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# Coherent Chaos Interest Rate Models

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**Abstract** The Wiener chaos approach to interest rate modelling arises from the observation that in the general context of an arbitrage-free model with a Brownian filtration, the pricing kernel admits a representation in terms of the conditional variance of a square-integrable generator, which in turn admits a chaos expansion. When the expansion coefficients of the random generator factorise into multiple copies of a single function, the resulting interest rate model is called 'coherent', whereas a generic interest rate model is necessarily 'incoherent'. Coherent representations are of fundamental importance because an incoherent generator can always be expressed as a linear superposition of coherent elements. This property is exploited to derive general expressions for the pricing kernel and the associated bond price and short rate processes in the case of a generic $n$th order chaos model, for each $n \in \mathbb{N}$. Pricing formulae for bond options and swaptions are obtained in closed form for a number of examples. An explicit representation for the pricing kernel of a generic incoherent model is then obtained by use of the underlying coherent elements. Finally, finite-dimensional realisations of coherent chaos models are investigated and we show that a class of highly tractable models can be constructed having the characteristic feature that the discount bond price is given by a piecewise-flat (simple) process.

**Keywords** Pricing kernel · Conditional variance representation · Wiener chaos expansion · Fock space · Coherent states

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## 1 Introduction

For more than four decades now, interest rate modelling has been developed, leading to a range of approaches embodying different points of emphasis (see, e.g., James & Webber 2000, Cairns 2004, Hunt & Kennedy 2004, Brigo & Mercurio 2006, Filipovic 2009, Björk 2009, Carmona & Tehranchi 2010). One of the more recent approaches that has attracted attention is based on the specification of the pricing kernel, from which interest rate dynamics are deduced. One of the advantages of the pricing kernel method is that it allows a wide range of financial derivatives, across a variety of different asset classes, to be treated and priced in a consistent and transparent manner (see, for instance, Cochrane 2005). An early example is that of Flesaker & Hughston (1996, 1997, 1998), who introduced a framework that incorporates interest rate positivity in a canonical way. Extensions of the positive interest approach include Rutkowski (1997) and Jin & Glasserman (2001). Also within the positive-interest context, Rogers (1997, 2006) proposed a 'potential approach' for the modelling of the pricing kernel, based on the observation that the pricing kernel belongs to a certain class of probabilistic potentials.

The purpose of the present paper is to develop a new class of interest-rate models, called 'coherent interest-rate models', within the pricing kernel formalism. Coherent interest rate models emerge in the context of the Wiener-chaos expansion for the pricing kernel, originally introduced by Hughston & Rafailidis (2005) who made the observation that, in the context of a Brownian filtration, the class of potentials adequate for the characterisation of the pricing kernel admits a representation in terms of the conditional variance of a certain random variable, and proposed the use of the Wiener-Ito chaos expansion to model and calibrate the underlying random variable. The chaotic approach to interest-rate modelling was extended further in Brody & Hughston (2004), where the most general form for the arbitrage-free dynamics of a positive-interest term structure, within the context of Brownian filtration, is obtained by exploiting calculus on function spaces. See also Rafailidis (2005), Grasselli & Hurd (2005), Tsujimoto (2010) and Grasselli & Tsujimoto (2011) for further important contributions in the 'chaotic' approach to interest rate modelling.

In the present paper, we shall extend the chaos-based models for the pricing kernel in two distinct ways: (a) by working out general representations for chaos models for each chaos order; and (b) by introducing finite-dimensional realisations of chaos models based on function spaces. With these objectives in mind, the paper is organised as follows. In Sections 2–4 we briefly review some of the background material for the benefit of readers less acquainted with the theory of chaos models, so as to make the present paper reasonably self-contained. Specifically, in Section 2 we give a brief overview of the pricing kernel and its role in financial modelling, based on the axiomatic framework of Hughston & Rafailidis (2005). Section 3 summarises the argument leading to the conditional variance representation of Hughston & Rafailidis (2005) and the associated chaos expansion for calibration. In Section 4 we explain the definition of the coherent chaos representation introduced in Brody & Hughston (2004), and its role in interest-rate modelling.

In Section 5 we introduce the notion of an $n^{\text{th}}$-order coherent chaos model, and derive the general representation for the pricing kernel, the short rate, the bond price, and the risk premium in this model. Explicit examples of derivative pricing formulae are then obtained in Sections 6-8 for coherent interest rate models. Specifically, bond option and swaption prices are obtained in closed form. Coherent chaos models are important because they act as building blocks for general interest rate models. That is, a generic interest rate model can be expressed in the form of a linear superposition of the underlying coherent components (in the sense explained in Section 4), and thus are incoherent. By exploiting the linearity structure we are able to derive a general expression in Section 9 for the pricing kernel for $n^{\text{th}}$-order incoherent chaos models.

In Section 10 we investigate finite-dimensional realisations of coherent chaos models. We note that a chaos coefficient is given by an element of an infinite-dimensional Hilbert space of square-integrable functions. When this coefficient is replaced by the square-root of a *finite* number of Dirac delta functions, then the resulting system can in effect be treated in a finite-dimensional Hilbert space. The corresponding interest rate models are 'simple' in the sense that the pricing kernel and the bond price processes are given by piecewise step functions, where the step sizes are sampled from nonlinear functions of independent Gaussian random variables. The idea of a model based on a finite-dimensional realisation is that it can be used to fit a finite-number of initial bond prices, without relying on interpolation methods. We conclude in Section 11 with a brief discussion and further comments.

## 2 The pricing kernel approach to interest rate modelling

The pricing kernel approach to valuation and risk management provides perhaps the most direct route towards deriving various familiar results in financial modelling. It also provides an insight into the relation between the risk, risk aversion, and return arising from risky investments when asset prices can jump—an idea that is perhaps difficult to grasp from other approaches available (Brody, Hughston & Mackie 2012). For these reasons here we shall adapt the pricing kernel approach to interest rate modelling. We shall begin in this section with a brief review of the idea of a pricing kernel. In particular, we find it convenient to follow the axiomatic scheme introduced in Hughston & Rafailidis (2005).

The axioms listed below are neither minimum nor unique (see also Brody & Hughston 2004, Rogers 2006, Hughston & Macrina 2008, 2010, Hughston & Mina 2012); however, we find that the approach taken in Hughston & Rafailidis (2005) leads to the desired representation of the pricing kernel needed for our purposes in the most expedient manner, and thus we shall take this as our starting point.

We proceed as follows. We model the economy with a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the 'physical' probability measure. We equip this space with the standard augmented filtration $\{\mathcal{F}_t\}_{0\leq t<\infty}$ generated by a system of one or more independent Brownian motions $\{W_t^\alpha\}_{0\leq t<\infty}$, $\alpha = 1, \ldots, k$. We assume that asset prices are continuous semimartingales on $(\Omega, \mathcal{F}, \mathbb{P})$, which will enable us to use various standard results of stochastic calculus. With this setup in mind, Hughston & Rafailidis (2005) characterise the absence of arbitrage in the economy by assuming the existence of a pricing kernel, a strictly positive Ito process $\{\pi_t\}_{t\geq 0}$ (also known variously as a state-price deflator, stochastic discount factor, or state-price density), such that the following set of axioms hold:

(a) There exists a money-market account asset with a strictly-increasing and absolutely-continuous price process $\{B_t\}_{t\geq 0}$.
(b) Given any asset with price process $\{S_t\}_{t\geq 0}$ and with $\{\mathcal{F}_t\}$-adapted dividend rate process $\{D_t\}_{t\geq 0}$, the process $\{M_t\}_{t\geq 0}$ defined by

$$M_t = \pi_t S_t + \int_0^t \pi_s D_s \mathrm{d}s \tag{1}$$

is a $\mathbb{P}$-martingale.
(c) There exists an asset—a floating rate note—that offers a continuous dividend rate such that its value remains constant over time.
(d) There exists a system of discount bonds $\{P_{tT}\}_{0\leq t\leq T\leq\infty}$ with the asymptotic property that

$$\lim_{T\to\infty} P_{tT} = 0. \tag{2}$$

From axiom (a), we deduce the existence of an $\{\mathcal{F}_t\}$-adapted short rate process $\{r_t\}_{t\geq 0}$ such that $r_t > 0$ for all $t \geq 0$ and

$$\mathrm{d}B_t = r_t B_t \mathrm{d}t. \tag{3}$$

Since the money market account is an asset that pays no dividend, it follows from axiom (b) that the process $\{\rho_t\}_{t\geq 0}$ defined by

$$\rho_t = \pi_t B_t \tag{4}$$

is a martingale. Note that at most one process $B_t$ satisfying (a) and (b) can exist. Due to the fact that $\pi_t > 0$ and $B_t > 0$, $\rho_t$ is strictly positive for all $t \geq 0$. Since, $\{\rho_t\}$ is a positive martingale, we can write

$$\mathrm{d}\rho_t = -\rho_t \lambda_t \mathrm{d}W_t \tag{5}$$

for some $\{\mathcal{F}_t\}$-adapted vector-valued process $\{\lambda_t\}_{t\geq 0}$. Here, for simplicity of notation, we write $\lambda_t \mathrm{d}W_t$ to denote the vector inner product $\sum_{\alpha=1}^k \lambda_t^\alpha \mathrm{d}W_t^\alpha$. As a consequence of (3), (4), and (5), the dynamical equation satisfied by the pricing kernel takes the form

$$\mathrm{d}\pi_t = -r_t \pi_t \mathrm{d}t - \lambda_t \pi_t \mathrm{d}W_t. \tag{6}$$

Equivalently, we can write

$$\pi_t = \exp\left(-\int_0^t r_s \mathrm{d}s - \int_0^t \lambda_s \mathrm{d}W_s - \tfrac{1}{2}\int_0^t \lambda_s^2 \mathrm{d}s\right). \tag{7}$$

Since the pricing kernel is a product of a martingale and a strictly decreasing process, it follows that $\{\pi_t\}$ is a supermartingale.

