REMARKS ON EXPLICIT STRONG ELLIPTICITY CONDITIONS FOR ANISOTROPIC OR PRE-STRESSED INCOMPRESSIBLE SOLIDS

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Summary

We present a set of explicit conditions, involving the components of the elastic stiffness tensor, which are necessary and sufficient to ensure the strong ellipticity of an orthorhombic incompressible medium. The derivation is based on the procedure developed by Zee and Sternberg (Arch. Rat. Mech. Anal. **83** (1983)) and, consequently, is also applicable to the case of the homogeneously pre-stressed incompressible isotropic solids. This allows us to reformulate the results by Zee and Sternberg in terms of components of the incremental stiffness tensor. In addition, the resulting conditions are specialised to higher symmetry classes and compared with strong ellipticity conditions for plane strain, commonly used in the literature.

1. Introduction

Components C_{ijkl} of the elasticity tensor of every anisotropic solid possess a number of symmetries (for example, in the linearly elastic case $C_{ijkl} = C_{klij} = C_{ijlk}$, $i, j, k, l = 1, 2, 3$). In addition, the tensor $\boldsymbol{C}$ may possess the symmetries necessary to confirm to the appropriate symmetry class of the material. The remaining non-zero components are also not completely arbitrary and must, in fact, satisfy certain conditions ensuring the physicality of the material response. Certainly, the definition of what 'physicality' means exactly tends to be imprecise, especially in the case of constrained media and/or finite deformations, see the discussion in (**1**, Chapter 3). However, certain types of conditions proved useful in applications; the strong ellipticity is one such condition.

A practical motivation to study constitutive inequalities, such as the strong ellipticity conditions, is provided by the modern advances in the design and fabrication of composites, in particular, micro- and nano-structured materials. These materials often behave as anisotropic elastic solids at the macro-scale. Due to the man-made nature of such solids, the implication is that it is now possible to design and fabricate anisotropic materials with the prescribed sets of material parameters. With this in mind, it is important to develop a better understanding of the theoretical limits on anisotropic parameters.

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For linearly elastic anisotropic solids the strong ellipticity condition may be formally introduced as the requirement that

$$C_{ijkl}a_ib_ja_kb_l > 0 \tag{1.1}$$

for any real vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (**2**, p. 56). Here and henceforth we assume summation over repeated indices unless otherwise stated. The condition (1.1) is not explicit, because it must be enforced for arbitrary $\boldsymbol{a}$ and $\boldsymbol{b}$. Nevertheless, when the tensor $\boldsymbol{C}$ belongs to sufficiently high symmetry class, it is often possible to formulate explicit necessary and sufficient conditions for components of $\boldsymbol{C}$ that are equivalent to (1.1). For example, the strong ellipticity of a linear elastic isotropic material is equivalent to the following inequalities

$$\mu > 0 \quad \text{and} \quad (\nu < 1/2 \quad \text{or} \quad \nu > 1)\,, \tag{1.2}$$

within which μ is the shear modulus and ν the Poisson ratio (**3**, Sect. 51). Explicit strong ellipticity conditions are also known for transversely isotropic elastic solids (**4–6**) and orthorhombic elastic solids (**7**). A semi-numerical procedure that verifies the strong ellipticity of a general linear anisotropic solid has been described recently, see (**8**). In addition, explicit strong ellipticity conditions are known for finitely deformed isotropic elastic solids (**9–11**). Applications to the stability analysis of constrained and unconstrained fibre-reinforced media are discussed in (**12**, **13**).

Inequalities (1.2) are weak. They do not guarantee the positive definiteness of the strain energy density, which in linear isotropic case is equivalent to the familiar pair of inequalities

$$\mu > 0 \quad \text{and} \quad -1 < \nu < 1/2\,. \tag{1.3}$$

Nevertheless, inequalities (1.2) are still useful, because they ensure global stability in the sense of Hadamard, see (**1**, Sect. III.8.B). A good example of the situation where this distinction matters is provided by papers (**14**, **15**), whose authors recognised that one can create composite materials with extreme effective properties by using inclusions that are strongly elliptic, but do not necessarily have a positive definite strain energy density.

The introduction of a kinematic constraint, such as the incompressibility, makes it harder to define conditions that ensure a plausible material response. For example, condition (1.1) is, essentially, the requirement of a positive definiteness of the acoustic tensor $Q_{ik} \equiv C_{ijkl}n_jn_l$, where unit vector $\mathbf{n}$ is the wave normal. However, the acoustic tensor of an incompressible linearly elastic material is at best positive semi-definite (**16**). Fortunately, one can show that condition (1.1) is equivalent to saying that squares of body wave speeds (that is, the eigen-values of the acoustic tensor) must all be real and positive. This makes it possible to formulate a weaker version of the strong ellipticity condition for incompressible materials, which only ensures that the squares of body wave speeds are real and positive, see (**17**). Conditions of this kind for incompressible linear isotropic solids are trivial (they simply give $\mu > 0$). A highly non-trivial generalisation of these conditions to the incremental elasticity of a homogeneously pre-stressed isotropic media is described in (**18**). Unfortunately, the explicit strong ellipticity conditions given in (**18**) are obtained in terms of the derivatives of the strain energy function, which is not always the most convenient representation.

The main goal of the present article is to obtain a set of explicit strong ellipticity conditions for incompressible materials in terms of components of the stiffness tensor $\boldsymbol{C}$. This is achieved in two steps. During the first step, we derive the explicit necessary and sufficient conditions for the strong ellipticity of the incompressible linearly elastic orthorhombic solids. The obtained conditions complement strong ellipticity conditions for unconstrained orthorhombic solids derived in (**7**). During the second step, we generalise our procedure for the incremental stiffness tensor of a homogeneously pre-stressed isotropic medium. Although the acoustic tensor considered at the second step is more general than the one considered at the first step (**3**, Sect. 68), the generalisation turns out to be simple due to a hidden symmetry of the tensor of incremental stiffnesses.

In addition, we compare the newly derived three-dimensional strong ellipticity conditions for orthorhombic linear elastic media with their two-dimensional counterparts that are commonly used when solving plane strain problems. The simplifications occurring when one deals with higher elastic symmetries are also analysed. In addition, we make comparisons between the obtained strong ellipticity conditions and the conditions that ensure the positive definiteness of the strain energy density.

