# Almost Sure State Estimation with $H_2$-type Performance Constraints for Nonlinear Hybrid Stochastic Systems

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**Abstract**

This paper is concerned with the problem of almost sure state estimation for general nonlinear hybrid stochastic systems whose coefficients only satisfy local Lipschitz conditions. By utilizing the stopping time method combined with martingale inequalities, a theoretical framework is established for analyzing the so-called almost surely asymptotic stability of the addressed system. Within such a theoretical framework, some sufficient conditions are derived under which the estimation dynamics is almost sure asymptotically stable and the upper bound of estimation error is also determined. Furthermore, a suboptimal state estimator is obtained by solving an optimization problem in the $H_2$ sense. According to the obtained results, for a class of special nonlinear hybrid stochastic systems, the corresponding conditions reduce to a set of matrix inequalities for the purpose of easy implementation. Finally, two numerical simulation examples are used to demonstrate the effectiveness of the results derived.

**Keywords**

State estimation, suboptimal estimator, almost surely asymptotic stability, nonlinear hybrid stochastic systems

## I. Introduction

The past few decades have seen an ever increasing attention in the optimal filtering or state estimation in various application areas including target tracking, image processing, signal processing and control engineering [25]. Among a variety of existing approaches, the Kalman filtering approach has proven to be an optimal one on the assumptions that the linear model is exact and the noise statistics is known. On the other hand, the $H_\infty$ filtering can be regarded as a suboptimal filtering due to its capability of providing a smaller bound for the worst-case estimation error [1, 2]. It might be worth emphasizing that, in most of the existing results, the estimators are designed to guarantee that the dynamics of the estimation error converges in terms of certain stability concepts such as the stochastic asymptotic stability [23], the mean-square stability [6–8, 11], the mean-square exponential stability [29], the $p$th moment stability [14], and the asymptotic stability in probability [24].

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In comparison with the aforementioned stability concepts, the almost surely asymptotic stability [20, 21] describes the dynamical performance from the sample state trajectories. More specifically, the almost sure stability requires that the probability of the system's largest excursion from equilibrium goes to zero as time tends to infinity. In addition, as mentioned in [13], there are some cases where the stochastic system is not mean-square stable but stable in almost sure sense since the noise can play a role for stabilizing the systems. Therefore, the performance of almost sure stability could be suitable to evaluate some real-world engineering situations with high reliability requirements where the impact from the noises cannot be ignored. For example, the performance requirement of the controlled rocket should be guaranteed with the probability 1. Very recently, the analysis and design problems in almost sure sense have attracted considerable attention, see [3, 4, 13, 14, 30] and the references therein. For instance, a Razumikhin-type theorem on $p$th moment input-to-state stability has been developed in [14] for a class of nonlinear stochastic systems with Markovian switching. In [13], a desired controller has been designed to stabilize a stochastic system almost surely which may not be stabilized in mean-square sense. Also, from a practical point of view, in addition to the traditional stability, the estimation performance is also vitally important and should been taken into account in the estimator design. For example, in engineering applications, one would expect the effects from the initial states and the jumping modes on the estimation accuracy to be attenuated or even minimized at a satisfactory level and such kind of $H_2$-type performance consideration would give rise to considerable difficulty in the selection of the estimator structure selection and the subsequent analysis. Unfortunately, to the best of the authors' knowledge, *the almost sure state estimation problem with $H_2$-type performance constraints* for general nonlinear stochastic systems has not been properly investigated so far, and this constitutes one of the motivations for the present research.

On the other hand, stochastic systems with Markovian switching parameters, which are usually referred to as hybrid stochastic systems, provide much convenience for modeling system plants whose structures are subject to abrupt changes such as component failures or repairs, changing subsystem interconnections, abrupt environmental disturbances, see [9] for more details. As a result, the performance analysis and synthesis issues for such systems have received much research attention [12, 26]. In particular, with regard to $H_\infty$ state estimation, some effective methods have been proposed in the literature [5, 17–19, 27]. For instance, attention has focused on the design of state estimator to estimate the semi-nonlinear Markovian jump system in [12] where a maximum likelihood solution has been obtained in terms of expectation-maximization algorithm. In addition, Since nonlinearities are ubiquitous in practice, much effort has been devoted to deal with the state estimation or filtering problems for nonlinear systems [22, 31, 32]. Up to now, in most reported results concerning nonlinear filtering, a common assumption is that the coefficients satisfy both the local Lipschitz condition and the linear growth condition. Such an assumption, however, would inevitably limit the application potentials of the established results. Therefore, there is a practical need to deal with the filtering problem with relaxed assumptions and another motivation of this paper is to shorten such a gap.

In this paper, we study the suboptimal almost sure state estimation problem for a class of nonlinear hybrid stochastic systems whose coefficients are assumed to satisfy the local Lipschitz condition only. For the concerned problem in this paper, the fundamental questions we are going to answer are identified as follows: 1) what kind of methods can be developed to overcome the difficultly stemming from the assumption of local Lipschitz conditions? and 2) how can we establish a suitable framework to analyze the estimation performance in almost sure sense? It is, therefore, the main motivation of this paper to provide satisfactory answers to the above two questions and also propose a design scheme of the suboptimal state estimator. The main contribution of this paper lies in the following three aspects: *1) in the plant under consideration, the coefficients are only assumed to satisfy local Lipschitz conditions and the traditional linear growth conditions are no longer needed; 2) a sufficient condition on almost sure stability is established in order to facilitate the later suboptimal state estimation problem; and 3) a suboptimal state estimator is obtained by solving an optimization problem.*

The rest of this paper is organized as follows. In Section II, a class of nonlinear hybrid stochastic systems (NHSSs) are presented, and some preliminaries are briefly outlined. In Section III, the main results are established by using the stochastic analysis techniques. These obtained results are then applied to a class of special NHSSs in Section IV. Furthermore, two examples are proposed to demonstrate the effectiveness of the proposed results in Section V. Finally, conclusions are drawn in Section VI.

**Notation** The notation used here is fairly standard unless otherwise specified. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the $n$ dimensional Euclidean space and the set of all $n\times m$ real matrices, and $\mathbb{R}_+ = [0,+\infty)$. $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete probability space with a natural filtration $\{\mathscr{F}\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). $M^T$ represents the transpose of the matrix $M$. $|\cdot|$ denotes the Euclidean norm. $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation. $\mathbb{P}\{\cdot\}$ means the probability. $C([-\tau,0];\mathbb{R}^n)$ denotes the family of all continuous $\mathbb{R}^n$-valued function $\varphi$ on $[-\tau,0]$ with the norm $|\varphi| = \sup\{|\varphi(\theta)| : -\tau \leq \theta \leq 0\}$. $C^b_{\mathcal{F}_0}([-\tau,0);\mathbb{R}^n)$ is the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau,0);\mathbb{R}^n)$-value random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$. $L^1(\mathbb{R}_+;\mathbb{R}_+)$ denotes the family of functions $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty \lambda(t)dt < \infty$. $\mathcal{K}$ denotes a class of continuous (strictly) increasing functions $\mu$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $\mu(0) = 0$. $\mathcal{K}_\infty$ denotes a class of functions $\mu$ in $\mathcal{K}$ with $\mu(r) \to \infty$ as $r \to \infty$. $L_2(\mathbb{R}_+, \mathbb{R}^p)$ denotes the space of nonanticipative stochastic process $y(t) \in \mathbb{R}^p$ with respect to the filtration $\mathscr{F}_t$ satisfying $|y(t)|^2_{L_2} := \mathbb{E}\int_0^\infty |y(t)|^2 dt < \infty$.