Consider now a system of default-free discount (zero coupon) bonds. We assume, in accordance with axiom (d), that the economy supports such a system of bonds over all time horizons. We write $P_{tT}$ for the value at time $t$ of a $T$-maturity bond that pays one unit of currency at maturity $T$. We deduce from axiom (b) that for each maturity $T$ the process $\{\pi_t P_{tT}\}_{0\le t\le T}$ is a martingale, which we denote by $\{M_{tT}\}_{0\le t\le T}$. It follows from equation (4) that

$$P_{tT} = \frac{M_{tT} B_t}{\rho_t}. \tag{8}$$

Since $\{M_{tT}\}$ is a parametric family of martingales, there exists a family of vector-valued processes $\{\sigma_{tT}\}_{0\le t\le T}$ such that we can write

$$\mathrm{d}M_{tT} = M_{tT}(\sigma_{tT} - \lambda_t)\mathrm{d}W_t. \tag{9}$$

By use of Ito's product and quotient rules, it follows from equations (3), (5), (8) and (9) that the dynamical equation satisfied by the discount-bond system is given by

$$\mathrm{d}P_{tT} = P_{tT}(r_t + \lambda_t \sigma_{tT})\mathrm{d}t + P_{tT}\sigma_{tT}\mathrm{d}W_t, \tag{10}$$

which implies that $P_{tT}$ can be written in the form

$$P_{tT} = P_{0T} B_t \exp\left(\int_0^t \sigma_{sT}\,(\mathrm{d}W_s + \lambda_s\,\mathrm{d}s) - \tfrac{1}{2}\int_0^t \sigma_{sT}^2\,\mathrm{d}s\right). \tag{11}$$

We recognise the process $\{\sigma_{tT}\}_{0\le t\le T}$ for a fixed $T$ as the $T$-maturity discount bond volatility, while the vector process $\{\lambda_t\}_{t\ge 0}$ is the associated market price of risk.

As remarked above, the pricing kernel approach provides an effective method for the valuation of contingent claims. Thus, for example, if $H_T$ is the payout of a derivative at time $T$, then on account of axiom (b) we find that the value of the derivative at time $t$ is given by

$$H_t = \frac{1}{\pi_t}\mathbb{E}_t[\pi_T H_T]. \tag{12}$$

In particular, for a unit cash flow $H_T = 1$ we obtain the bond pricing formula

$$P_{tT} = \frac{1}{\pi_t}\mathbb{E}_t[\pi_T]. \tag{13}$$

Since $\{\pi_t\}$ is a supermartingale, it follows that $\pi_T \ge \mathbb{E}_T[\pi_U]$ for $T \le U$, and hence $\mathbb{E}_t[\pi_T] \ge \mathbb{E}_t[\pi_U]$ by the tower property. Thus $\mathbb{E}_t[\pi_T]$ is monotonically decreasing as a function of $T$, and we deduce that the limit $\lim_{T\to\infty}\mathbb{E}_t[\pi_T]$ exists. It follows then by use of (2) that $\lim_{T\to\infty}\mathbb{E}_t[\pi_T] = 0$, $\forall\ t \geq 0$, and hence $\lim_{T\to\infty}\mathbb{E}[\pi_T] = 0$. This shows, under the axioms assumed, that the pricing kernel is necessarily a potential, i.e. a right-continuous positive supermartingale whose expectation vanishes as $T\to\infty$ (Meyer 1966).

The specification of the pricing kernel therefore leads on the one hand to the bond price dynamics as well as the associated short rate process, while on the other hand to the pricing of general contingent claims. It is for these reasons that we prefer to model the pricing kernel directly.

## 3 Conditional variance and the Wiener chaos expansion

To proceed we follow the observation made in Hughston & Rafailidis (2005) that the pricing kernel can be expressed in the form of a conditional variance of an $\mathcal{F}_\infty$-measurable square-integrable random variable. The idea is as follows. We write (6) in the integral form

$$\pi_T - \pi_t = -\int_t^T r_s\pi_s\mathrm{d}s - \int_t^T \lambda_s\pi_s\mathrm{d}W_s, \tag{14}$$

and take the $\mathcal{F}_t$-conditional expectation on each side of (14) to obtain

$$\pi_t = \mathbb{E}_t[\pi_T] + \mathbb{E}_t\left[\int_t^T r_s\pi_s\mathrm{d}s\right]. \tag{15}$$

Now taking the limit $T\to\infty$ we deduce the following implicit relation satisfied by the pricing kernel:

$$\pi_t = \mathbb{E}_t\left[\int_t^\infty r_s\pi_s\mathrm{d}s\right]. \tag{16}$$

This relation can also be seen as following from axiom (c). Note that since $\{\pi_t\}$ and $\{r_t\}$ are both $\{\mathcal{F}_t\}$-adapted, the vector valued process $\{\eta_t\}_{t\geq 0}$ defined by the relation

$$\eta_t^2 = \sum_{\alpha=1}^k \eta_t^\alpha\eta_t^\alpha = r_t\pi_t \tag{17}$$

is also $\{\mathcal{F}_t\}$-adapted. Evidently, the vector $\eta_t$ for each $t\geq 0$ is unique only up to SO($k$)-rotational degrees of freedom. For any given representative element $\{\eta_t\}$ of this equivalence class we define an $\{\mathcal{F}_t\}$-martingale $\{X_t\}$ by setting

$$X_t = \int_0^t \eta_s\mathrm{d}W_s. \tag{18}$$

Then we have

$$X_\infty - X_t = \int_t^\infty \eta_s\mathrm{d}W_s, \tag{19}$$

from which, on account of the conditional Wiener-Ito isometry we deduce that

$$\pi_t = \mathbb{E}_t\left[ (X_\infty - \mathbb{E}_t[X_\infty])^2 \right]. \tag{20}$$

We shall refer to this identity as the conditional-variance representation for the pricing kernel.

The foregoing analysis shows that to model the pricing kernel, it suffices to model the generating random variable $X_\infty$ that is an element of the Hilbert space $\mathcal{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$ of square integrable random variables on $(\Omega, \mathcal{F}_\infty, \mathbb{P})$. The proposal of Hughston & Rafailidis (2005) is to employ the Wiener chaos expansion to 'parametrise' the generator $X_\infty$, work out $\{\pi_t\}$, and use the result to obtain pricing formulae for various derivatives; which in turn allows one to calibrate the functional parameters in the chaos expansion (see also Björk 2007). Specifically, the generator $X_\infty$ admits a unique expansion of the form

$$X_\infty = \phi + \int_0^\infty \phi(s_1)\mathrm{d}W_{s_1} + \int_0^\infty \left( \int_0^{s_1} \phi(s_1, s_2)\mathrm{d}W_{s_2} \right) \mathrm{d}W_{s_1} + \cdots, \tag{21}$$

where $\phi = \mathbb{E}[X_\infty]$, $\phi(s) \in \mathcal{L}^2(\mathbb{R}_+)$ is a square-integrable function of one variable on the positive half-line $\mathbb{R}_+$, $\phi(s, s') \in \mathcal{L}^2(\mathbb{R}_+) \otimes \mathcal{L}^2(\mathbb{R}_+)$ is a square-integrable function of two variables, and so on. Notice that the antisymmetric part of the expansion coefficient $\phi(s_1, s_2, \cdots, s_n)$ makes null contribution to the stochastic integral over the range $\mathbb{R}_+^n$; thus we may assume that the functions are defined over symmetric tensor product spaces; additionally, we consider the subspace $\Delta_+^n \subset \mathbb{R}_+^n$ defined by the triangular region

$$\Delta_+^n = \left\{ (s_1, s_2, \cdots, s_n) \in \mathbb{R}_+^n \,|\, 0 \leq s_n \leq \cdots \leq s_2 \leq s_1 \leq \infty \right\}, \tag{22}$$

and consider square-integrable functions on $\Delta_+^n$ (Hughston & Rafailidis 2005).

The idea of a chaos expansion was introduced by Wiener in his seminal paper entitled "Homogeneous chaos" (Wiener 1938); the specific representation (21) in terms of the stochastic integral is due to Ito (1951). We refer to Nualart (2006), Janson (1997), Malliavin (1997), Øksendal (1997), and Di Nunno *et al.* (2009) for further discussion of the chaos expansion technique and its role in stochastic analysis.