2. Strong ellipticity conditions for incompressible orthorhombic solids

Equations that govern the motion of a linearly elastic body subjected to the incompressibility constraint are given by

$$\sigma_{ij,j} = \rho \ddot{u}_i\,, \qquad u_{i,i} = 0\,, \qquad i, j \in \{1, 2, 3\}\,, \tag{2.1}$$

where ρ stands for the material density, and u_i and σ_{ij} are the components of displacement and Cauchy stress, respectively. We use overdots to denote differentiation with respect to time, and comma suffices to denote differentiation with respect to implied spatial coordinate. The constitutive relations for an incompressible orthorhombic solid may be written using the contracted (Voigt) notation for the components of the stiffness tensor (**2**, p. 35), yielding

$$\begin{aligned}
\sigma_{11} &= c_{11}\varepsilon_{11} + c_{12}\varepsilon_{22} + c_{13}\varepsilon_{33} - p\,, & \sigma_{12} &= 2c_{66}\varepsilon_{12}\,,\\
\sigma_{22} &= c_{12}\varepsilon_{11} + c_{22}\varepsilon_{22} + c_{23}\varepsilon_{33} - p\,, & \sigma_{13} &= 2c_{55}\varepsilon_{13}\,,\\
\sigma_{33} &= c_{13}\varepsilon_{11} + c_{23}\varepsilon_{22} + c_{33}\varepsilon_{33} - p\,, & \sigma_{23} &= 2c_{44}\varepsilon_{23}\,,
\end{aligned} \tag{2.2}$$

in which $\varepsilon_{ij} \equiv \frac{1}{2}(u_{i,j} + u_{j,i})$, $i, j = 1, 2, 3$, are components of the strain tensor. Scalar p is pressure, a Lagrange multiplier necessary to accommodate the incompressibility constraint. The plane waves propagating in such media may be sought in the form

$$\mathbf{u} = Af(\mathbf{x}\cdot\mathbf{n} - vt)\mathbf{e}\,, \qquad p = Asf'(\mathbf{x}\cdot\mathbf{n} - vt)\,, \tag{2.3}$$

where A is the wave amplitude, $\mathbf{n}$ the unit vector along the direction of propagation, $\mathbf{e}$ the unit vector along the polarisation direction, v the wave speed and s to be determined. If equations (2.2) and (2.3) are inserted into governing equation (2.1), one obtains

$$Q_{ik}e_k - sn_i = \rho v^2 e_i\,, \qquad e_i n_i = 0\,, \tag{2.4}$$

with tensor $\mathbf{Q} = (Q_{ik})$ usually referred to as the acoustic tensor. In the case of orthorhombic media, the acoustic tensor may be written in the following form

$$\mathbf{Q} = \begin{pmatrix}
c_{11}n_1^2 + c_{66}n_2^2 + c_{55}n_3^2 & \frac{1}{2}(c_{11}+c_{22}-2\beta_3)\,n_1n_2 & \frac{1}{2}(c_{11}+c_{33}-2\beta_2)\,n_1n_3\\
\frac{1}{2}(c_{11}+c_{22}-2\beta_3)\,n_1n_2 & c_{66}n_1^2 + c_{22}n_2^2 + c_{44}n_3^2 & \frac{1}{2}(c_{22}+c_{33}-2\beta_1)\,n_2n_3\\
\frac{1}{2}(c_{11}+c_{33}-2\beta_2)\,n_1n_3 & \frac{1}{2}(c_{22}+c_{33}-2\beta_1)\,n_2n_3 & c_{55}n_1^2 + c_{44}n_2^2 + c_{33}n_3^2
\end{pmatrix}, \tag{2.5}$$

within which we introduced the shorthand notation

$$2\beta_1 = c_{22} + c_{33} - 2(c_{23} + c_{44}), \quad 2\beta_2 = c_{11} + c_{33} - 2(c_{13} + c_{55}), \\ 2\beta_3 = c_{11} + c_{22} - 2(c_{12} + c_{66}), \tag{2.6}$$

that will prove convenient in subsequent analysis. Equations (2.4) can be manipulated to establish that $s = \mathbf{n} \cdot (\mathbf{Qe})$, which allows one to re-write propagation condition $(2.4)_1$ as

$$\left(\mathbf{PQ} - \rho v^2 \mathbf{I}\right) \mathbf{e} = 0, \tag{2.7}$$

where $\mathbf{P} \equiv \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$ is a symmetric tensor.

The strong ellipticity conditions may be defined as the requirement that the squares of body wave speeds must be real and positive for all directions of propagation. In unconstrained materials, this requirement is equivalent to the positive definiteness of the acoustic tensor $\mathbf{Q}$. For incompressible materials this is not true, because $\det \mathbf{P} = 0$, so that $\det \mathbf{PQ} = 0$ and one of the squared wave speeds given by propagation condition (2.7) is always zero. The requirement that squares of two remaining wave speeds are real and positive is formulated by noting that (2.7) is equivalent to the quadratic equation in ρv^2:

$$2\rho^2 v^4 - 2\,\mathrm{tr}(\mathbf{PQ})\rho v^2 + (\mathrm{tr}\,\mathbf{PQ})^2 - \mathrm{tr}(\mathbf{PQ})^2 = 0, \tag{2.8}$$

see (**19**). Inspection of (2.8) leads to the conclusion that the strong ellipticity conditions for incompressible materials must have the form

$$\mathrm{tr}\,\mathbf{PQ} > 0, \qquad (\mathrm{tr}\,\mathbf{PQ})^2 - \mathrm{tr}(\mathbf{PQ})^2 > 0. \tag{2.9}$$

