

H_∞ Control for 2-D Time-Delay Systems with Randomly Occurring Nonlinearities under Sensor Saturation and Missing Measurements

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Abstract

In this paper, the H_∞ output-feedback control problem is investigated for a class of two-dimensional (2-D) nonlinear systems with time-varying delays under imperfect measurements. Randomly occurring nonlinearities (RONs) are introduced in the system to account for probabilistic nonlinear disturbances typically caused by networked environments and governed by a sequence of random variables obeying the Bernoulli distribution. The imperfect measurement outputs are subject to both data missing and randomly occurring sensor saturations (ROSSs), which are put forward to characterize the network-induced phenomena such as probabilistic communication failures and limited capacity of the communication devices. The aim of this paper is to design an output-feedback controller such that the closed-loop system is globally asymptotically stable in the mean square and the prescribed H_∞ performance index is satisfied. Sufficient conditions are presented by resorting to intensive stochastic analysis and matrix inequality techniques, which not only guarantee the existence of the desired controllers for all possible time-delays, RONs, missing measurements and ROSSs but also lead to the explicit expressions of such controllers. Finally, a numerical simulation example is given to demonstrate the applicability of the proposed control scheme.

Index Terms

Two-dimensional (2-D) systems; Output-feedback control; Sensor saturation; Randomly occurring nonlinearities (RONs); Missing measurements.

I. INTRODUCTION

Two-dimensional (2-D) systems have received tremendous research attention since they have extensive applications in image processing, seismographic data processing, thermal processes and water stream heating [4], [11], [17], [19], [24], [26], [29]. In the past decade, many important methodologies and techniques have been developed for analysis and synthesis problems of 2-D systems, which include, but are not limited to, the stability and performance analysis problems [3], [5], [10], [13], [18], [25], [31], [39], [41], robust and/or H_∞ control problems [9], [21], [34], [40], [43], robust and/or H_∞ filtering problems [4], [8], [28], as well as the H_∞ model reduction problems [12]. Since time delays frequently occur in practical systems and are often the source of instability, 2-D systems with

This work was supported in part by the National Natural Science Foundation of China under Grants 61174136, 61134009 and 61329301, the Natural Science Foundation of Jiangsu Province of China under Grant BK20130017, the “333 Project” Foundation of Jiangsu Province, the Programme for New Century Excellent Talents in University under Grant NCET-12-0117, the Royal Society of the U.K., and the Alexander von Humboldt Foundation of Germany.

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various types of delays have also been a research focus in the past few years, and a great number of results have been reported in the literature, see e.g. [3], [4], [31], [41]. Note that, in the context of 2-D systems, the stochastic perturbation issue has been taken into account in [13], [44] and the saturated nonlinearities have been studied in [3], [18], [33].

Virtually, almost all real-world systems are influenced by certain nonlinear disturbances and therefore nonlinear analysis has been a main stream of research for several decades. Traditionally, nonlinearities have been treated as a deterministic function of the system states. In today's pervasive networked environments, however, the nonlinearities may occur in a random way due probably to the random fluctuation of the network load and the unreliability of the wireless links. In other words, the nonlinearities themselves could experience random abrupt changes in their type or intensity because of abrupt phenomena such as random failures, repairs of the components as well as the changes in the interconnections of subsystems, see [7], [15], [32] for more details. Such network-induced nonlinearities are customarily referred to as randomly occurring nonlinearities (RONs), see [6], [16], [37]. Although RONs have received some initial research attention for 1-D systems, the corresponding results for 2-D systems have been scattered, and this constitutes one of the motivations for the present research.

It is worth mentioning that, in the aforementioned literature, the control and filtering synthesis approaches rely on the ideal assumption that there is a continuous flow of measurement signals with infinite precision. Unfortunately, such an assumption is not always true especially under networked environments [23], [27], [38]. For example, the sensor output often suffers from probabilistic signal missing due to multi-path fading, channel congestion, rejection in-transit, faulty networking hardware or faulty network drivers, etc. Therefore, the missing measurement (also called packet dropout or packet loss) problem has gained a growing research interest in the past few years leading to a wealth of published results. On the other hand, network-induced sensor saturations often occur randomly because of physical limitations of system components as well as the difficulties in ensuring high fidelity and timely arrival of the control and sensing signals through a possibly unreliable network of limited bandwidth. In networked control systems, the randomly occurring sensor saturation (ROSS) can be regarded as a random phenomenon in which physical entities or processes cannot, due to probabilistic fluctuations of the network loads, transmit energy and power without bounds on the magnitude or rate [35], [36]. So far, for 1-D networked control systems, important features such as control and sensing under limited capacity and missing measurement have been incorporated in the design approaches, and much attention has been drawn on the network-induced phenomena including signal quantization/saturation and stochastic loss/degradation of measurement data in the feedback loop, see [7], [32] for more details. For example, the state estimation and control problems in the case of sensor saturations and/or missing measurements have been well studied, see e.g. [2], [30], [35], [42]. However, little effort has been devoted to the corresponding control problems for 2-D systems despite their practical significance.

Summarizing the above discussion, it can be concluded that: 1) the H_∞ control problem has attracted persistently increasing research attention for the 2-D time-delay systems because of their wide applications; 2) RONs occur sometimes in networked systems and should not be overlooked in the analysis of system performances; 3) both the sensor saturations and the missing measurements may appear simultaneously during the signal transmission due to the limited bandwidth of the networks; and 4) it is of both theoretical significance and practical importance to investigate how the RONs, ROSSs and missing measurements affect the dynamic behavior of the controlled systems. It is, therefore, the main purpose of this paper to design an output-feedback controller such that, in the simultaneous presence of RONs, ROSSs and missing measurements, the closed-loop 2-D system is globally asymptotically stable in the mean square and the prescribed H_∞ performance index is satisfied. It is noticeable that such a design problem is rather challenging due to its mathematical difficulty in both system modeling and performance analysis, and this gives rise to the main motivation for our current research.

In this paper, we aim to investigate the H_∞ output-feedback control problem for a class of 2-D nonlinear systems with RONS and time-varying delays under imperfect measurements including ROSSs and missing data. We are interested in deriving sufficient conditions under which the existence of the desired controllers is guaranteed and the explicit expression of such controllers is given. *The main contribution of this paper is mainly fourfold: 1) a “comprehensive” 2-D model is proposed to describe RONS in the system states as well as sensor saturations and missing measurements in the system outputs, all of which are governed by Bernoulli distributed white sequences; 2) a combination of important factors contributing to the complexity of networked systems are investigated within an unified framework that caters for RONS, ROSSs and missing measurements; 3) a new energy-like quadratic function is employed to analyze the system stability and performance; and 4) intensive stochastic analysis is conducted to enforce the H_∞ performance for the addressed comprehensive systems in addition to the stochastic stability constraint.*

The remaining part of this paper is organized as follows. In Section II, the H_∞ output-feedback control problem is formulated for the 2-D time-delay systems with RONS, ROSS and missing measurements, and some preliminaries are briefly outlined. In Section III, the global asymptotic stability in the mean square for the closed-loop system is analyzed, the H_∞ performance level is investigated, and the output-feedback controller is also explicitly designed. In Section IV, an illustrative example is provided to verify the effectiveness of the designed control scheme. Finally, conclusions are drawn in Section V.

Notations. The notations used throughout this paper are fairly standard except where otherwise stated. \mathbb{N} is used to be the set $\{0, 1, 2, \dots\}$. \mathbb{R}^n and $\mathbb{R}^{m \times n}$ denote, respectively, the n -dimensional Euclidean space and the set of all $m \times n$ real matrices. I and 0 represent the identity matrix and the zero matrix with appropriate dimensions, respectively. The notation $X \geq 0$ (respectively, $X > 0$) means that matrix X is real, symmetric and positive semidefinite (respectively, positive definite). $\text{diag}(\dots)$ stands for the block-diagonal matrix with blocks given by the matrices in (\dots) . For matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$, their Kronecker product is a matrix in $\mathbb{R}^{mp \times nq}$ and denoted as $A \otimes B$. The superscript “T” is used to represent the matrix transposition, and “*” in a matrix stands for the term which is induced by symmetry. For integers m and n with $m \leq n$, $[m, n]$ denotes the integers set $\{m, m+1, \dots, n\}$ and $\llbracket m, n \rrbracket$ means the integers set $\{m, m+1, \dots, n-1\}$. $(\Omega, \mathcal{F}, \text{Prob})$ is a complete probability space, where the probability measure Prob has total mass 1. $\mathbb{E}\{\alpha\}$ and $\mathbb{E}\{\alpha|\beta\}$ represent, respectively, the mathematical expectation of the stochastic variable α and the expectation of α conditional on β with respect to the given probability measure Prob . For $v \in l_2(\mathbb{N} \times \mathbb{N})$, similar as in [8], define its norm $\|v\|_2^2 = \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \mathbb{E}\{\|v(k, h)\|^2\} - \frac{1}{2} \sum_{k=0}^{\infty} \mathbb{E}\{\|v(k, 0)\|^2\} - \frac{1}{2} \sum_{h=0}^{\infty} \mathbb{E}\{\|v(0, h)\|^2\}$ where $\|\cdot\|$ refers to the Euclidean vector norm. Matrices, if not stated, are assumed to have compatible dimensions for algebraic operations.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider a 2-D system along two directions described by the general Fornasini-Marchesini state-space model of the following form:

$$\left\{ \begin{array}{l} x(k+1, h+1) = A_1 x(k+1, h) + A_2 x(k, h+1) + D_1 x(k+1, h-\tau(h)) + D_2 x(k-\sigma(k), h+1) \\ \quad + \hat{\gamma}(k, h) G f(x(k+1, h), x(k, h+1)) + B_1 u(k+1, h) + B_2 u(k, h+1) \\ \quad + E_1 v(k+1, h) + E_2 v(k, h+1), \\ y(k, h) = \Xi(k, h) \Lambda(k, h) C x(k, h) + (I - \Lambda(k, h)) g(C x(k, h)) + M_1 v(k, h), \\ z(k, h) = W_1 x(k, h) + W_2 u(k, h) + M_2 v(k, h) \end{array} \right. \quad (1)$$

where $k, h \in \mathbb{N}$, $x(k, h) \in \mathbb{R}^n$ is the state vector, $y(k, h) \in \mathbb{R}^m$ is the measured output vector and $z(k, h) \in \mathbb{R}^r$ is the signal to be controlled. $u(k, h) \in \mathbb{R}^q$ is the control input vector, $v(k, h) \in \mathbb{R}^p$ is the exogenous disturbance

input which belongs to $l_2(\mathbb{N} \times \mathbb{N})$. $A_i, D_i, B_i, E_i, M_i, W_i, C$ and G ($i = 1, 2$) are known system matrices with compatible dimensions. $\sigma(k)$ and $\tau(h)$ are time-varying positive scalars denoting the delays, respectively, along the horizontal direction and along the vertical direction, which satisfy

$$\sigma_1 \leq \sigma(k) \leq \sigma_2, \quad \tau_1 \leq \tau(h) \leq \tau_2, \quad \forall k, h \in \mathbb{N} \quad (2)$$

where σ_i and τ_i ($i = 1, 2$) are positive known integers denoting, respectively, the lower and upper bounds of the time-varying delays. $\hat{\gamma}(k, h) \in \mathbb{R}$ is a Bernoulli distributed white sequence accounting for the phenomena of randomly occurring nonlinearities and taking values of either 1 or 0 with

$$\text{Prob}\{\hat{\gamma}(k, h) = 1\} = \bar{\gamma}, \quad \text{Prob}\{\hat{\gamma}(k, h) = 0\} = 1 - \bar{\gamma}, \quad (3)$$

where $\bar{\gamma} \in [0, 1]$ is a known constant. Obviously, for all $k, h \in \mathbb{N}$, the stochastic variable $\hat{\gamma}(k, h)$ has the variance $\bar{\gamma}(1 - \bar{\gamma})$. $f(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a known nonlinear function satisfying $f(0, 0) = 0$ and the following inequality

$$(f(u, v) - F_1\varsigma)^T (f(u, v) - F_2\varsigma) \leq 0 \quad (4)$$

where $u, v \in \mathbb{R}^n$, $\varsigma = (u^T \ v^T)^T$, $F_1 = [F_{11} \ F_{12}]$ and $F_2 = [F_{21} \ F_{22}] \in \mathbb{R}^{n \times 2n}$.

The saturation function $g(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ has the following form:

$$g(w) = \left[\begin{array}{cccc} g_1(w_1) & g_2(w_2) & \cdots & g_m(w_m) \end{array} \right]^T \quad (5)$$

with $w = (w_1, w_2, \dots, w_m)^T \in \mathbb{R}^m$ and, for $i = 1, 2, \dots, m$, $g_i(w_i) = \text{sign}(w_i) \min\{|w_i|, w_{i,\max}\}$ where $w_{i,\max}$ is the i th element of the saturation level vector w_{\max} .

In system (1), $\Xi(k, h) = \text{diag}(\xi_1(k, h), \xi_2(k, h), \dots, \xi_m(k, h))$ and $\xi_i(k, h)$ ($i = 1, 2, \dots, m$) are mutually independent scalar random signals on the probability space $(\Omega, \mathcal{F}, \text{Prob})$ taking values on the interval $[0, 1]$ and satisfying

$$\mathbb{E}\{\xi_i(k, h)\} = \bar{\xi}_i, \quad \mathbb{E}\{\xi_i^2(k, h)\} = \sigma_i^2. \quad (6)$$

$\Lambda(k, h) = \text{diag}(\lambda_1(k, h), \lambda_2(k, h), \dots, \lambda_m(k, h))$ and $\lambda_i(k, h)$ ($i = 1, 2, \dots, m$) are Bernoulli distributed white sequences taking values on 0 and 1 with

$$\text{Prob}\{\lambda_i(k, h) = 1\} = \bar{\lambda}_i, \quad \text{Prob}\{\lambda_i(k, h) = 0\} = 1 - \bar{\lambda}_i, \quad (7)$$

where $i = 1, 2, \dots, m$; $k, h \in \mathbb{N}$ and $\bar{\lambda}_i \in [0, 1]$ is known. It is further assumed that, in this paper, $\hat{\gamma}(k, h)$, $\xi_i(k, h)$ and $\lambda_i(k, h)$ ($i = 1, 2, \dots, m$) are mutually independent.