## II. Problem formulation

In this paper, let $r(t), t \geq 0$ be a right-continuous Markov chain taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if} \quad i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if} \quad i = j, \end{cases}$$

where $\triangle > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from mode $i$ to mode $j$ if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. It is known that almost all sample paths of $r(\cdot)$ are right-continuous step functions with a finite number of simple jumps in any finite subinterval of $R_+ := [0,\infty)$.

Let us consider the nonlinear hybrid stochastic systems of the form

$$\begin{cases} dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))d\omega(t) \\ \ \ y(t) = h(x(t), t, r(t)) \\ \ \ z(t) = m(x(t), t, r(t)) \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the actual measurement output, $z(t) \in \mathbb{R}^q$ is the state combination to be estimated, and $\omega(t)$ represents a scalar Wiener process (Brownian motion) on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. Furthermore, assume that $\omega(t)$ is independent of Markov chain $r(t)$ and satisfies:

$$\mathbb{E}\{d\omega(t)\} = 0, \quad \mathbb{E}\{d\omega(t)^2\} = dt.$$

$f, g : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n$, $h : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^p$ and $m : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^q$ with $f(0, t, i) = 0$, $g(0, t, i) = 0$, $h(0, t, i) = 0$ and $m(0, t, i) = 0$ are vector-valued nonlinear functions.

In this paper, we are concerned with the estimate $\hat{x}(t)$ of the state $x(t)$ by utilizing the measurement output $y(t)$. The state estimator is of the following form

$$\begin{cases} d\hat{x}(t) = f(\hat{x}(t), t, r(t))dt + K(r(t))y(t)dt \\ \hat{z}(t) = m(\hat{x}(t), t, r(t)), \quad \hat{x}(0) = 0 \end{cases} \tag{2.2}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of state $x(t)$, $\hat{z}(t) \in \mathbb{R}^q$ is the estimate of the output $z(t)$, and $K(i)$ is the estimator gain matrix to be designed.

*Assumption 1:* For each $i \in S$, the function $f$, $g$, $h$ and $m$ satisfy the local Lipschitz condition. That is, for any $H > 0$, there is $L_H > 0$ such that

$$|\phi(x, t, i) - \phi(\bar{x}, t, i)| \leq L_H |x - \bar{x}|, \tag{2.3}$$

for all $(x, \bar{x}, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$ with $|x| \vee |\bar{x}| \leq H$, and the function $\phi$ could be $f$, $g$, $h$ or $m$.

Denoting $\eta(t) = [x^T(t),\ \hat{x}^T(t)]^T$ and the estimation error $\tilde{z}(t) = z(t) - \hat{z}(t)$, we can obtain the following augmented system

$$\begin{cases} d\eta(t) = f_e(\eta(t), t, r(t))dt + g_e(\eta(t), t, r(t))d\omega(t) \\ \tilde{z}(t) = m(x(t), t, r(t)) - \hat{m}(\hat{x}(t), t, r(t)) \end{cases} \tag{2.4}$$

where

$$\begin{aligned} g_e(\eta(t), t, r(t)) &= \begin{bmatrix} g^T(x(t), t, r(t)) & 0 \end{bmatrix}^T, \\ f_e(\eta(t), t, r(t)) &= \begin{bmatrix} f(x(t), t, r(t)) \\ f(\hat{x}(t), t, r(t)) + K(r(t))h(x(t), t, r(t)) \end{bmatrix}. \end{aligned}$$

Let $C^{2,1}(\mathbb{R}^{2n} \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ be the family of all nonnegative functions $V(\eta, t, i)$ on $\mathbb{R}^{2n} \times \mathbb{R}_+ \times S$ that are twice continuously differentiable in $\eta$ and once in $t$. If $V \in C^{2,1}(\mathbb{R}^{2n} \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, define an operator $\mathcal{L}$ associated with (2.4) from $\mathbb{R}^{2n} \times \mathbb{R}_+ \times S$ to $\mathbb{R}$ by

$$\begin{aligned} \mathcal{L}V(\eta, t, i) &= V_t(\eta, t, i) + V_\eta(\eta, t, i) f_e(\eta, t, i) \\ &\quad + \frac{1}{2}\text{trace}[g_e^T(\eta, t, i) V_{\eta\eta}(\eta, t, i) g_e(\eta, t, i)] + \sum_{j=1}^N \gamma_{ij} V(\eta, t, j) \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} V_t(\eta, t, i) &= \frac{\partial V(\eta, t, i)}{\partial t}, \quad V_\eta(\eta, t, i) = \left( \frac{\partial V(\eta, t, i)}{\partial \eta_1}, \cdots, \frac{\partial V(\eta, t, i)}{\partial \eta_{2n}} \right), \\ V_{\eta\eta}(\eta, t, i) &= \left( \frac{\partial^2 V(\eta, t, i)}{\partial \eta_i \partial \eta_j} \right)_{2n \times 2n}. \end{aligned}$$

Before proceeding further, we introduce the following definition.

*Definition 1:* For $\eta_0 \in \mathbb{R}^{2n}$ and $r_0 \in S$, the augmented system (2.4) with the equilibrium $\eta = 0$ is said to be almost surely asymptotically stable if

$$\mathbb{P}\left( \lim_{t \to \infty} \eta(t; \eta_0, r_0) = 0 \right) = 1. \tag{2.6}$$

Our aim in this paper is to develop techniques to deal with the suboptimal almost sure state estimation problem for nonlinear hybrid stochastic systems. In other words, we are going to determine the estimator parameter $K_i$ such that the estimation error output $\tilde{z}$ satisfies the following performance requirement

R1) the augmented system (2.4) is almost surely asymptotically stable;

R2) for every $i \in S$, the estimation error output $\tilde{z}$ satisfies

$$\mathbb{E}\int_0^\infty |\tilde{z}|^2 dt \leq \min_{s.t.K_i} V(\eta_0, 0, i) \tag{2.7}$$

where $V(\eta, t, i)$ is a positive definite function to be determined. Furthermore, such an estimator is said to be a suboptimal almost sure state estimator. Note that the second requirement is closely related to the $H_2$ performance of the designed estimator against the initial states.

*Remark 1:* The traditional $H_2$ performance has been considered for linear time-invariance systems based on the $H_2$-norm, see [10] for the definition. In this paper, we use the interpretations of the $H_2$-norm to describe the impact on the total output energy of the error dynamics from the nonzero initial values as well as the jumping mode. The $H_2$-type performance constraint given in (2.7) is a very helpful measure to quantify the capability of the design estimator that attenuates the effects from the initial states.

*Remark 2:* In [15], the almost sure asymptotic stability has been investigated for the state estimation problem of a general class of nonlinear stochastic systems with Markovian switching. Compared to [15], this paper exhibits the following distinctive features: 1) the controlled output is included in the underlying model that facilitates the investigation on the estimation performance; 2) the $H_2$-type performance index is introduced to reflect the attenuation level of the estimation performance against the initial states as well as the jumping modes; 3) an optimization problem is considered to design a suboptimal estimator in order to minimize the impact from the initial states as well as the jumping modes on the estimation accuracy; and 4) the consideration of the performance constraints in the almost sure sense demands more comprehensive mathematical analysis, for example, the feasibility analysis of the suboptimal estimator design is more challenging.