For interest-rate modelling, we see that by substituting (21) in (20), the pricing kernel is parameterised by a vector $\Phi$ of a set of deterministic quantities:

$$\Phi = \left( \phi,\ \phi(s),\ \phi(s, s_1),\ \phi(s, s_1, s_2),\ \cdots \right), \tag{23}$$

where $\Phi$ is itself an element of a Fock space $\mathfrak{F}$ of the direct sum of the Hilbert spaces of square-integrable functions. The elements of $\Phi$ are called the *Wiener-Ito chaos coefficients*. These coefficients fully characterise the information in $X_\infty$. Since $X_\infty$ determines the pricing kernel $\{\pi_t\}$, which in turn generates the bond price process $\{P_{tT}\}$, each interest rate model can be viewed as depending on the specification of its Wiener-Ito chaos coefficients.

## 4 Coherent chaos expansion

There is a special class of vectors in $\mathfrak{F}$ called 'coherent vectors'. Such vectors admit a number of desirable characteristics. Let us consider the symmetric tensor product $\mathcal{H}^{(n)} = \mathcal{L}^2(\mathbb{R}_+)\otimes\mathcal{L}^2(\mathbb{R}_+)\otimes\cdots\otimes\mathcal{L}^2(\mathbb{R}_+)$ of $n$ copies of Hilbert spaces. A generic element of $\mathcal{H}^{(n)}$ may be written in the form $\phi(s_1, s_2, \cdots, s_n)$; whereas a coherent vector of $\mathfrak{F}$ is generated by a map from an element of $\mathcal{H}^{(1)}$ to an element of $\mathcal{H}^{(n)}$. Specifically, given $\phi(s) \in \mathcal{H}^{(1)} = \mathcal{L}^2(\mathbb{R}_+)$, we consider an element of $\mathcal{H}^{(n)}$ of the 'degenerate' form

$$\phi(s_1, s_2, \cdots, s_n) = \phi(s_1)\phi(s_2)\ldots\phi(s_n). \tag{24}$$

The importance of coherent vectors is that the totality of such vectors in $\mathcal{H}^{(n)}$ constitutes a resolution of the identity, i.e. these vectors are in general not orthogonal but nevertheless are complete. Therefore, an arbitrary element of $\mathcal{H}^{(n)}$ can be expressed in the form of a linear combination of (possibly uncountably many) coherent vectors.

More generally, given an element $\phi(s) \in \mathcal{H}^{(1)}$ we can generate a coherent Fock vector of the form

$$\Phi_\phi = \big(1, \phi(s), \phi(s_1)\phi(s_2), \cdots, \phi(s_1)\phi(s_2)\ldots\phi(s_n)\big). \tag{25}$$

Then an arbitrary element of $\mathfrak{F}$ can likewise be expressed as a linear combination of coherent vectors. The significance of the completeness of the system of coherent vectors for the chaos expansion, noted in Brody & Hughston (2004), is as follows. First we observe that if $\Phi_\phi$ is coherent, the associated generator $X_\infty^\phi$ arising from the chaos expansion (21) takes the form

$$X_\infty^\phi = \sum_{n=0}^{\infty}\left(\int_0^\infty\int_0^{s_1}\cdots\int_0^{s_{n-1}} \phi(s_1)\phi(s_2)\cdots\phi(s_n)\mathrm{d}W_{s_n}\cdots\mathrm{d}W_{s_2}\mathrm{d}W_{s_1}\right). \tag{26}$$

Here the $n = 0$ term is assumed to be unity. Then we make use of the following identity due to Ito (1951):

$$\int_0^T\int_0^{s_1}\cdots\int_0^{s_{n-1}} \phi(s_1)\phi(s_2)\cdots\phi(s_n)\mathrm{d}W_{s_n}\cdots\mathrm{d}W_{s_1} = \frac{Q_T^{n/2}}{n!}H_n\left(\frac{R_T}{Q_T^{1/2}}\right), \tag{27}$$

where

$$R_t = \int_0^t \phi(u)\mathrm{d}W_u \quad \text{and} \quad Q_t = \int_0^t \phi^2(u)\mathrm{d}u, \tag{28}$$

and where

$$H_n(x) = n!\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k\, x^{n-2k}}{k!\,(n-2k)!\,2^k} \tag{29}$$

denotes the $n^{\text{th}}$ Hermite polynomial, determined by the generating function

$$\exp\left(tx - \tfrac{1}{2}t^2\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x). \tag{30}$$

The role of the Hermite polynomials in relation to Gaussian random variables is of course well known (see, e.g., Schoutens 2000). We remark that if $Y$ and $Z$ are standard normal random variables, then

$$\mathbb{E}[H_n(Y)H_m(Z)] = (\mathbb{E}[YZ])^n \ \delta_{mn}, \tag{31}$$

which follows from the fact that Hermite polynomials are orthogonal with respect to the standard normal density function. Here $\delta_{nm}$ denotes the Kronecker delta, i.e. $\delta_{nm} = 0$ if $n \neq m$ and $\delta_{nn} = 1$ for all $n$. By comparing (27) and (30) we thus deduce that

$$X_{\infty}^{\phi} = \exp\left(\int_0^{\infty} \phi(s)\mathrm{d}W_s - \tfrac{1}{2}\int_0^{\infty} \phi^2(s)\mathrm{d}s\right). \tag{32}$$

On account of the completeness of the coherent vectors and linearity, therefore, it follows that an arbitrary $\mathcal{F}_{\infty}$-measurable square-integrable random variable $X_{\infty}$ admits a representation of the form

$$X_{\infty} = \sum_j c_j \exp\left(\int_0^{\infty} \phi_j(s)\mathrm{d}W_s - \tfrac{1}{2}\int_0^{\infty} \phi_j^2(s)\mathrm{d}s\right), \tag{33}$$

where $\{c_j\}$ are constants satisfying $\sum_j c_j^2 < \infty$, where $\phi_j(s) \in \mathcal{L}^2(\mathbb{R}_+)$ for each $j$ is a deterministic square-integrable function, and where the summation in (33) is formal and may be replaced by an appropriate integration in the uncountable case. (For example, both the expansion coefficient $c(\theta)$ and the square-integrable function $\phi(s,\theta)$ may depend on a parameter $\theta$, in which case the summation over the index $j$ in (33) is replaced by an integration over the parameter $\theta$.) It should be evident that an analogous result holds more generally for an arbitrary $\mathcal{F}_t$-measurable square-integrable random variable—that is, any such random variable can be expressed in terms of a linear combination of log-normally distributed random variables. Putting the matter differently, log-normal random variables are dense in the space of square-integrable random variables. This fact was applied in Brody & Hughston (2004) to identify the general expressions for the pricing kernel and other quantities (such as the bond price, risk premium, and the various interest rates) in the Brownian-based setting, owing to the fact that the conditional variance of $X_{\infty}$ in (33) is easily calculated in closed form.

## 5 Coherent chaos interest-rate models

The point of departure in the present investigation is to examine the $n^{\text{th}}$-order chaos models for each $n$ more closely. That is to say, we are concerned with interest-rate models that arise from the chaos expansion of the form

$$\int_0^\infty \int_0^{s_1} \cdots \int_0^{s_{n-1}} \phi(s_1, s_2, \ldots, s_n)\, \mathrm{d}W_{s_n} \cdots \mathrm{d}W_{s_2} \mathrm{d}W_{s_1} \tag{34}$$

for each $n$, where $\phi(s_1, s_2, \ldots, s_n) \in \mathcal{H}^{(n)}$. As indicated above, any such function $\phi(s_1, s_2, \ldots, s_n)$ can be expressed as a linear combination of a (possibly uncountable) collection of coherent functions of the form $\phi(s_1)\phi(s_2)\cdots\phi(s_n)$. Let us therefore introduce:

**Definition 1** An $n^{\text{th}}$-*order coherent chaos model* is generated by a random variable of the form

$$X_\infty^{[n]} = \int_0^\infty \int_0^{s_1} \cdots \int_0^{s_{n-1}} \phi(s_1)\phi(s_2)\cdots\phi(s_n)\, \mathrm{d}W_{s_n} \cdots \mathrm{d}W_{s_2} \mathrm{d}W_{s_1}. \tag{35}$$

Our strategy therefore is first to work out the interest rate model arising from such a coherent element of $\mathcal{H}^{(n)}$, i.e. an $n^{\text{th}}$-order coherent chaos model, and then to consider their linear combinations for more general $n^{\text{th}}$-order chaos models for the pricing kernel.

To proceed, we recall that from (27) it follows that for a coherent element we have

$$X_\infty^{[n]} = \frac{Q_\infty^{n/2}}{n!} H_n\left(\frac{R_\infty}{Q_\infty^{1/2}}\right) \tag{36}$$

and

$$X_t^{[n]} = \frac{Q_t^{n/2}}{n!} H_n\left(\frac{R_t}{Q_t^{1/2}}\right), \tag{37}$$

where we have written

$$X_t^{[n]} = \mathbb{E}_t\left[X_\infty^{[n]}\right]. \tag{38}$$

Note that the argument $R_t/Q_t^{1/2}$ appearing in the Hermite polynomial is, for each $t$, a standard normally distributed random variable. Thus on account of (31) we find that the martingales $\{X_t^{[n]}\}_{0\leq t\leq\infty}$ for different values of $n$ are mutually orthogonal. This property will be exploited in a calculation below when we analyse the more general 'incoherent' models.