Remark 1: In reality, the RONS, the missing measurements and the ROSS are three main important issues that have been investigated extensively for various systems such as networked control systems, sensor networks, power grid networks and coupled mechanical systems. Random abrupt changes in the environmental circumstances result in the nonlinear disturbances occurring in a probabilistic way. In system (1), the random variable $\hat{\gamma}(k, h)$ is introduced to regulate the nonlinear influence $f(x(k+1, h), x(k, h+1))$ on the structure and dynamics of the 2-D system. Such kind of phenomenon has been named as RONS in [37] and has drawn some attention ever since then, see e.g. [6]. Moreover, due to the physical or technical limitations of the system components, the sensor measurement cannot provide unlimited amplitude signals, and hence the random diagonal matrix $\Lambda(k, h)$ is used in (1) to account for the sensor saturation case which might occur randomly [35], [36]. On the other hand, missing measurements are ubiquitous due to the limited bandwidth of the channels for signal transmission or the sensor aging/temporal failure in the sensor networks. In model (1), we use random diagonal matrix $\Xi(k, h)$ to characterize such unavoidable phenomena. It should be noted that the aforementioned RONS, missing measurements and ROSS

have been frequently considered for the 1-D (one-dimensional) systems. When it comes to the 2-D dynamical systems, the related results have been very few.

The initial boundary condition associated with the discrete 2-D system (1) is given by

$$x(k, h) = \begin{cases} \varphi(k, h), & k \in [-\sigma_2, 0], h \in [0, \kappa_1] \\ 0, & k \in [-\sigma_2, 0], h \in [\kappa_1 + 1, \infty) \\ \phi(k, h), & k \in [0, \kappa_2], h \in [-\tau_2, 0] \\ 0 & k \in [\kappa_2 + 1, \infty), h \in [-\tau_2, 0] \end{cases} \quad (8)$$

with $\varphi(0, 0) = \phi(0, 0)$, where κ_1 and κ_2 are two finite positive integers, $\varphi(k, h)$ and $\phi(k, h)$ are known vectors belonging to \mathbb{R}^n with finite norm.

Before presenting the main aim of this paper, we introduce the following definitions for the 2-D system (1) with initial condition (8) which are illuminated by the ideas in [13], [26].

Definition 1: For the unforced system (1) (i.e., $u(k, h) \equiv 0$ in (1)) and every initial boundary condition in (8), the trivial solution of (1) is said to be globally asymptotically stable in the mean square if, in the case of $v(k, h) \equiv 0$, the trivial solution of (1) is stable in the mean square (in the sense of Lyapunov) and the following equality holds:

$$\lim_{k+h \rightarrow \infty} \mathbb{E}\{\|x(k, h)\|\} = 0.$$

Definition 2: For the given scalar $\gamma > 0$, the discrete 2-D system (1) is said to be globally asymptotically stable in the mean square with an H_∞ disturbance attenuation level γ if it is globally asymptotically stable in the mean square, and under zero-initial condition, i.e., $\phi(k, h) \equiv 0 \equiv \varphi(k, h)$, the controlled output $z(k, h)$ satisfies

$$\|z\|_2 \leq \gamma \|v\|_2$$

for all nonzero $v \in l_2(\mathbb{N} \times \mathbb{N}, \mathbb{R}^p)$.

In this paper, the following output-feedback controller is adopted:

$$\begin{cases} \hat{x}(k+1, h+1) = A_{1f}\hat{x}(k+1, h) + A_{2f}\hat{x}(k, h+1) + K_{1f}y(k+1, h) + K_{2f}y(k, h+1), \\ u(k, h) = H_f\hat{x}(k, h) \end{cases} \quad (9)$$

where $\hat{x}(k, h) \in \mathbb{R}^n$ is the state of the controller, A_{if} , K_{if} and H_f ($i = 1, 2$) are the controller parameters to be designed. It is assumed that the initial boundary condition for (9) is taken to be $\hat{x}(0, h) = 0 = \hat{x}(k, 0)$ for $k, h \in [0, \infty)$.

By letting $\eta(k, h) = (x^T(k, h), \hat{x}^T(k, h))^T$ and substituting (9) into (1), we get the augmented closed-loop system as follows:

$$\begin{cases} \eta(k+1, h+1) = (\mathcal{A}_1 + \Delta\mathcal{A}_1(k, h))\eta(k+1, h) + (\mathcal{A}_2 + \Delta\mathcal{A}_2(k, h))\eta(k, h+1) \\ \quad + \mathcal{D}_1L\eta(k+1, h - \tau(h)) + \mathcal{D}_2L\eta(k - \sigma(k), h+1) \\ \quad + (\mathcal{G} + \Delta\mathcal{G}(k, h))\mathcal{F}(\eta(k+1, h), \eta(k, h+1)) + \mathcal{E}_1v(k+1, h) + \mathcal{E}_2v(k, h+1), \\ z(k, h) = \mathcal{W}\eta(k, h) + M_2v(k, h) \end{cases} \quad (10)$$

where $\bar{\Xi} = \text{diag}(\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_m)$, $\bar{\Lambda} = \text{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_m)$,

$$A_1 = \begin{bmatrix} A_1 & B_1H_f \\ K_{1f}\bar{\Xi}\bar{\Lambda}C & A_{1f} \end{bmatrix}, \quad \Delta A_1(k, h) = \begin{bmatrix} 0 & 0 \\ K_{1f}(\Xi(k+1, h)\Lambda(k+1, h) - \bar{\Xi}\bar{\Lambda})C & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} A_2 & B_2H_f \\ K_{2f}\bar{\Xi}\bar{\Lambda}C & A_{2f} \end{bmatrix}, \quad \Delta A_2(k, h) = \begin{bmatrix} 0 & 0 \\ K_{2f}(\Xi(k, h+1)\Lambda(k, h+1) - \bar{\Xi}\bar{\Lambda})C & 0 \end{bmatrix},$$

$$\begin{aligned}
\mathcal{D}_1 &= \begin{bmatrix} D_1 \\ 0 \end{bmatrix}, & \mathcal{D}_2 &= \begin{bmatrix} D_2 \\ 0 \end{bmatrix}, & \mathcal{E}_1 &= \begin{bmatrix} E_1 \\ K_{1f}M_1 \end{bmatrix}, & \mathcal{E}_2 &= \begin{bmatrix} E_2 \\ K_{2f}M_1 \end{bmatrix}, \\
\mathcal{G} &= \begin{bmatrix} \bar{\gamma}G & 0 & 0 \\ 0 & K_{1f}(I - \bar{\Lambda}) & K_{2f}(I - \bar{\Lambda}) \end{bmatrix}, & L &= \begin{bmatrix} I & 0 \end{bmatrix}, \\
\Delta\mathcal{G}(k, h) &= \begin{bmatrix} (\hat{\gamma}(k, h) - \bar{\gamma})G & 0 & 0 \\ 0 & K_{1f}(\bar{\Lambda} - \Lambda(k+1, h)) & K_{2f}(\bar{\Lambda} - \Lambda(k, h+1)) \end{bmatrix}, \\
\mathcal{F}(\eta(k+1, h), \eta(k, h+1)) &= \begin{bmatrix} f(L\eta(k+1, h), L\eta(k, h+1)) \\ g(CL\eta(k+1, h)) \\ g(CL\eta(k, h+1)) \end{bmatrix}, & \mathcal{W} &= \begin{bmatrix} W_1 & W_2H_f \end{bmatrix}.
\end{aligned}$$

The main objective of this paper is to design an output-feedback controller in the form of (9) for the discrete 2-D time-delay system (1) such that the closed-loop 2-D system (10) is globally asymptotically stable in the mean square with a prescribed H_∞ disturbance attenuation level γ .

Remark 2: The output-feedback control problem has been extensively investigated in the literature for 1-D systems. Compared with the rich literature for the output-feedback control of 1-D systems, the corresponding results for the 2-D systems are relatively few [39]–[41]. On the other hand, most of the existing output-feedback control results have been established for the *linear* 2-D systems only. When referring to the case with stochastic disturbances such as RONs, missing measurements and ROSS, the corresponding research problem remains *unsolved*.

III. ANALYSIS AND SYNTHESIS FOR THE 2-D TIME-DELAY SYSTEM

In this section, the H_∞ output-feedback control problem formulated in the previous section is to be investigated. First, by employing an energy-like functional and some intensive stochastic analysis, the stability and H_∞ performance issues are discussed. Then, the controller synthesis problem is considered and two design schemes are given ensuring the closed-loop 2-D system (10) to be globally asymptotically stable in the mean square with a prescribed H_∞ disturbance attenuation level γ .

As for the saturation function $g(\cdot)$ in (5), with the similar techniques employed in [20], [36], [42], it is assumed that there exists a certain diagonal matrix S such that $0 < S < I$ and the following sector condition holds:

$$(g(Cx(k, h)) - Cx(k, h))^T (g(Cx(k, h)) - SCx(k, h)) \leq 0. \quad (11)$$

In the following discussion, for simplicity, the first equation in the closed-loop system (10) can be written as

$$\begin{aligned}
\eta(k+1, h+1) &= \mathcal{Y}(k, h) + \Delta\mathcal{A}_1(k, h)\eta(k+1, h) + \Delta\mathcal{A}_2(k, h)\eta(k, h+1) \\
&\quad + \Delta\mathcal{G}(k, h)\mathcal{F}(\eta(k+1, h), \eta(k, h+1)) + \mathcal{E}_1v(k+1, h) + \mathcal{E}_2v(k, h+1)
\end{aligned} \quad (12)$$

where $\mathcal{Y}(k, h) = \mathcal{A}\zeta(k, h)$ and $\mathcal{A} = [\mathcal{A}_1 \ \mathcal{A}_2 \ \mathcal{D}_1L \ \mathcal{D}_2L \ \mathcal{G}]$, $\zeta(k, h) = (\eta^T(k+1, h) \ \eta^T(k, h+1) \ \eta^T(k+1, h-\tau(h)) \ \eta^T(k-\sigma(k), h+1) \ \mathcal{F}^T(\eta(k+1, h), \eta(k, h+1)))^T$.

Theorem 1: Let the output-feedback controller parameters A_{if} , K_{if} , H_f ($i = 1, 2$) and the H_∞ performance level $\gamma > 0$ be given. Then, the closed-loop 2-D system (10) is globally asymptotically stable in the mean square with disturbance attenuation level γ if there exist matrices $\mathcal{P}_i > 0$, $\mathcal{Q}_i > 0$ ($i = 1, 2$) and positive scalars ε_j ($j = 1, 2, 3$) such that the following matrix inequality holds:

$$\begin{aligned}
\tilde{\Phi} &:= \Phi + \mathcal{G}(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{G}^T + \mathcal{A}\mathcal{E}^T(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{A}\mathcal{E} \\
&\quad + \sum_{j=1}^2 \left(\mathcal{W}_j\mathcal{W}_j^T + \sum_{i=1}^m \left(\iota_{1i}\mathcal{C}_{ji}(\mathcal{P}_1 + \mathcal{P}_2)^{-1}\mathcal{C}_{ji}^T + \iota_{4i}\hat{\mathcal{C}}_{ji}(\mathcal{P}_1 + \mathcal{P}_2)^{-1}\hat{\mathcal{C}}_{ji}^T \right) \right) < 0,
\end{aligned} \quad (13)$$

where $\iota_{1i} = \bar{\lambda}_i(1 - \bar{\lambda}_i)$, $\iota_{4i} = \bar{\lambda}_i(\sigma_i^2 - \bar{\xi}_i^2)$, $\mathcal{A}_\mathcal{E} = [\mathcal{A} \ \mathcal{E}_1 \ \mathcal{E}_2]$, $\mathcal{G} = [\tilde{\mathcal{G}}^T \ 0 \ 0]^T$ with $\tilde{\mathcal{G}} = [0 \ 0 \ 0 \ 0 \ \mathbf{G}]^T$,

$$\begin{aligned} \mathcal{W}_1 &= \begin{bmatrix} \mathcal{W} & 0 & 0 & 0 & 0 & M_2 & 0 \end{bmatrix}^T, & \mathcal{W}_2 &= \begin{bmatrix} 0 & \mathcal{W} & 0 & 0 & 0 & 0 & M_2 \end{bmatrix}^T, \\ \mathcal{C}_{1i} &= \begin{bmatrix} \tilde{\mathcal{C}}_{1i}^T & 0 & 0 \end{bmatrix}^T, & \tilde{\mathcal{C}}_{1i} &= \begin{bmatrix} \bar{\xi}_i(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{C}_{1i}L & 0 & 0 & 0 & -(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{K}_{1i} \end{bmatrix}^T, \\ \mathcal{C}_{2i} &= \begin{bmatrix} \tilde{\mathcal{C}}_{2i}^T & 0 & 0 \end{bmatrix}^T, & \tilde{\mathcal{C}}_{2i} &= \begin{bmatrix} 0 & \bar{\xi}_i(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{C}_{2i}L & 0 & 0 & -(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{K}_{2i} \end{bmatrix}^T, \\ \hat{\mathcal{C}}_{1i} &= \begin{bmatrix} \vec{\mathcal{C}}_{1i}^T & 0 & 0 \end{bmatrix}^T, & \vec{\mathcal{C}}_{1i} &= \begin{bmatrix} (\mathcal{P}_1 + \mathcal{P}_2)\mathcal{C}_{1i}L & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\ \hat{\mathcal{C}}_{2i} &= \begin{bmatrix} \vec{\mathcal{C}}_{2i}^T & 0 & 0 \end{bmatrix}^T, & \vec{\mathcal{C}}_{2i} &= \begin{bmatrix} 0 & (\mathcal{P}_1 + \mathcal{P}_2)\mathcal{C}_{2i}L & 0 & 0 & 0 \end{bmatrix}^T, \\ \Phi &= \begin{bmatrix} \hat{\Phi} & 0 & 0 \\ * & -\gamma^2 I & 0 \\ * & * & -\gamma^2 I \end{bmatrix}, & \hat{\Phi} &= \begin{bmatrix} \Phi_{11} & \Phi_{12} & 0 & 0 & \Phi_{15} \\ * & \Phi_{22} & 0 & 0 & \Phi_{25} \\ * & * & -\mathcal{Q}_1 & 0 & 0 \\ * & * & * & -\mathcal{Q}_2 & 0 \\ * & * & * & * & \Phi_{55} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \Phi_{11} &= (\tau_2 - \tau_1 + 1)\mathcal{Q}_1 - \mathcal{P}_1 - \frac{\varepsilon_1}{2}L^T(F_{11}^T F_{21} + F_{21}^T F_{11})L - \varepsilon_2 L^T C^T S C L, \\ \Phi_{22} &= (\sigma_2 - \sigma_1 + 1)\mathcal{Q}_2 - \mathcal{P}_2 - \frac{\varepsilon_1}{2}L^T(F_{12}^T F_{22} + F_{22}^T F_{12})L - \varepsilon_3 L^T C^T S C L, \\ \Phi_{12} &= -\frac{\varepsilon_1}{2}L^T(F_{11}^T F_{22} + F_{21}^T F_{12})L, & \Phi_{15} &= \frac{\varepsilon_1}{2}L^T(F_{11}^T + F_{21}^T)\mathcal{I}_1 + \frac{\varepsilon_2}{2}L^T C^T (I + S)\mathcal{I}_2, \\ \Phi_{25} &= \frac{\varepsilon_1}{2}L^T(F_{12}^T + F_{22}^T)\mathcal{I}_1 + \frac{\varepsilon_3}{2}L^T C^T (I + S)\mathcal{I}_3, & \Phi_{55} &= -\varepsilon_1 \mathcal{I}_1^T \mathcal{I}_1 - \varepsilon_2 \mathcal{I}_2^T \mathcal{I}_2 - \varepsilon_3 \mathcal{I}_3^T \mathcal{I}_3, \\ \mathcal{C}_{1i} &= \mathbf{K}_1 \bar{E}_i C, & \mathcal{K}_{1i} &= \mathbf{K}_1 \mathcal{E}_{2i}, & \mathcal{C}_{2i} &= \mathbf{K}_2 \bar{E}_i C, & \mathcal{K}_{2i} &= \mathbf{K}_2 \mathcal{E}_{1i}, & \mathcal{E}_{1i} &= [0 \ 0 \ \bar{E}_i], & \mathcal{E}_{2i} &= [0 \ \bar{E}_i \ 0], \\ \mathbf{K}_1 &= \begin{bmatrix} 0 \\ K_{1f} \end{bmatrix}, & \mathbf{K}_2 &= \begin{bmatrix} 0 \\ K_{2f} \end{bmatrix}, & \mathbf{G} &= \sqrt{\bar{\gamma}(1 - \bar{\gamma})} \begin{bmatrix} G & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \end{aligned}$$