## III. Main Results

In this section, the suboptimal almost sure state estimation problem of nonlinear hybrid stochastic systems is discussed, and the results are specialized to a special case for practical convenience.

In the following theorem, a sufficient condition is derived to obtain a suboptimal almost sure state estimator.

*Theorem 3.1:* If there are functions $V \in C^{2,1}(\mathbb{R}^{2n} \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, $\lambda \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, $\mu_1 \in \mathcal{K}_\infty$, $n \in \mathcal{K}$ and constant matrices $K_i$ such that

$$\mu_1(|\eta|) \leq V(\eta, t, i), \tag{3.1}$$

$$\mathcal{L}V(\eta, t, i) \leq \lambda(t) - n(V(\eta, t, i)), \tag{3.2}$$

hold for every $i \in S$ and all $(\eta, t, i) \in \mathbb{R}^{2n} \times \mathbb{R}_+ \times S$, then one has

$$\lim_{t\to\infty} \eta(t; \eta_0, r_0) = 0 \quad \text{almost surely (a.s.)} \tag{3.3}$$

for all $\eta_0 \in \mathbb{R}^{2n}$ and $r_0 \in S$, that is, the augmented system in (2.4) is almost surely asymptotically stable.

Before proving the main results in Theorem 3.1, let us present the following three lemmas.

*Lemma 1:* If $V \in C^{2,1}(\mathbb{R}^{2n} \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, then for any $t \geq 0$, the generalized Itô's formula is given as

$$\begin{aligned} dV(\eta(t), t, r(t)) =& \mathcal{L}V(\eta(t), t, r(t))dt + V_\eta(\eta(t), t, r(t))g_e(\eta(t), t, r(t))d\omega(t) \\ &+ \int_{\mathbb{R}} [V(\eta(t), t, r(t) + n(r(t), \alpha)) - V(\eta(t), t, r(t))] \times \mu(dt, d\alpha) \end{aligned}$$

where the function $n(\cdot, \cdot)$ and the martingale measure $\mu(\cdot, \cdot)$ are the same as (2.6) and (2.7) in [30].

It is well known that both the local Lipschitz condition and the linear growth condition are generally required on the coefficients of a stochastic differential equation in order to make use of the classical existence-and-uniqueness theorem. Unfortunately, in some practical situations, the linear growth condition might be difficult to be satisfied. In this paper, however, only the local Lipschitz condition (i.e. Assumption 1) is utilized to guarantee a unique maximal local solution. Furthermore, under such a condition, the maximal local solution is in fact a unique global solution, which is to be proved in the following lemma.

*Lemma 2:* Under the conditions of Theorem 3.1 and Assumption 1, for any initial data $\eta_0 \in \mathbb{R}^{2n}$ and $r(0) = r_0 \in S$, the system (2.4) has a unique global solution.

*Proof:* The proof is conducted following a similar line to [30]. For each $i \in S$ and any integer $k > b \geq |\eta_0|$, we define

$$f_e^{(k)}(\eta, t, i) = f_e(m_{\eta,k}, t, i), \quad g_e^{(k)}(\eta, t, i) = g_e(m_{\eta,k}, t, i)$$

where

$$m_{\eta,k} = \begin{cases} \dfrac{|\eta| \wedge k}{|\eta|}\eta, & \eta \neq 0, \\ 0, & \eta = 0. \end{cases}$$

Then, in light of Assumption 1, it can be found that both $f_e^{(k)}(\eta, t, i)$ and $g_e^{(k)}(\eta, t, i)$ satisfy the global Lipschitz condition as well as the linear growth condition. By utilizing the well-known existence-and-uniqueness theorem, it can be seen that there is a unique global solution $\eta_k(t; \eta_0, r_0)$ to the system (2.4) with coefficients $f_e^{(k)}(\eta, t, i)$ and $g_e^{(k)}(\eta, t, i)$.

Define the stopping time

$$\sigma_k \triangleq \inf\{t \geq 0 : |\eta_k(t)| \geq k\}$$

with $\inf \phi = \infty$. It is easy to see that $\sigma_k$ is increasing in $k$. The property aforementioned also enables us to define $\eta(t)$ as follow:

$$\eta(t) = \eta_k(t), \quad 0 \leq t \leq \sigma_k.$$

It is clear that $\eta(t)$ is a unique solution to system (2.4) for $t \in [0,\ \sigma)$ where $\sigma = \lim_{k\to\infty} \sigma_k$. Obviously, to complete the proof, we only need to show $\mathbb{P}\{\sigma = \infty\} = 1$.

Applying the general Itô's formula in Lemma 1, for any $t > 0$, we have

$$\mathbb{E}V(\eta_k(t \wedge \sigma_k), t \wedge \sigma_k, r(t \wedge \sigma_k)) = \mathbb{E}V(\eta_k(0), 0, r(0)) + \mathbb{E}\int_0^{t\wedge\sigma_k} \mathcal{L}^{(k)}V(\eta_k(s), s, r(s))ds$$

where the operator $\mathcal{L}^{(k)}V$ is similar to the definition $\mathcal{L}V$. Specifically, $f_e$ and $g_e$ are replaced by $f_e^{(k)}$ and $g_e^{(k)}$, respectively. Note the definitions of $f_e^{(k)}$ and $g_e^{(k)}$, one has

$$\mathcal{L}^{(k)}V(\eta_k(s), s, r(s)) = \mathcal{L}V(\eta(s), s, r(s)), \quad 0 \leq s \leq t \wedge \sigma_k.$$

Consequently, it follows from (3.2) that

$$\mathbb{E}V(\eta_k(t \wedge \sigma_k), t \wedge \sigma_k, r(t \wedge \sigma_k))$$

$$\leq V(\eta(0),0,r_0)+\int_0^t \lambda(s)ds+\mathbb{E}\int_0^t -n(V(\eta_k(s),s,r(s)))ds$$
$$\leq V(\eta(0),0,r_0)+\int_0^t \lambda(s)ds.$$

Noting the fact

$$\begin{aligned}\mathbb{E}V(\eta_k(t\wedge\sigma_k),t\wedge\sigma_k,r(t\wedge\sigma_k)) &\geq \int_{\{\sigma_k\leq t\}} V(\eta_k(t\wedge\sigma_k),t\wedge\sigma_k,r(t\wedge\sigma_k))d\mathbb{P}\\ &\geq \mathbb{P}\{\sigma_k\leq t\}\mathrm{inf}_{|\eta|\geq k,t\geq 0,i\in S}V(\eta,t,i),\end{aligned}$$

one has

$$\mathbb{P}\{\sigma_k\leq t\}\leq \frac{V(\eta(0),0,r_0)+\int_0^t\lambda(s)ds}{\inf_{|\eta|\geq k,t\geq 0,i\in S}V(\eta,t,i)}.$$

Taking $\lambda(t)\in L^1(\mathbb{R}_+;\mathbb{R}_+)$ and (3.1) into consideration, we can obtain that

$$\mathbb{P}\{\sigma\leq t\}=0.$$

Since $t$ is arbitrary, we have $\mathbb{P}\{\sigma=\infty\}=1$, which completes the proof. ■