Let us turn to the determination of the pricing kernel $\{\pi_t^{[n]}\}$ defined by

$$\pi_t^{[n]} = \mathbb{E}_t\left[\left(X_\infty^{[n]}\right)^2\right] - \left(X_t^{[n]}\right)^2. \tag{39}$$

**Proposition 1** *The pricing kernel* $\{\pi_t^{[n]}\}$ *associated with an* $n^{\text{th}}$*-order coherent chaos generator* $X_\infty^{[n]}$ *of the form* (36) *is given by an expression of the form*

$$\pi_t^{[n]} = \sum_{k=0}^{n} \frac{[2(n-k)]!\,(1-Q_t^k)\,X_t^{[2n-2k]}}{k!\,[(n-k)!]^2}. \tag{40}$$

*Proof* To analyse the expressions in (39) for each $n \in \mathbb{N}$, we make use of the following product identity for a pair of Hermite polynomials of different order (see, e.g., Janson 1997, p. 28):

$$H_n(x)H_m(x) = \sum_{k=0}^{m\wedge n} \binom{m}{k}\binom{n}{k} k!\, H_{m+n-2k}(x), \tag{41}$$

where $m \wedge n = \min(m, n)$. Setting $m = n$ we obtain

$$H_n^2(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} k!\, H_{2n-2k}(x). \tag{42}$$

Since, from (29), we have

$$X_t^{[n]} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k\, R_t^{n-2k}\, Q_t^k}{k!\,(n-2k)!\,2^k}, \tag{43}$$

it follows that by squaring $X_t^{[n]}$ of (37) and making use of (42), we deduce that

$$\left(X_t^{[n]}\right)^2 = \sum_{k=0}^{n} \frac{Q_t^k\,[2(n-k)]!\,X_t^{[2n-2k]}}{k!\,[(n-k)!]^2}. \tag{44}$$

Taking the limit $t \to \infty$ in (44) and substituting the result in (39), we obtain (40), after unit-normalising the function $\phi(s)$ such that $Q_\infty = 1$. □

We observe from (37) that $\{\pi_t^{[n]}\}$ in this case is given by a polynomial of order $2n-2$ of the Gaussian process $\{R_t\}$. Having obtained an expression for the pricing kernel for each $n \in \mathbb{N}$ we are in a position to determine representations for the various quantities of interest. To this end, let us first derive expressions for the short rate and risk premium processes.

**Proposition 2** *Let* $\{r_t^{[n]}\}$ *be the short rate process in the coherent chaos model of order* $n$ *and* $\{\lambda_t^{[n]}\}$ *be the associated risk premium process. Then*

$$r_t^{[n]} = \frac{\phi^2(t)}{\pi_t^{[n]}} \sum_{k=0}^{n} \frac{[2(n-k)]!\,Q_t^{k-1}\,X_t^{[2n-2k]}}{(k-1)!\,[(n-k)!]^2} \tag{45}$$

*and*

$$\lambda_t^{[n]} = -\frac{\phi(t)}{\pi_t^{[n]}} \sum_{k=0}^{n} \frac{(1-Q_t^k)\,[2(n-k)]!}{k!\,[(n-k)!]^2} X_t^{[2n-2k-1]}. \tag{46}$$

*Proof* On account of (6) we observe that the processes are given by the drift and the volatility of the pricing kernel. If the pricing kernel takes the form (40), then an application of the Ito calculus along with the the relation

$$\mathrm{d}X_t^{[2n-2k]} = X_t^{[2n-2k-1]}\phi(t)\mathrm{d}W_t \tag{47}$$

gives us the desired expressions at once. □

As for the discount bonds, it follows from (13) that the bond price process takes the form of a ratio of polynomials of the Gaussian process $\{R_t\}$:

**Proposition 3** *Let $P_{tT}^{[n]}$ be the discount bond process for the coherent chaos model of order $n$. Then*

$$P_{tT}^{[n]} = \frac{\sum_{k=0}^{n} A_{k,n}(1-Q_T^k)\; X_t^{[2n-2k]}}{\sum_{k=0}^{n} A_{k,n}(1-Q_t^k)\; X_t^{[2n-2k]}}, \tag{48}$$

*where*

$$A_{k,n} = \frac{[2(n-k)]!}{k![(n-k)!]^2}. \tag{49}$$

*The initial term structure is given by the discount function*

$$P_{0T}^{[n]} = 1 - Q_T^n. \tag{50}$$

*Proof* Substituting (40) into (13) and making use of the martingale property of $\{X_t^{[k]}\}$ we deduce (48) at once. The expression (50) can be seen by noting that $X_0^{[2n-2k]} = \delta_{nk}$. □

The initial term structure (50) shows, in particular, that in the case of a single-factor setup, coherent chaos models are fully characterised by the initial term structure. Such a strong constraint, of course, is expected in the case of a single factor model for which the only model 'parameter' is a deterministic scalar function $\phi(t)$, and it is indeed natural that these functional degrees of freedom should be fixed unambiguously by the initial yield curve. On the other hand, the general cases are obtained simply by taking linear superpositions of the coherent chaos models in an appropriate manner. Before we examine these general cases, let us first work out a number of derivative-pricing formulae in the case of single-factor coherent chaos models.

## 6 Bond option pricing for second-order coherent chaos models

We recall expression (12) for the price of a generic derivative. The purpose of this section is to derive explicit formulae for the prices of options on zero-coupon bonds in a second order coherent chaos model. In this case, it follows from (40) that the pricing kernel is given by a quadratic function of a single Gaussian state variable $R_t$:

$$\pi_t^{[2]} = (1-Q_t)(R_t^2 - Q_t) + \tfrac{1}{2}(1-Q_t^2). \tag{51}$$

The bond price process can then be written as a rational function of $R_t$:

$$P_{tT}^{[2]} = \frac{(1-Q_T)(R_t^2 - Q_t) + \frac{1}{2}(1-Q_T^2)}{(1-Q_t)(R_t^2 - Q_t) + \frac{1}{2}(1-Q_t^2)}. \tag{52}$$

We shall make use of (51) and (52) to find the pricing formula for a European-style call option on a discount bond. In particular, let $t$ be the option maturity, let $T \geq t$ be the bond maturity and let $K$ be the option strike. Then bearing in mind that $\pi_0^{[2]} = \frac{1}{2}$ under the convention $Q_\infty = 1$ that we have chosen here, we find that the initial price of the option is given by

$$\begin{aligned} C_0^{[2]}(t,T,K) &= 2\,\mathbb{E}\left[\pi_t^{[2]}\left(P_{tT}^{[2]} - K\right)^+\right] \\ &= 2\,\mathbb{E}\Big[\Big(R_t^2\left[(1-Q_T) - K(1-Q_t)\right] - Q_t(1-Q_T) \\ &\qquad + \tfrac{1}{2}(1-Q_T^2) - \tfrac{1}{2}K\left(1-Q_t\right)^2\Big)^+\Big] \\ &= 2\,\mathbb{E}\left[\left(AZ_t^2 + B\right)^+\right], \end{aligned} \tag{53}$$

where $Z_t = R_t/\sqrt{Q_t}$ is a standard normal random variable, and where

$$A = Q_t\left[(1-Q_T) - K(1-Q_t)\right] \tag{54}$$

and

$$B = \tfrac{1}{2}\left[(1-Q_T^2) - K(1-Q_t^2)\right] - Q_t\left[(1-Q_T) - K(1-Q_t)\right]. \tag{55}$$

The problem of pricing the call option in this model thus reduces to finding the roots of a quadratic equation $Az^2 + B = 0$. Letting $z_1, z_2$ denote the roots:

$$z_1 = -\sqrt{-\frac{B}{A}} \quad \text{and} \quad z_2 = +\sqrt{-\frac{B}{A}}, \tag{56}$$

we must consider different scenarios depending on the signatures of the coefficients $A$ and $B$. We shall proceed to examine this case by case.

(i) If $A = 0$ then $B = \frac{1}{2}(1-Q_T)(Q_T - Q_t) > 0$ since $Q_T > Q_t$, so the option is always in the money and we have

$$C_0^{[2]} = P_{0T}^{[2]} - KP_{0t}^{[2]} = (1-Q_T)(Q_T - Q_t), \tag{57}$$

which is equal to $2B$, as is evident from (53).

(ii) If $A > 0$ then $B > 0$. This follows from the fact that $A > 0$ implies $K < (1-Q_T)/(1-Q_t)$. Since $B = \frac{1}{2}(1-Q_T^2) - Q_t(1-Q_T) - \frac{1}{2}K(1-Q_t)^2$, the bound on $K$ implies $B > \frac{1}{2}(1+Q_T)(Q_T - Q_t)$. But since $Q_T > Q_t$, we have $B > 0$. In this case, $Az^2 + B$ is always positive, so again the call option will always expire in the money. The price of such an option is thus

$$\begin{aligned} C_0^{[2]} &= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} (Az^2 + B)\, \mathrm{e}^{-\frac{1}{2}z^2} \mathrm{d}z \\ &= P_{0T}^{[2]} - K P_{0t}^{[2]} \\ &= (1 - Q_T^2) - K(1 - Q_t^2), \end{aligned} \tag{58}$$

which is $2(A+B)$.

(iii) If $A < 0$ and $B \leq 0$, then the payoff is never positive, resulting in an option whose value is zero.

