THE IMPACT OF NONLINEAR DYNAMICS ON THE RESILIENCE OF A GROCERY SUPPLY CHAIN

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ABSTRACT
Purpose of this paper
The resilience of supply chain replenishment systems is an important attribute and especially so in the retail sector where initiatives such as Efficient Consumer Response have led to lower inventory holding while attempting to maintain high levels of on-shelf availability. A common approach to testing for resilience of such systems would be through simulation modelling, especially where batching of orders occurs, for example. However, with developments in non-linear control theory, there is an opportunity to use more sophisticated analytical approaches to evaluate and improve resilience. The aim of this paper is to demonstrate the value of an analytical approach with empirical testing on a replenishment system used by a grocery retailer.

Design/methodology/approach
An Industrial Dynamics (ID) approach is used for framing and building a credible representation of the grocery retailer's replenishment system. Initially a nonlinear causal loop and block diagram representations of the actual system were developed based on empirical data collection. Mathematical analysis of the model, based on nonlinear control engineering techniques in combination with ID simulation, have been used to understand the behaviour of stock and shipment output responses in the distribution centre given step and periodic demand signals.

Findings
Mathematical analysis through nonlinear control theory techniques has led to insights into the dynamic behaviour of the replenishment control model. This allowed the identification of specific behavioural changes in the supply chain stock and shipment responses, which are key indicators for assessing supply chain resilience, without going through a time-consuming simulation process. Transfer function and describing function analyses served as guidelines for undertaking ID simulation.

Value
The integrated method we have used combines to best advantage the knowledge generated via the twin approaches of non-linear control systems engineering analysis plus ID simulation. This duality maximises insight into the resultant causal relationships output from these procedures and hence enables the engineering of the optimal design for a real-world supply chain. The consequence is the development of a robust system based approach which brings together two mutually supportive components, simulation and non-linear control theory, to enhance supply chain resilience. The approach is illustrated using data concomitantly with a comprehensive grocery supply chain case study.

Research limitations/implications
This research is limited to the dynamics of single-echelon supply chain system. Although the electronic point of sales data and the store replenishment system have been considered in the validation process, this study has focused on analysing the resilience performance of a replenishment system only. Future research will consider a multi-echelon supply chain.

Practical implications
The systems based method is readily transferable to other industrial settings and environments, thereby enabling insights into resilience. A number of lessons for the case study are identified and these may also be applicable in other practical contexts.

**INTRODUCTION**

In recent years, successful businesses have moved from a product-driven strategy to a more market-driven one. In the retail environment, strong competition creates constant pressure on retailers to continually improve performance. To achieve this, grocery retailers have modernised their supply chains (Hingley et al., 2011). The distribution centre has becoming increasingly important in decrease lead-times and taking inventory out of the retail operations. Moreover, with the growth of internet ordering for groceries and the use of store-based picking strategies for e-fulfilment to home shoppers, DCs now have to deliver stock to meet demands of both store and home shoppers (Fernie and Grant, 2008). Hence, this resulting complex retail business created the necessity for DCs to have effective replenishment systems, not only to meet the requirements of the supply chain but to be resilient to disturbances.

The dynamic behaviour of these systems plays a significant role in supply chain resilience performance. These dynamics are normally driven by the application of different control system policies and can be considered as a source of supply chain disruption depending on the control system design (Mason-Jones and Towill, 1998). More often than not, such resilience may be evaluated through simulation, given the complexity of the system. However, developments in computing capabilities allow non-linear control theory to now be effectively used instead. In this paper, we aim to analyse the resilience performance of a DC replenishment system within one of the largest grocery retailers in the UK, using an approach that combines nonlinear control theory and simulation modelling.

**SUPPLY CHAIN RESILIENCE AND SYSTEM DYNAMICS**

In the supply chain literature, the idea of resilience has recently emerged (Christopher and Peck, 2004), and is defined as "the adaptive capability of the supply chain to prepare for unexpected events, respond to disruptions, and recover from them by maintaining continuity of operations at desired levels of connectedness and control over structure and function" (Ponomarov and Holcomb, 2009). This definition implies achieving three properties: readiness (being prepared or available for service), response (reaction to a specific stimulus) and recovery (a return to ‘normal’ stable or steady state conditions).