*Lemma 3:* [16] Let $A_1(t)$ and $A_2(t)$ be two continuous adapted increasing processes on $t\geq 0$ with $A_1(0)=A_2(0)=0$ a.s., $M(t)$ be a real-valued continuous local martingale with $M(0)=0$ a.s., and $\phi$ satisfying $\mathbb{E}\phi<\infty$ be a nonnegative $\mathcal{F}_0$-measurable random variable. Denote $X(t)=\phi+A_1(t)-A_2(t)+M(t)$ for all $t\geq 0$ . If $X(t)$ is nonnegative, then

$$\{\lim_{t\to\infty}A_1(t)<\infty\}\subset\{\lim_{t\to\infty}X(t)<\infty\}\cap\{\lim_{t\to\infty}A_2(t)<\infty\}\quad \text{a.s.}$$

where $C\subset D$ a.s. means $\mathbb{P}(C\cap D^c=0)=0$. In particular, if $\lim_{t\to\infty}A_1(t)<\infty$ a.s., then,

$$\lim_{t\to\infty}X(t)<\infty,\quad \lim_{t\to\infty}A_2(t)<\infty \textit{ and } -\infty<\lim_{t\to\infty}M(t)<\infty\quad \text{a.s.}$$

That is, all of the three processes $X(t)$, $A_2(t)$ and $M(t)$ converge to finite random variables with probability one.

After presenting the previous three lemmas, we are now in a position to prove the Theorem 3.1.

**Proof of Theorem 3.1**.

According to Lemma 1 and the condition (3.2), we have

$$\begin{aligned}V(\eta(t),t,r(t)) &= V(\eta_0,0,r_0)+\int_0^t \mathcal{L}V(\eta(s),s,r(s))ds\\ &\quad+\int_0^t V_\eta(\eta(s),s,r(s))g_e(\eta(s),s,r(s))d\omega(s)\\ &\quad+\int_0^t\int_{\mathbb{R}}[V(\eta(s),s,r_0+n(r(s),\alpha))-V(\eta(s),s,r(s))]\mu(ds,d\alpha)\\ &\leq V(\eta_0,0,r_0)+\int_0^t\lambda(s)ds-\int_0^t n(V(\eta(s),s,r(s)))ds\\ &\quad+\int_0^t V_\eta(\eta(s),s,r(s))g_e(\eta(s),s,r(s))d\omega(s)\\ &\quad+\int_0^t\int_{\mathbb{R}}[V(\eta(s),s,r_0+n(r(s),\alpha)-V(\eta(s),s,r(s))]\mu(ds,d\alpha).\end{aligned}\tag{3.4}$$

Since $\int_0^\infty \lambda(s)ds < \infty$, applying Lemma 3, it follows that there is an $\bar{\Omega} \subset \Omega$ with $\mathbb{P}(\bar{\Omega}) = 1$ such that for every $\omega \in \bar{\Omega}$

$$\lim_{t\to\infty} \sup V(\eta(t), t, r(t)) < \infty, \tag{3.5}$$

and

$$\lim_{t\to\infty} \int_0^t n(V(\eta(s), s, r(s)))ds = \int_0^\infty n(V(\eta(s), s, r(s)))ds < \infty. \tag{3.6}$$

In what follows, for every $\omega \in \bar{\Omega}$, we claim that

$$\lim_{t\to\infty} n(V(\eta(t,\omega), t, r(t))) = 0. \tag{3.7}$$

If (3.7) is false, then for some $\hat{\omega} \in \bar{\Omega}$ and $i \in S$, the following inequality

$$\lim_{t\to\infty} \sup n(V(\eta(t,\hat{\omega}), t, i)) > 0$$

holds. Therefore, there exist a number $\varepsilon > 0$ and a positive sequence $\{t_k\}_{k\geq 1}$ with $t_k + 1 < t_{k+1}$ such that

$$n(V(\eta(t_k,\hat{\omega}), t_k, i)) > \varepsilon \quad \text{for all} \quad k \geq 1. \tag{3.8}$$

On the other hand, it is straightforward to see from (3.5) that $n(V(\eta(t_k,\hat{\omega}), t_k, i))$ and $V(\eta(t,\hat{\omega}), t, i$ are bounded and uniformly continuous. Let $h$ be the bound of $n(V(\eta(t_k,\hat{\omega}), t_k, i))$ and $\delta_1$ be a positive number to satisfy

$$|n(u) - n(v)| \leq \frac{\varepsilon}{2} \quad \text{if} \quad 0 \leq u, v \leq h, \quad |u - v| < \delta_1. \tag{3.9}$$

Furthermore, there is a $\delta_2 \in (0,\ 1/2)$ such that

$$|V(\eta(t,\hat{\omega}), t, i) - V(\eta(s,\hat{\omega}), s, i)| < \delta_1 \quad \text{if} \quad 0 \leq t, s < \infty, \quad |t - s| \leq \delta_2. \tag{3.10}$$

Moreover, for every $k \geq 1$, it follows from (3.9) and (3.10) that

$$|n(V(\eta(t_k,\hat{\omega}), t_k, i)) - n(V(\eta(t,\hat{\omega}), t, i))| \leq \frac{\varepsilon}{2} \quad \text{if} \quad t_k \leq t \leq t_k + \delta_2.$$

For $t_k \leq t \leq t_k + \delta_2$, it follows from (3.8) that

$$n(V(\eta(t,\hat{\omega}), t, i)) \geq n(V(\eta(t_k,\hat{\omega}), t_k, i)) - |n(V(\eta(t_k,\hat{\omega}), t_k, i)) - n(V(\eta(t,\hat{\omega}), t, i))| \geq \frac{\varepsilon}{2}$$

which results in

$$\int_0^\infty n(V(\eta(t,\hat{\omega}), t, i))dt \geq \sum_{k=1}^\infty \int_{t_k}^{t_k+\delta_2} n(V(\eta(t,\hat{\omega}), t, i))dt \geq \sum_{k=1}^\infty \frac{\varepsilon\delta_2}{2} = \infty.$$

This contradicts (3.6), which implies that (3.7) must be true.

Finally, noting $n \in \mathcal{K}$, we have

$$\lim_{t\to\infty} V(\eta(t;\eta_0, r_0), t, i) = 0 \quad a.s.$$

which, together with (3.1), yields

$$\lim_{t\to\infty} \mu_1(|\eta(t;\eta_0, r_0)|) = 0 \quad a.s.$$

Because $\mu_1(u) = 0$ if and only if $u = 0$, we must therefore have

$$\lim_{t\to\infty} |\eta(t;\eta_0, r_0)| = 0 \quad a.s.$$

which completes the proof of Theorem 3.1.

In the following theorem, a sufficient condition is derived to obtain a suboptimal almost sure state estimator.

