Observer-based $\mathcal{H}_\infty$ Control of Networked Systems with Stochastic Communication Protocol: the Finite-Horizon Case ⋆

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Abstract

This paper is concerned with the $\mathcal{H}_\infty$ control problem for a class of linear time-varying networked control systems (NCSs) with stochastic communication protocol (SCP). The sensor-to-controller network (the controller-to-actuator network) is considered where only one sensor (one actuator) obtains access to the communication network at each transmission instant. The SCP is applied to determine which sensor (actuator) should be given the access to the network at a certain instant. The aim of the problem addressed is to design an observer-based controller such that the $\mathcal{H}_\infty$ performance of the closed-loop system is guaranteed over a given finite horizon. For the purpose of simplifying the NCS model, a new Markov chain is constructed to model the SCP scheduling of communication networks. Then, both the methods of stochastic analysis and completing squares are utilized to establish the sufficient conditions for the existence of the desired controller. The controller parameters are characterized by solving two coupled backward recursive Riccati difference equations subject to the scheduled SCP. Finally, a numerical example is given to illustrate the effectiveness of the proposed controller design scheme.

Key words: Stochastic communication protocol; $\mathcal{H}_\infty$ control; Time-varying systems; Networked control systems; Recursive Riccati difference equations.

1 Introduction

Networked control systems (NCSs) are control systems in which the signal transmission between system components (e.g. sensors, actuators and controller) is implemented through the communication networks. Since NCSs possess many advantages such as low cost, simple installation, reduced system wiring and high reliability, they have found successful applications in a wide range of areas including environmental monitoring, industrial automation, smart grids and distributed/mobile communications. Accordingly, the control and filtering issues of NCSs have gained ever-increasing research attention, see e.g. [1,12,16]. For instance, the reliable control problem has been investigated in [16] for unreliable NCSs with probabilistic actuator failures, measurement distortions, random network-induced delays and packet dropouts. The design problem of the optimal $\mathcal{H}_\infty$ filtering has been dealt with in [12] for NCSs with multiple packet dropouts.

In reality, almost all systems have certain time-varying parameters since the system parameters may be changeable in time due to a variety of reasons such as temperature fluctuation, operating point shifting, graduate aging of system components, etc. Because of the time-varying nature of the underlying systems, one would be naturally more interested in analyzing their *transient* dynamics over a finite horizon than the traditional *steady-state* behaviors over the infinite horizon, see e.g. [8, 10, 14]. In recent years, considerable research attention has been devoted to the $\mathcal{H}_\infty$ control/filtering problems for time-varying systems, see e.g. [2–4, 6–8, 13] and the references therein. From a technical point of view, there are generally two effective approaches to solving the $\mathcal{H}_\infty$ control/filtering problems for time-varying systems: the so-called recursive linear matrix inequality (RLMI) approach [3,4,7,8] and the Riccati differential/difference equation (RDE) approach [2]. For example, in [4], a finite-horizon $\mathcal{H}_\infty$ fault estimator has been designed for a class of nonlinear stochastic time-varying systems with both randomly occurring faults and fading channels based on the RLMI approach. The probability-guaranteed $\mathcal{H}_\infty$ finite-horizon filtering problem has been considered in [7] for a class of nonlinear time-varying systems with uncertain parameters and sensor saturations by using the RLMI approach. The $\mathcal{H}_\infty$ control problem has been investigated in [2] for discrete time-varying nonlinear systems with both randomly occurring nonlinearities and fading measurements over a finite-horizon by using the backward recursive RDE approach.