(iv) The nontrivial case is when we have that $A < 0$ and $B > 0$. Then the quadratic polynomial is positive over the interval $(z_1, z_2)$, and we have

$$\begin{aligned} C_0^{[2]} &= \sqrt{\frac{2}{\pi}} \int_{z_1}^{z_2} \left(Az^2 + B\right)\, \mathrm{e}^{-\frac{1}{2}z^2} \mathrm{d}z \\ &= (P_{0T}^{[2]} - K P_{0t}^{[2]})\, [1 - 2N(z_1)] + 4Az_1 \rho(z_1) Q_t, \end{aligned} \tag{59}$$

where $N(z)$ denotes the standard cumulative normal distribution function, and $\rho(z)$ the associated density function. A short calculation shows that the option delta that gives the position on the underlying bond required to hedge the option, in this case, is given by

$$\Delta = 1 - 2N(z_1) + \frac{1}{2Az_1}\, \rho(z_1)(P_{0T} - KP_{0t})(4Az_1 - 1). \tag{60}$$

We see that in a coherent chaos model of the second order, the pricing of options on discount bonds is entirely tractable and so is the determination of the associated hedge portfolio. As an illustrative example, let us consider the case in which we choose $\phi^2(s) = \lambda^{-1} \exp(-\lambda s)$ for some constant $\lambda$. As indicated above, in a coherent chaos model, this is all we need in order to calculate both bond prices and call prices. In Figure 1 on the left panel we show the relation between the call price and its strike. Here, the bond maturity is fixed at $T = 10$ and the option maturity at $t = 3$, and we set $\lambda = 0.1$. On the right-side of Figure 1 we show how the call price changes with respect to the bond values for different maturities. Here, the option maturity has been chosen to be $t = 5$, the bond maturities vary from 100 to five, the strike has been fixed at 0.7, and we have set $\lambda = 0.03$.

## 7 Pricing swaptions in second-order coherent chaos models

It turns out that swaptions can also be valued in closed form, in a manner analogous to the factorisable second-order chaos models of Hughston & Rafailidis (2005). In this case, the payout of the contract can be expressed in the

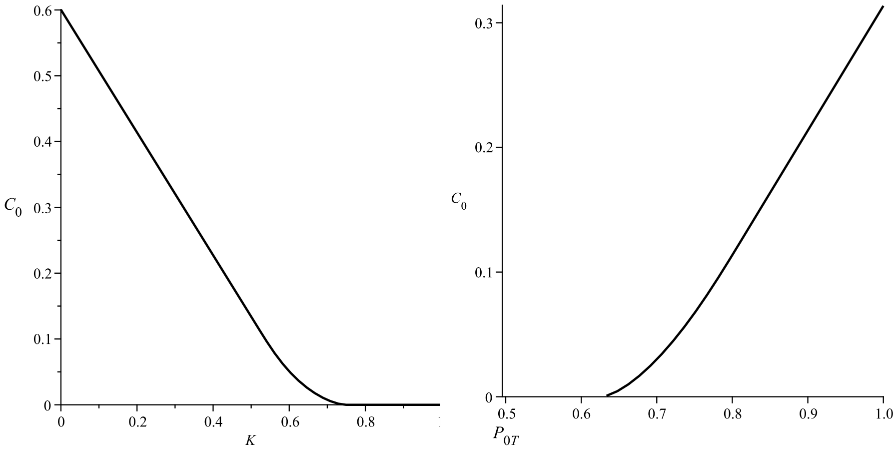


Fig. 1: Left panel: the call price as a function of the strike for fixed bond maturity ($T = 10$) and fixed option maturity ($t = 3$), in the second-order coherent chaos model with $\phi^2(s) = (0.1)^{-1} \exp(-0.1s)$. Right panel: the call price as a function of the initial bond price, where the bond price is varied by changing its mathrity $T$ from 100 to 5. Other parameters are set as $t = 5$, $K = 0.7$, and $\phi^2(s) = (0.03)^{-1} \exp(-0.03s)$.

form:

$$H_t^{[2]} = \left(1 - P_{tT_n}^{[2]} - K \sum_{i=1}^{n} P_{tT_i}^{[2]}\right)^+ , \tag{61}$$

where $T_i,\ i = 1, \ldots, n$, are specified future payment dates. The initial value of the swaption is then given by the expectation

$$\begin{aligned} H_0^{[2]} &= 2\,\mathbb{E}\left[\pi_t^{[2]} \left(1 - P_{tT_n}^{[2]} - K \sum_{i=1}^{n} P_{tT_i}^{[2]}\right)^+\right] \\ &= 2\,\mathbb{E}\left[\left(AZ_t^2 + B\right)^+\right], \end{aligned} \tag{62}$$

where $Z_t = R_t/\sqrt{Q_t}$ is defined as before. In the present case, however, we have

$$A = Q_t[(Q_{T_n} - Q_t) - K \sum_{i=1}^{n} (1 - Q_{T_i})] \tag{63}$$

and

$$B = \tfrac{1}{2}(Q_{T_n} - Q_t)^2 + KQ_t \sum_{i=1}^{n} (1 - Q_{T\,i}) - \frac{K}{2} \sum_{i=1}^{n} (1 - Q_{T_i}^2). \tag{64}$$

In other words, the valuation of a swaption proceeds in the manner as that of a call option, except that the coefficients defined as $A$ and $B$ are a little more complicated. Writing $z_1$ and $z_2$ for the roots of $Az^2 + B = 0$, we thus have the following:

(i) When $A < 0$ and $B \leq 0$, the swaption has value zero.
(ii) When $A > 0$ and $B \geq 0$, we have

$$H_0^{[2]} = P_{0t}^{[2]} - P_{0T_n}^{[2]} - K \sum_{i=1}^{n} P_{0T_i}^{[2]}. \tag{65}$$

(iii) When $A < 0$ and $B > 0$ so that $-B/A > 0$, we have

$$\begin{aligned} H_0^{[2]} &= 2(A+B)\left[N(z_2) - N(z_1)\right] + 2A\left[z_1\rho(z_1) - z_2\rho(z_2)\right] \\ &= \left(P_{0t}^{[2]} - P_{0T_n}^{[2]} - K \sum_{i=1}^{n} P_{0T_i}^{[2]}\right)\left[1 - 2N(z_1)\right] + 4Az_1\rho(z_1). \end{aligned} \tag{66}$$

(iv) When $A > 0$ and $B < 0$ so that $-B/A > 0$, we have

$$H_0^{[2]} = 2N(z_1)\left(P_{0t}^{[2]} - P_{0T_n}^{[2]} - K \sum_{i=1}^{n} P_{0T_i}^{[2]}\right) - 4Az_1\rho(z_1). \tag{67}$$

## 8 Bond option pricing for third-order coherent chaos models

The pricing kernel for a third-order pure coherent chaos model can be expressed in the form

$$\pi_t^{[3]} = 6(1 - Q_t)X_t^{[4]} + (1 - Q_t^2)X_t^{[2]} + \frac{1}{6}(1 - Q_t^3). \tag{68}$$

The bond price process is consequently given by

$$P_{tT}^{[3]} = \frac{36(1 - Q_T)X_t^{[4]} + 6(1 - Q_T^2)X_t^{[2]} + (1 - Q_T^3)}{36(1 - Q_t)X_t^{[4]} + 6(1 - Q_t^2)X_t^{[2]} + (1 - Q_t^3)}. \tag{69}$$

The value at time zero of a European-style call option on a discount bond in this model is thus expressible as

$$\begin{aligned} C_0^{[3]} &= \mathbb{E}\left[\frac{\pi_t^{[3]}}{\pi_0^{[3]}}\left(P_{tT}^{[3]} - K\right)^+\right] \\ &= 3!\,\mathbb{E}\left[\left(AZ_t^4 + BZ_t^2 + C\right)^+\right], \end{aligned} \tag{70}$$

where

$$A = \frac{1}{4}Q_t^2\left[(1 - Q_T) - K(1 - Q_t)\right], \tag{71}$$

$$B = \frac{1}{2} Q_t \left[(1 - Q_T^2) - K(1 - Q_t^2)\right] - \frac{3}{2} Q_t^2 \left[(1 - Q_T) - K(1 - Q_t)\right], \quad (72)$$

and

$$C = \frac{1}{6}\left[(1 - Q_T^3) - K(1 - Q_t^3)\right] - \frac{1}{2} Q_t \left[(1 - Q_T^2) - K(1 - Q_t^2)\right] \\ + \frac{3}{4} Q_t^2 \left[(1 - Q_T) - K(1 - Q_t)\right]. \quad (73)$$

The roots of the quartic polynomial $Az^4 + Bz^2 + C$ are given by

$$z_1 = -\sqrt{\frac{-B - \sqrt{\delta}}{2A}}, \quad z_2 = -\sqrt{\frac{-B + \sqrt{\delta}}{2A}}, \quad z_3 = -z_2, \quad z_4 = -z_1, \quad (74)$$

where $\delta = B^2 - 4AC$. Thus, we have the following:

(i) If $A = 0$ then $B = \frac{1}{2} Q_t(1 - Q_T)(Q_T - Q_t)$ and $C = \frac{1}{6}(1 - Q_T)(Q_T - Q_t)(Q_T - Q_t + 1 - Q_t)$. Since both $B$ and $C$ are positive, the option is always in the money and we have

$$C_0^{[3]} = \frac{6}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Bz^2 + C)\mathrm{e}^{-\frac{1}{2}z^2} \mathrm{d}z \\ = (1 - Q_T)(Q_T - Q_t)(1 + Q_t + Q_T). \quad (75)$$