*Theorem 3.2:* For each $i \in S$, assume that there are functions $V \in C^{2,1}(\mathbb{R}^{2n} \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ with $V(0,t,i) = 0$, $\mu_1 \in \mathcal{K}_\infty$, $n \in \mathcal{K}$ and the constant matrix $K_i$ such that

$$\mu_1(|\eta|) \leq V(\eta, t, i), \tag{3.11}$$

$$\mathcal{L}V(\eta, t, i) + |\tilde{z}|^2 \leq -n(V(\eta, t, i)), \tag{3.12}$$

hold for all $(\eta, t, i) \in \mathbb{R}^{2n} \times \mathbb{R}_+ \times S$. In this case, the state estimator (2.2) is a suboptimal almost sure state estimator in which the gain matrix is obtained by solving the following optimization problem:

$$\min_{s.t. \quad K_i,(3.11),(3.12)} V(\eta(0), 0, i). \tag{3.13}$$

*Proof:* Firstly, in terms of Theorem 3.1, the system (2.4) is almost sure asymptotically stable, that is

$$\lim_{t \to \infty} \eta(t) = 0 \quad a.s. \tag{3.14}$$

For every $i \in S$ and $t \geq 0$, by means of the condition of $V(0,t,i) = 0$, it is not difficult to verify that

$$\lim_{t \to \infty} V(\eta(t), t, i) = 0 \quad a.s. \tag{3.15}$$

Moreover, for any $T > 0$, it follows from Lemma 1 and (3.12) that

$$\begin{aligned} \mathbb{E}\int_0^T |\tilde{z}|^2 dt &= \mathbb{E}\int_0^T [|\tilde{z}|^2 + dV(\eta(t), t, i)]dt + V(\eta_0, 0, i) - \mathbb{E}V(\eta(T), T, i) \\ &= V(\eta_0, 0, i) - \mathbb{E}V(\eta(T), T, i) + \mathbb{E}\int_0^T (|\tilde{z}|^2 + \mathcal{L}V(\eta(t), t, i))dt + \mathbb{E}\int_0^T V_\eta g_e d\omega \\ &\leq V(\eta_0, 0, i) - \mathbb{E}V(\eta(T), T, i) + \mathbb{E}\int_0^T -n(V(\eta, t, i))dt \\ &\leq V(\eta_0, 0, i) - \mathbb{E}V(\eta(T), T, i). \end{aligned} \tag{3.16}$$

Finally, taking $T \to \infty$ in (3.16), it follows from (3.15) that

$$\min_{s.t. \quad K_i,(3.11),(3.12)} \mathbb{E}\int_0^T |\tilde{z}|^2 dt \leq \min_{s.t. \quad K_i,(3.11),(3.12)} V(\eta_0, 0, i) \quad \forall\, i \in S$$

which completes the proof. ■

*Remark 3:* It can be found from the proof of Theorem 3.2 that, for each $i \in S$, $V(\eta_0, 0, i)$ is not a tight upper bound of $\mathbb{E}\int_0^\infty |\tilde{z}|^2 dt$. Obviously, it is a difficult task to determine the tightest (minimal) upper bound due to difficulty in choosing $n(V(\eta, t, i))$. Instead, a more realistic way is to take $V(\eta_0, 0, i)$ as a smaller upper bound of $\mathbb{E}\int_0^\infty |\tilde{z}|^2 dt$. It should be pointed out that, as discussed in [32], $V(\eta_0, 0, i)$ is a tighter upper bound of $\mathbb{E}\int_0^\infty |\tilde{z}|^2 dt$ when the condition (3.12) reduces to $\mathcal{L}V(\eta, t, i) \leq -|\tilde{z}|^2$.

Obviously, the sufficient conditions given in Theorem 3.2 are difficult to be verified/solved. For application convenience, a special Lyapunov function $V(\eta) = V(x) + V(\hat{x})$ can be utilized to decoupled the complicated nonlinear inequality (3.12) into a seemingly simple form in the following corollary.

*Corollary 1:* Assume that there are two positive definite matrices $P_i$, $Q_i$, a set of matrices $K_i$ $(i \in S)$, and a positive scalar $\varepsilon_i$ such that

$$2x^T P_i f + \varepsilon_i^{-1} h^2 + g^T P_i g + 2m^2 + x^T P_i x + 2\hat{x}^T Q_i \hat{f} + \varepsilon_i \hat{x}^T Q_i K_i K_i^T Q_i \hat{x} + 2\hat{m}^2 + \hat{x}^T Q_i \hat{x} < 0 \tag{3.17}$$

holds for all $x \in \mathbb{R}^n, \hat{x} \in \mathbb{R}^n$. In this case, the state estimator (2.2) is a suboptimal almost sure state estimator in which the gain matrix is obtained by solving the following optimization problem:

$$\min_{s.t.\ \ K_i,(3.17)} P_i x_0^2. \tag{3.18}$$

*Proof:* Define the following Lyapunov function

$$V(\eta,t,i) = x^T P_i x + \hat{x}^T Q_i \hat{x}$$

and let $n(V(\eta,t,i)) = V(\eta,t,i)$. Then, the condition (3.12) associated with the system (2.4) becomes

$$2x^T P_i f + 2\hat{x}^T Q_i \hat{f} + 2\hat{x}^T Q_i K_i h + g^T P_i g + |\tilde{z}|^2 + x^T P_i x + \hat{x}^T Q_i \hat{x} < 0. \tag{3.19}$$

Applying the inequality

$$X^T Y + Y^T X \leq \varepsilon X^T X + \varepsilon^{-1} Y^T Y, \quad \forall \varepsilon > 0, \tag{3.20}$$

one has

$$2\lambda \hat{x}^T K_i h \leq \varepsilon_i \hat{x}^T Q_i K_i K_i^T Q_i \hat{x} + \varepsilon_i^{-1} h^2, \tag{3.21}$$

and

$$|\tilde{z}|^2 = |m - \hat{m}|^2 \leq 2m^2 + 2\hat{m}^2. \tag{3.22}$$

On the other hand, it follows from (3.19)-(3.22) that the condition (3.12) is true. Therefore, it follows from Theorem 3.2 that the state estimator (2.2) is a suboptimal almost sure state estimator and its gain matrix is obtained by solving the following optimization problem (3.18). ■

To illustrate the application insights of the main results we have obtained so far, a linear state estimator is adopted to study the suboptimal almost sure state estimation problem for a class of special NHSSs in the next section.

## IV. A Class of Special NHSSs with Linear Estimator

In the sequel, we denote the matrix associated with the $i$th mode by

$$\Gamma_i \triangleq \Gamma(r(t) = i)$$

where the matrix $\Gamma$ could be $A, B, C, L, F, H$ or $K$.

Consider a class of special nonlinear hybrid stochastic system as follows:

$$\begin{cases} dx(t) = [A(r(t))x(t) + f(x(t), r(t))]dt + B(r(t))x(t)d\omega(t) \\ y(t) = C(r(t))x(t) + \phi(x(t), r(t)) \\ z(t) = L(r(t))x(t) \end{cases} \tag{4.1}$$

where $A_i, B_i, C_i$ and $L_i$ are known matrices with appropriate dimensions, $f(\cdot,\cdot)$ and $\phi(\cdot,\cdot)$ are nonlinear functions which are assumed to satisfy the following conditions for every $i \in S$:

$$|f(x(t), i)|^2 \leq x^T F_i x, \quad |\phi(x(t), i)|^2 \leq x^T H_i x$$

with known positive definite matrices $F_i$ and $H_i$.