in which $\mathcal{I}_1 = [I \ 0 \ 0]$, $\mathcal{I}_2 = [0 \ I \ 0]$, $\mathcal{I}_3 = [0 \ 0 \ I]$ and \bar{E}_i is the matrix in $\mathbb{R}^{m \times m}$ with only the diagonal (i, i) -component as 1 and all the other elements as 0.

Proof: Consider the following energy-like functional

$$V(k, h) = V_1(k, h) + V_2(k, h), \quad (14)$$

where $V_1(k, h) = \sum_{i=1}^3 V_{1i}(k, h)$ and $V_2(k, h) = \sum_{j=1}^3 V_{2j}(k, h)$ with

$$\begin{aligned} V_{11}(k, h) &= \eta^T(k, h)\mathcal{P}_1\eta(k, h), & V_{12}(k, h) &= \sum_{i=h-\tau(h)}^{h-1} \eta^T(k, i)\mathcal{Q}_1\eta(k, i), \\ V_{13}(k, h) &= \sum_{j=1-\tau_2}^{-\tau_1} \sum_{i=h+j}^{h-1} \eta^T(k, i)\mathcal{Q}_1\eta(k, i); & V_{21}(k, h) &= \eta^T(k, h)\mathcal{P}_2\eta(k, h), \\ V_{22}(k, h) &= \sum_{i=k-\sigma(k)}^{k-1} \eta^T(i, h)\mathcal{Q}_2\eta(i, h), & V_{23}(k, h) &= \sum_{j=1-\sigma_2}^{-\sigma_1} \sum_{i=k+j}^{k-1} \eta^T(i, h)\mathcal{Q}_2\eta(i, h); \end{aligned}$$

where $k, h \in \mathbb{N}$, matrices $\mathcal{P}_i > 0$ and $\mathcal{Q}_i > 0$ ($i = 1, 2$) are to be determined from the matrix inequality (13).

Denote $\aleph(k, h) = \{\eta(k+1, h), \eta(k+1, h-1), \dots, \eta(k+1, h-\tau_2), \eta(k, h+1), \eta(k-1, h+1), \dots, \eta(k-\sigma_2, h+1)\}$, and define the index as follows:

$$\mathcal{I} := \mathbb{E} \left\{ \left(\sum_{i=1}^3 \Delta V_{1i}(k, h) + \sum_{j=1}^3 \Delta V_{2j}(k, h) \right) | \aleph(k, h) \right\} \quad (15)$$

with $\Delta V_{1i}(k, h) = V_{1i}(k+1, h+1) - V_{1i}(k+1, h)$ and $\Delta V_{2j}(k, h) = V_{2j}(k+1, h+1) - V_{2j}(k, h+1)$. Calculating (15) along the solutions of the closed-loop system (12), we can obtain

$$\mathbb{E}\{\Delta V_{11}(k, h)|\aleph(k, h)\} = \mathbb{E}\{\eta^T(k+1, h+1)\mathcal{P}_1\eta(k+1, h+1)|\aleph(k, h)\} - \eta^T(k+1, h)\mathcal{P}_1\eta(k+1, h), \quad (16)$$

$$\begin{aligned} \mathbb{E}\{\Delta V_{12}(k, h)|\aleph(k, h)\} &= \mathbb{E}\left\{\left(\sum_{i=h+1-\tau(h+1)}^h - \sum_{i=h-\tau(h)}^{h-1}\right)\eta^T(k+1, i)\mathcal{Q}_1\eta(k+1, i)|\aleph(k, h)\right\} \\ &= \mathbb{E}\left\{\left(\eta^T(k+1, h)\mathcal{Q}_1\eta(k+1, h) - \eta^T(k+1, h-\tau(h))\mathcal{Q}_1\eta(k+1, h-\tau(h))\right.\right. \\ &\quad \left.\left.+ \left(\sum_{i=h+1-\tau_1}^{h-1} + \sum_{i=h+1-\tau(h+1)}^{h-\tau_1} - \sum_{i=h+1-\tau(h)}^{h-1}\right)\eta^T(k+1, i)\mathcal{Q}_1\eta(k+1, i)\right)|\aleph(k, h)\right\} \\ &\leq \mathbb{E}\left\{\left(\eta^T(k+1, h)\mathcal{Q}_1\eta(k+1, h) - \eta^T(k+1, h-\tau(h))\mathcal{Q}_1\eta(k+1, h-\tau(h))\right.\right. \\ &\quad \left.\left.+ \sum_{i=h+1-\tau(h+1)}^{h-\tau_1} \eta^T(k+1, i)\mathcal{Q}_1\eta(k+1, i)\right)|\aleph(k, h)\right\} \\ &\leq \mathbb{E}\left\{\left(\eta^T(k+1, h)\mathcal{Q}_1\eta(k+1, h) - \eta^T(k+1, h-\tau(h))\mathcal{Q}_1\eta(k+1, h-\tau(h))\right.\right. \\ &\quad \left.\left.+ \sum_{i=h+1-\tau_2}^{h-\tau_1} \eta^T(k+1, i)\mathcal{Q}_1\eta(k+1, i)\right)|\aleph(k, h)\right\}, \end{aligned} \quad (17)$$

$$\begin{aligned} \mathbb{E}\{\Delta V_{13}(k, h)|\aleph(k, h)\} &= \mathbb{E}\left\{\sum_{j=1-\tau_2}^{-\tau_1} \left(\sum_{i=h+1+j}^h - \sum_{i=h+j}^{h-1}\right)\eta^T(k+1, i)\mathcal{Q}_1\eta(k+1, i)|\aleph(k, h)\right\} \\ &= \mathbb{E}\left\{\sum_{j=1-\tau_2}^{-\tau_1} \left(\eta^T(k+1, h)\mathcal{Q}_1\eta(k+1, h) - \eta^T(k+1, h+j)\mathcal{Q}_1\eta(k+1, h+j)\right)|\aleph(k, h)\right\} \\ &= \mathbb{E}\left\{\left((\tau_2 - \tau_1)\eta^T(k+1, h)\mathcal{Q}_1\eta(k+1, h)\right.\right. \\ &\quad \left.\left.- \sum_{j=1-\tau_2}^{-\tau_1} \eta^T(k+1, h+j)\mathcal{Q}_1\eta(k+1, h+j)\right)|\aleph(k, h)\right\} \end{aligned} \quad (18)$$

and

$$\mathbb{E}\{\Delta V_{21}(k, h)|\aleph(k, h)\} = \mathbb{E}\{\eta^T(k+1, h+1)\mathcal{P}_2\eta(k+1, h+1)|\aleph(k, h)\} - \eta^T(k, h+1)\mathcal{P}_2\eta(k, h+1), \quad (19)$$

$$\begin{aligned} \mathbb{E}\{\Delta V_{22}(k, h)|\aleph(k, h)\} &\leq \mathbb{E}\left\{\left(\eta^T(k, h+1)\mathcal{Q}_2\eta(k, h+1) - \eta^T(k-\sigma(k), h+1)\mathcal{Q}_2\eta(k-\sigma(k), h+1)\right.\right. \\ &\quad \left.\left.+ \sum_{i=k+1-\sigma_2}^{k-\sigma_1} \eta^T(i, h+1)\mathcal{Q}_2\eta(i, h+1)\right)|\aleph(k, h)\right\}, \end{aligned} \quad (20)$$

$$\begin{aligned} \mathbb{E}\{\Delta V_{23}(k, h)|\aleph(k, h)\} &= \mathbb{E}\left\{\left((\sigma_2 - \sigma_1)\eta^T(k, h+1)\mathcal{Q}_2\eta(k, h+1)\right.\right. \\ &\quad \left.\left.- \sum_{j=1-\sigma_2}^{-\sigma_1} \eta^T(k+j, h+1)\mathcal{Q}_2\eta(k+j, h+1)\right)|\aleph(k, h)\right\}. \end{aligned} \quad (21)$$

Substituting (16)-(21) into (15), one has

$$\begin{aligned} \mathcal{J} &\leq \mathbb{E}\left\{\left(\eta^T(k+1, h+1)(\mathcal{P}_1 + \mathcal{P}_2)\eta(k+1, h+1) + \eta^T(k+1, h)((\tau_2 - \tau_1 + 1)\mathcal{Q}_1 - \mathcal{P}_1)\eta(k+1, h)\right.\right. \\ &\quad \left.- \eta^T(k+1, h-\tau(h))\mathcal{Q}_1\eta(k+1, h-\tau(h)) + \eta^T(k, h+1)((\sigma_2 - \sigma_1 + 1)\mathcal{Q}_2 - \mathcal{P}_2)\eta(k, h+1)\right. \\ &\quad \left.- \eta^T(k-\sigma(k), h+1)\mathcal{Q}_2\eta(k-\sigma(k), h+1)\right)|\aleph(k, h)\right\}. \end{aligned} \quad (22)$$

In the following, we first prove the global asymptotic stability in the mean square of the closed-loop 2-D system (10) with $v(k, h) \equiv 0$. It follows from (10) that

$$\begin{aligned}
& \mathbb{E}\{\eta^T(k+1, h+1)(\mathcal{P}_1 + \mathcal{P}_2)\eta(k+1, h+1)|\aleph(k, h)\} \\
&= \mathbb{E}\left\{\mathcal{Y}^T(k, h)(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{Y}(k, h) + \eta^T(k+1, h)(\Delta\mathcal{A}_1(k, h))^T(\mathcal{P}_1 + \mathcal{P}_2)(\Delta\mathcal{A}_1(k, h))\eta(k+1, h) \right. \\
&\quad + \eta^T(k, h+1)(\Delta\mathcal{A}_2(k, h))^T(\mathcal{P}_1 + \mathcal{P}_2)(\Delta\mathcal{A}_2(k, h))\eta(k, h+1) \\
&\quad + \mathcal{F}^T(\eta(k+1, h), \eta(k, h+1))(\Delta\mathcal{G}(k, h))^T(\mathcal{P}_1 + \mathcal{P}_2)(\Delta\mathcal{G}(k, h))\mathcal{F}(\eta(k+1, h), \eta(k, h+1)) \\
&\quad + 2\eta^T(k+1, h)(\Delta\mathcal{A}_1(k, h))^T(\mathcal{P}_1 + \mathcal{P}_2)\Delta\mathcal{G}(k, h)\mathcal{F}(\eta(k+1, h), \eta(k, h+1)) \\
&\quad \left. + 2\eta^T(k, h+1)(\Delta\mathcal{A}_2(k, h))^T(\mathcal{P}_1 + \mathcal{P}_2)\Delta\mathcal{G}(k, h)\mathcal{F}(\eta(k+1, h), \eta(k, h+1))\right\}|\aleph(k, h)\}, \quad (23)
\end{aligned}$$

where the mutual independence property of the random variables $\hat{\gamma}(k, h)$, $\xi_i(k, h)$ and $\lambda_i(k, h)$ ($i = 1, 2, \dots, m$) has been utilized when deriving the equality (23).