When expressions for the components of the tensors $\mathbf{Q}$ and $\mathbf{P}$, specified in (2.5) and immediately below (2.7), respectively, are inserted into inequalities (2.9), one arrives at

$$(c_{55} + c_{66})\, n_1^4 + (c_{44} + c_{55} + 2\beta_3)\, n_1^2 n_2^2 + (c_{44} + c_{66})\, n_2^4 \\ + (c_{44} + c_{66} + 2\beta_2)\, n_1^2 n_3^2 + (c_{55} + c_{66} + 2\beta_1)\, n_2^2 n_3^2 + (c_{44} + c_{55})\, n_3^4 > 0 \tag{2.10}$$

and

$$c_{55}c_{66}n_1^6 + (c_{44}c_{55} + 2\beta_1 c_{66})\, n_2^4 n_3^2 + (c_{44}c_{66} + 2\beta_1 c_{55})\, n_2^2 n_3^4 + \\ + c_{44}c_{66}n_2^6 + (c_{44}c_{55} + 2\beta_2 c_{66})\, n_1^4 n_3^2 + (c_{55}c_{66} + 2\beta_2 c_{44})\, n_1^2 n_3^4 + \\ + c_{44}c_{55}n_3^6 + (c_{44}c_{66} + 2\beta_3 c_{55})\, n_1^4 n_2^2 + (c_{55}c_{66} + 2\beta_3 c_{44})\, n_1^2 n_2^4 + \\ + \left(c_{44}^2 + c_{55}^2 + c_{66}^2 - (\beta_1 + \beta_2 - \beta_3)^2 + 4\beta_1\beta_2\right) n_1^2 n_2^2 n_3^2 > 0. \tag{2.11}$$

The strong ellipticity conditions (2.10), (2.11) are still implicit in the sense that they involve components of the propagation vector $\mathbf{n}$. Our goal now is to formulate conditions on material parameters occurring in (2.10), (2.11) such that they will ensure strong ellipticity along an arbitrary propagation direction $\mathbf{n}$.

2.1 *Necessary and sufficient conditions for coordinate planes*

First, we consider waves that propagate along coordinate axes. Since two components of the corresponding wave vectors **n** are equal to zero, the substitution of these directions into conditions (2.10) and (2.11) yields

$$c_{44}+c_{55} > 0\,, \qquad c_{44}+c_{66} > 0\,, \qquad c_{55}+c_{66} > 0\,,$$
$$c_{44}c_{55} > 0, \qquad c_{44}c_{66} > 0\,, \qquad c_{55}c_{66} > 0\,,$$

which are equivalent to the following three explicit inequalities

$$c_{44} > 0\,, \qquad c_{55} > 0\,, \qquad c_{66} > 0\,. \tag{2.12}$$

It is worth pointing out that these conditions are the necessary and sufficient strong ellipticity conditions for waves propagating along the coordinate axes.

Second, we consider waves that propagate in coordinate planes and have wave vectors with one zero component. For example, vectors $\mathbf{n} = (n_1, n_2, 0)$, where $n_1^2 + n_2^2 = 1$, describe the propagation in the plane Ox_1x_2. In this case condition (2.11) assumes the simplified form

$$c_{55}c_{66}n_1^6 + (c_{44}c_{66} + 2\beta_3 c_{55})\, n_1^4 n_2^2 + (c_{55}c_{66} + 2\beta_3 c_{44})\, n_1^2 n_2^4 + c_{44}c_{66}n_2^6 > 0\,, \tag{2.13}$$

which can be made explicit by invoking a lemma proved by Zee and Sternberg (**18**). The latter states that the inequalities

$$ax^3 + bx^2y + cxy^2 + dy^3 > 0\,, \qquad \text{where} \quad a > 0\,, \;\; d > 0\,, \tag{2.14}$$

are satisfied for all $x \geqslant 0$, $y \geqslant 0$, such that $x + y = 1$, *iff*

$$\text{either} \quad 27a^2d^2 + 4c^3a + 4b^3d - b^2c^2 - 18abcd > 0\,, \quad \text{or} \quad (b > 0 \quad \text{and} \quad c > 0)\,. \tag{2.15}$$

When specialised to our case, the lemma states that inequality (2.13) is equivalent to

$$\begin{aligned} &\text{either} \quad 4\left(c_{66}^2 - \beta_3^2\right)\left(c_{66}(c_{44}^2 + c_{55}^2) - 2c_{44}c_{55}\beta_3\right)^2 > 0\,, \\ &\text{or} \quad (c_{44}c_{66} + 2c_{55}\beta_3 > 0 \quad \text{and} \quad c_{55}c_{66} + 2c_{44}\beta_3 > 0)\,. \end{aligned} \tag{2.16}$$

A straightforward analysis, supplemented by the use of already established conditions (2.12), reduces inequalities (2.16) to

$$\text{either} \quad |\beta_3| < c_{66}\,, \quad \text{or} \quad \left(\beta_3 > -\frac{c_{44}}{c_{55}}\frac{c_{66}}{2} \quad \text{and} \quad \beta_3 > -\frac{c_{55}}{c_{44}}\frac{c_{66}}{2}\right). \tag{2.17}$$

Altogether, this means that

$$\beta_3 > -c_{66}\,. \tag{2.18}$$

Similar considerations applied to the coordinate planes Ox_1x_3 and Ox_2x_3 lead to the additional conditions

$$\beta_2 > -c_{55}\,, \qquad \beta_1 > -c_{44}\,. \tag{2.19}$$

In our forthcoming derivations it will be convenient to use additional material constants

$$\nu_1 = \beta_1 + c_{44}\,, \qquad \nu_2 = \beta_2 + c_{55}\,, \qquad \nu_3 = \beta_3 + c_{66}\,, \tag{2.20}$$

which allow us to re-write inequalities (2.18) and (2.19) as

$$\nu_i > 0\,, \qquad i \in \{1, 2, 3\}\,. \tag{2.21}$$

Effectively, we just proved that, when the direction of propagation is confined to coordinate planes, condition (2.11) is equivalent to inequalities (2.18), (2.19). In fact, inequalities (2.12) used in conjunction with inequalities (2.18), (2.19) are also sufficient to satisfy condition (2.10) for an *arbitrary* direction of propagation. Indeed,