Despite the growing importance of the field of supply chain resilience, most existing studies are qualitative in nature. An exception is Wilson (2007), who analysed the impact caused by disruptions in transport processes on customer service levels, inventory levels and goods in transit and how a more collaborative supply chain can help to overcome this problem. Spiegler et al. (2012) investigated how different control policies and system dynamics in supply chains affect resilience, which is measured by calculating the integral of time absolute error (ITAE) of inventory and shipment responses. Both of these works are conceptual, exploratory and no empirical data were considered. However, in this study, we extend Spiegler et al.’s (2012) analytical framework for assessing supply chain resilience by enriching it with empirical research. In doing so, there is a need to consider how non-linearities have been incorporated into the analysis of dynamic systems.

It has been claimed that in order to improve supply chain performance, dynamics in production-inventory control systems should be reduced (Torres and Maltz, 2010). Hence, there is a plethora of literature researching the bullwhip effect and its impact on different supply chain performances, from both a quantitative modelling perspective (Fransoo and Wouters, 2000; Dejonckheere et al., 2004), either conceptually or based on empirical studies, and a descriptive perspective in the form of case studies (Lee et al., 1997; Kumar and Nigmatullin, 2011). However, so far, emphasis has been given to financial performance measures. For instance, most research focuses upon the impact of system dynamics on inventory, production and transport costs. Even when service levels and customer satisfaction are considered, these have been seen as service penalty costs.
Moreover, most of the system dynamics studies that use mathematical modelling still focus on linear models (such as Zhou et al., 2010). Forrester’s work (1968) on industrial dynamics calls attention to the importance of considering nonlinear models to represent industrial and social processes: “Nonlinearity can introduce unexpected behaviour in a system”. Such unexpected behaviour can cause instability and uncertainty. Despite many analytical methods being cited and already recommended by system dynamics scholars 30 years ago to examine nonlinear models (for example Cuypers and Rademaker, 1974; Mohapatra, 1980), they have been disregarded by recent studies where simulation techniques still dominate. Simulating complex systems without having first done some preliminary analysis can be exhaustive and unrewarding (Atherton, 1975). We use both mathematical and simulation modelling, highlighting how combining these provides greater insight than just simulation alone.

**RESEARCH METHOD**

The first stage of this research project involved developing a conceptualisation of the system through input-output and block diagrams. These were then used to create the mathematical and simulation models, with the aim of building a simple but credible representation of the real system. Nonlinear control engineering and spreadsheet simulation methods have then been used to analyse the resilience of the systems. The first stage in formulating the empirical model was defining the overall scope as well as identifying the following assumptions to be included:

- Store orders are aggregated, rather than being placed on store-by-store basis.
- Only a single product is modelled, with no promotions.
- The products are unaffected by unpredictable external factors (e.g. the weather).
- All supplier deliveries are made in full when compared with the ordered volume.

The results of conceptualisation were converted into a block diagram in the Laplace domain, ‘s’ (Figure 1). Block Diagrams are a useful and simple method for analysing a system graphically. By using a block diagram, it is relatively straightforward to create the simulation model when transforming the Laplace frequency domain into difference equations by using a sampling period of $\Delta t=1$. The parameter $Ti$ has been included in the block diagram representation and it determines the time actual and safety stocks take to balance. In the ‘As Is’ scenario this is set equal to 1.

**Figure 1. Block diagram of the DC replenishment system**

In Figure 1, the presence of CLIP and ROUNGING functions make the model nonlinear. The CLIP function denotes that shipments to the store will depend upon stock levels and deliveries from suppliers. When the shipments are not equal to the retail store orders, then backlog builds up. Hence, the desired shipment in the next replenishment period will

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be the store demand, store orders plus any backlog. The ROUNDDING function regards to
the Buying Quantity and Truckload Constraint already mentioned before.

Having developed the simulation model, the next stage was to verify and validate the
findings, using Sterman (1984)’s validation process. The model was verified by talking to
the system manager through the equations entered into a spreadsheet. Then, tests using
extreme input and parameter values and eliminating assumptions were undertaken.
Finally, actual data obtained for three different products has been used to test the model.
Results show that our model represents well the response of the real system, especially
for products with high volume and lower standard deviations.

RESILIENCE PERFORMANCE ANALYSIS
In this section, analysis of the DC replenishment system’s resilience performance will be
undertaken. Transfer functions, describing functions and system dynamics simulations
will be used will be used to estimate the performance of this nonlinear system. The ITAE
(Towill, 1970) of the DC stock response and the system’s natural frequency (ωn) and
damping ratio (ζ) will be used as indicators of resilience.