For simplicity, in the following analysis, denote $\mathcal{P}_1 + \mathcal{P}_2$ as $[(\mathcal{P}_1 + \mathcal{P}_2)_{ij}]_{2 \times 2}$ where $(\mathcal{P}_1 + \mathcal{P}_2)_{ij} \in \mathbb{R}^n$ means the (i, j) -block of matrix $\mathcal{P}_1 + \mathcal{P}_2$. By resorting to the conditions (3), (6) and (7), we have

$$\begin{aligned}
& \mathbb{E}\{2\eta^T(k, h+1)(\Delta\mathcal{A}_2(k, h))^T(\mathcal{P}_1 + \mathcal{P}_2)\Delta\mathcal{G}(k, h)\mathcal{F}(\eta(k+1, h), \eta(k, h+1))|\aleph(k, h)\} \\
&= 2\mathbb{E}\left\{\left[(\hat{\gamma}(k, h) - \bar{\gamma})(K_{2f}(\Xi(k, h+1)\Lambda(k, h+1) - \bar{\Xi}\bar{\Lambda})Cx(k, h+1))^T(\mathcal{P}_1 + \mathcal{P}_2)_{12}^T G \right. \right. \\
&\quad \times f(x(k+1, h), x(k, h+1)) + (K_{2f}(\Xi(k, h+1)\Lambda(k, h+1) - \bar{\Xi}\bar{\Lambda})Cx(k, h+1))^T(\mathcal{P}_1 + \mathcal{P}_2)_{22} \\
&\quad \left. \left. \times (K_{1f}(\bar{\Lambda} - \Lambda(k+1, h))g(Cx(k+1, h)) + K_{2f}(\bar{\Lambda} - \Lambda(k, h+1))g(Cx(k, h+1)))\right]\right\}|\aleph(k, h)\} \\
&= 2\mathbb{E}\left\{(K_{2f}(\Xi(k, h+1)\Lambda(k, h+1) - \bar{\Xi}\bar{\Lambda})Cx(k, h+1))^T(\mathcal{P}_1 + \mathcal{P}_2)_{22} \right. \\
&\quad \left. \times K_{2f}(\bar{\Lambda} - \Lambda(k, h+1))g(Cx(k, h+1))\right\}|\aleph(k, h)\} \\
&= 2\mathbb{E}\left\{x^T(k, h+1)C^T\left(\sum_{i=1}^m(\lambda_i(k, h+1)\xi_i(k, h+1) - \bar{\xi}_i\bar{\lambda}_i)\bar{E}_i^T K_{2f}^T\right)(\mathcal{P}_1 + \mathcal{P}_2)_{22} \right. \\
&\quad \left. \times \left(\sum_{j=1}^m(\bar{\lambda}_j - \lambda_j(k, h+1))K_{2f}\bar{E}_j\right)g(Cx(k, h+1))\right\}|\aleph(k, h)\} \\
&= 2\mathbb{E}\left\{x^T(k, h+1)C^T\left[\sum_{i=1}^m(\lambda_i(k, h+1)\xi_i(k, h+1) - \bar{\xi}_i\bar{\lambda}_i)(\bar{\lambda}_i - \lambda_i(k, h+1))\bar{E}_i^T K_{2f}^T \right. \right. \\
&\quad \left. \left. \times (\mathcal{P}_1 + \mathcal{P}_2)_{22}K_{2f}\bar{E}_i\right]g(Cx(k, h+1))\right\}|\aleph(k, h)\} \\
&= 2\mathbb{E}\left\{x^T(k, h+1)C^T\left(\sum_{i=1}^m\bar{\lambda}_i(\bar{\lambda}_i - 1)\bar{\xi}_i\bar{E}_i^T K_{2f}^T(\mathcal{P}_1 + \mathcal{P}_2)_{22}K_{2f}\bar{E}_i\right)g(Cx(k, h+1))\right\}|\aleph(k, h)\} \\
&= -2\mathbb{E}\left\{\sum_{i=1}^m\iota_{2i}\eta^T(k, h+1)L^T\mathcal{C}_{2i}^T(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{K}_{2i}\mathcal{F}(\eta(k+1, h), \eta(k, h+1))\right\}|\aleph(k, h)\} \quad (24)
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E}\{2\eta^T(k+1, h)(\Delta\mathcal{A}_1(k, h))^T(\mathcal{P}_1 + \mathcal{P}_2)\Delta\mathcal{G}(k, h)\mathcal{F}(\eta(k+1, h), \eta(k, h+1))|\aleph(k, h)\} \\
&= 2\mathbb{E}\left\{x^T(k+1, h)C^T\left(\sum_{i=1}^m\bar{\lambda}_i(\bar{\lambda}_i - 1)\bar{\xi}_i\bar{E}_i^T K_{1f}^T(\mathcal{P}_1 + \mathcal{P}_2)_{22}K_{1f}\bar{E}_i\right)g(Cx(k+1, h))\right\}|\aleph(k, h)\} \\
&= -2\mathbb{E}\left\{\sum_{i=1}^m\iota_{2i}\eta^T(k+1, h)L^T\mathcal{C}_{1i}^T(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{K}_{1i}\mathcal{F}(\eta(k+1, h), \eta(k, h+1))\right\}|\aleph(k, h)\} \quad (25)
\end{aligned}$$

where $\iota_{2i} = \bar{\lambda}_i(1 - \bar{\lambda}_i)\bar{\xi}_i$. Similarly, it can be obtained that

$$\begin{aligned}
& \mathbb{E}\{\eta^T(k+1, h)(\Delta\mathcal{A}_1(k, h))^T(\mathcal{P}_1 + \mathcal{P}_2)(\Delta\mathcal{A}_1(k, h))\eta(k+1, h)|\aleph(k, h)\} \\
&= \mathbb{E}\{x^T(k+1, h)C^T \sum_{i=1}^m (\xi_i(k+1, h)\lambda_i(k+1, h) - \bar{\xi}_i\bar{\lambda}_i)^2 \bar{E}_i^T K_{1f}^T (\mathcal{P}_1 + \mathcal{P}_2)_{22} K_{1f} \bar{E}_i Cx(k+1, h)|\aleph(k, h)\} \\
&= \mathbb{E}\{x^T(k+1, h)C^T \sum_{i=1}^m (\bar{\lambda}_i\sigma_i^2 - \bar{\xi}_i^2\bar{\lambda}_i^2) \bar{E}_i^T K_{1f}^T (\mathcal{P}_1 + \mathcal{P}_2)_{22} K_{1f} \bar{E}_i Cx(k+1, h)|\aleph(k, h)\} \\
&= \mathbb{E}\left\{\sum_{i=1}^m \iota_{3i} \eta^T(k+1, h) L^T C_{1i}^T (\mathcal{P}_1 + \mathcal{P}_2) C_{1i} L \eta(k+1, h) | \aleph(k, h)\right\}, \tag{26}
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E}\{\eta^T(k, h+1)(\Delta\mathcal{A}_2(k, h))^T(\mathcal{P}_1 + \mathcal{P}_2)(\Delta\mathcal{A}_2(k, h))\eta(k, h+1)|\aleph(k, h)\} \\
&= \mathbb{E}\left\{\sum_{i=1}^m \iota_{3i} \eta^T(k, h+1) L^T C_{2i}^T (\mathcal{P}_1 + \mathcal{P}_2) C_{2i} L \eta(k, h+1) | \aleph(k, h)\right\} \tag{27}
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E}\{\mathcal{F}^T(\eta(k+1, h), \eta(k, h+1))(\Delta\mathcal{G}(k, h))^T(\mathcal{P}_1 + \mathcal{P}_2)(\Delta\mathcal{G}(k, h))\mathcal{F}(\eta(k+1, h), \eta(k, h+1))|\aleph(k, h)\} \\
&= \mathbb{E}\left\{\left[\begin{aligned} & (\hat{\gamma}(k, h) - \bar{\gamma})^2 f^T(x(k+1, h), x(k, h+1)) G^T (\mathcal{P}_1 + \mathcal{P}_2)_{11} G f(x(k+1, h), x(k, h+1)) \\ & + 2(\hat{\gamma}(k, h) - \bar{\gamma}) f^T(x(k+1, h), x(k, h+1)) G^T (\mathcal{P}_1 + \mathcal{P}_2)_{12} \\ & \times (K_{1f}(\bar{\Lambda} - \Lambda(k+1, h))g(Cx(k+1, h)) + K_{2f}(\bar{\Lambda} - \Lambda(k, h+1))g(Cx(k, h+1))) \\ & + (K_{1f}(\bar{\Lambda} - \Lambda(k+1, h))g(Cx(k+1, h)) + K_{2f}(\bar{\Lambda} - \Lambda(k, h+1))g(Cx(k, h+1)))^T (\mathcal{P}_1 + \mathcal{P}_2)_{22} \\ & \times (K_{1f}(\bar{\Lambda} - \Lambda(k+1, h))g(Cx(k+1, h)) + K_{2f}(\bar{\Lambda} - \Lambda(k, h+1))g(Cx(k, h+1))) \end{aligned}\right] |\aleph(k, h)\right\} \\
&= \mathbb{E}\left\{\left[\begin{aligned} & \bar{\gamma}(1 - \bar{\gamma}) f^T(x(k+1, h), x(k, h+1)) G^T (\mathcal{P}_1 + \mathcal{P}_2)_{11} G f(x(k+1, h), x(k, h+1)) \\ & + g^T(Cx(k+1, h))(\bar{\Lambda} - \Lambda(k+1, h))^T K_{1f}^T (\mathcal{P}_1 + \mathcal{P}_2)_{22} K_{1f} (\bar{\Lambda} - \Lambda(k+1, h))g(Cx(k+1, h)) \\ & + g^T(Cx(k, h+1))(\bar{\Lambda} - \Lambda(k, h+1))^T K_{2f}^T (\mathcal{P}_1 + \mathcal{P}_2)_{22} K_{2f} (\bar{\Lambda} - \Lambda(k, h+1))g(Cx(k, h+1)) \end{aligned}\right] |\aleph(k, h)\right\} \\
&= \mathbb{E}\left\{\left[\begin{aligned} & \bar{\gamma}(1 - \bar{\gamma}) f^T(x(k+1, h), x(k, h+1)) G^T (\mathcal{P}_1 + \mathcal{P}_2)_{11} G f(x(k+1, h), x(k, h+1)) \\ & + \sum_{i=1}^m \bar{\lambda}_i(1 - \bar{\lambda}_i) g^T(Cx(k+1, h)) \bar{E}_i^T K_{1f}^T (\mathcal{P}_1 + \mathcal{P}_2)_{22} K_{1f} \bar{E}_i g(Cx(k+1, h)) \\ & + \sum_{i=1}^m \bar{\lambda}_i(1 - \bar{\lambda}_i) g^T(Cx(k, h+1)) \bar{E}_i^T K_{2f}^T (\mathcal{P}_1 + \mathcal{P}_2)_{22} K_{2f} \bar{E}_i g(Cx(k, h+1)) \end{aligned}\right] |\aleph(k, h)\right\} \\
&= \mathbb{E}\left\{\mathcal{F}^T(\eta(k+1, h), \eta(k, h+1)) \left[\sum_{i=1}^m \iota_{1i} (\mathcal{K}_{1i}^T (\mathcal{P}_1 + \mathcal{P}_2) \mathcal{K}_{1i} + \mathcal{K}_{2i}^T (\mathcal{P}_1 + \mathcal{P}_2) \mathcal{K}_{2i}) + \mathbf{G}^T (\mathcal{P}_1 + \mathcal{P}_2) \mathbf{G} \right] \right. \\
&\quad \left. \times \mathcal{F}(\eta(k+1, h), \eta(k, h+1)) | \aleph(k, h)\right\} \tag{28}
\end{aligned}$$

where $\iota_{1i} = \bar{\lambda}_i(1 - \bar{\lambda}_i)$ and $\iota_{3i} = \bar{\lambda}_i\sigma_i^2 - \bar{\xi}_i^2\bar{\lambda}_i^2$, C_{1i} , C_{2i} , \mathcal{K}_{1i} , \mathcal{K}_{2i} and \mathbf{G} are matrices defined in (13).

On the other hand, it follows from inequality (4) that for any given scalar $\varepsilon_1 > 0$, the following inequality holds:

$$\begin{aligned}
& \varepsilon_1 \left(\mathcal{I}_1 \mathcal{F}(\eta(k+1, h), \eta(k, h+1)) - (F_{11} L \eta(k+1, h) + F_{12} L \eta(k, h+1)) \right)^T \\
& \quad \times \left(\mathcal{I}_1 \mathcal{F}(\eta(k+1, h), \eta(k, h+1)) - (F_{21} L \eta(k+1, h) + F_{22} L \eta(k, h+1)) \right) \leq 0. \tag{29}
\end{aligned}$$

Similarly, for any given scalars $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$, the condition (11) infers directly the validity of the following

two inequalities:

$$\begin{aligned} & \varepsilon_2 \left(\mathcal{I}_2 \mathcal{F}(\eta(k+1, h), \eta(k, h+1)) - CL\eta(k+1, h) \right)^T \\ & \quad \times \left(\mathcal{I}_2 \mathcal{F}(\eta(k+1, h), \eta(k, h+1)) - SCL\eta(k+1, h) \right) \leq 0, \end{aligned} \quad (30)$$

$$\begin{aligned} & \varepsilon_3 \left(\mathcal{I}_3 \mathcal{F}(\eta(k+1, h), \eta(k, h+1)) - CL\eta(k, h+1) \right)^T \\ & \quad \times \left(\mathcal{I}_3 \mathcal{F}(\eta(k+1, h), \eta(k, h+1)) - SCL\eta(k, h+1) \right) \leq 0, \end{aligned} \quad (31)$$

where matrices \mathcal{I}_1 , \mathcal{I}_2 and \mathcal{I}_3 are defined in (13).

Substituting (24)-(28) into (23), and from inequalities (22), (23), (29)-(31), we have that when $v(k, h) \equiv 0$,

$$\mathcal{J} \leq \mathbb{E}\{\zeta^T(k, h) \vec{\Phi} \zeta(k, h) | \mathfrak{N}(k, h)\}, \quad (32)$$

where $\vec{\Phi} = \hat{\Phi} + \mathcal{A}^T(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{A} + \tilde{\mathcal{G}}(\mathcal{P}_1 + \mathcal{P}_2)\tilde{\mathcal{G}}^T + \sum_{j=1}^2 \sum_{i=1}^m (\iota_{1i} \tilde{\mathcal{E}}_{ji}(\mathcal{P}_1 + \mathcal{P}_2)^{-1} \tilde{\mathcal{E}}_{ji}^T + \iota_{4i} \vec{\mathcal{E}}_{ji}(\mathcal{P}_1 + \mathcal{P}_2)^{-1} \vec{\mathcal{E}}_{ji}^T)$ and $\zeta(k, h)$ is defined in (12). By the Schur's lemma [1], $\vec{\Phi} < 0$ if the matrix inequality (13) holds, which infers that there exists a constant $\mu > 0$ such that

$$\mathbb{E}\{(V(k+1, h+1) - \sum_{i=1}^3 V_{1i}(k+1, h) - \sum_{j=1}^3 V_{2j}(k, h+1)) | \mathfrak{N}(k, h)\} \leq -\mu \|\eta(k, h+1)\|^2. \quad (33)$$