(ii) If $A > 0$ and $-B - \sqrt{\delta} \geq 0$ then this implies that $B \leq -\sqrt{\delta}$. Since there are four roots, the call option in this case is

$$C_0^{[3]} = \frac{3!}{\sqrt{2\pi}} \left( \int_{-\infty}^{z_2} + \int_{z_1}^{z_4} + \int_{z_3}^{\infty} \right) \left(Az^4 + Bz^2 + C\right) \mathrm{e}^{-\frac{1}{2}z^2} \mathrm{d}z \\ = 6(3A + B + C)(2N(z_2) - 2N(z_1) + 1) \\ +12(3A + B)(z_1\rho(z_1) - z_2\rho(z_2)) \\ +12A(z_1^3\rho(z_1) - z_2^3\rho(z_2)). \quad (76)$$

(iii) If $A > 0$ and $-B - \sqrt{\delta} < 0$ but $-B + \sqrt{\delta} \geq 0$ then if $-\sqrt{\delta} < B \leq 0$, the initial value of the call option is

$$C_0^{[3]} = \frac{3!}{\sqrt{2\pi}} \left( \int_{-\infty}^{z_2} + \int_{z_3}^{\infty} \right) \left(Az^4 + Bz^2 + C\right) \mathrm{e}^{-\frac{1}{2}z^2} \mathrm{d}z \\ = 12(3A + B + C)N(z_2) - 12(Az_2^2 + 3A + B)z_2\rho(z_2). \quad (77)$$

If $0 < B \leq \sqrt{\delta}$, then

$$C_0^{[3]} = \frac{3!}{\sqrt{2\pi}} \int_{z_2}^{z_3} \left(Az^4 + Bz^2 + C\right) \mathrm{e}^{-\frac{1}{2}z^2} \mathrm{d}z \\ = 6(3A + B + C)\left(1 - 2N(z_2)\right) + 12(Az_2^2 + 3A + B)z_2\rho(z_2). \quad (78)$$

(iv) If $A > 0$ and $-B + \sqrt{\delta} < 0$ then the value of the option is zero.

(v) If $A<0$ and $-B-\sqrt{\delta}>0$ then

$$C_0^{[3]} = \frac{3!}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Az^4+Bz^2+C)\,\mathrm{e}^{-\frac{1}{2}z^2}\mathrm{d}z \tag{79}$$

$$= 6(3A+B+C). \tag{80}$$

(vi) If $A<0$ and $-B+\sqrt{\delta}>0$ but $-B-\sqrt{\delta}\leq 0$ then if $-\sqrt{\delta}<B\leq 0$,

$$C_0^{[3]} = \frac{3!}{\sqrt{2\pi}} \left(\int_{-\infty}^{z_1} + \int_{z_4}^{\infty}\right) (Az^4+Bz^2+C)\,\mathrm{e}^{-\frac{1}{2}z^2}\mathrm{d}z \tag{81}$$

$$= 12(3A+B+C)N(z_1) - 12(Az_1^2+3A+B)z_1\rho(z_1). \tag{82}$$

If $0<B\leq\sqrt{\delta}$,

$$C_0^{[3]} = \frac{3!}{\sqrt{2\pi}} \int_{z_1}^{z_4} (Az^4+Bz^2+C)\,\mathrm{e}^{-\frac{1}{2}z^2}\mathrm{d}z \tag{83}$$

$$= 6(3A+B+C)(1-2N(z_1)) + 12(Az_1^2+3A+B)z_1\rho(z_1). \tag{84}$$

(vii) Finally, if $A<0$ and $-B+\sqrt{\delta}\leq 0$ then

$$C_0^{[3]} = \frac{3!}{\sqrt{2\pi}} \left(\int_{z_1}^{z_2} + \int_{z_3}^{z_4}\right) (Az^4+Bz^2+C)\,\mathrm{e}^{-\frac{1}{2}z^2}\mathrm{d}z$$

$$= 12(3A+B+C)(N(z_2)-N(z_1)) + 12(Az_1^2+3A+B)z_1\rho(z_1) - 12(Az_2^2+3A+B)z_2\rho(z_2). \tag{85}$$

In summary, we observe that in general for a coherent chaos model of order $n$ the pricing kernel is a polynomial of order $2n-2$ in the single Gaussian process $\{R_t\}$. It follows that for the valuation of an option or a swaption, the relevant calculation reduces to that of taking the expectation of the positive part of a polynomial of the same order in the standard normal random variable $Z_t = R_t/\sqrt{Q_t}$. In other words, the problem reduces to the identification of the roots of a polynomial of order $2n-2$. Since such a root-finder can be carried out numerically, we find that semi-analytic expressions for both option and swaption prices are available in the case of an $n^{\text{th}}$-order coherent chaos model.

## 9 Incoherent chaos models

We have examined the coherent chaos interest-rate models in some detail with the view to generalise these models to more generic cases, which we call *incoherent chaos models*. As indicated above, our key observation is that the $\mathcal{F}_\infty$-measurable square-integrable generator $X_\infty^{[n]}$ associated with an $n^{\text{th}}$-order chaos expansion can be expressed as a linear combination of 'coherent' log-normal random variables $X_\infty^{[n]}(\phi_i)$ for different structure functions $\phi_i(s)$. Putting the matter differently, we have the representation

$$X_\infty^{[n]} = \sum_i c_i\, X_\infty^{[n]}(\phi_i), \tag{86}$$

where for clarity we have written

$$X_\infty^{[n]}(\phi_i) = \int_0^\infty \int_0^{s_1} \cdots \int_0^{s_{n-1}} \phi_i(s_1)\phi_i(s_2)\cdots\phi_i(s_n)\,\mathrm{d}W_{s_n}\cdots\mathrm{d}W_{s_2}\mathrm{d}W_{s_1} \quad (87)$$

for each $\phi_i$ for the coherent random variables. The pricing kernel associated with a general $n^{\text{th}}$-order chaos model can thus be obtained by working out the following expectation:

$$\pi_t^{[n]} = \mathbb{E}_t\left[\left(\sum_i c_i\, X_\infty^{[n]}(\phi_i)\right)^2\right] - \left(\mathbb{E}_t\left[\sum_i c_i\, X_\infty^{[n]}(\phi_i)\right]\right)^2. \quad (88)$$

The second term on the right side of (88) can be evaluated on account of the martingale relation

$$\mathbb{E}_t[X_\infty^{[n]}(\phi_i)] = X_t^{[n]}(\phi_i). \quad (89)$$

Calculating the first term on the right side of (88) is evidently a little more complicated. We proceed as follows. First, observe from (87) that the random variable $X_\infty^{[n]}(\phi_i)$ can be written in the recursive form

$$X_\infty^{[n]}(\phi_i) = \int_0^\infty \phi_i(t) X_t^{[n-1]}(\phi_i)\,\mathrm{d}W_t. \quad (90)$$

It follows, by use of the conditional form of the Wiener-Ito isometry, that

$$\begin{aligned}\mathbb{E}_t\left[X_\infty^{[n]}(\phi_i)X_\infty^{[n]}(\phi_j)\right] &= X_t^{[n]}(\phi_i)X_t^{[n]}(\phi_j) \\ &\quad + \int_t^\infty \phi_i(u)\phi_j(u)\mathbb{E}_t\left[X_u^{[n-1]}(\phi_i)X_u^{[n-1]}(\phi_j)\right]\mathrm{d}u. \quad (91)\end{aligned}$$

An application of the recursion relation (90) on $X_u^{[n-1]}$ together with another use of the conditional form of the Wiener-Ito isometry then shows that what remains in the conditional expectation on the right side of (91) is reduced to order $n-2$. By iteration, we then find that

$$\mathbb{E}_t\left[X_\infty^{[n]}(\phi_i)X_\infty^{[n]}(\phi_j)\right] = \sum_{k=0}^{n} \langle\phi_i,\phi_j\rangle_t^{[k]} X_t^{[n-k]}(\phi_i)X_t^{[n-k]}(\phi_j), \quad (92)$$

where for simplicity of notation we have written

$$\langle\phi_i,\phi_j\rangle_t^{[k]} = \int_t^\infty\int_t^{s_1}\cdots\int_t^{s_{k-1}} \phi_i(s_1)\phi_j(s_1)\cdots\phi_i(s_k)\phi_j(s_k)\,\mathrm{d}s_k\cdots\mathrm{d}s_1, \quad (93)$$

with $\langle\phi_i,\phi_j\rangle_t^{[0]} = 1$. Putting these together, we obtain

$$\pi_t^{[n]} = \sum_{i,j} c_i c_j \left[\sum_{k=0}^{n}\langle\phi_i,\phi_j\rangle_t^{[k]} X_t^{[n-k]}(\phi_i)X_t^{[n-k]}(\phi_j) - X_t^{[n]}(\phi_i)X_t^{[n]}(\phi_j)\right], \quad (94)$$

from which we conclude:

**Proposition 4** *The pricing kernel in a $n^{\text{th}}$-order coherent model can be expressed in the form*

$$\pi_t^{[n]} = \sum_{i,j} c_i c_j \sum_{k=1}^{n} \langle \phi_i, \phi_j \rangle_t^{(k)} X_t^{[n-k]}(\phi_i) X_t^{[n-k]}(\phi_j). \tag{95}$$