We are interested in constructing a linear estimator of the following form for system (4.1)

$$\begin{cases} d\hat{x}(t) = A(r(t))\hat{x}(t)dt + K(r(t))y(t)dt, \\ d\hat{z}(t) = L(r(t))\hat{x}(t), \quad \hat{x}(0) = 0. \end{cases} \tag{4.2}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of state $x(t)$, $\hat{z}(t) \in \mathbb{R}^q$ is the estimate of the output $z(t)$, and $K(i)$ is the estimator gain matrix to be designed. The augmented system can be written as follows:

$$\begin{cases} d\eta(t) = f_e(\eta(t), r(t)) + g_e(\eta(t), r(t))d\omega(t) \\ \tilde{z}(t) = L(r(t))x(t) - L(r(t))\hat{x}(t), \quad \hat{x}(0) = 0 \end{cases} \tag{4.3}$$

where

$$f_e(\eta(t), r(t)) = \begin{bmatrix} A(r(t))x(t) + f(x(t), r(t)) \\ A(r(t))\hat{x}(t) + K(r(t))(C(r(t))x(t) + \phi(x(t), r(t))) \end{bmatrix}$$

$$g_e(\eta(t), r(t)) = \begin{bmatrix} B^T(r(t))x(t) & 0 \end{bmatrix}^T.$$

Applying Theorem 3.2, we have the following result for which only the sketch of the proof is given in order to ensure the conciseness.

*Theorem 4.1:* For the augmented system (4.3), if there exist matrices $K_i$ $(i \in S)$, and positive scalars $\varepsilon_{1i}, \varepsilon_{2i}$ and $\lambda$ satisfying the following matrices inequalities:

$$\begin{bmatrix} \Omega_i & \lambda K_i C_i & \Pi_i & 0 & 0 \\ * & \lambda A_i + \lambda A_i^T + \lambda I & 0 & \lambda L_i & B_{fi} \\ * & * & \Lambda_i & 0 & 0 \\ * & * & * & -\frac{1}{2}I & 0 \\ * & * & * & * & -\varepsilon_{2i}I \end{bmatrix} < 0, \quad \forall\, i \in S \tag{4.4}$$

where

$$\begin{aligned} \Omega_i &= \lambda A_i + \lambda A_i^T + \varepsilon_{1i}\lambda^2 I + \varepsilon_{2i}\lambda^2 H_i + \lambda I, \\ \Pi_i &= [F_i \quad \lambda B_i \quad L_i], \quad \Lambda_i = \mathrm{diag}\{-\varepsilon_{1i}F_i, \quad -I, \quad -\frac{1}{2}I\}, \end{aligned}$$

then the state estimator (4.2) is a suboptimal almost sure state estimator in which the gain matrix is obtain by solving the following optimization problem

$$\min_{s.t. \quad K_i, (4.4)} \lambda x^2(0). \tag{4.5}$$

*Proof:* Define the following Lyapunov function for system (4.3) as follows:

$$V(\eta(t), t, i) = \lambda\eta^T(t)\eta(t) = \lambda x^T(t)x(t) + \lambda\hat{x}^T(t)\hat{x}(t), \quad \forall i \in S \tag{4.6}$$

and let $n(V(\eta, t, i)) = \lambda\eta^T(t)\eta(t) = V(\eta(t), t, i)$.

Based on Lemma 1, the condition (3.12) associated with the system (4.3) becomes

$$\begin{aligned} &V_t + V_\eta f_e + \frac{1}{2}g_e^T V_{\eta\eta} g_e + \sum_{j=1}^{N} \gamma_{ij} V_j + |\tilde{z}|^2 + \lambda\eta^2 \\ &= 2\lambda x^T(t)A_i x(t) + 2\lambda x^T(t)f(x(t), i) + 2\lambda\hat{x}^T(t)A\hat{x}(t) + 2\lambda\hat{x}^T(t)K_i(C_i x(t) + \phi(x(t), i)) \\ &\quad + \lambda x^T(t)B_i^T B_i x(t) + |L_i x(t) - L_i\hat{x}(t)|^2 + \lambda x^T(t)x(t) + \lambda\hat{x}^T(t)\hat{x}(t) \\ &\le x^T(t)[\lambda A_i + \lambda A_i^T + \varepsilon_{1i}\lambda^2 I + \varepsilon_{1i}^{-1}F_i\varepsilon_{2i}\lambda^2 H_i + \lambda B_i^T B_i + 2\lambda^2 L_i^T L_i + \lambda I]x(t) \\ &\quad + \hat{x}^T(t)[\lambda K_i C_i + \lambda C_i^T K_i^T]x(t) + \hat{x}^T(t)[\lambda A_i + \lambda A_i^T + \varepsilon_{2i}^{-1}K_i^T K_i + 2L_i^T L_i + \lambda I]\hat{x}(t). \end{aligned} \tag{4.7}$$

By using the Schur Complement Lemma, it follows directly from (4.4) that the right-hand side of the inequality in (4.7) is less than zero for all $\eta(t) \neq 0$, which means that the condition (3.12) associated with the system (4.3) holds. Furthermore, it is easy to verify that (3.11) is also true. Therefore, it follows from Theorem 3.2 that the state estimator (4.2) is a suboptimal almost sure one with its gain matrix obtained by solving the optimization problem (4.5). ■

## V. Examples

To illustrate the effectiveness of the proposed methods, two numerical examples are given in this section. For this purpose, let $\omega(t)$ be a scalar Wiener process, $\gamma(t)$ be a right-continuous Markov chain which is independent on $\omega(t)$ and takes values in $S=\{1,2\}$, and the step size be $\triangle$=0.005.

*Example 1:* **The analysis of almost surely asymptotic stability**

Let the generator $\Gamma$ be

$$\Gamma=(\gamma_{ij})_{2\times 2}=\begin{pmatrix} -0.9 & 0.9 \\ 0.5 & -0.5 \end{pmatrix}.$$

Consider the system (2.1) with parameters:

$$\begin{aligned} &f(x(t),t,1)=-\frac{3}{2}x(t)+e^{-t/2}, \quad g(x(t),t,1)=x\sin(t), \quad h(x(t),t,1)=x(t);\\ &f(x(t),t,2)=-2x(t)+\sin(x(t)), \quad g(x(t),t,2)=\frac{1}{\sqrt{2}}x\cos(t), \quad h(x(t),t,2)=x(t). \end{aligned}$$

By selecting $K_1=1$, $K_2=0.5$, $\lambda(t)=e^{-t}$ and $V(\eta,t,r(t))=x^2(t)+\hat{x}^2(t)$, and utilizing Theorem 3.1, we can check that the augmented system (2.4) is almost surely asymptotically stable. The simulation results of the actual state trajectory $x(t)$ and its estimate $\hat{x}(t)$ are shown in Fig. 1. It can be found that they tend to zeros when $t\to+\infty$.