Taking mathematical expectation on both sides of (33) and summing up both sides of the inequality with k, h varying from 0 to N , we get

$$\begin{aligned} & \sum_{k=0}^N \sum_{h=0}^N \mathbb{E}\left\{V(k+1, h+1) - \sum_{i=1}^3 V_{1i}(k+1, h) - \sum_{j=1}^3 V_{2j}(k, h+1)\right\} \\ & = \sum_{k=0}^N \sum_{i=1}^3 \mathbb{E}\{V_{1i}(k+1, N+1) - V_{1i}(k+1, 0)\} + \sum_{h=0}^N \sum_{j=1}^3 \mathbb{E}\{V_{2j}(N+1, h+1) - V_{2j}(0, h+1)\} \\ & \leq -\mu \sum_{k=0}^N \sum_{h=0}^N \mathbb{E}\{\|\eta(k, h+1)\|^2\}, \end{aligned} \quad (34)$$

where N is a constant integer satisfying $N > \max\{\kappa_1, \kappa_2\} + \max\{\sigma_2, \tau_2\}$ with $\sigma_2, \tau_2, \kappa_1$ and κ_2 defined, respectively, in (2) and (8). From the above inequality (34), it is not difficult to obtain the following inequality

$$\begin{aligned} \sum_{k=0}^N \sum_{h=0}^N \mathbb{E}\{\|\eta(k, h+1)\|^2\} & \leq \frac{1}{\mu} \left(\sum_{k=0}^N \sum_{i=1}^3 \mathbb{E}\{V_{1i}(k+1, 0) - V_{1i}(k+1, N+1)\} \right. \\ & \quad \left. + \sum_{h=0}^N \sum_{j=1}^3 \mathbb{E}\{V_{2j}(0, h+1) - V_{2j}(N+1, h+1)\} \right) \\ & \leq \frac{1}{\mu} \left(\sum_{k=0}^N \sum_{i=1}^3 \mathbb{E}\{V_{1i}(k+1, 0)\} + \sum_{h=0}^N \sum_{j=1}^3 \mathbb{E}\{V_{2j}(0, h+1)\} \right) < \infty \end{aligned} \quad (35)$$

where the last inequality holds under the bounded initial condition (8). From the necessary condition for the convergent positive series, (35) further means

$$\lim_{k+h \rightarrow \infty} \mathbb{E}\{\|\eta(k, h)\|\} = 0.$$

To draw the conclusion that the closed-loop 2-D system (10) with $v(k, h) \equiv 0$ is globally asymptotically stable in the mean square, we still need to show that the trivial solution of (10) with $v(k, h) \equiv 0$ is stable in the mean square (in the sense of Lyapunov). Taking mathematical expectation on both sides of (33) leads to

$$\mathbb{E}\{V(k+1, h+1)\} \leq \mathbb{E}\{V_1(k+1, h) + V_2(k, h+1)\}.$$

Illuminated by the ideas introduced firstly in [26], the above inequality and the definition of $V(k, h)$ in (14) infer that for any $d \geq N$ where N is defined in (34), the following inequality holds:

$$\begin{aligned} \sum_{(k,h) \in \mathcal{N}(d+1)} \mathbb{E}\{V(k, h)\} &= \mathbb{E}\{V(d+1, 0) + V(d, 1) + \dots + V(1, d) + V(0, d+1)\} \\ &\leq \mathbb{E}\left\{ (V_1(d+1, 0) + V_2(d+1, 0)) + (V_1(d, 0) + V_2(d-1, 1)) + \dots \right. \\ &\quad \left. + (V_1(1, d-1) + V_2(0, d)) + (V_1(0, d+1) + V_2(0, d+1)) \right\} \\ &= \mathbb{E}\left\{ V_1(d, 0) + (V_1(d-1, 1) + V_2(d-1, 1)) + \dots + (V_1(1, d-1) + V_2(1, d-1)) + V_2(0, d) \right\} \\ &= \mathbb{E}\left\{ (V_1(d, 0) + V_2(d, 0)) + (V_1(d-1, 1) + V_2(d-1, 1)) + \dots \right. \\ &\quad \left. + (V_1(1, d-1) + V_2(1, d-1)) + (V_1(0, d) + V_2(0, d)) \right\} \\ &= \mathbb{E}\{V(d, 0) + V(d-1, 1) + \dots + V(1, d-1) + V(0, d)\} \\ &= \sum_{(k,h) \in \mathcal{N}(d)} \mathbb{E}\{V(k, h)\}, \end{aligned} \tag{36}$$

where $\mathcal{N}(d)$ is defined to be the index set $\{(k, h) | k + h = d; k, h \in \mathbb{N}\}$. It should be noted that when deriving the third and the fourth steps of (36), the initial conditions $\varphi(k, h) = 0$ for $(k, h) \in [-\sigma_2, 0] \times [\kappa_1 + 1, \infty)$ and $\phi(k, h) = 0$ for $(k, h) \in [\kappa_2 + 1, \infty) \times [-\tau_2, 0]$ in (8) have been utilized. For any given scalar $\tilde{\varepsilon} > 0$, by resorting to the boundary initial condition (8), there must exist one small scalar $\delta \in (0, \tilde{\varepsilon})$ such that

$$\max_{d \in [0, N]} \sum_{(k,h) \in \mathcal{N}(d)} \mathbb{E}\{V(k, h)\} \leq \tilde{\varepsilon}^2$$

whenever $\|\varphi(k, h)\| \leq \delta$ for $(k, h) \in [-\sigma_2, 0] \times [0, \kappa_1]$ and $\|\phi(k, h)\| \leq \delta$ for $(k, h) \in [0, \kappa_2] \times [-\tau_2, 0]$ in (8). This together with (36) guarantee that the closed-loop system (10) with $v(k, h) \equiv 0$ is stable in the mean square. From Definition 1, we know that the closed-loop 2-D system (10) with $v(k, h) \equiv 0$ is globally asymptotically stable in the mean square.

Let us now deal with the H_∞ performance for the closed-loop 2-D system (10). In the following, assume that in (8) $\varphi(\cdot, \cdot) \equiv 0$ and $\phi(\cdot, \cdot) \equiv 0$. Consider the index as follows:

$$\tilde{\mathcal{J}} := \mathcal{J} + \mathbb{E}\{(\tilde{z}^T(k, h)\tilde{z}(k, h) - \gamma^2\tilde{v}^T(k, h)\tilde{v}(k, h)) | \aleph(k, h)\}, \tag{37}$$

where $\tilde{z}(k, h) = (z^T(k+1, h) \ z^T(k, h+1))^T$, $\tilde{v}(k, h) = (v^T(k+1, h) \ v^T(k, h+1))^T$ and \mathcal{J} is defined in (15). Computing the index $\tilde{\mathcal{J}}$ along the solutions of the closed-loop system (10), we have

$$\begin{aligned} \tilde{\mathcal{J}} &\leq \mathbb{E}\left\{ \left[\zeta^T(k, h)\tilde{\Phi}\zeta(k, h) + v^T(k+1, h)\mathcal{E}_1^T(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{E}_1v(k+1, h) + v^T(k, h+1)\mathcal{E}_2^T(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{E}_2v(k, h+1) \right. \right. \\ &\quad \left. \left. + 2v^T(k+1, h)\mathcal{E}_1^T(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{E}_2v(k, h+1) + 2\mathcal{Y}^T(k, h)(\mathcal{P}_1 + \mathcal{P}_2)(\mathcal{E}_1v(k+1, h) + \mathcal{E}_2v(k, h+1)) \right. \right. \\ &\quad \left. \left. + \eta^T(k+1, h)\mathcal{W}^T\mathcal{W}\eta(k+1, h) + 2\eta^T(k+1, h)\mathcal{W}^T M_2v(k+1, h) + v^T(k+1, h)M_2^T M_2v(k+1, h) \right. \right. \\ &\quad \left. \left. + \eta^T(k, h+1)\mathcal{W}^T\mathcal{W}\eta(k, h+1) + 2\eta^T(k, h+1)\mathcal{W}^T M_2v(k, h+1) + v^T(k, h+1)M_2^T M_2v(k, h+1) \right. \right. \\ &\quad \left. \left. - \gamma^2v^T(k+1, h)v(k+1, h) - \gamma^2v^T(k, h+1)v(k, h+1) \right] | \aleph(k, h) \right\} \\ &= \mathbb{E}\{\tilde{\zeta}^T(k, h)\tilde{\Phi}\tilde{\zeta}(k, h) | \aleph(k, h)\}, \end{aligned} \tag{38}$$

where $\bar{\zeta}(k, h) = (\zeta^T(k, h) \ v^T(k+1, h) \ v^T(k, h+1))^T$ and matrix $\tilde{\Phi}$ is defined in (13). The condition (13) assures that for all $\bar{\zeta}(k, h) \neq 0$, $\tilde{\mathcal{J}} < 0$, i.e.,

$$\begin{aligned} \mathbb{E}\{V(k+1, h+1)|\aleph(k, h)\} &< \mathbb{E}\{[(V_1(k+1, h) + V_2(k, h+1)) - (\|z(k+1, h)\|^2 + \|z(k, h+1)\|^2) \\ &+ \gamma^2(\|v(k+1, h)\|^2 + \|v(k, h+1)\|^2)]|\aleph(k, h)\}. \end{aligned}$$

Taking mathematical expectation on both sides of the above inequality, we have the validity of the following $k+2$ inequalities:

$$\begin{aligned} \mathbb{E}\{V(k+1, 0)\} &= \mathbb{E}\{V_1(k+1, 0) + V_2(k+1, 0)\}, \\ \mathbb{E}\{V(k, 1)\} &\leq \mathbb{E}\{(V_1(k, 0) + V_2(k-1, 1)) - (\|z(k, 0)\|^2 + \|z(k-1, 1)\|^2) + \gamma^2(\|v(k, 0)\|^2 + \|v(k-1, 1)\|^2)\}, \\ \mathbb{E}\{V(k-1, 2)\} &\leq \mathbb{E}\{(V_1(k-1, 1) + V_2(k-2, 2)) - (\|z(k-1, 1)\|^2 + \|z(k-2, 2)\|^2) \\ &+ \gamma^2(\|v(k-1, 1)\|^2 + \|v(k-2, 2)\|^2)\}, \\ &\vdots \\ \mathbb{E}\{V(2, k-1)\} &\leq \mathbb{E}\{(V_1(2, k-2) + V_2(1, k-1)) - (\|z(2, k-2)\|^2 + \|z(1, k-1)\|^2) \\ &+ \gamma^2(\|v(2, k-2)\|^2 + \|v(1, k-1)\|^2)\}, \\ \mathbb{E}\{V(1, k)\} &\leq \mathbb{E}\{(V_1(1, k-1) + V_2(0, k)) - (\|z(1, k-1)\|^2 + \|z(0, k)\|^2) + \gamma^2(\|v(1, k-1)\|^2 + \|v(0, k)\|^2)\}, \\ \mathbb{E}\{V(0, k+1)\} &= \mathbb{E}\{V_1(0, k+1) + V_2(0, k+1)\}. \end{aligned}$$

Adding up both sides of the above inequalities and considering the zero-initial boundary condition, we obtain

$$\begin{aligned} \sum_{j=0}^{k+1} \mathbb{E}\{V(k+1-j, j)\} &\leq \mathbb{E}\{V_2(k+1, 0) + V_1(k, 0)\} + \sum_{j=1}^{k-1} \mathbb{E}\{V(k-j, j)\} + \mathbb{E}\{V_1(0, k+1) + V_2(0, k)\} \\ &+ \mathbb{E}\{V_1(k+1, 0) + V_2(0, k+1)\} - \mathbb{E}\{2 \sum_{j=1}^{k-1} \|z(k-j, j)\|^2 + \|z(k, 0)\|^2 + \|z(0, k)\|^2\} \\ &+ \gamma^2 \mathbb{E}\{2 \sum_{j=1}^{k-1} \|v(k-j, j)\|^2 + \|v(k, 0)\|^2 + \|v(0, k)\|^2\} \\ &= \sum_{j=0}^k \mathbb{E}\{V(k-j, j)\} - \mathbb{E}\{2 \sum_{j=0}^k \|z(k-j, j)\|^2 - \|z(k, 0)\|^2 - \|z(0, k)\|^2\} \\ &+ \gamma^2 \mathbb{E}\{2 \sum_{j=0}^k \|v(k-j, j)\|^2 - \|v(k, 0)\|^2 - \|v(0, k)\|^2\}, \end{aligned} \tag{39}$$

which leads to the following inequality:

$$\begin{aligned} &\sum_{k=0}^{N_1} \sum_{j=0}^k \mathbb{E}\{\|z(k-j, j)\|^2\} - \frac{1}{2} \sum_{k=0}^{N_1} \mathbb{E}\{\|z(k, 0)\|^2\} - \frac{1}{2} \sum_{k=0}^{N_1} \mathbb{E}\{\|z(0, k)\|^2\} \\ &\leq \sum_{k=0}^{N_1} \sum_{j=0}^k \mathbb{E}\{V(k-j, j)\} - \sum_{k=0}^{N_1} \sum_{j=0}^{k+1} \mathbb{E}\{V(k+1-j, j)\} \\ &+ \gamma^2 \left(\sum_{k=0}^{N_1} \sum_{j=0}^k \mathbb{E}\{\|v(k-j, j)\|^2\} - \frac{1}{2} \sum_{k=0}^{N_1} \mathbb{E}\{\|v(k, 0)\|^2\} - \frac{1}{2} \sum_{k=0}^{N_1} \mathbb{E}\{\|v(0, k)\|^2\} \right) \end{aligned}$$

$$\begin{aligned}
&= V(0,0) - \sum_{j=0}^{N_1+1} \mathbb{E}\{V(N_1+1-j, j)\} \\
&\quad + \gamma^2 \left(\sum_{k=0}^{N_1} \sum_{j=0}^k \mathbb{E}\{\|v(k-j, j)\|^2\} - \frac{1}{2} \sum_{k=0}^{N_1} \mathbb{E}\{\|v(k, 0)\|^2\} - \frac{1}{2} \sum_{k=0}^{N_1} \mathbb{E}\{\|v(0, k)\|^2\} \right) \\
&\leq \gamma^2 \left(\sum_{k=0}^{N_1} \sum_{j=0}^k \mathbb{E}\{\|v(k-j, j)\|^2\} - \frac{1}{2} \sum_{k=0}^{N_1} \mathbb{E}\{\|v(k, 0)\|^2\} - \frac{1}{2} \sum_{k=0}^{N_1} \mathbb{E}\{\|v(0, k)\|^2\} \right),
\end{aligned}$$

where $N_1 \in \mathbb{N}$. Subsequently, by letting $N_1 \rightarrow \infty$, one has

$$\begin{aligned}
&\sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \mathbb{E}\{\|z(k, h)\|^2\} - \frac{1}{2} \sum_{k=0}^{\infty} \mathbb{E}\{\|z(k, 0)\|^2\} - \frac{1}{2} \sum_{h=0}^{\infty} \mathbb{E}\{\|z(0, h)\|^2\} \\
&\leq \gamma^2 \left\{ \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \mathbb{E}\{\|v(k, h)\|^2\} - \frac{1}{2} \sum_{k=0}^{\infty} \mathbb{E}\{\|v(k, 0)\|^2\} - \frac{1}{2} \sum_{h=0}^{\infty} \mathbb{E}\{\|v(0, h)\|^2\} \right\},
\end{aligned}$$

or equivalently,

$$\|z\|_2^2 \leq \gamma^2 \|v\|_2^2,$$

which completes the proof of Theorem 1. \blacksquare

Remark 3: It is well known that Lyapunov function and the relating Lyapunov stability theory are always utilized when investigating the 1-D dynamical systems. When referring to the 2-D time-delay system (10) which evolves in *two* independent directions, an energy-like function $V(k, h)$ in the form of (14) is constructed here firstly, and then an index \mathcal{J} based on this defined quadratic function is introduced in (15), which is just in the role of the difference of the Lyapunov function when studying the 1-D systems. The main work in Theorem 1 is to find some sufficient conditions under which the index \mathcal{J} is negative along the trajectories of the 2-D system (10). It is further shown in Theorem 1 that such kind of negativeness guarantees the global asymptotic stability for the 2-D system (10) in the mean square sense.