$$\begin{aligned}
&(c_{55}+c_{66})\,n_1^4 + (c_{44}+c_{55}+2\beta_3)\,n_1^2n_2^2 + (c_{44}+c_{66})\,n_2^4 \\
&\qquad + (c_{44}+c_{66}+2\beta_2)\,n_1^2n_3^2 + (c_{55}+c_{66}+2\beta_1)\,n_2^2n_3^2 + (c_{44}+c_{55})\,n_3^4 \\
&> (c_{55}+c_{66})\,n_1^4 + (c_{44}+c_{55}-2c_{66})\,n_1^2n_2^2 + (c_{44}+c_{66})\,n_2^4 \\
&\qquad + (c_{44}+c_{66}-2c_{55})\,n_1^2n_3^2 + (c_{55}+c_{66}-2c_{44})\,n_2^2n_3^2 + (c_{44}+c_{55})\,n_3^4 \\
&= \left[n_1^2(n_2^2+n_3^2) + (n_2^2-n_3^2)^2\right] c_{44} + \left[n_2^2(n_1^2+n_3^2) + (n_1^2-n_3^2)^2\right] c_{55} \\
&\qquad\qquad + \left[n_3^2(n_1^2+n_2^2) + (n_1^2-n_2^2)^2\right] c_{66} > 0\,.
\end{aligned} \tag{2.22}$$

This means that condition (2.10) does not need to be considered any more.

To conclude this subsection, we remark that, as long as one of the components of vector $\mathbf{n}$ is equal to zero, the explicit inequalities (2.12), (2.18) and (2.19) are both necessary and sufficient to satisfy implicit strong ellipticity conditions (2.10) and (2.11). Consequently, these inequalities ensure the strong ellipticity in plane strain problems. In particular, the inequalities (2.18) and $(2.12)_3$ were previously obtained as the strong ellipticity conditions in a plane strain problem for incompressible orthotropic media, see (**20**).

2.2 *Necessary conditions for arbitrary directions*

The inequalities we derived thus far are insufficient to satisfy condition (2.11) for an arbitrary unit vector $\mathbf{n}$. Therefore, we need to construct additional inequalities that, together with inequalities (2.12), (2.18) and (2.19), would be both necessary and sufficient to satisfy (2.11) for a general direction of wave propagation. This is best done by introducing the following new functions of $\mathbf{n}$:

$$\begin{aligned}
\psi_1 &= 2\beta_1 + \frac{n_2^4+n_3^4}{n_2^2n_3^2}\,c_{44} + \frac{n_1^2}{n_3^2}\,c_{55} + \frac{n_1^2}{n_2^2}\,c_{66}\,, \\
\psi_2 &= 2\beta_2 + \frac{n_1^4+n_3^4}{n_1^2n_3^2}\,c_{55} + \frac{n_2^2}{n_3^2}\,c_{44} + \frac{n_2^2}{n_1^2}\,c_{66}, \\
\psi_3 &= 2\beta_3 + \frac{n_1^4+n_2^4}{n_1^2n_2^2}\,c_{66} + \frac{n_3^2}{n_2^2}\,c_{44} + \frac{n_3^2}{n_1^2}\,c_{55}\,.
\end{aligned} \tag{2.23}$$

Because of the already established inequalities (2.12), (2.18) and (2.19), it is easy to see that $\psi_i > 0$, $i = 1, 2, 3$, for all unit vectors $\mathbf{n}$. Written in terms of these functions, condition (2.11) takes the form

$$(\psi_3 - \psi_1 - \psi_2)^2 < 4\psi_1\psi_2 \,. \tag{2.24}$$

Since the product $\psi_1\psi_2$ is positive, it is clear that condition (2.24) necessitates that

$$\psi_3 < \psi_1 + \psi_2 + 2\sqrt{\psi_1\psi_2} \,. \tag{2.25}$$

This condition is simpler than (2.11) or (2.24), but is implicit still. Hence, we are now going to consider particular unit vectors $\mathbf{n}$ such that the square root on the right-hand side of (2.25) can be evaluated explicitly,[1] that is, that $\sqrt{\psi_1\psi_2} = \psi$. For such directions there exists a positive constant W such that

$$\psi_1 = \frac{\psi}{W}\,, \qquad \psi_2 = \psi W \,. \tag{2.26}$$

Straightforward algebraic manipulations allow one to conclude that this can be achieved provided

$$n_1^2 = \frac{n_3^2}{1+W}\,, \quad n_2^2 = \frac{Wn_3^2}{1+W}\,, \quad \text{where} \quad W = \frac{\nu_2}{\nu_1} \tag{2.27}$$

and definitions (2.20) were used. When direction (2.27) is inserted into inequality (2.25), it takes the following explicit form

$$\nu_3 < 2c_{66} + (\sqrt{\nu_1} + \sqrt{\nu_2})^2 \,. \tag{2.28}$$

The original condition (2.24) is symmetric in ψ_1, ψ_2 and ψ_3. By applying the outlined procedure to different permutations of indices, one can obtain two more explicit conditions

$$\nu_1 < 2c_{44} + (\sqrt{\nu_2} + \sqrt{\nu_3})^2 \,, \qquad \nu_2 < 2c_{55} + (\sqrt{\nu_1} + \sqrt{\nu_3})^2 \,. \tag{2.29}$$

The newly obtained inequalities (2.28) and (2.29) are necessary conditions that follow from strong ellipticity condition (2.11). In the following Section 2.3 we will prove that, when considered together with inequalities (2.12) and (2.21), they also happen to be sufficient.

2.3 *Sufficient conditions for an arbitrary direction*

Suppose that explicit inequalities for the components of stiffness tensor (2.12), (2.21), (2.28) and (2.29) hold true. We already mentioned in Section 2.1 that inequalities (2.12) and (2.21) are sufficient to ensure that condition (2.10) holds for an arbitrary direction $\mathbf{n}$. In addition, these inequalities are sufficient to satisfy condition (2.11) for directions confined to coordinate planes. Therefore, we are going to assume that vector $\mathbf{n}$ does not belong to a coordinate plane and will prove that inequalities (2.12), (2.21), (2.28) and (2.29) are sufficient to ensure that condition (2.11) is satisfied.

[1] In their derivation of the strong ellipticity conditions for the pre-stressed media, Zee and Sternberg (**18**) use a conventional procedure for finding the critical point of (2.25). However, the algebraic complexity of underlying expressions prevents them from computing the second derivative and proving that their solution delivers a local minimum. Thus, their procedure only serves to find the critical point in a constructive way (as opposed to specifying the critical point explicitly). We use another, simpler constructive procedure to find the relevant point.