*Example 2:* **The design of suboptimal almost sure state estimator**

Let the generator $\Gamma$ be

$$\Gamma=(\gamma_{ij})_{2\times 2}=\begin{pmatrix} -1.2 & 1.2 \\ 0.6 & -0.6 \end{pmatrix}.$$

Consider the system (4.1) with parameters

$$\begin{aligned} &A_1=-2, \quad f(x(t),1)=\frac{1}{2}x(t)\sin(t), \quad B_1=\frac{1}{2},\\ &C_1=\frac{1}{2}, \quad \phi(x(t),1)=\frac{1}{2}x(t)\cos(t), \quad L_1=\frac{1}{2},\\ &A_2=-3, \quad f(x(t),2)=\frac{1}{3}x(t)\sin(t), \quad B_2=\frac{1}{3},\\ &C_2=\frac{1}{3}, \quad \phi(x(t),2)=\frac{1}{3}x(t)\cos(t), \quad L_2=\frac{1}{3}. \end{aligned}$$

By using Matlab software to solve (4.4) and the optimization problem (4.5) in Theorem 4.1, we obtain that

$$\begin{aligned} &K_1=0.52, \quad \varepsilon_{11}=2.33, \quad \varepsilon_{21}=1.17,\\ &K_2=1.12, \quad \varepsilon_{21}=6.3, \quad \varepsilon_{22}=1.32, \quad \lambda=0.412. \end{aligned}$$

The simulation results are shown in Fig. 2 and Fig. 3, where Fig. 2 plots the real state $x$ and its estimate $\hat{x}$, and Fig. 3 depicts the estimate error $x-\hat{x}$. The simulation results have confirmed that the designed estimator performs very well.

## VI. Conclusions

In this paper, we have investigated the suboptimal almost sure state estimation problem for a class of general nonlinear hybrid stochastic systems with the assumption that coefficients only satisfy local Lipschitz conditions. By utilizing the stopping time method combined with martingale inequalities, sufficient conditions have been obtained such that the estimation error process is almost surely asymptotically stable and the upper bound of estimation error is also determined. Then, a linear suboptimal state estimator has been obtained by solving an optimal problem. Based on the proposed results, for a class of special nonlinear hybrid stochastic systems, the corresponding conditions have been decoupled into a set of matrix inequalities which can be easily solved by using Matlab software. Finally, the results derived in this paper have been demonstrated by using two numerical examples.

## References

[1] M. Aliyu and E. Boukas, Mixed $H_2/H_\infty$ nonlinear filtering, *International Journal of Robust and Nonlinear Control*, Vol. 19, No. 4, pp. 394-417, 2009.

[2] M. Basin, P. Shi, D. Calderon-Alvarez and J. Wang, Central suboptimal $H_\infty$ filter design for linear time-varying systems with state or measurement delay, *Circuits, Systems and Signal Processing*, Vol. 28, No. 2, pp. 305-330, 2009.

[3] B. Bercu, F. Dufour and G. G. Yin, Almost sure stabilization for feedback controls of regime-switching linear systems with a hidden Markov chain, *IEEE Transaction on Automatic Control*, Vol. 54, No. 9, pp. 2114-2125, 2009.

[4] M. Tanelli, B. Picasso, P. Bolzern and P. Colaneri, Almost sure stabilization of uncertain continuous-time Markov jump linear systems, *IEEE Transaction on Automatic Control*, Vol. 55, No. 1, pp. 195-201, 2010.

[5] B. Chen, L. Yu and W. Zhang, $H_\infty$ filtering for Markovian switcing genetic regulatory networks with time-delays and stochastic disturbances, *Circuits, Systems, and Signal Processing*, Vol. 30, No. 6, pp. 1231-1252, 2011.

[6] D. Ding, Z. Wang, H. Dong and H. Shu, Distributed $H_\infty$ state estimation with stochasitc parameters and nonlinearities through sensor networks: the finite-horizon case, *Automatica*, Vol. 48, No. 8, pp. 1575-1585, 2012.

[7] D. Ding, Z. Wang, J. Lam and B. Shen, Finite-Horizon $H_\infty$ control for discrete time-varying systems with randomly occurring nonlinearities and fading measurements, *IEEE Transactions on Automatic Control*, In press DOI: 10.1109/TAC.2014.2380671

[8] D. Ding, Z. Wang, B. Shen and H. Dong, Envelope-constrained $H_\infty$ filtering with fading measurements and randomly occurring nonlinearities: the finite horizon case, *Automatica*, Vol. 55, pp. 37–45, 2015.

[9] H. Dong, Z. Wang , W. C. Daniel and H. Gao, Robust $H_\infty$ filtering for Markovian jump systems with randomly occurring nonlinearities and sensor saturation: the finite-horizon case, *IEEE Transactions on Signal Processing*, Vol. 59, No. 7, pp. 3048-3057, 2011.

[10] J. C. Doyle, K. Glover, P. P. Khargonekar and B. A. Francis, State-space solutions to standard $H_2$ and $H_\infty$ control problems, *IEEE Transactions on Automatic Control*, Vol. 34, No. 8, pp. 831-847, 1989.

[11] H. Dong, Z. Wang, D. W. C. Ho and H. Gao, Variance-constrained $\mathcal{H}_\infty$ filtering for a class of nonlinear time-varying systems with multiple missing measurements: the finite-horizon case, *IEEE Transactions on Signal Processing*, Vol. 58, No. 5, pp. 2534-2543, 2010.

[12] D. Hong and H. Leung, Maximum likelihood state estimation of semi-Markovian switching system in non-Gaussian measurement noise, *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 46, No. 1, pp. 133-146, 2010.

[13] L. Hu and X. Mao, Almost sure exponential stabilisation of stochastic systems by state-feedback control,*Automatica*, Vol. 44, No. 2, pp. 465-471, 2008.

[14] L. Huang and X. Mao, On almost sure stability of hybrid stochastic systems with mode-dependent interval delays, *IEEE Transaction on Automatic Control*, Vol. 55, No. 8, pp. 1946-1952, 2010.

[15] X. Kan, H. Shu and Z. Li, Almost sure state estimation for nonlinear stochastic systems with Markovian switching, *Nonlinear Dynamics*, Vol. 76, No. 2, pp. 1591-1602, 2014.

[16] R. S. Lipster and A. N. Shiryayev, Theory of Martingales, Dordrecht. Kluwer Academic, The Netherlands, 1989.

[17] J. Liang, F. Sun and X. Liu, Finite-horizon $H_\infty$ filtering for time-varying delay systems with randomly varying nonlinearities and sensor saturations, *Systems Science and Control Engineering: An Open Access Journal*, Vol. 2, No. 1, pp. 108-118, 2014.

[18] J. Liu, B. Yao and Z. Gu, Delay-dependent $H_\infty$ filtering for Markovian jump time-delay systems: a piecewise analysis method, *Circuits, Systems, and Signal Processing*, Vol. 30, pp. 1253-1273, 2011.

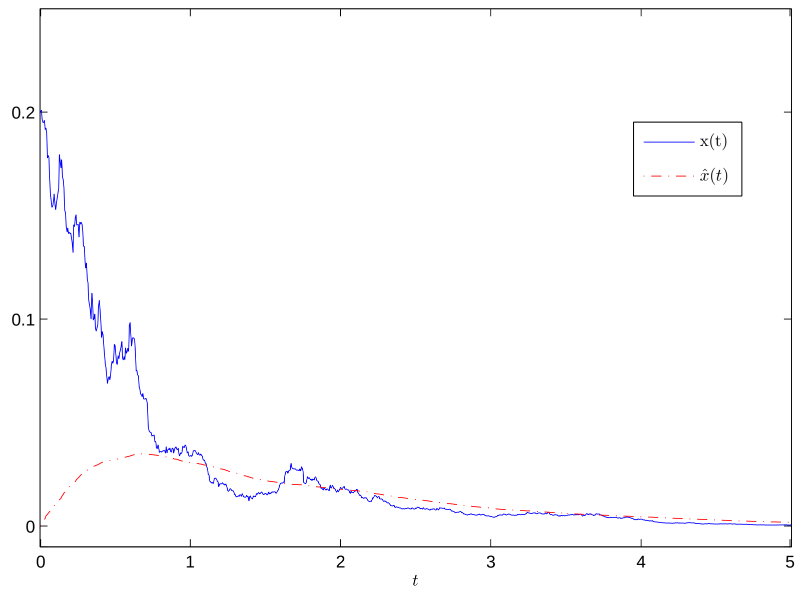


Fig. 1. The trajectories of $x$ and $\hat{x}$ of Example 1.