After establishing the analysis result, we are now in a position to solve the output-feedback controller design problem for the system (10).

Theorem 2: For the given H_∞ performance level $\gamma > 0$, the closed-loop 2-D system (10) is globally asymptotically stable in the mean square with disturbance attenuation level γ if there exist matrices $\mathcal{P}_i > 0$, $\mathcal{Q}_i > 0$, $\mathcal{X} > 0$, A_{if} , K_{if} , H_f ($i = 1, 2$) and positive scalars ε_j ($j = 1, 2, 3$) such that the following matrix inequalities hold:

$$(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{X} = I \quad \text{and} \quad \Psi := \begin{bmatrix} \Phi & \mathcal{W}_1 & \mathcal{W}_2 & \Psi_{14} \\ * & -I & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & \Psi_{44} \end{bmatrix} < 0, \quad (40)$$

where $\Psi_{44} = \text{diag}(1, \nu_1, \nu_1, \nu_4, \nu_4, 1) \otimes (-\mathcal{X})$,

$$\Psi_{14} = [\mathcal{A}_\varepsilon^T \quad \mathcal{C}_{11}^{(1)} \quad \dots \quad \mathcal{C}_{1m}^{(1)} \quad \mathcal{C}_{21}^{(1)} \quad \dots \quad \mathcal{C}_{2m}^{(1)} \quad \mathcal{C}_{11}^{(2)} \quad \dots \quad \mathcal{C}_{1m}^{(2)} \quad \mathcal{C}_{21}^{(2)} \quad \dots \quad \mathcal{C}_{2m}^{(2)} \quad \mathcal{G}]$$

with $\nu_1 = \text{diag}(1/\nu_{11}, 1/\nu_{12}, \dots, 1/\nu_{1m})$, $\nu_4 = \text{diag}(1/\nu_{41}, 1/\nu_{42}, \dots, 1/\nu_{4m})$,

$$\mathcal{C}_{1i}^{(1)} = \begin{bmatrix} \bar{\xi}_i \mathcal{C}_{1i} L & 0 & 0 & 0 & -\mathcal{K}_{1i} & 0 & 0 \end{bmatrix}^T, \quad \mathcal{C}_{2i}^{(1)} = \begin{bmatrix} 0 & \bar{\xi}_i \mathcal{C}_{2i} L & 0 & 0 & -\mathcal{K}_{2i} & 0 & 0 \end{bmatrix}^T,$$

$$\mathcal{C}_{1i}^{(2)} = \begin{bmatrix} \mathcal{C}_{1i} L & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \quad \mathcal{C}_{2i}^{(2)} = \begin{bmatrix} 0 & \mathcal{C}_{2i} L & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T; \quad (i = 1, 2, \dots, m)$$

and the other symbols are the same as defined in Theorem 1.

Proof: With the first equality in (40), it is known that $\mathcal{X} = (\mathcal{P}_1 + \mathcal{P}_2)^{-1}$. From this fact and the Schur's lemma [1], it is easy to find that the second inequality in condition (40) is equivalent to that of (13) in Theorem 1 which further infers the validity of Theorem 2. ■

Theorem 3: For the given H_∞ performance level $\gamma > 0$, the closed-loop 2-D system (10) is globally asymptotically stable in the mean square with disturbance attenuation level γ if there exist matrices $Y > 0$, $\overline{\mathcal{Q}}_i > 0$, \tilde{H} , \tilde{A}_i , \tilde{K}_i ($i = 1, 2$), and positive scalars ε_j ($j = 1, 2, 3$) such that for the given matrix $X > 0$, the following matrix inequality holds:

$$\overline{\Psi} := \begin{bmatrix} \overline{\Psi}_{11} & \overline{\mathcal{W}}_1 & \overline{\mathcal{W}}_2 & \overline{\Psi}_{14} \\ * & -I & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & \overline{\Psi}_{44} \end{bmatrix} < 0, \quad (41)$$

where $\overline{\Psi}_{44} = \text{diag}(\frac{1}{2}, \frac{1}{2}\iota_1, \frac{1}{2}\iota_1, \frac{1}{2}\iota_4, \frac{1}{2}\iota_4, \frac{1}{2}) \otimes (-\mathcal{Y})$,

$$\begin{aligned} \overline{\mathcal{W}}_1 &= \begin{bmatrix} \overline{\mathcal{W}} & 0 & 0 & 0 & 0 & M_2 & 0 \end{bmatrix}^T, & \overline{\mathcal{W}}_2 &= \begin{bmatrix} 0 & \overline{\mathcal{W}} & 0 & 0 & 0 & 0 & M_2 \end{bmatrix}^T, \\ \mathcal{Y} &= \begin{bmatrix} X & I \\ I & Y \end{bmatrix}, & \overline{\Psi}_{11} &= \begin{bmatrix} \Theta & 0 & 0 \\ * & -\gamma^2 I & 0 \\ * & * & -\gamma^2 I \end{bmatrix}, & \Theta &= \begin{bmatrix} \Theta_{11} & \Theta_{12} & 0 & 0 & \Theta_{15} \\ * & \Theta_{22} & 0 & 0 & \Theta_{25} \\ * & * & -\overline{\mathcal{Q}}_1 & 0 & 0 \\ * & * & * & -\overline{\mathcal{Q}}_2 & 0 \\ * & * & * & * & \Theta_{55} \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \overline{\Psi}_{14} &= [\overline{\mathcal{A}}_\varepsilon^T \quad \overline{\mathcal{C}}_{11}^{(1)} \quad \dots \quad \overline{\mathcal{C}}_{1m}^{(1)} \quad \overline{\mathcal{C}}_{21}^{(1)} \quad \dots \quad \overline{\mathcal{C}}_{2m}^{(1)} \quad \overline{\mathcal{C}}_{11}^{(2)} \quad \dots \quad \overline{\mathcal{C}}_{1m}^{(2)} \quad \overline{\mathcal{C}}_{21}^{(2)} \quad \dots \quad \overline{\mathcal{C}}_{2m}^{(2)} \quad \overline{\mathcal{G}}], \\ \overline{\mathcal{C}}_{1i}^{(1)} &= [\xi_i \overline{\mathcal{C}}_{1i} \quad 0 \quad 0 \quad 0 \quad -\overline{\mathcal{K}}_{1i} \quad 0 \quad 0]^T, & \overline{\mathcal{C}}_{2i}^{(1)} &= [0 \quad \xi_i \overline{\mathcal{C}}_{2i} \quad 0 \quad 0 \quad -\overline{\mathcal{K}}_{2i} \quad 0 \quad 0]^T, \\ \overline{\mathcal{C}}_{1i}^{(2)} &= [\overline{\mathcal{C}}_{1i} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T, & \overline{\mathcal{C}}_{2i}^{(2)} &= [0 \quad \overline{\mathcal{C}}_{2i} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T, \\ \overline{\mathcal{G}} &= [0 \quad 0 \quad 0 \quad 0 \quad \overline{\mathbf{G}} \quad 0 \quad 0]^T, & \overline{\mathcal{A}}_\varepsilon &= [\overline{\mathcal{A}}_1 \quad \overline{\mathcal{A}}_2 \quad \overline{\mathcal{D}}_1 \quad \overline{\mathcal{D}}_2 \quad \overline{\mathcal{G}} \quad \overline{\mathcal{E}}_1 \quad \overline{\mathcal{E}}_2] \end{aligned}$$

with

$$\begin{aligned} \overline{\mathcal{W}} &= [W_1 X + W_2 \tilde{H} \quad W_1], & \overline{\mathcal{C}}_{1i} &= \begin{bmatrix} 0 & 0 \\ \tilde{K}_1 \tilde{E}_i C X & \tilde{K}_1 \tilde{E}_i C \end{bmatrix}, & \overline{\mathcal{C}}_{2i} &= \begin{bmatrix} 0 & 0 \\ \tilde{K}_2 \tilde{E}_i C X & \tilde{K}_2 \tilde{E}_i C \end{bmatrix}, \\ \overline{\mathcal{K}}_{1i} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tilde{K}_1 \tilde{E}_i & 0 \end{bmatrix}, & \overline{\mathcal{K}}_{2i} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tilde{K}_2 \tilde{E}_i \end{bmatrix}, & \overline{\mathbf{G}} &= \sqrt{\gamma(1-\gamma)} \begin{bmatrix} G & 0 & 0 \\ YG & 0 & 0 \end{bmatrix}, \\ \overline{\mathcal{A}}_1 &= \begin{bmatrix} A_1 X + B_1 \tilde{H} & A_1 \\ \tilde{A}_1 & Y A_1 + \tilde{K}_1 \tilde{E}_1 \tilde{\Lambda} C \end{bmatrix}, & \overline{\mathcal{D}}_1 &= \begin{bmatrix} D_1 X & D_1 \\ Y D_1 X & Y D_1 \end{bmatrix}, & \overline{\mathcal{E}}_1 &= \begin{bmatrix} E_1 \\ Y E_1 + \tilde{K}_1 M_1 \end{bmatrix}, \\ \overline{\mathcal{A}}_2 &= \begin{bmatrix} A_2 X + B_2 \tilde{H} & A_2 \\ \tilde{A}_2 & Y A_2 + \tilde{K}_2 \tilde{E}_2 \tilde{\Lambda} C \end{bmatrix}, & \overline{\mathcal{D}}_2 &= \begin{bmatrix} D_2 X & D_2 \\ Y D_2 X & Y D_2 \end{bmatrix}, & \overline{\mathcal{E}}_2 &= \begin{bmatrix} E_2 \\ Y E_2 + \tilde{K}_2 M_1 \end{bmatrix}, \\ \overline{\mathcal{G}} &= \begin{bmatrix} \gamma G & 0 & 0 \\ \gamma Y G & \tilde{K}_1 (I - \tilde{\Lambda}) & \tilde{K}_2 (I - \tilde{\Lambda}) \end{bmatrix}, \end{aligned}$$

and

$$\Theta_{11} = (\tau_2 - \tau_1 + 1) \overline{\mathcal{Q}}_1 - \mathcal{Y} - \varepsilon_2 \begin{bmatrix} X C^T S C X & X C^T S C \\ C^T S C X & C^T S C \end{bmatrix} - \frac{\varepsilon_1}{2} \begin{bmatrix} X \\ I \end{bmatrix} (F_{11}^T F_{21} + F_{21}^T F_{11}) \begin{bmatrix} X & I \end{bmatrix},$$

$$\begin{aligned}
\Theta_{22} &= (\sigma_2 - \sigma_1 + 1)\bar{\mathcal{Q}}_2 - \mathcal{Y} - \varepsilon_3 \begin{bmatrix} XC^T SCX & XC^T SC \\ C^T SCX & C^T SC \end{bmatrix} - \frac{\varepsilon_1}{2} \begin{bmatrix} X \\ I \end{bmatrix} (F_{12}^T F_{22} + F_{22}^T F_{12}) \begin{bmatrix} X & I \end{bmatrix}, \\
\Theta_{12} &= -\frac{\varepsilon_1}{2} \begin{bmatrix} X \\ I \end{bmatrix} (F_{11}^T F_{22} + F_{21}^T F_{12}) \begin{bmatrix} X & I \end{bmatrix}, \quad \Theta_{15} = \frac{\varepsilon_1}{2} \begin{bmatrix} X \\ I \end{bmatrix} (F_{11}^T + F_{21}^T) \mathcal{L}_1 + \frac{\varepsilon_2}{2} \begin{bmatrix} X \\ I \end{bmatrix} C^T (I + S) \mathcal{L}_2, \\
\Theta_{25} &= \frac{\varepsilon_1}{2} \begin{bmatrix} X \\ I \end{bmatrix} (F_{12}^T + F_{22}^T) \mathcal{L}_1 + \frac{\varepsilon_3}{2} \begin{bmatrix} X \\ I \end{bmatrix} C^T (I + S) \mathcal{L}_3, \quad \Theta_{55} = -\varepsilon_1 \mathcal{L}_1^T \mathcal{L}_1 - \varepsilon_2 \mathcal{L}_2^T \mathcal{L}_2 - \varepsilon_3 \mathcal{L}_3^T \mathcal{L}_3,
\end{aligned}$$

and the other symbols are the same as defined in Theorem 1 and Theorem 2. Moreover, the controller parameters in (9) can be designed as follows:

$$A_{i_f} = T^{-1}(\tilde{A}_i - Y A_i X - \tilde{K}_i \tilde{\Xi} \tilde{\Lambda} C X - Y B_i \tilde{H}) R^{-T}, \quad K_{i_f} = T^{-1} \tilde{K}_i, \quad H_f = \tilde{H} R^{-T} \quad (i = 1, 2) \quad (42)$$

in which R and T are any nonsingular matrices satisfying the following equality:

$$TR^T = I - YX. \quad (43)$$

Proof: From the definition of matrix \mathcal{Y} and the condition (41), it is guaranteed that $I - YX$ is nonsingular which further infers the existence of nonsingular matrices R and T satisfying (43). Define two nonsingular matrices Γ_1 and Γ_2 as follows:

$$\Gamma_1 = \begin{bmatrix} X & I \\ R^T & 0 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} I & Y \\ 0 & T^T \end{bmatrix}.$$

By taking matrices \mathcal{P}_1 and \mathcal{P}_2 in Theorem 2 as $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}$ with $\mathcal{P} = \Gamma_1^{-T} \mathcal{Y} \Gamma_1^{-1}$, we get that the matrix \mathcal{X} in (40) has the form $\mathcal{X} = \frac{1}{2} \mathcal{P}^{-1} = \frac{1}{2} \Gamma_2^{-T} \mathcal{Y} \Gamma_2^{-1}$. Performing a congruence transformation to the inequality in (40) by block diagonal matrix $\text{diag}(I_4 \otimes \Gamma_1, I, I_2 \otimes I, I_2 \otimes I, \Gamma_2, I_{4m} \otimes \Gamma_2, \Gamma_2)$, and then taking $\bar{\mathcal{Q}}_i = \Gamma_1^T \mathcal{Q}_i \Gamma_1$ ($i = 1, 2$), after some tedious computation, we will get the inequality in (41) by just noticing that the equalities in (42) are equivalent to the fact that

$$\tilde{H} = H_f R^T, \quad \tilde{A}_i = Y A_i X + T K_{i_f} \tilde{\Xi} \tilde{\Lambda} C X + Y B_i H_f R^T + T A_{i_f} R^T, \quad \tilde{K}_i = T K_{i_f} \quad (i = 1, 2).$$

It follows immediately from Theorem 2 that the closed-loop 2-D system (10) is globally asymptotically stable in the mean square with disturbance attenuation level γ , and the proof of this theorem is complete. \blacksquare

Remark 4: Two kinds of design schemes are given in Theorem 2 and Theorem 3, respectively, for the output-feedback controller parameters of (9). Compared with Theorem 3, by noticing the first equality constraint in (40), the conditions presented in Theorem 2 are not strict linear matrix inequalities (LMIs), and hence cannot be solved directly by the convex optimization algorithm. However, by resorting to the cone complementary linearization (CCL) method [14] and the sequential linear programming matrix method (SLPMM) [22], such kind of difficulty can be overcome effectively. To further deal with such a non-convex problem, we have enforced slight restriction on the matrices \mathcal{P}_1 and \mathcal{P}_2 (i.e., $\mathcal{P}_1 = \mathcal{P}_2$) and, accordingly, the conditions in (41) of Theorem 3 are now strict LMIs which can be solved directly and effectively by the Matlab LMI Toolbox.