Consider a function

$$X_3 = 2\psi_1\psi_2 + \left(1 - \frac{n_2^2}{n_3^2}\right) c_{44} + \left(1 - \frac{n_1^2}{n_3^2}\right) c_{55} + c_{66}\,. \tag{2.30}$$

It is not difficult to verify that

$$\tfrac{1}{2}(\psi_1\psi_2 - X_3^2) = \frac{\left((\sqrt{\nu_1} + \sqrt{\nu_2})n_2^2 - \sqrt{\nu_2}n_3^2\right)^2}{n_2^2 n_3^2} c_{44} + \frac{\left((\sqrt{\nu_1} + \sqrt{\nu_2})n_1^2 - \sqrt{\nu_1}n_3^2\right)^2}{n_1^2 n_3^2} c_{55}$$
$$+ \frac{\left(\sqrt{\nu_2}n_1^2 - \sqrt{\nu_1}n_2^2\right)^2}{n_1^2 n_2^2} c_{66} + \frac{(n_1^2 + n_2^2 - n_3^2)^2 \left(c_{55}c_{66}n_1^2 + c_{44}c_{66}n_2^2 + c_{44}c_{55}n_3^2\right)}{2n_1^2 n_2^2 n_3^2} \geqslant 0, \tag{2.31}$$

which immediately leads to the conclusion that

$$X_3^2 \leqslant \psi_1\psi_2\,. \tag{2.32}$$

Inequality (2.32) implies, in particular, that

$$\left(\sqrt{\psi_1} + \sqrt{\psi_2}\right)^2 - \psi_3 \geqslant \psi_1 + 2|X_3| + \psi_2 - \psi_3 \geqslant \psi_1 + 2X_3 + \psi_2 - \psi_3\,. \tag{2.33}$$

After some rather tedious algebra it may be additionally shown that

$$\psi_1 + 2X_3 + \psi_2 - \psi_3 = 2\left(2c_{66} + (\sqrt{\nu_1} + \sqrt{\nu_2})^2 - \nu_3\right) > 0\,, \tag{2.34}$$

where we used inequality (2.28). Altogether, this means that

$$\psi_3 < \left(\sqrt{\psi_1} + \sqrt{\psi_2}\right)^2. \tag{2.35}$$

Similar arguments can be constructed for other combinations of indices, which leads us to the following conclusion:

$$\psi_i < \left(\sqrt{\psi_j} + \sqrt{\psi_k}\right)^2, \qquad i, j, k \in \{1, 2, 3\}, \quad i \neq j \neq k \neq i. \tag{2.36}$$

All of ψ_i, $i = 1, 2, 3$, are positive (see definitions (2.23), as well as inequalities (2.12) and (2.21)). Therefore, inequalities (2.36) imply, in particular, that

$$\sqrt{\psi_1} < \sqrt{\psi_2} + \sqrt{\psi_3} \quad \text{and} \quad \sqrt{\psi_2} < \sqrt{\psi_1} + \sqrt{\psi_3}\,, \tag{2.37}$$

which is equivalent to

$$-\sqrt{\psi_3} < \sqrt{\psi_1} - \sqrt{\psi_2} < \sqrt{\psi_3}\,, \tag{2.38}$$

which, in turn, is equivalent to

$$\psi_3 > \left(\sqrt{\psi_1} - \sqrt{\psi_2}\right)^2. \tag{2.39}$$

The inequalities (2.35) and (2.39) considered together are equivalent to inequality (2.24) that is equivalent to the strong ellipticity condition (2.11). Therefore, we have now proved that the explicit inequalities (2.12), (2.21), (2.28) and (2.29) are both necessary and sufficient conditions for the strong ellipticity of orthorhombic, incompressible, elastic solids.

3. Strong ellipticity of finitely and homogeneously pre-stressed solids

Now we generalise our strong ellipticity conditions to the case of a pre-stressed incompressible elastic media, that is, to the case previously analysed by Zee and Sternberg (**18**). It should be mentioned that the pre-stressed case is formally rather similar to the previously considered case of orthorhombic symmetry. However, the response of a finitely deformed material is slightly more general, due to the loss of symmetry of the Cauchy stress tensor, see (**3**, p. 250) for more details.

The pre-deformed state is assumed to be isotropic, and is referred to as the initial configuration. A static homogeneous deformation is then imposed, transferring the initial configuration to a reference equilibrium state. Finally, small-amplitude motions are superimposed upon the reference configuration. The appropriate propagation condition turns out to have the same form as (2.7). It can be shown, using the appropriate constitutive relations given by Ogden (**21**), that the associated acoustic tensor is given by

$$\mathbf{Q} = \begin{pmatrix} \gamma_{11}n_1^2+\gamma_{21}n_2^2+\gamma_{31}n_3^2 & \frac{1}{2}\left(\gamma_{11}+\gamma_{22}-2\bar{\beta}_3\right)n_1n_2 & \frac{1}{2}\left(\gamma_{11}+\gamma_{33}-2\bar{\beta}_2\right)n_1n_3 \\ \frac{1}{2}\left(\gamma_{11}+\gamma_{22}-2\bar{\beta}_3\right)n_1n_2 & \gamma_{12}n_1^2+\gamma_{22}n_2^2+\gamma_{32}n_3^2 & \frac{1}{2}\left(\gamma_{22}+\gamma_{33}-2\bar{\beta}_1\right)n_2n_3 \\ \frac{1}{2}\left(\gamma_{11}+\gamma_{33}-2\bar{\beta}_2\right)n_1n_3 & \frac{1}{2}\left(\gamma_{22}+\gamma_{33}-2\bar{\beta}_1\right)n_2n_3 & \gamma_{13}n_1^2+\gamma_{23}n_2^2+\gamma_{33}n_3^2 \end{pmatrix}. \tag{3.1}$$

