + \frac{L_0}{1 + \tau \lambda_0} \right)^{-1} \left( \eta_0 I + \frac{1}{1 + \tau \lambda_0} \right),
\]
\[
F = \begin{pmatrix} \xi_1^2 + L_0 \eta_0 \\ \vdots \\ \xi_K^2 + L_0 \eta_0 \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix},
\]
and \( I \) is the identity matrix.

Diagonalizing the matrix \( B \), we are lead to the study of the following single NLEPs:
\[
\text{(NLEP)} \quad \Delta \phi - \phi + 2w\phi - 2\mu_i(\tau \lambda_0) \int_{R^2} \frac{w\phi}{w^2} w^2 = \lambda_0 \phi, \quad i = 1, 2, \phi \in H^2(R^2),
\]
where \( \mu(z) \) is an analytic function of \( z = \tau \lambda_0 \).

We need a key result about (NLEP) from [25]: if \( \tau = 0 \), then (NLEP) is stable if \( 2\mu_i(0) > 1 \). To prove instability, we first show that (NLEP) admits a positive real eigenvalue under the condition \( 2\mu_i(0) < 1 \) and then apply a compactness argument of Dancer [1] to show the original eigenvalue problem has also a positive eigenvalue provided that \( \epsilon \) is sufficiently small.

References