In most existing literature concerning the control problems of NCSs, it has been assumed that all the sensors (or actuators) could simultaneously get access to the communication network to transmit/receive signal-

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s. However, this assumption is generally unrealistic since real-world networks unavoidably suffer from limited bandwidth which is likely to give rise to data collisions in case of simultaneous multiple accesses. As such, many communication protocols have been introduced in industry in order to prevent the data from collisions by determining which sensors (or actuators) should obtain access to the communication networks. These protocols include, but are not limited to, the Round-Robin protocol [17], the try-once-discard protocol [18] and the stochastic communication protocol (SCP) [9, 15, 20]. So far, the analysis and synthesis problems of NCSs subject to various communication protocols have begun to stir some initial research interest. For example, in [17], the distributed $\mathcal{H}_\infty$ estimation problem has been studied for sensor networks subject to the Round-Robin protocol scheduling by using the time-delay system approach. The optimal linear estimation problem has been investigated in [20] for networked systems subject to a random media access control (MAC) protocol. In [5], the stability issue has been investigated for NCSs with time-varying transmission intervals, time-varying transmission delays, packet dropouts subject to various communication protocols (e.g. Round-Robin protocol, try-once-discard protocol and SCP) by using a switching system approach.

The SCP serves as a widely used model describing a certain class of carrier-sense multiple access with collision avoidance (CSMA/CA) protocols. The CSMA/CA protocols have been implemented in a variety of communication systems (e.g. IEEE 802.11-based wireless local area networks and IEEE 802.15.4-based wireless sensor networks). Recently, the analysis issue of NCSs subject to SCP has drawn some refreshed research attention, see e.g. [5, 9, 15]. In particular, a linear time-invariant (LTI) continuous-time NCS with the SCP has been modeled in [5] by utilizing the properties of the Markov process. It should be mentioned that the communication protocol would inevitably complicate the dynamics analysis of the NCS especially when the NCS exhibits the time-varying. To this end, a seemingly interesting research problem is to investigate the control problem for the time-varying NCS with SCP constraints owing to its clear engineering insight in both control and communication areas. Nevertheless, this is a non-trivial problem with three challenges identified as follows: 1) how to develop a recursive algorithm accounting for the time-varying nature of the SCP-constrained NCS? 2) how to obtain the sufficient conditions for the existence of the desired time-varying controllers? and 3) how to examine the impact from the SCP on the control performance of the overall system? It is, therefore, the main purpose of this paper to offer satisfactory answers to the aforementioned three questions.

In response to the above discussion, in this paper, we aim to investigate the finite-horizon $\mathcal{H}_\infty$ control problem for the NCS with the SCP constraints. More specifically, the objective of this paper is to design an observer-based controller for the NCS subject to SCP such that the $\mathcal{H}_\infty$ performance of the closed-loop system is guaranteed over a given finite horizon. *The main contributions of this paper are highlighted as follows. 1) The control problem is, for the first time, investigated for time-varying systems with the SCP. 2) Both sensor-to-controller network and controller-to-actuator network featured with the SCPs are simultaneously considered in the controller design. 3) A novel coupled RDE approach is developed to solve the addressed finite-horizon $\mathcal{H}_\infty$ control problem. 4) The impact from the SCP on the structure of the controller gain matrix is revealed.*

The rest of this paper is organized as follows. In Section 2, the NCS with time-varying parameters and two communication networks are introduced and the problem under consideration is formulated. In Section 3, the design problem of observer-based controller is solved in terms of the solution to two coupled backward recursive RDEs. Furthermore, a numerical simulation example is given in Section 4 to illustrate the effectiveness of the controller design scheme. Finally, we conclude the paper in Section 5.

Notations: The notation used here is fairly standard except where otherwise stated. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and set of all $n \times m$ real matrices. The notation $X \geq Y$ ($X > Y$), where X and Y are real symmetric matrices, means that $X - Y$ is positive semi-definite (positive definite). Prob$\{\cdot\}$ means the occurrence probability of the event "$\cdot$". $\mathbb{E}\{x\}$ and $\mathbb{E}\{x|y\}$ will, respectively, denote the expectation of the stochastic variable x and expectation of x conditional on y. 0 represents the zero matrix of compatible dimensions. The n-dimensional identity matrix is denoted as I_n or simply I, if no confusion is caused. The shorthand diag$\{\cdots\}$ stands for a block-diagonal matrix. $\|v\|$ refers to the Euclidean norm of a vector v. M^T and $M^\dagger \in \mathbb{R}^{n\times m}$ represent the transpose and the Moore-Penrose pseudo inverse of $M \in \mathbb{R}^{m\times n}$. $\|M\|_F$ denotes the Frobenius norm of the matrix M. Matrices, if they are not explicitly specified, are assumed to have compatible dimensions. Let a be an integer and b be a positive integer. The function mod(a, b) represents the unique nonnegative remainder on division of the integer a by the positive integer b. The floor function $\lfloor b \rfloor$ denotes the largest integer not greater than b. The Kronecker delta function $\delta(a)$ is a binary function that equals 1 if $a = 0$ and equals 0 otherwise.