[19] Y. Luo, G. Wei, Y. Liu and X. Ding, Reliable $H_\infty$ state estimation for 2-D discrete systems with infinite distributed delays and incomplete observations, *International Journal of General Systems*, Vol. 44, No. 2, pp. 155-168, 2015.

[20] X. Mao, Stability of stochastic differential equations with Markovian switching, *Stochastic Processes and their Applications*, Vol. 79, No. 1, pp. 45-67, 1999.

[21] X. Mao, A note on the LaSalle-type theorems for stochastic differential delay equations, *Mathematical Analysis and Applications*, Vol. 268, pp. 125-142, 2002.

[22] M. N$\phi$rgaard, N. K. poulsen and O. Ravn, New developments in state estimation for nonlinear systems, *Automactica*, Vol. 36, No. 11, pp. 1627-1638, 2000.

[23] B. Shen, Z. Wang, H. Shu and G. Wei, on Nonlinear $H_\infty$ filtering for discrete-time stochastic systems with missing measurements, *IEEE Transactions on Automatic Control*, Vol. 53, No. 9, pp. 2170-2180, 2008.

[24] B. Shen, Z. Wang, Y. S. Hung and G. Chesi, Distributed $H_\infty$ filtering for polynomial nonlinear stochastic systems in sensor networks, *IEEE Transaction on Industrial Electronics*, Vol. 58, No. 5, pp. 1971-1979, 2011.

[25] B. Shen, A survey on the applications of the Krein-space theory in signal estimation, *Systems Science and Control Engineering: An Open Access Journal*, Vol. 2, No. 1, pp. 143-149, 2014.

[26] H. Shen, S. Xu, X. Song and Y. Chu, Delay-dependent $H_\infty$ filtering for stochastic systems with Markovian switching and mixed mode-dependent delays, *Nonlinear Analysis: Hybrid Systems*, Vol. 4, No. 1, pp. 122-133, 2010.

[27] C. E. de Souza, A. Trofino and K. A. Barbosa, Mode-independent $H_\infty$ filters for Markkovian jump linear systems, *IEEE Transation on Automatic Control*, Vol. 51, No. 11, pp. 1837-1841, 2006.

[28] Z. Wang, H. Wu, J. Liang and X. Liu, On modeling and state estimation for genetic regulatory networks with polytopic uncertainties, *IEEE Transaction on NanoBioscience*, Vol. 12, No. 1, pp. 13-20, 2013.

[29] G. Wei, Z. Wang, B. Shen and M. Li, Probability-dependent gain-scheduled filtering for stochasitc systems with missing measurements, *IEEE Transactions on Circuits and Systems: Part II*, Vol. 58, No. 11, pp. 753-757, 2011.

[30] C. Yuan and X. Mao, Robust stability and controllability of stochastic differential delay equations with Markovian switching, *Automatica*, Vol. 40, No. 3, pp. 343-354, 2004.

[31] H. Yan, M. Meng, H. Zhang and H. Shi, Robust $H_\infty$ exponential filtering for uncertain stochastic time-delay systems with Markovian switching and nonlinearities, *Applied Mathematics and Computation*, Vol. 215, No. 12, pp. 4358-4369, 2010.

[32] W. Zhang, B. S. Chen and C. S. Tseng, Robust $H_\infty$ filtering for nonlinear stochastic systems, *IEEE Transactions on Signal Processing*, Vol. 53, No. 2, pp. 589-598, 2005.

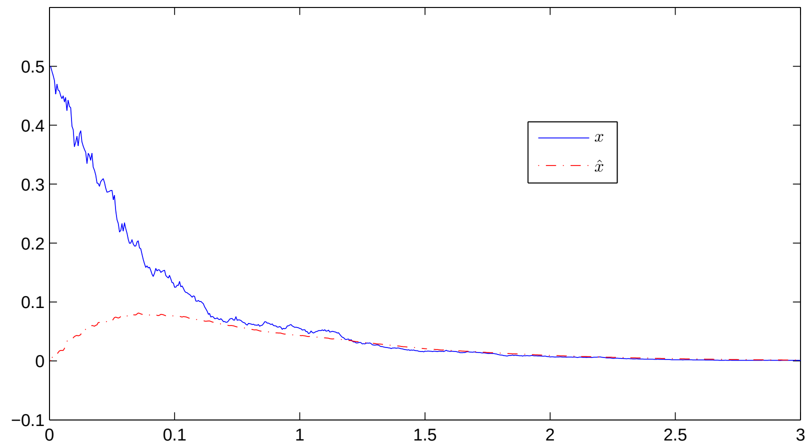


Fig. 2. The trajectories of $x$ and $\hat{x}$ of Example 2.

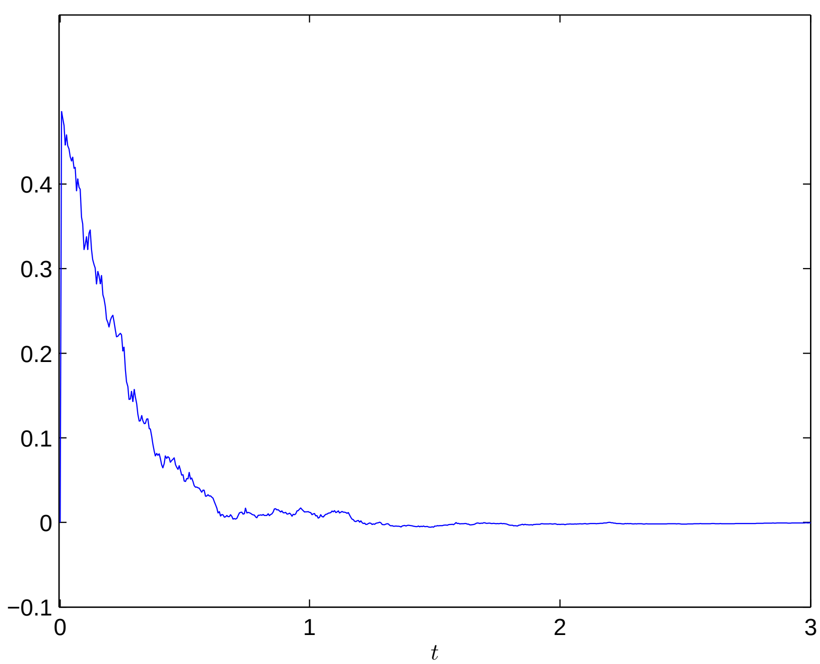


Fig. 3. Estimation error $x - \hat{x}$ of Example 2.