The newly introduced material constants γ_{ij} and $\bar{\beta}_k$

$$\gamma_{ij} = B_{ijij}, \quad 2\bar{\beta}_k = \gamma_{ii} + \gamma_{jj} - 2\left(B_{iijj} + B_{ijji}\right), \quad i, j, k \in \{1, 2, 3\}, \quad k \neq i, \quad k \neq j, \tag{3.2}$$

are expressed through the components of the fourth order elasticity tensor, with the non-zero components having the form B_{iijj}, B_{ijij} and B_{ijji},

$$B_{iijj} = \lambda_i\lambda_j \frac{\partial^2 W_0}{\partial\lambda_i\partial\lambda_j}, \tag{3.3}$$

$$B_{ijij} = \begin{cases} \dfrac{\lambda_i^2}{\lambda_i^2-\lambda_j^2}\left(\lambda_i\dfrac{\partial W_0}{\partial\lambda_i} - \lambda_j\dfrac{\partial W_0}{\partial\lambda_j}\right) & i \neq j, \quad \lambda_i \neq \lambda_j, \\ \dfrac{1}{2}\left(B_{iiii} - B_{iijj} + \lambda_i\dfrac{\partial W_0}{\partial\lambda_i}\right) & i \neq j, \quad \lambda_i = \lambda_j, \end{cases} \tag{3.4}$$

$$B_{ijji} = B_{ijij} - \lambda_i\frac{\partial W_0}{\partial\lambda_i} \qquad i \neq j, \tag{3.5}$$

where W_0 is the strain-energy function, and λ_i are the principal stretches of the homogeneous static pre-deformation, see (**21**). The formal parallels between (2.5) and (3.1) are obvious. The associated (implicit) strong ellipticity conditions, arising from the propagation condition for pre-stressed elastic media, may then be written as

$$\begin{aligned}(\gamma_{12}+\gamma_{13})\,n_1^4 &+ \left(\gamma_{13}+\gamma_{23}+2\bar{\beta}_3\right)n_1^2n_2^2 + (\gamma_{21}+\gamma_{23})\,n_2^4 \\ &+ \left(\gamma_{12}+\gamma_{32}+2\bar{\beta}_2\right)n_1^2n_3^2 + \left(\gamma_{21}+\gamma_{31}+2\bar{\beta}_1\right)n_2^2n_3^2 + (\gamma_{31}+\gamma_{32})\,n_3^4 > 0,\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&\gamma_{12}\gamma_{13}n_1^6 + \left(\gamma_{23}\gamma_{31} + 2\bar{\beta}_1\gamma_{21}\right)n_2^4n_3^2 + \left(\gamma_{21}\gamma_{32} + 2\bar{\beta}_1\gamma_{31}\right)n_2^2n_3^4 + \\
&\quad + \gamma_{21}\gamma_{23}n_2^6 + \left(\gamma_{13}\gamma_{32} + 2\bar{\beta}_2\gamma_{12}\right)n_1^4n_3^2 + \left(\gamma_{12}\gamma_{31} + 2\bar{\beta}_2\gamma_{32}\right)n_1^2n_3^4 + \\
&\quad + \gamma_{31}\gamma_{32}n_3^6 + \left(\gamma_{12}\gamma_{23} + 2\bar{\beta}_3\gamma_{13}\right)n_1^4n_2^2 + \left(\gamma_{13}\gamma_{21} + 2\bar{\beta}_3\gamma_{23}\right)n_1^2n_2^4 + \\
&\quad + \left(\gamma_{12}\gamma_{21} + \gamma_{13}\gamma_{31} + \gamma_{23}\gamma_{32} - \left(\bar{\beta}_1^2 + \bar{\beta}_2^2 - \bar{\beta}_3^2\right)^2 + 4\bar{\beta}_1\bar{\beta}_2\right)n_1^2n_2^2n_3^2 > 0\,.
\end{aligned} \tag{3.7}$$

The loss of symmetries of the orthorhombic case is apparent when conditions (3.6), (3.7) are compared with the previously considered conditions. For example, c_{66} in (2.10), (2.11) formally corresponds to both γ_{12} and γ_{21}, which are not, generally speaking, equal to each other. Nevertheless, the inspection of definitions (3.4) reveals the presence of an additional symmetry

$$\frac{\lambda_i}{\lambda_j}\gamma_{ji} = \frac{\lambda_j}{\lambda_i}\gamma_{ij} = \chi_k\,, \qquad i,j,k \in \{1,2,3\}\,, \quad i \neq j \neq k \neq i, \tag{3.8}$$

where no summation is assumed over repeated suffices. Identity (3.8) is a generalisation of the identity $\gamma_{12}\lambda_2^2 = \gamma_{21}\lambda_1^2$, often used in the analysis of the plane strain problems (**22**).

If, as in Section 2.1, one considers wave vectors $\mathbf{n}$ aligned with the coordinate axes, it may be shown that the conditions (3.6), (3.7) imply six inequalities

$$\gamma_{ij} > 0\,, \qquad i,j \in \{1,2,3\}\,, \quad i \neq j\,. \tag{3.9}$$

Moreover, using the symmetries (3.8), we may reduce (3.9) to three conditions

$$\chi_k > 0\,, \qquad k \in \{1,2,3\}\,. \tag{3.10}$$

The analysis for wave vectors confined to coordinate planes is also rather similar to that presented in great detail in the previous section. For example, in the Ox_1x_2 plane the condition (3.7) reduces to

$$\gamma_{12}\gamma_{13}n_1^6 + \left(\gamma_{12}\gamma_{23} + 2\bar{\beta}_3\gamma_{13}\right)n_1^4n_2^2 + \left(\gamma_{13}\gamma_{21} + 2\bar{\beta}_3\gamma_{23}\right)n_1^2n_2^4 + \gamma_{21}\gamma_{23}n_2^6 > 0\,. \tag{3.11}$$