2 Problem Formulation and Preliminaries

In this section, we introduce some preliminaries related to the communication of NCSs and then describe the problem setup.

2.1 Stochastic Communication Protocol (SCP)

Consider a NCS with N transmission nodes labeled as $\{1, 2, \cdots, N\}$. The main idea of the SCP for *discrete-time systems* is that *only one node* is selected to transmit/receive data via the communication network at each transmission instant. Let $\xi(k)$ denote the selected node obtaining access to the network at time k. Then, as shown in [5], under the SCP scheduling, $\xi(k) \in \{1, 2, \cdots, N\}$ can be regarded as a stochastic process which could be modeled by a Markov chain. The occurrence probability of $\xi(k + 1) = j$ conditioned on $\xi(k) = i$ is given by

$$\mathcal{P}\{\xi(k + 1) = j|\xi(k) = i\} = \pi_{ij}(k)$$

where $\pi_{ij}(k) \geq 0$ ($i, j \in \{1, 2, \cdots, N\}$) is the transition probability from i to j at time instant k and $\sum_{j=1}^{N} \pi_{ij}(k) = 1$ ($i \in \{1, 2, \cdots, N\}$).

Remark 1 *The so-called stochastic communication protocol (SCP) has been first investigated in [15] for continuous-time systems and [5] for discrete-time systems. Such a protocol actually belongs to the category of the CSMA/CA protocols. The CSMA protocol is implemented based on the principle "sense before transmit" or "listen before talk". The CSMA/CA protocol can be seen as an improved version of the CSMA protocol. In practice, by relying on the acknowledgements from the communication network to indicate each transmission, the packet collisions could be avoided. Specifically, in a NCS with the CSMA/CA protocol, each node of the NCS should have a sense of the communication network for its status (idle or busy), where a node is permitted to transmit/receive data only when the network is idle. If the network is known to be busy, the node intending to send/receive data waits for a random interval (random backoff time) and then checks again to see if the network is idle. Because of the "random switch" behavior of such a node scheduling procedure, the Markov chain has been employed to characterize the CSMA/CA protocol [5]. On the other hand, the co-design issue of the node scheduling scheme and the $\mathcal{H}_\infty$ controller is important yet complicated especially when the trade-off between the scheduler performance and the controller performance becomes a concern, and this is one of our future research topics.*

2.2 Problem formulation

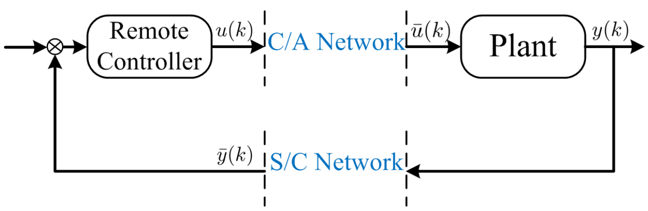

Fig. 1. Structure of the control system with communication networks

Consider a NCS with two communication networks shown in Fig. 1. The signal transmission is implemented between the plant and remote controller via two communication networks: the S/C (sensor-to-controller) network and the C/A (controller-to-actuator) network. The plant is a discrete time-varying system defined on the finite horizon ($k \in [0, N-1]$) of the form

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)\bar{u}(k) + D(k)\nu(k) \\ \quad y(k) = C(k)x(k) + E(k)\nu(k) \\ \quad z(k) = M(k)x(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$, $\bar{u}(k) \in \mathbb{R}^{n_u}$, $y(k) \in \mathbb{R}^{n_y}$ and $z(k) \in \mathbb{R}^{n_z}$ denote, respectively, the state vector, the control input after transmitted through the C/A network, the measurement output before transmitted through the S/C network and the signal to be controlled. $\nu(k) \in l_2([0, N-1]; \mathbb{R}^{n_\nu})$ is the disturbance input where $l_2([0, N-1], \mathbb{R}^{n_\nu})$ is the space of square-summable n_ν-dimensional vector functions over the interval $[0, N-1]$. $A(k)$, $B(k)$, $C(k)$, $D(k)$, $E(k)$ and $M(k)$ are known time-varying matrices with appropriate dimensions.