Remark 5: In this paper, the H_∞ output-feedback control problem is investigated for a class of two-dimensional (2-D) nonlinear systems with RONS and time-varying delays under imperfect measurements including ROSSs and missing data. The main novelty lies in that 1) the proposed 2-D system is general enough to model the phenomena of RONS, ROSSs and missing measurements; 2) a new energy-like quadratic function is employed to analyze the system stability and performance; and 3) intensive stochastic analysis is conducted to enforce the H_∞ performance for the addressed comprehensive systems in addition to the stochastic stability constraint. It should be pointed out that the main results established in Theorem 2 and Theorem 3 contain all the information about the system

parameters, the occurring probabilities of RONS, ROSSs and missing measurements, as well as the bounds of the time-varying delays.

IV. ILLUSTRATIVE EXAMPLES

In this section, an illustrative example is given to demonstrate the effectiveness of the proposed output-feedback control design schemes.

Consider the 2-D time-delay model (1) with system parameters as follows:

$$\begin{aligned}
 A_1 &= \begin{bmatrix} 0.2 & 0.025 & 0 \\ -0.005 & 0.1 & 0.15 \\ 0.05 & -0.25 & 0.1 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0.1 & 0.105 & 0.055 \\ 0.155 & 0 & -0.15 \\ 0.25 & -0.1 & -0.2 \end{bmatrix}, & B_1 &= \begin{bmatrix} 0.04 & 0 \\ -0.02 & 0.04 \\ 0.02 & 0 \end{bmatrix}, \\
 D_1 &= \begin{bmatrix} 0.0504 & 0.036 & 0.09 \\ 0.072 & 0.036 & 0 \\ -0.072 & 0.018 & 0 \end{bmatrix}, & D_2 &= \begin{bmatrix} 0.036 & 0.0612 & 0.036 \\ 0 & 0.072 & 0.0468 \\ 0.0348 & 0.036 & 0.0864 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0.06 & 0.04 \\ 0.02 & 0.04 \\ 0 & 0.02 \end{bmatrix}, \\
 G &= \begin{bmatrix} -0.01 & 0.11 & 0.2 \\ 0 & -0.3 & 0.5 \\ 0.4 & -0.2 & 0.3 \end{bmatrix}, & E_1 &= \begin{bmatrix} 0.1 \\ 0.1 \\ 0.04 \end{bmatrix}, & E_2 &= \begin{bmatrix} 0.04 \\ 0.2 \\ -0.06 \end{bmatrix}, & C &= \begin{bmatrix} 0.02 & 0.04 & 0.06 \\ 0.06 & 0.08 & 0.04 \end{bmatrix}, \\
 M_1 &= \begin{bmatrix} 0.12 \\ 0.08 \end{bmatrix}, & W_1 &= [0.02 \quad -0.06 \quad 0.08], & W_2 &= [0.14 \quad 0.06], & M_2 &= [0.1320].
 \end{aligned}$$

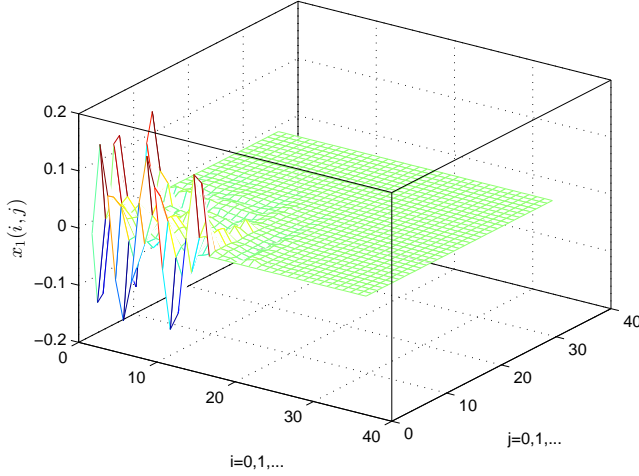
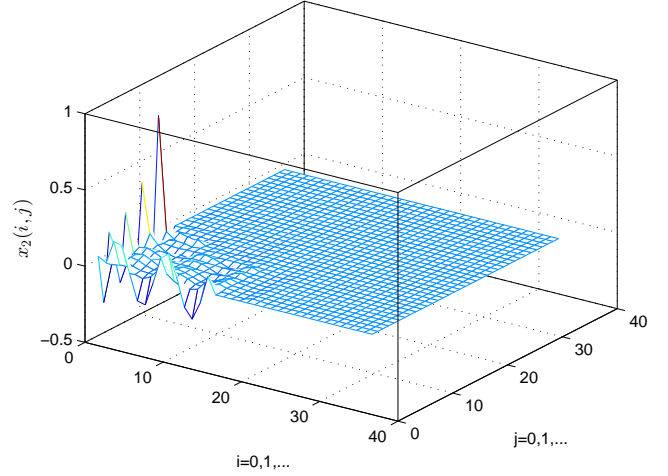
The time-varying delays in the horizontal direction and the vertical direction are, respectively, $\sigma(k) = 3 + 4|\sin(\frac{k}{2}\pi)|$ and $\tau(h) = 4 + 2|\cos(\frac{h}{2}\pi)|$. Obviously, the upper (lower) bounds of the time-varying delays are $\sigma_2 = 7$, $\sigma_1 = 2$, $\tau_2 = 6$ and $\tau_1 = 3$. The nonlinear function $f(\cdot, \cdot)$ occurs with a probability $\bar{\gamma} = 0.68$ and it is in the form of $f(u, v) = (f_1(u_1, v_1), f_2(u_2, v_2), f_3(u_3, v_3))^T$ with $u = (u_1, u_2, u_3)^T \in \mathbb{R}^3$, $v = (v_1, v_2, v_3)^T \in \mathbb{R}^3$, $f_1(u_1, v_1) = 0.2u_1 + \tanh(0.04u_1) + 0.2v_1 - \tanh(0.1v_1)$, $f_2(u_2, v_2) = 0.2u_2 - \tanh(0.1u_2) + 0.2v_2 + \tanh(0.04v_2)$ and $f_3(u_3, v_3) = 0.1u_3 + \tanh(0.05u_3) + 0.2v_3 + \tanh(0.04v_3)$. It can be shown that $f(\cdot, \cdot)$ satisfies the condition (4) with

$$F_1 = \begin{bmatrix} 0.2 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & 0.2 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.24 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0.24 & 0 \\ 0 & 0 & 0.15 & 0 & 0 & 0.24 \end{bmatrix}.$$

The other related data accounting for the ROSSs and the missing measurements are taken to be $\bar{\lambda}_1 = 0.7$, $\bar{\lambda}_2 = 0.8$, $\bar{\xi}_1 = 0.86$, $\bar{\xi}_2 = 0.76$, $\sigma_1^2 = 0.8$ and $\sigma_2^2 = 0.66$. The saturation level vector w_{\max} for the saturation function $g(\cdot)$ is taken to be $(6, 8)^T$. In this paper the saturation function is assumed to satisfy condition (11) with $S = \text{diag}(0.1, 0.2)$.

With the parameters given above, we aim to design an output-feedback controller (9) for the 2-D time-delay system (1) such that the closed-loop 2-D system (10) is globally asymptotically stable in the mean square with a given H_∞ disturbance attenuation level γ . In the following, let the H_∞ performance level $\gamma = 0.8$ and the matrix $X = \text{diag}(1.7, 1.2, 1.6)$ in Theorem 3, by solving the matrix inequality (41) via Matlab Toolbox, we can obtain a feasible solution as follows (for space consideration, only part of the solution is listed here): $\varepsilon_1 = 20.9901$, $\varepsilon_2 = 24.2991$, $\varepsilon_3 = 20.1111$,

$$\tilde{A}_1 = \begin{bmatrix} 0.1289 & -0.0369 & -0.0512 \\ -0.0020 & 0.1850 & 0.0419 \\ -0.1576 & -0.1152 & 0.0146 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 0.0315 & -0.0136 & 0.0720 \\ 0.0911 & 0.1002 & -0.2984 \\ 0.0238 & 0.0291 & -0.1763 \end{bmatrix},$$

Fig. 1. State trajectory $x_1(k, h)$ of the controlled system (1).Fig. 2. State trajectory $x_2(k, h)$ of the controlled system (1).

$$\tilde{K}_1 = \begin{bmatrix} 0.4095 & -2.5345 \\ -1.1643 & -1.2066 \\ -0.6595 & 1.4734 \end{bmatrix}, \quad \tilde{K}_2 = \begin{bmatrix} 0.7722 & -2.6675 \\ -2.5481 & 0.3004 \\ 1.3063 & 0.1409 \end{bmatrix},$$

$$\tilde{H} = \begin{bmatrix} -0.2727 & -0.7414 & -0.4222 \\ -4.6907 & 1.8610 & -0.1680 \end{bmatrix}, \quad Y = \begin{bmatrix} 2.2708 & -0.8335 & -0.2343 \\ -0.8335 & 2.6414 & -0.4427 \\ -0.2343 & -0.4427 & 1.7706 \end{bmatrix}.$$

To design the specific output-feedback controller (9), decomposing the matrix $I - YX$ shown in (43) with the nonsingular matrices T and R as

$$T = \begin{bmatrix} 1.6667 & -0.4294 & -0.1430 \\ -0.4294 & 1.9135 & -0.2381 \\ -0.1430 & -0.2381 & 1.4797 \end{bmatrix}, \quad R = \begin{bmatrix} -1.5965 & 0.4046 & 0.1800 \\ 0.3554 & -1.0257 & 0.2283 \\ 0.1927 & 0.2669 & -1.1771 \end{bmatrix};$$

then the output-feedback controller gains in (9) can be designed as follows according to (42):

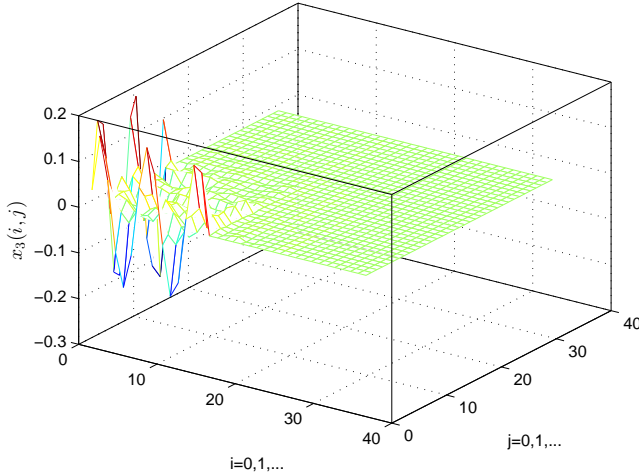
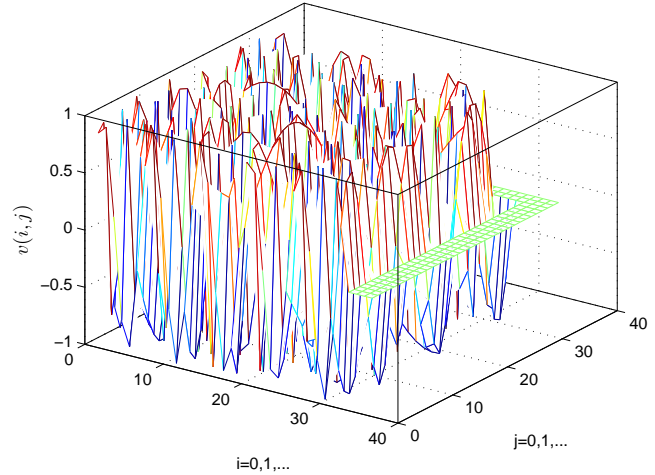
$$A_{1f} = \begin{bmatrix} 0.1518 & -0.0710 & -0.0861 \\ -0.1894 & 0.1195 & 0.1801 \\ 0.0868 & -0.2481 & 0.0759 \end{bmatrix}, \quad A_{2f} = \begin{bmatrix} -0.1035 & 0.0818 & 0.0869 \\ -0.0747 & -0.1155 & -0.2214 \\ 0.1897 & -0.0879 & -0.1337 \end{bmatrix},$$

$$K_{1f} = \begin{bmatrix} 0.0254 & -1.7011 \\ -0.6714 & -0.9274 \\ -0.5513 & 0.6821 \end{bmatrix}, \quad K_{2f} = \begin{bmatrix} 0.2166 & -1.6686 \\ -1.1945 & -0.2303 \\ 0.7115 & -0.1031 \end{bmatrix},$$

$$H_f = \begin{bmatrix} 0.5145 & 1.0527 & 0.6816 \\ 2.7967 & -0.7495 & 0.4306 \end{bmatrix}.$$

It follows immediately from Theorem 3 that the 2-D time-delay system (1) with parameters given above is globally asymptotically stable in the mean square with a given H_∞ disturbance attenuation level 0.8, if cooperated with the output-feedback controller (9) with controller gains designed as above. Moreover, by resorting to the Matlab LMI toolbox, it is obtained that the minimum H_∞ performance level γ can be taken as $\gamma^* = 0.4924$.