*Example 1*: As an illustration, consider the case in which $n = 2$, and suppose that the chaos expansion involves the superposition of a pair of functions. Thus, we have

$$\Phi(s_1, s_2) = \sum_{i=1}^{2} c_i \; \phi_i(s_1) \; \phi_i(s_2). \tag{96}$$

In this case, after some calculation, we obtain the following expression for the pricing kernel:

$$\begin{aligned} \pi_t^{[2]} = {} & 2c_1^2 \left( R_t^2\,(\phi_1)(1 - Q_t(\phi_1)) + \tfrac{1}{2}\,(1 - Q_t(\phi_1))^2 \right) \\ & + 2c_2^2 \left( R_t^2\,(\phi_2)(1 - Q_t(\phi_2)) + \tfrac{1}{2}\,(1 - Q_t(\phi_2))^2 \right) \\ & + 2c_1 c_2 R_t(\phi_1) R_t(\phi_2) \int_t^\infty \phi_1(s)\phi_2(s)\mathrm{d}s \\ & + c_1 c_2 \left( \int_t^\infty \phi_1(s)\phi_2(s)\mathrm{d}s \right)^2 . \end{aligned} \tag{97}$$

Here, we have written $R_t(\phi_i) = \int_0^t \phi_i(s)\mathrm{d}W_s, \; i = 1, 2$, for the two Gaussian state-variable processes. We see that the pricing kernel in this case is a simple quadratic cross-polynomial of the two state variables. Analogous results hold for higher $n$ and for a larger number of terms in the expansion.

*Example 2*: More generally, consider an 'incoherent' model that consists of a combination of coherent terms of different chaos orders. Suppose that $X_\infty$ is given by a combination of a first order and an $n^{\text{th}}$ order coherent chaos element:

$$X_\infty = X_\infty^{[1]}(\phi_1) + X_\infty^{[n]}(\phi_2). \tag{98}$$

Taking the conditional variance and making use of (92), we find that the associated pricing kernel takes the form

$$\pi_t^{[n]} = 1 - Q_t(\phi_1) + \sum_{k=1}^{n} \frac{\left[ X_t^{[n-k]}(\phi_2) \right]^2}{(k!)^2} + 2X_t^{[n-1]}(\phi_2) \int_t^\infty \phi_1(s)\phi_2(s)\mathrm{d}s, \tag{99}$$

from which the corresponding term-structure dynamics can easily be derived.

## 10 Finite dimensional realisations of coherent chaos models

In this section we consider the case in which the Hilbert space of square-integrable functions is approximated by (or replaced with) a finite-dimensional Hilbert space. To facilitate the calculation, we make use of the Dirac delta function so that the function $\phi(x)$ is given by the square root of a weighted sum of a finite number of delta functions. The idea of finite-dimensional realisations is to generate a set of models that can be calibrated purely in terms of a finite number of available market data, without evoking the assumption of hypothetical initial bond prices across all maturities.

To proceed, we recall first that we have made the normalisation convention such that

$$Q_\infty = \int_0^\infty \phi^2(s)\mathrm{d}s = 1. \tag{100}$$

Hence, $\phi^2(s)$ can be interpreted as defining a probability density function, except that now in effect in a finite-dimensional space. More precisely, and strictly speaking, we continue to work in the infinite-dimensional Hilbert space, but by facilitating a finite number of distributions rather than functions, the analysis effectively reduces to that based on a finite-dimensional Hilbert space. We therefore choose $\phi^2$ to take the following form:

$$\phi^2(s) = \sum_{i=1}^{N+1} p_i \delta(s - T_i), \tag{101}$$

where $\delta(s)$ denotes the Dirac delta function, and the $\{T_i\}_{i=1,\ldots,N}$ represent different maturity dates of bonds for which prices are available in the market; $N$ indicates how many of these there are. The coefficients $\{p_i\}$ are probability weights so that $0 \le p_i \le 1$ and that $\sum_{i=1}^{N+1} p_i = 1$. For this choice of $\phi$ the integral $Q_t$ then takes the following form of a piecewise step function:

$$Q_t = \sum_{i=1}^{N+1} p_i \mathbb{1}_{\{T_i \le t\}} = \sum_{i=1}^{N+1} \mathbb{1}_{\{T_i \le t < T_{i+1}\}} \sum_{j=1}^{i} p_j. \tag{102}$$

We remark that in (101) and (102) we have artificially introduced $T_{N+1}$, where $T_{N+1} > T_N$, such that the bond price is assumed to become zero at this point. Note that $T_{N+1}$ is not an expiry date, but rather an arbitrary time that is beyond the maturity date of the bond in the market with the longest lifetime. The choice of $T_{N+1}$ will not affect the valuation of contracts whose maturities are $\le T_N$; thus the analysis below will not be affected by the arbitrariness of the choice of $T_{N+1}$. The reason for introducing $T_{N+1}$ is to fulfil the asymptotic condition (2) of axiom (d) in the finite-dimensional setup. Conversely, had we not introduced $T_{N+1}$, the discussion below would not be affected.

Let us now consider the $n^{\text{th}}$-order coherent chaos model associated with the positive square-root of (101). A short calculation using expression (102) in the result (50) for $P_{0t}^{[n]}$ shows that the corresponding initial bond prices with maturity $t$ admit the following representation:

$$P_{0t}^{[n]} = 1 - \sum_{i=1}^{N+1} \mathbb{1}_{\{T_i \le t < T_{i+1}\}} \left( \sum_{j=1}^{i} p_j \right)^n . \qquad (103)$$

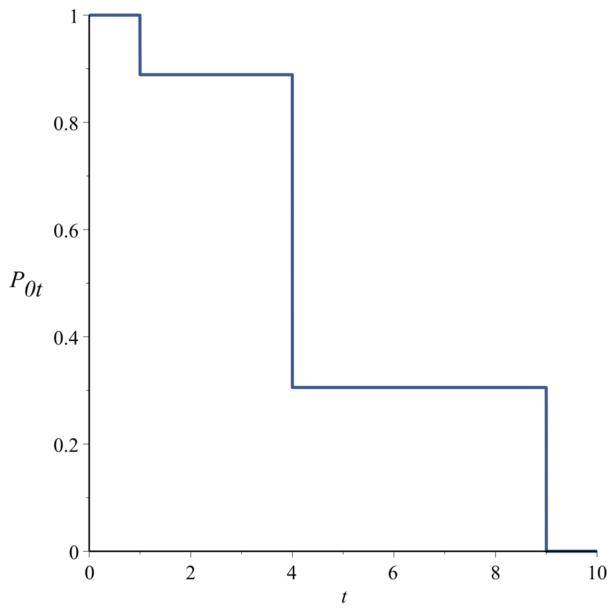


Fig. 2: Initial bond prices $P_{0t}^{[n]}$ as a function of maturity time for $n = 2$. Two bond prices are assumed given at times $T_1 = 1$ and $T_2 = 4$; while at time $T_3 = 9$ the bond price is assumed to jump to zero. The parameter values chosen here are $p_1 = \frac{1}{6}$, $p_2 = \frac{1}{2}$, and $p_3 = \frac{1}{3}$.

To illustrate the initial term structure in the present model we sketch in Figure 2 an example of the initial bond prices as a function of the maturity time for $n = 2$. Three maturities have been used here: $T_1 = 1$, $T_2 = 4$ and the 'artificially' chosen $T_3 = 9$. According to (103) the initial value $P_{0T_1}$ of a bond expiring at time $T_1$ (where the first jump occurs) is $P_{0T_1} = 1 - p_1^n$, the price at time $T_2$ is $P_{0T_2} = 1 - (p_1 + p_2)^n$, and so on. Hence in the pure second-order coherent chaos model with only two data points specified, we have the values $P_{0T_1} = 1 - p_1^2$, $P_{0T_2} = 1 - (p_1 + p_2)^2$, and $P_{0T_3} = 1 - (p_1 + p_2 + p_3)^2 = 0$.