Let us now discuss the effect induced by communication networks (i.e., S/C network and C/A network). In this paper, it is assumed that there is no packet dropout occurring during the data transmissions through communication networks. For technical analysis, we first write

$$y(k) = \left[y_1^T(k) \ y_2^T(k) \ \cdots \ y_{n_y}^T(k) \right]^T,$$
$$\bar{u}(k) = \left[\bar{u}_1^T(k) \ \bar{u}_2^T(k) \ \cdots \ \bar{u}_{n_u}^T(k) \right]^T.$$

where $y_i(k)$ is the measurement of the i-th sensor, $\bar{u}_i(k)$ is the input signal of the j-th actuator.

In this work, the SCP is utilized to determine which sensor (or actuator) obtains access to the S/C network (C/A network). According to the SCP scheduling, only one sensor is allowed to get access to the S/C network and only one actuator obtains access to the C/A network. For the sake of examining the influence of the SCP constraints, let $\sigma(k) \in \{1, 2, \cdots, n_y\}$ ($\theta(k) \in \{1, 2, \cdots, n_u\}$) denote the selected sensor (actuator) obtaining access to the S/C network (C/A network) at the time instant k, where $\theta(k)$ and $\sigma(k)$ are independent of each other. Under the SCP scheduling, $\theta(k)$ and $\sigma(k)$ can be governed by Markov chains with the transition probability matrices $\mathcal{P}_1(k)$ and $\mathcal{P}_2(k)$, respectively. The transition probability matrices $\mathcal{P}_1(k) = \left[p_1^{ij}(k)\right]_{n_y \times n_y}$ and $\mathcal{P}_2(k) = \left[p_2^{ij}(k)\right]_{n_u \times n_u}$ are defined as follows:

$$\begin{cases} p_1^{ij}(k) \triangleq \text{Prob}\left(\theta(k+1) = j | \theta(k) = i\right) \\ p_2^{ij}(k) \triangleq \text{Prob}\left(\sigma(k+1) = j | \sigma(k) = i\right) \end{cases} \tag{2}$$

Let $\bar{y}(k) \triangleq \left[\bar{y}_1^T(k) \ \bar{y}_2^T(k) \ \cdots \ \bar{y}_{n_y}^T(k) \right]^T \in \mathbb{R}^{n_y}$ be the measurement output after transmitted through S/C network and $u(k) \triangleq \left[u_1^T(k) \ u_2^T(k) \ \cdots \ u_{n_u}^T(k) \right] \in \mathbb{R}^{n_u}$ be the control input before transmitted through C/A network. The updating rule for $\bar{y}_i(k)$ ($i = 1, 2, \cdots, n_y$) and $\bar{u}_i(k)$ ($i = 1, 2, \cdots, n_u$) subject to the SCP is set to be

$$\bar{y}_i(k) = \begin{cases} y_i(k) + E_{1,i}\omega(k), & \text{if } i = \sigma(k) \\ \bar{y}_i(k-1), & \text{otherwise} \end{cases} \tag{3}$$

$$\bar{u}_i(k) = \begin{cases} u_i(k) + E_{2,i}\omega(k), & \text{if } i = \theta(k) \\ \bar{u}_i(k-1), & \text{otherwise} \end{cases} \tag{4}$$

where $\omega(k) \in l_2\left([0, N-1]; \mathbb{R}^{n_\omega}\right)$ is the network-induced disturbance, and $E_{1,i}(k) \in \mathbb{R}^{1 \times n_\omega}$ ($i = 1, 2, \cdots, n_y$) and $E_{2,j}(k) \in \mathbb{R}^{1 \times n_\omega}$ ($j = 1, 2, \cdots, n_u$) are known, real, time-varying matrices. Defining