Applying Zee and Sternberg's lemma, just as in Section 2.1, we obtain that condition (3.11) holds *iff*

$$\text{either} \quad |\bar{\beta}_3| < \sqrt{\gamma_{12}\gamma_{21}} \quad \text{or} \quad \left(\bar{\beta}_3 > -\frac{\gamma_{23}}{\gamma_{13}}\frac{\gamma_{12}}{2} \quad \text{and} \quad \bar{\beta}_3 > -\frac{\gamma_{13}}{\gamma_{23}}\frac{\gamma_{21}}{2}\right), \tag{3.12}$$

which also satisfies condition (3.6). Once again, using identity (3.8), one can rewrite (3.12) as

$$\text{either} \quad |\bar{\beta}_3| < \chi_3 \quad \text{or} \quad \left(\bar{\beta}_3 > -\frac{\chi_1}{\chi_2}\frac{\chi_3}{2} \quad \text{and} \quad \bar{\beta}_3 > -\frac{\chi_2}{\chi_1}\frac{\chi_3}{2}\right), \tag{3.13}$$

which is equivalent to $\bar{\beta}_3 > -\chi_3$. Similar analysis for the other two coordinate planes may be performed, and the well-known necessary and sufficient strong ellipticity conditions for plane strain deformations of a pre-stressed elastic solid may then be written as

$$\bar{\nu}_k > 0\,, \qquad k \in \{1, 2, 3\}\,, \tag{3.14}$$

where $\bar{\nu}_k = \bar{\beta}_k + \chi_k$. Conditions (3.10), and (3.14) turn out to be the necessary and sufficient strong ellipticity conditions for wave vectors confined to coordinate planes.

The analysis for general three-dimensional wave vectors is more involved, but can also be reproduced if steps in Sections 2.2 and 2.3 are reproduced using the definitions

$$\bar{\psi}_k = 2\bar{\beta}_k + \gamma_{ki}\frac{n_k^2}{n_i^2} + \gamma_{kj}\frac{n_k^2}{n_j^2} + \gamma_{ij}\frac{n_i^2}{n_j^2} + \gamma_{ji}\frac{n_j^2}{n_i^2}, \qquad i, j, k \in \{1, 2, 3\}\,, \quad i \neq j \neq k \neq i, \tag{3.15}$$

instead of ψ_k for the orthorhombic case. The rest of the derivations are fully identical, see Pichugin (**23**, Chapter 1) for more details, and the counterpart of (2.28) and (2.29) may be shown to have the form

$$\bar{\nu}_k < 2\chi_k + \left(\sqrt{\bar{\nu}_i} + \sqrt{\bar{\nu}_j}\right)^2, \qquad i, j, k \in \{1, 2, 3\}\,, \quad i \neq j \neq k \neq i. \tag{3.16}$$

Together, conditions (3.10), (3.14) and (3.16) constitute the necessary and sufficient conditions for the strong ellipticity of a homogeneously pre-stressed incompressible solid. In contrast to the results obtained in (**18**), these formulae are given directly in terms of components of the incremental stiffness tensor.

4. Discussion of the obtained conditions

Since explicit strong ellipticity conditions were typically unavailable, or available in inconvenient form, many parametric analyses of boundary value problems in elasticity were performed under some other form of constitutive assumptions. A number of plane strain results were obtained under the assumption of two-dimensional strong ellipticity conditions, equivalent to inequalities (2.12) and (2.21), which are always easy to derive. The positive definiteness of the strain energy density is also sometimes considered, even though the physical significance of such restriction in a constrained medium is unclear (**1**, p. 238). For example, Nair and Sotiropoulos in their analysis of plane wave reflection in the plane strain problem showed that the two-dimensional strong ellipticity conditions in their case are identical to the conditions that ensure positive definiteness of the associated strain energy density (**20**). Therefore, it seems worthwhile to assess relative strength of the mentioned constitutive inequalities, especially when applied to solids belonging to higher symmetry classes.

We begin by constructing explicit conditions that ensure positive definiteness of the strain energy density. The strain energy density is defined as

$$U = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}\,. \tag{4.1}$$

Incompressibility constraint $(2.1)_2$ may be written as $\varepsilon_{ii} = 0$. Using the constitutive relations (2.2), quadratic form (4.1) can be re-written in one of the three essentially equivalent forms, omitting either ε_{11} or ε_{22} or ε_{33}:

$$\begin{aligned} U &= \nu_3\varepsilon_{22}^2 + (\nu_2 + \nu_3 - \nu_1)\varepsilon_{22}\varepsilon_{33} + \nu_2\varepsilon_{33}^2 + 2c_{66}\varepsilon_{12}^2 + 2c_{55}\varepsilon_{13}^2 + 2c_{44}\varepsilon_{23}^2 \\ &= \nu_3\varepsilon_{11}^2 + (\nu_1 + \nu_3 - \nu_2)\varepsilon_{11}\varepsilon_{33} + \nu_1\varepsilon_{33}^2 + 2c_{66}\varepsilon_{12}^2 + 2c_{55}\varepsilon_{13}^2 + 2c_{44}\varepsilon_{23}^2 \\ &= \nu_2\varepsilon_{11}^2 + (\nu_1 + \nu_2 - \nu_3)\varepsilon_{11}\varepsilon_{22} + \nu_1\varepsilon_{22}^2 + 2c_{66}\varepsilon_{12}^2 + 2c_{55}\varepsilon_{13}^2 + 2c_{44}\varepsilon_{23}^2\,, \end{aligned} \tag{4.2}$$

Table 1 Explicit strong ellipticity conditions specialized to higher symmetry classes and viewed in the context of two-dimensional (plane strain) and three-dimensional deformations

Material symmetry	Strong ellipticity	Positive definiteness of the strain energy density
Isotropic (2D+3D)	$\mu = \frac{1}{2}(c_{11} - c_{12}) > 0$	
Cubic (2D+3D)	$c_{44} > 0,\ c_{12} < c_{11}$	
Hexagonal (2D)	$c_{44} > 0,\ c_{12} < c_{11},\ 2c_{13} < c_{11} + c_{33}$	
Hexagonal (3D)	—	$4c_{13} - 2c_{33} - c_{11} < c_{12}$
Tetragonal (2D)	$c_{44} > 0,\ c_{66} > 0,\ c_{12} < c_{11},\ 2c_{13} < c_{11} + c_{33}$	
Tetragonal (3D)	$4c_{13} - 2c_{33} - 2c_{66} - c_{11} < c_{12}$	$4c_{13} - 2c_{33} - c_{11} < c_{12}$

see definitions (2.20). If one's attention is restricted to plane strain deformations, the strong ellipticity conditions for the coordinate axes and planes turn out to be necessary and sufficient for the positive definiteness of U. For example, for problems confined to the plane Ox_1x_2 when $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$ and $\varepsilon_{11} + \varepsilon_{22} = 0$, the elementary analysis of the third line of (4.2) gives $c_{66} > 0$ and $\nu_3 > 0$. Consideration of all three coordinate planes results in the full set of conditions (2.12) and (2.21). Therefore, for plane strain problems, the strong ellipticity conditions are fully equivalent to the requirement of positive definiteness of the strain energy density U.