Transfer Functions in ideal operating conditions
Under ideal operating conditions, a backlog situation would not occur and therefore
shipments to the store would be made in full every period. Moreover, if buying quantity
and truckload constraints could be overcome, then nonlinearities in the model of Figure 1
could be eliminated. In doing so, the DC replenishment system is simplified to a linear
dynamic model and simple block diagram algebra can be used to find the system transfer
functions:

\[
\begin{align*}
\text{Supplier Order } & \quad \frac{\text{OS}}{1 + s (K + Ta + Ti + Tp) + s^2 (K + Ta + Ti) Tp} \\
\text{DC Stock } & \quad \frac{\text{OS}}{1 + s (Ti + Ta + Ti + Tp) + s^2 (Ti + Ti) Tp} \\
\text{GIT } & \quad \frac{\text{OS}}{1 + s (K + Ta + Ti) Tp}
\end{align*}
\]

In order to find out how the outputs will respond to a step change in the input, initial and
final value theorems can be used. Table 1 presents the results for the final values of
supplier order, DC stock and GIT when store orders undergo a unit step change (from 0
to 1 unit). The initial values of all responses are zero, including their targets.

<table>
<thead>
<tr>
<th>Response</th>
<th>Final value</th>
<th>Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>Supplier Order</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>DC stock</td>
<td>K-Ti.Tp</td>
<td>K</td>
</tr>
<tr>
<td>GIT</td>
<td>Tp</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1. Results from the Final Value Theorem

The results in Table 1 demonstrate that there is a permanent offset occurring in both DC
stock and GIT responses since none of them will ever reach target values. The stock drift
has a significant impact on the resilience performance since the system has been
designed in such way that stock levels will never recover from changes in the orders
received from the store. The company had not recognised this problem since the
replenishment system is continuously fed with erratic orders from the store. Hence, a
proposed solution for the target GIT is to make it a variable with function of demand and
the lead-time, assuming that lead-time is always known. In the case application,
suppliers have consistent delivery lead-times, making this is a reasonable assumption.
In order to calculate ITAE of the DC stock response, the final value (Table 1) of DC Stock
can be used so as to estimate which parameter values would increase the system’s
response and recovery. From Eqn (2), partial fraction expansion method can be used to
determine the time function for the DC Stock. In this way, ITAE can be estimated as:
\[ ITAE = \int_0^\infty t |e(t)| dt = \int_0^\infty t |K - Ti \cdot Tp - DC Stock(t)| dt = \\
= KT a^2 - Ti^3 Tp (1 + Tp)^2 + Ti (Ta^2 - KTp + KTa (1 + Tp)) + Ti^2 (Tp + 2Tp^2 + Ta (1 + Tp)) + K (1 + Tp)^2 \]  

Note that Eqn (4) is only valid for Ta > 0 and K > Ti Tp. It is also assumed that after the step change, the stock level drops and recovers without overshooting again. Hence it should be used only for exploratory analysis. Eqn (4) shows that ITAE can be minimised, hence resilience can be maximised, by decreasing Ti, Ta and Tp.

**Effect of nonlinearities**

In order to investigate the impact of nonlinearities in feedback systems, we can use the describing function method, which is a quasi-linear representation for a nonlinear element subjected to a sinusoidal input. A sinusoidal input raw order quantity (ROQ) with an angular frequency \( \omega \), amplitude \( A \) and mean \( B \), to the Rounding nonlinearity will produce an output (Supplier Order) of same frequency and phase but different amplitude and mean. Although the function is nonlinear, it can be represented by multiple piecewise linear equations. In order to simplify calculations, the lookup table equation can be linearised beforehand and then we obtain the two piecewise linear equations:

\[
Supplier Order(t) = \begin{cases} 
ROQ(t), & \text{if } ROQ > 0 (\pi \leq t \leq \gamma) \\
0, & \text{if } ROQ < 0 (\pi < t < \gamma) 
\end{cases}
\]

The basic idea of the describing function is to represent a nonlinear element by a transfer function, or gain, derived from its effect on a sinusoidal input signal. Given ROQ as a sinusoidal input, the output Supplier Order can be approximated to:

\[
Supplier Order(t) \approx N_a \cos(\omega t + \phi) + N_b B, 
\]

where \( \phi \) is the phase angle. In order to determine the gain of the describing function \( N_a \), we need to expand the series and determine its first harmonic coefficients. Fourier series expansion method is used to represent the output Supplier Order as a series. For the describing function, only the first, or fundamental harmonic is usually used to approximate the periodic series. If we approximate the piecewise linear output Supplier Order to the first harmonic, we have that:

\[
N_a = \frac{\gamma \cos(\gamma) \sin(\gamma)}{\pi}, \text{ where } \gamma = \cos^{-1}\left(\frac{-B}{A}\right). 
\]