Since it is assumed that the maturities $\{T_i\}_{i=1,\ldots,N}$ are the points where we have market data for the bond prices, we see that the probability weights $\{p_i\}$ can be calibrated from the initial market prices of the discount bonds. This

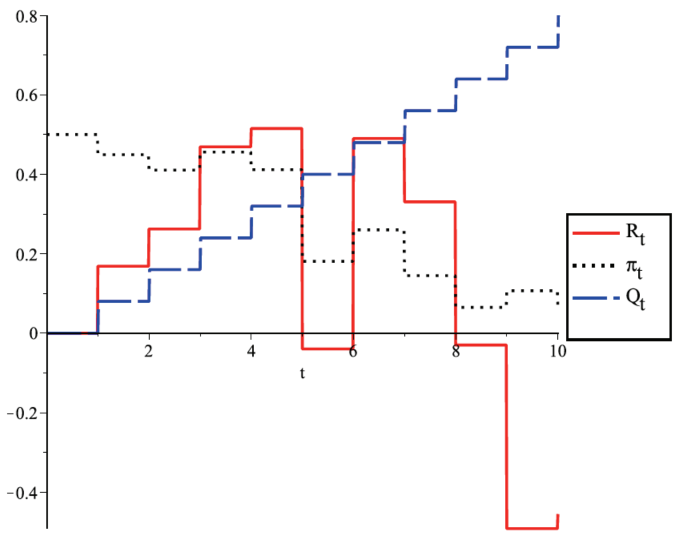


Fig. 3: Sample path simulations for the Gaussian driver $\{R_t\}$, its quadratic variation $\{Q_t\}$, and the resulting pricing kernel $\{\pi_t\}$, in the case of a finite-dimensional coherent chaos model with $n = 2$. The parameters chosen here are $p_i = 0.08$ and $T_i = i$ for $i = 1 \ldots 10$, and $p_{11} = 0.2$ with $T_{11} > 10$ arbitrary.

in turn determines the function $\phi(s)$, which in turn determines the subsequent dynamics for the term structure. Let us proceed to analyse the term structure dynamics in the finite-dimensional models. To begin, recall that $R_t \sim \mathcal{N}(0, Q_t)$, so in order to determine the Gaussian process $\{R_t\}$ whose probabilistic characteristics are identical to that of $\int_0^t \phi(s)\mathrm{d}W_s$, where $\phi(s)$ is of the form (101), let us consider a family of independent Gaussian random variables

$$n_i \sim \mathcal{N}\left(0, \sum_{j=1}^{i} p_j\right) \qquad \text{for } i = 1, \ldots, N+1. \tag{104}$$

Then we can represent the Gaussian process $\{R_t\}$ according to

$$R_t = \sum_{i=1}^{N+1} \mathbb{1}_{\{T_i \le t < T_{i+1}\}}\, n_i \tag{105}$$

where equality here holds in probability. By making use of (40) and (43), the corresponding pricing kernel can be identified in the finite-dimensional models. In figure 3 we illustrate typical sample paths of $\{Q_t\}$, $\{R_t\}$, and $\{\pi_t^{[2]}\}$ for $n = 2$

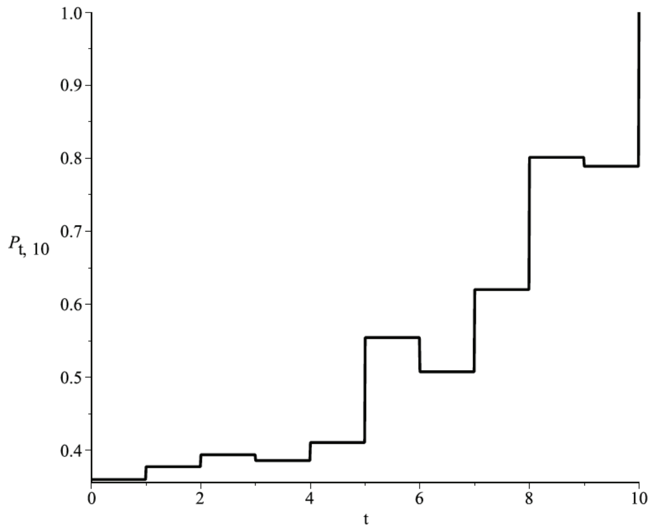


Fig. 4: A sample path simulation for the discount bond price process $\{P_{tT}\}$ in the coherent chaos model for order $n = 2$. The parameters chosen here as $p_i = 0.08$ for $i = 1, \ldots, 10$, and the bond maturity is set as $T = 10$. The bond price fluctuates up and down, but eventually converges to its terminal value one. The prices jump at the points at which the initial bond prices are assumed given in accordance with (103).

and $N = 10$. For simplicity we have chosen a uniform probability and equal spacing: $p_i = 0.08$ and $T_i = i$ for $i = 1, \ldots, 10$. Similarly, in figure 4, we show a sample path of a corresponding bond price process, where we have again chosen $p_i = 0.08$ for $i = 1, \ldots, 10$ with bond maturity $T = 10$.

Summing up the section, we see that using the finite-dimensional approach, a class of elementary, highly tractable and easily calibrated models can be constructed. These models have the characteristic feature that the discount bond process is piecewise flat and that the distribution of the bond price at any time is determined by a ratio of polynomials of standard normal random variables. The bond price processes emerging in these models can alternatively be viewed as representing ‘simple’ approximations to more sophisticated continuous processes. To illustrate how a typical bond price in these models behaves, in figure 5 we show an example of the bond price dynamics where the ‘grid size’ is made ten times finer than the one sketched in figure 4. It is interesting to note in this connection that although the Gaussian process $R_t = \int_0^t \phi(s)\mathrm{d}W_s$ formally appears to be continuous, owing to the appearance of distributions in (101) the resulting process contains jumps, as is evident in figure 3.

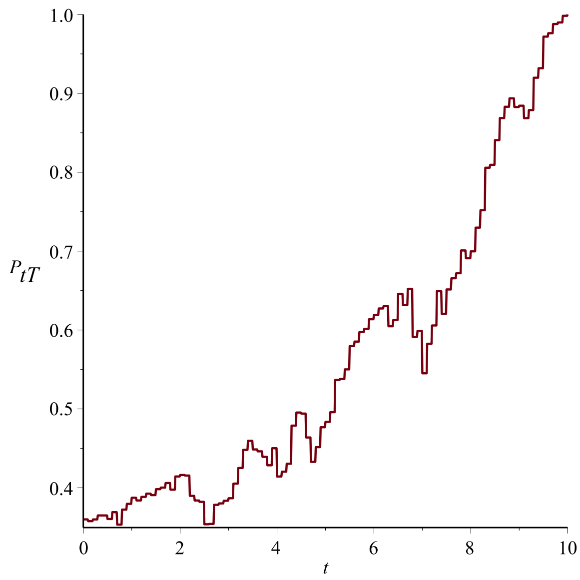


Fig. 5: A sample path simulation for the discount bond price process $\{P_{tT}\}$ in the coherent chaos model for order $n = 2$. The parameters chosen here are $p_i = 0.008$ for $i = 1, \ldots, 100$, and the bond maturity is set as $T = 10$. The bond price fluctuates up and down, but eventually converges to its terminal value one. The prices jump at the points at which the initial bond prices are assumed given in accordance with (103).

## 11 Conclusion and discussion

The purpose of this paper is to offer an in-depth analysis of the coherent chaos interest-rate models of each order with the view that they form the basis for general (incoherent) interest-rate models. We have found that for a pure $n^{\text{th}}$ order coherent chaos model, the pricing kernel is given by a polynomial of order $2n - 2$ in a Gaussian state variable process. This leads to tractable expressions for the bond price, the risk premium, and the short rate processes. Additionally, we have shown that in all these models it is possible to derive either analytic or semianalytic formulae for the pricing of both bond options and swaptions, involving at most the numerical determination of the roots of basic polynomials.

Single-factor coherent chaos models, although themselves considerably richer than elementary Gaussian interest rate models, are too restrictive on account of the fact that each coherent chaos model depends on a single functional degree of freedom. For more realistic models, it suffices to take linear superpositions of coherent chaos elements in the sense described in Section 4. The resulting expression for the pricing kernel, in the most generic case, is tractable albeit cumbersome. For practical purposes it seems sufficient to consider a small number of (two or three) coherent vectors to generate rich and flexible interest-rate models. We hope that the results presented here will form a foundation for further investigation into this direction.

By way of comparison, we draw attention to the implementation of Wiener chaos models that has been carried out in Grasselli & Tsujimoto (2011) in the case of a third-order chaos model, with the choices $\phi(s_1) = \alpha(s_1)$, $\phi(s_1, s_2) = \beta(s_1)$, and $\phi(s_1, s_2, s_3) = \gamma(s_1)$. Their model has been shown to fit the forward curve more closely, and with less parameters, than various currently preferred models employed in industry. An interesting extension would therefore be to consider the implementation of the following generalisation: $\phi(s_1) = \alpha(s_1)$, $\phi(s_1, s_2) = \beta(s_1)\beta(s_2)$, and $\phi(s_1, s_2, s_3) = \gamma(s_1)\gamma(s_2)\gamma(s_3)$, in line with the foregoing discussion, and examine how well the generalised model fits both the forward and volatility curves.

In relation to the analysis on finite-dimensional models, it is worth making the following observation. For simplicity of discussion, if we set $N = 1$, then formally we are led to an expression of the form

$$R_t = \int_0^t \sqrt{\delta(s-T)}\,\mathrm{d}W_s \tag{106}$$

for the Gaussian process $\{R_t\}$, the meaning of which *a priori* is not easily interpreted. In the present paper we have circumvented the direct handling of processes of the form (106) and its generalisations by means of identifying for each $t$ an alternative random variable whose probability law is identical to that of $R_t$, and used this alternative representation to characterise interest-rate dynamics. For a more direct analysis of stochastic integrals of the form (106), calculus on the multiplication of distributions due to Colombeau (1990) might prove useful. Alternatively, the analysis of Hanzon (1983) on the square-root of the Dirac delta function, or that of Judge (1966) on the eigenfunction of the coordinate multiplication operator, might become relevant in this respect. We defer such an analysis to another occasion.

We conclude by remarking on potential extensions and applications of the present work. These include, in particular, the consideration of negative interest rates; multiple yield curves; modelling of the spread between the LIBOR rate and the overnight index swap rate; and the calibration of the finite-dimensional models against swaption-implied volatility.

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