More general conditions may be obtained by considering strain configurations for which $\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}$, and one of ε_{11} or ε_{22} or ε_{33}, are equal to zero. The positive definiteness of U in this case can be shown to be equivalent to the following inequalities

$$4\nu_2\nu_3 > (\nu_2 + \nu_3 - \nu_1)^2, \quad 4\nu_1\nu_3 > (\nu_1 + \nu_3 - \nu_2)^2, \quad 4\nu_1\nu_2 > (\nu_1 + \nu_2 - \nu_3)^2. \tag{4.3}$$

Incidentally, these conditions are also sufficient to ensure the positive definiteness of U, which can be seen by analysing the first three terms of each quadratic form in (4.2).

Using the previously established conditions (2.12) and (2.21), one can rearrange the inequalities (4.3) in a more explicit form and conclude that the positive definiteness of U is equivalent to the following set of necessary and sufficient conditions

$$\begin{gathered} c_{44} > 0\,, \quad c_{55} > 0\,, \quad c_{66} > 0\,, \\ (\sqrt{\nu_2} - \sqrt{\nu_3})^2 < \nu_1 < (\sqrt{\nu_2} + \sqrt{\nu_3})^2\,, \\ (\sqrt{\nu_1} - \sqrt{\nu_3})^2 < \nu_2 < (\sqrt{\nu_1} + \sqrt{\nu_3})^2\,, \\ (\sqrt{\nu_1} - \sqrt{\nu_2})^2 < \nu_3 < (\sqrt{\nu_1} + \sqrt{\nu_2})^2\,. \end{gathered} \tag{4.4}$$

Written in this form, conditions (4.4) are obviously stronger than the strong ellipticity conditions we derived earlier.

How much of a difference do the additional, non-plane strain (three-dimensional) strong ellipticity conditions make for the materials that belong to higher symmetry classes? We derived explicit versions of conditions (2.12), (2.21), (2.28) and (2.29), as well as explicit versions of conditions (4.4), for principal symmetry classes and presented them in Table 1. This analysis showed that

- For isotropic, cubic and hexagonal media the strong ellipticity conditions obtained for plane strain deformations are also correct for general deformations;
- For hexagonal and tetragonal media, the positive definiteness of the strain energy density requires to satisfy, in addition to the usual plane strain conditions, one more inequality $4c_{13} - 2c_{33} - c_{11} < c_{12}$;
- Only for tetragonal (and orthorhombic) materials newly obtained strong ellipticity conditions (2.28) and (2.29) result in non-trivial additional requirements compared to the strong ellipticity conditions known for plane strain deformations.

5. Conclusion

In this article, we obtained two sets of explicit strong ellipticity conditions for incompressible elastic media. More specifically, the following conditions, equivalent to inequalities (2.12), (2.18), (2.19), (2.28) and (2.29), are the necessary and sufficient conditions for the strong ellipticity of the orthorhombic incompressible solids:

$$c_{44} > 0\,, \quad c_{55} > 0\,, \quad c_{66} > 0\,, \tag{5.1}$$

$$-c_{44} < \beta_1 < c_{44} + \left(\sqrt{\beta_2 + c_{55}} + \sqrt{\beta_3 + c_{66}}\right)^2, \tag{5.2}$$

$$-c_{55} < \beta_2 < c_{55} + \left(\sqrt{\beta_1 + c_{44}} + \sqrt{\beta_3 + c_{66}}\right)^2, \tag{5.3}$$

$$-c_{66} < \beta_3 < c_{66} + \left(\sqrt{\beta_1 + c_{44}} + \sqrt{\beta_2 + c_{55}}\right)^2, \tag{5.4}$$

with the definitions for β_1, β_2 and β_3 given by (2.6).

The left-hand side inequalities in (5.2)–(5.4) are not new and may be obtained by considering plane strain deformations, whereas the right-hand side inequalities are new and result from considering three-dimensional deformations. It is worth stressing that our analysis of higher symmetry classes, summarized in Table 1, indicates that the newly derived three-dimensional strong ellipticity conditions are trivial for isotropic, cubic and hexagonal incompressible media. Non-trivial consequences of the right-hand side inequalities in (5.2)–(5.4) only become apparent in tetragonal and orthorhombic incompressible media.

In the case of homogeneously pre-stressed incompressible elastic media the necessary and sufficient strong ellipticity conditions are given by

$$\gamma_{12} > 0, \qquad \gamma_{13} > 0, \qquad \gamma_{23} > 0, \tag{5.5}$$

$$-\sqrt{\gamma_{23}\gamma_{32}} < \bar{\beta}_1 < \sqrt{\gamma_{23}\gamma_{32}} + \left(\sqrt{\bar{\beta}_2 + \sqrt{\gamma_{13}\gamma_{31}}} + \sqrt{\bar{\beta}_3 + \sqrt{\gamma_{12}\gamma_{21}}}\right)^2, \tag{5.6}$$

$$-\sqrt{\gamma_{13}\gamma_{31}} < \bar{\beta}_2 < \sqrt{\gamma_{13}\gamma_{31}} + \left(\sqrt{\bar{\beta}_1 + \sqrt{\gamma_{23}\gamma_{32}}} + \sqrt{\bar{\beta}_3 + \sqrt{\gamma_{12}\gamma_{21}}}\right)^2, \tag{5.7}$$

$$-\sqrt{\gamma_{12}\gamma_{21}} < \bar{\beta}_3 < \sqrt{\gamma_{12}\gamma_{21}} + \left(\sqrt{\bar{\beta}_1 + \sqrt{\gamma_{23}\gamma_{32}}} + \sqrt{\bar{\beta}_2 + \sqrt{\gamma_{13}\gamma_{31}}}\right)^2, \tag{5.8}$$

with the constants γ_{ij} and β_k, $i, j, k \in \{1, 2, 3\}$, defined in (3.2).

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