Figure 2a illustrates how the describing function gain varies as the amplitude of the ROQ increases. For amplitudes lower than the mean B, the system behaves linearly and Supplier Order will be equal to the input ROQ corresponding to a describing function gain \( N_a \) equal to 1. However, when the amplitude of ROQ increases only a fraction of this rate will actually be ordered. So, the gain of the describing function varies from 0.5 to 1.

![Figure 2a: Describing function gains](image)

**Figure 2. Describing function gains**

The second nonlinearity in the model is the CLIP function in the shipment system, which is used to avoid any shipments being made to the store if no stock is available. While in the Rounding function all constraints (buying quantity, truckload and non-negative orders) were fixed, in the CLIP function the constraint is given by current responses of...
DC stock and delivery, which are variable values. Because of that, the nonlinearity caused by the CLIP function is not only amplitude-dependent but also frequency-dependent. Hence, there will be one describing function for each frequency. Matlab™ combined with Simulink™ has been used to automate calculations and find the describing function gains for a set of amplitudes and frequency resulting in Figure 2b. The figure demonstrates that the nonlinearity in the shipment process only occurs for very low frequencies and high amplitudes.

Although each nonlinearity in the DC replenishment system has different features, they both decrease their respective output gain, whose value is always between 0.5 and 1. Now, root locus techniques can be used to predict how these nonlinearities affect the system responses and the resilience performance. By replacing the ROUNDING and CLIP functions with the gains $N_{A(ROQ)}$ and $N_{A(Ship.)}$ respectively and using block diagram algebra again we find that the new system characteristic equation is equal to:

$$\left(1 + s \cdot Ta\right)\left[N_{A(ROQ)} + s \left(N_{A(ROQ)} \cdot Ti \cdot Tp + Ti\right) + s^2 Ti \cdot Tp\right],$$  

In this way the effect of the change in gains on the system natural frequency ($\omega_n$) and damping ratio ($\zeta$) can be calculated as:

$$\omega_n = \sqrt{\frac{N_{A(ROQ)}}{Ti \cdot Tp}}$$  

$$\zeta = \frac{1 + N_{A(ROQ)} \cdot Ti}{2N_{A(ROQ)}} \sqrt{\frac{N_{A(ROQ)}}{Ti \cdot Tp}}$$

Analysing Eqns 8 and 10, the nonlinearity will always decrease significantly the value of the $\omega_n$. This causes a negative impact on supply chain resilience since $\omega_n$ determines how fast the system oscillates during the transient response. On the other hand, the damping ratio, which describes how oscillations in the system decay with time, depends on the combining values of other parameters. When replacing the parameter values used by the DC replenishment system, the natural frequency decreases from 0.71 to 0.5 rad/day as the nonlinearity becomes active. The damping ratio decreases from 1.06 to 1, which means that the resilience performance is improved since the responses would switch from overdamped (slow decay of oscillations) to critically damped. $Ti$ can also be adjusted in order to achieve responses which ideally has a $\zeta=0.7$. The ROUNDING nonlinearity provokes complex behaviour such that the resilience performance depends upon the parameters values, buying quantities and truckload constraints.

Note that the CLIP function $N_{A(Ship.)}$ has no effect on the $\omega_n$ and $\zeta$. Hence, it does not influence the DC stock and supplier orders responses. The only impact that this nonlinearity has is in the shipment response. When the CLIP nonlinearity takes effect it means that DC Backlog is no longer zero and shipments to customers will be cut (ITAE in shipments will increase) and the supply chain will be less resilient.

**Simulation Results**

In order to understand the impact of both nonlinearities, we introduced a step input of 100% (from 1,000 to 2,000 cases/day). Figure 3a illustrates the ‘As Is’ scenario. When stock levels are negative (backlog situation), shipments to the stores are no longer made in full, which in consequence may lead to on-shelf stockouts. Hence, the replenishment system becomes less resilient to greater changes in demand and the stock offset is also intensified. By decreasing the values of the control parameter $Ti$ we observed an improvement in the resilience performance because: i) the stock offset decreases because the final value of stock oppositely depends on this parameter, as demonstrated in Table 1; and ii) inventory response and recovery times are reduced, therefore the system recovers from backlog situation quicker.
In order to quantify the results, ITAE values of the DC stock and shipment responses for a period of 180 days have been calculated and normalised by dividing all the ITAE performance by the ‘As Is’ index value. As Figure 3 shows, the integral time absolute value error decreases as Ti is adjusted to lower values. However, if Ti is too small (lower than 0.5), the DC stock response will start to oscillate and potentially becomes unstable. Similar results are obtained for the forecasting constant Ta and Tp. As Ta and Tp decrease, ITAE values of both stock and shipment are improved.