By taking the initial boundary condition $x(k, h)$ of the time-delay 2-D system (1) to be $(0.15 \tanh(k+8) \sin(h+7), -0.2 \tanh(k+8) \cos(h+7), 0.16 \tanh(k+8) \sin(h+7))^T$ for $(k, h) \in [-7, 0] \times [1, 14]$, $(0.15 \cos(k+8) \coth(h+7), -0.2 \tanh(k+8) \cos(h+7), 0.16 \tanh(k+8) \sin(h+7))^T$ for $(k, h) \in [0, 40] \times [1, 14]$, it is obtained that the state trajectories of the controlled system (1) are shown in Figs. 1 and 2.

Fig. 3. State trajectory $x_3(k, h)$ of the controlled system (1).Fig. 4. Exogenous disturbance input $v(k, h)$ of system (1).

$7)$, $0.18 \cot(k+8) \sin(h+7)$, $-0.2 \cos(k+8) \coth(h+7))^T$ for $(k, h) \in [1, 13] \times [-6, 0]$, $(-0.012, 0.06, 0.036)^T$ for $(k, h) = (0, 0)$, and $(0, 0, 0)^T$ for $(k, h) \in [-7, 0] \times [15, \infty) \cup [14, \infty) \times [-6, 0]$. With the output-feedback controller gains designed as above, the state trajectories of the closed-loop system (10) are given in Figs. 1-3. It can be seen from Figs. 1-3 that the time-delay 2-D system (1) under control without exogenous disturbance input is globally asymptotically stable in the mean square.

Furthermore, let the exogenous disturbance input $v(k, h)$ be $\sin((k+8)(h+7))$ for $(k, h) \in [0, 24] \times [0, 25]$ and 0 otherwise which have been explicitly shown in Fig. 4. Under the zero-initial condition, the dynamical behavior of the controlled output $z(k, h)$ of the time-delay 2-D system (1) is presented in Fig. 5 which further demonstrate the effectiveness of the design scheme proposed here.

V. CONCLUSIONS

This paper has investigated the H_∞ output-feedback control problem for the time-delay 2-D systems. Three sets of random variables have been introduced, respectively, to account for the phenomena of randomly occurring nonlinearities (RONs), missing measurements as well as randomly occurring sensor saturations (ROSSs). First, by resorting to an energy-like index and the intensive stochastic analysis, sufficient criteria have been given which guarantee the closed-loop 2-D system to be globally asymptotically stable in the mean square with an given H_∞ disturbance attenuation level. Then, two kinds of design schemes have been proposed separately which depend on the explicit solutions of certain matrix inequalities. A numerical example has also been given in the end of the paper to demonstrate the effectiveness of the proposed design schemes.

REFERENCES

- [1] S. Boyd, L. E. Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Philadelphia: SIAM, 1994.
- [2] Y.-Y. Cao, Z. Lin and B. M. Chen, An output dfeedback H_∞ controller design for linear systems subject to sensor nonlinearities, *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, Vol. 50, No. 7, pp. 914-921, 2003.
- [3] S.-F. Chen, Stability analysis for 2-D systems with interval time-varying delays and saturation nonlinearities, *Signal Processing*, Vol. 90, No. 7, pp. 2265-2275, 2010.
- [4] S.-F. Chen and I.-K. Fong, Delay-dependent robust H_∞ filtering for uncertain 2-D state-delayed systems, *Signal Processing*, Vol. 87, No. 11, pp. 2659-2672, 2007.

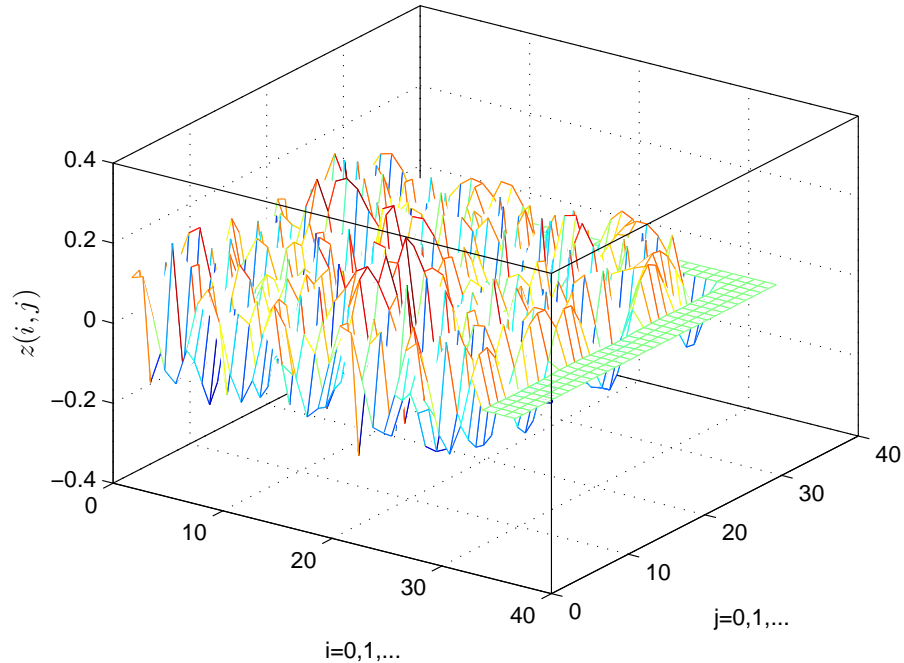


Fig. 5. Dynamical behavior of $z(k, h)$ for the controlled 2-D system (1).

- [5] A. Dey and H. Kar, Robust stability of 2-D discrete systems employing generalized overflow nonlinearities: an LMI approach, *Digital Signal Processing*, Vol. 21, No. 2, pp. 262-269, 2011.
- [6] D. Ding, Z. Wang, J. Hu and H. Shu, Dissipative control for state-saturated discrete time-varying systems with randomly occurring nonlinearities and missing measurements, *International Journal of Control*, Vol. 86, No. 4, pp. 674-688, 2013.
- [7] H. Dong, Z. Wang and H. Gao, *Filtering, Control and Fault Detection with Randomly Occurring Incomplete Information*, Wiley, Chichester, U.K., 2013.
- [8] C. Du, L. Xie and Y. C. Soh, H_∞ filtering of 2-D discrete systems, *IEEE Transactions on Signal Processing*, Vol. 48, No. 6, pp. 1760-1768, 2000.
- [9] C. Du, L. Xie and C. Zhang, H_∞ control and robust stabilization of two-dimensional systems in Roesser models, *Automatica*, Vol. 37, No. 2, pp. 205-211, 2001.
- [10] B. Dumitrescu, LMI stability tests for the Fornasini-Marchesini model, *IEEE Transactions on Signal Processing*, Vol. 56, No. 8, pp. 4091-4095, 2008.
- [11] E. Fornasini and G. Marchesini, Doubly-indexed dynamical systems: state-space models and structural properties, *Mathematical Systems Theory*, Vol. 12, No. 1, pp. 59-72, 1978.
- [12] H. Gao, J. Lam, C. Wang and S. Xu, H_∞ model reduction for uncertain two-dimensional discrete systems, *Optimal control Applications and Methods*, Vol. 26, No. 4, pp. 199-227, 2005.
- [13] H. Gao, J. Lam, S. Xu and C. Wang, Stability and stabilization of uncertain 2-D discrete systems with stochastic perturbation, *Multidimensional Systems and Signal Processing*, Vol. 16, No. 1, pp. 85-106, 2005.
- [14] L. El Ghaoui, F. Oustry and M. AitRami, A cone complementarity linearization algorithm for static output-feedback and related problems, *IEEE Transactions on Automatic Control*, Vol. 42, No. 8, pp. 1171-1176, 1997.
- [15] M. Hernandez-Gonzalez and M. V. Basin, Discrete-time optimal control for stochastic nonlinear polynomial systems, *International Journal of General Systems*, Vol. 43, Nos. 3-4, pp. 359-371, 2014.
- [16] M. Hernandez-Gonzalez and M. V. Basin, Discrete-time filtering for nonlinear polynomial systems over linear observations, *International Journal of Systems Science*, Vol. 45, No. 7, pp. 1461-1472, 2014.
- [17] T. Hinamoto, 2-D Lyapunov equation and filter design based on the Fornasini-Marchesini second model, *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, Vol. 40, No. 2, pp. 102-110, 1993.
- [18] A. Hmamed, F. Mesquine, F. Tadeo, M. Benhayoun and A. Benzaouia, Stabilization of 2D saturated systems by state feedback control, *Multidimensional Systems and Signal Processing*, Vol. 21, No. 3, pp. 277-292, 2010.
- [19] T. Kaczorek, *Two-Dimensional Linear Systems*, Springer-Verlag, Berlin, Germany, 1985.

- [20] H. Khalil, *Nonlinear Systems*, Upper Saddle River, NJ: Prentice-Hall, 1996.
- [21] J. Lam, S. Xu, Y. Zou, Z. Lin and K. Galkowski, Robust output feedback stabilization for two-dimensional continuous systems in Roesser form, *Applied Mathematics Letters*, Vol. 17, No. 12, pp. 1331-1341, 2004.
- [22] F. Leibfritz, An LMI-based algorithm for designing suboptimal static H_2/H_∞ output feedback controllers, *SIAM Journal on Control and Optimization*, Vol. 39, No. 6, pp. 1711-1735, 2001.
- [23] J. Liang, F. Sun and X. Liu, Finite-horizon H_∞ filtering for time-varying delay systems with randomly varying nonlinearities and sensor saturations, *Systems Science and Control Engineering: An Open Access Journal*, Vol. 2, No. 1, pp. 108-118, 2014.
- [24] Z. Lin, Feedback stabilization of multivariable two-dimensional linear systems, *International Journal of Control*, Vol. 48, No. 3, pp. 1301-1317, 1988.
- [25] Z. Lin, J. Lam, K. Galkowski and S. Xu, A constructive approach to stabilizability and stabilization of a class of nD systems, *Multidimensional Systems and Signal Processing*, Vol. 12, No. 3-4, pp. 329-343, 2001.
- [26] D. Liu and A. N. Michel, Stability analysis of state-space realizations for two-dimensional filters with overflow nonlinearities, *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, Vol. 41, No. 2, pp. 127-137, 1994.
- [27] Q. Liu, Z. Wang, X. He and D. Zhou, A survey of event-based strategies on control and estimation, *Systems Science and Control Engineering: An Open Access Journal*, Vol. 2, No. 1, pp. 90-97, 2014.
- [28] X. Liu, H. Gao and P. Shi, Robust H_∞ filtering for 2-D systems with intermittent measurements, *Circuits, Systems, and Signal Processing*, Vol. 28, No. 2, pp. 283-303, 2009.
- [29] W. Marszalek, Two-dimensional state-space discrete models for hyperbolic partial differential equations, *Applied Mathematical Modelling*, Vol. 8, No. 1, pp. 11-14, 1984.
- [30] M. Moayedi, Y. K. Foo and Y. C. Soh, Adaptive Kalman filtering in networked systems with random sensor delays, multiple packet dropouts and missing measurements, *IEEE Transactions on Signal Processing*, Vol. 58, No. 3, pp. 1577-1588, 2010.
- [31] W. Paszke, J. Lam, K. Galkowski, S. Xu and Z. Lin, Robust stability and stabilization of 2D discrete state-delayed systems, *Systems & Control Letters*, Vol. 51, No. 3-4, pp. 277-291, 2004.
- [32] B. Shen, Z. Wang and H. Shu, *Nonlinear Stochastic Systems with Incomplete Information: Filtering and Control*, Springer, London, U.K., 2013.
- [33] V. Singh, Robust stability of 2-D digital filters employing saturation, *IEEE Signal Processing Letters*, Vol. 12, No. 2, pp. 142-145, 2005.
- [34] Y. Tong, L. Zhang, M. V. Basin and C. Wang, Weighted H_∞ control with D-stability constraint for switched positive linear systems, *International Journal of Robust and Nonlinear Control*, Vol. 24, No. 4, pp. 758-774, 2014.
- [35] Z. Wang, D. W. C. Ho, H. Dong and H. Gao, Robust H_∞ finite-horizon control for a class of stochastic nonlinear time-varying systems subject to sensor and actuator saturations, *IEEE Transactions on Automatic Control*, Vol. 55, No. 7, pp. 1716-1722, 2010.
- [36] Z. Wang, B. Shen and X. Liu, H_∞ filtering with randomly occurring sensor saturations and missing measurements. *Automatica*, Vol. 48, No. 3, pp. 556-562, 2012.
- [37] Y. Wang, Z. Wang and J. Liang, Global synchronization for delayed complex networks with randomly occurring nonlinearities and multiple stochastic disturbances, *Journal of Physics A: Mathematical and Theoretical*, Vol. 42, No. 13, Art. No.: 135101, 2009.
- [38] G. Wei, L. Wang and F. Han, A gain-scheduled approach to fault-tolerant control for discrete-time stochastic delayed systems with randomly occurring actuator faults, *Systems Science and Control Engineering: An Open Access Journal*, Vol. 1, No. 1, pp. 82-90, 2013.
- [39] L. Wu, J. Lam and C. Wang, Robust H_∞ dynamic output feedback control for 2D linear parameter-varying systems, *IMA Journal of Mathematical Control and Information*, Vol. 26, No. 1, pp. 23-44, 2009.
- [40] L. Xie, C. Du, Y. Soh and C. Zhang, H_∞ and robust control of 2-D systems in FM second model, *Multidimensional Systems and Signal Processing*, Vol. 13, No. 3, pp. 265-287, 2002.
- [41] H. Xu, Y. Zou, S. Xu and L. Guo, Robust H_∞ control for uncertain two-dimensional discrete systems described by the general model via output feedback controllers, *International Journal of Control, Automation and Systems*, Vol. 6, No. 5, pp. 785-791, 2008.
- [42] F. Yang and Y. Li, Set-membership filtering for systems with sensor saturation, *Automatica*, Vol. 45, No. 8, pp. 1896-1902, 2009.
- [43] R. Yang, L. Xie and C. Zhang, H_2 and mixed H_2/H_∞ control of two-dimensional systems in Roesser model, *Automatica*, Vol. 42, No. 9, pp. 1507-1514, 2006.
- [44] J. Yao, W. Wang and S. Ye, Robust stabilization of 2-D state-delayed systems with stochastic perturbation, in: *11th International Conference on Control, Automation, Robotics and Vision*, Singapore, 7-10th December, pp. 1951-1956, 2010.