A set of simulations has been undertaken to investigate in depth the ROUNDING function. This nonlinearity is very complex for the system manager to handle, not only depending on the parameter values, buying quantity and truckload constraint but also the demand pattern and steps. Depending on the step sizes, certain rounding values will improve resilience by making inventory recovery faster. But a single unit change in the rounding value may change completely the inventory response. Another curious phenomenon discovered by employing the describing function technique was that the CLIP nonlinearity in the shipment process only occurs for low frequencies (but not too low) and high amplitudes. The effect of high amplitude demands has been...
demonstrated by increasing the step sizes in the ‘shock’ analysis previously. To confirm the effect of demand frequencies, simulations using sinusoidal inputs of same amplitude but different frequencies have been undertaken. Figure 3b demonstrates that as input frequency decreases from 62.82 (or higher) to 37.68 rad/day, backlogs start to occur. But as the input frequency decreases further, the shipment process behaves linearly.

**IMPLICATIONS AND CONCLUSIONS**

This research has analysed the resilience performance of a DC replenishment system within a large grocery retailer. Potential risks from a lack of resilience include a mismatch between supply and demand, and serving stores inefficiently. The findings identify several potential improvements to the DC replenishment system that can be made in order to become more resilient. These include:

- making the target GIT a variable related to demand and a function of the lead-time, to address the permanent offset in the DC stock response. Inventory drift is a problem for the supply chain to maintain its resilience performance, especially under multi-event disruptions and uncertain demand.
- automatically adjusting the control parameters to the resilience ‘mode’ in times of uncertain demand patterns and abrupt, sharp changes in stock levels.
- grouping products with the same demand pattern to determine the order quantity that maximises resilience without negatively impacting on warehousing and transportation costs. This is because the demand amplitude and frequency affects potential backlog situations.

Preliminary mathematical analysis through nonlinear control theory techniques has been undertaken in order to gain initial insights in the understanding of the replenishment control model. Table 2 summarises the insights gained by combining these methods.

This research is limited to the dynamics of single-echelon supply chain systems. Although the EPOS sales data and the store replenishment system have been considered in the validation process, this study has focused on analysing the resilience performance of the DC replenishment system only. Considering the multi-echelon supply chain is intended for further research activities.
Transfer function analysis

**Example 1:** The parameters $Ti$ was found to be important control parameters for resilience. It provokes opposite impacts on ITAE values. Hence it has been investigated more in-depth in the simulation process since its value may impact on system stability.

**Example 2:** Small values of $Ta$ will always benefit resilience. Simulations confirmed that a demand chase strategy ($Ta=0$ or $a=1$) is preferable and $Ta$ causes no problem to stability.

**Example 3:** Delivery lead-time $Tp$ is also important for resilience and should be minimised.

Describing Function

**Example 1:** Analysis showed that the Rounding function may cause a positive or negative impact on resilience depending on control parameters. In this way, simulation efforts were given to find such situations. Although simulations confirm this result and reveal an even more complex behaviour caused by this nonlinearity, the positive impact on resilience might have never been discovered when using only simulation techniques.

**Example 2:** The CLIP function does not cause any impact on other system’s responses. This effect was easily pointed out by describing function techniques and is confirmed via simulations.

Describing Function

**Example 1:** Initial and final value theorems revealed how parameters $Ti$ and $Tp$ influence in the inventory offset. This would have been impractical with numerical/simulation technique only.

**Example 2:** Simulations confirm that $Ta$ does not change the stock’s final value.

Analytical Insights | Resulting simulation experiments | If not carried out
--- | --- | ---
Possibility to find system’s transfer functions and ITAE estimated equations | Simulation process focused on important parameters for achieving supply chain resilience | A better understanding of each control parameter’s influence on resilience was achieved by using both analytical and simulation techniques.

Table 2. Insights gained from undertaking preliminary analysis of system dynamics models

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