$$\begin{aligned} E_1(k) &\triangleq \left[E_{1,1}^T(k) \ E_{1,2}^T(k) \ \cdots \ E_{1,n_y}^T(k) \right]^T, \\ E_2(k) &\triangleq \left[E_{2,1}^T(k) \ E_{2,2}^T(k) \ \cdots \ E_{2,n_u}^T(k) \right]^T. \end{aligned} \tag{5}$$

we have

$$\begin{cases} \bar{y}(k) = \Phi^y_{\sigma(k)}\big(y(k) + E_1(k)\omega(k)\big) \\ \qquad\quad + (I - \Phi^y_{\sigma(k)})\bar{y}(k-1) \\ \bar{u}(k) = \Phi^u_{\theta(k)}\big(u(k) + E_2(k)\omega(k)\big) \\ \qquad\quad + (I - \Phi^u_{\theta(k)})\bar{u}(k-1) \end{cases} \tag{6}$$

where

$$\begin{cases} \Phi^y_{\sigma(k)} = \text{diag}\big\{\bar{\delta}^1_{\sigma(k)}, \bar{\delta}^2_{\sigma(k)}, \cdots, \bar{\delta}^{n_y}_{\sigma(k)}\big\} \\ \Phi^u_{\theta(k)} = \text{diag}\big\{\bar{\delta}^1_{\theta(k)}, \bar{\delta}^2_{\theta(k)}, \cdots, \bar{\delta}^{n_u}_{\theta(k)}\big\} \end{cases}$$

and $\bar{\delta}_a^b \triangleq \delta(a-b)$ in which $\delta(\cdot)$ is the Kronecker delta function.

Denoting $\tilde{x}(k) \triangleq \left[x^T(k)\ \bar{y}^T(k-1)\ \bar{u}^T(k-1)\right]^T$, the time-varying system (1) with the SCP constraints can be reformulated as follows:

$$\begin{cases} \tilde{x}(k+1) = \bar{A}_{\sigma(k),\theta(k)}(k)\tilde{x}(k) + \bar{B}_{\theta(k)}(k)u(k) \\ \qquad\qquad + \bar{D}_{\sigma(k),\theta(k)}(k)\bar{\nu}(k) \\ \quad \bar{y}(k) = \bar{C}_{\sigma(k)}(k)\tilde{x}(k) + \bar{E}_{\sigma(k)}(k)\bar{\nu}(k) \\ \quad z(k) = \bar{M}(k)\tilde{x}(k) \end{cases} \tag{7}$$

where

$$\begin{aligned} \bar{A}_{\sigma(k),\theta(k)}(k) &= \begin{bmatrix} A(k) & 0 & B(k)\left(I-\Phi^u_{\theta(k)}\right) \\ \Phi^y_{\sigma(k)}C(k) & I-\Phi^y_{\sigma(k)} & 0 \\ 0 & 0 & I-\Phi^u_{\theta(k)} \end{bmatrix}, \\ \bar{B}_{\theta(k)}(k) &= \left[\left(\Phi^u_{\theta(k)}\right)^T B^T(k)\ 0\ \left(\Phi^u_{\theta(k)}\right)^T\right]^T, \\ \bar{D}_{\sigma(k),\theta(k)}(k) &= \begin{bmatrix} D(k) & B(k)\Phi^u_{\theta(k)}E_2(k) \\ \Phi^y_{\sigma(k)}E(k) & \Phi^y_{\sigma(k)}E_1(k) \\ 0 & \Phi^u_{\theta(k)}E_2(k) \end{bmatrix}, \\ \bar{C}_{\sigma(k)}(k) &= \left[\Phi^y_{\sigma(k)}C(k)\ \ I-\Phi^y_{\sigma(k)}\ \ 0\right], \\ \bar{E}_{\theta(k)}(k) &= \left[\Phi^y_{\sigma(k)}E(k)\ \ \Phi^y_{\sigma(k)}E_1(k)\right], \\ \bar{M}(k) &= \left[M(k)\ 0\ 0\right], \quad \bar{\nu}(k) = \left[\nu^T(k)\ \omega^T(k)\right]^T. \end{aligned}$$

For analysis convenience, we now reformulate the system (7) by mapping the two stochastic processes $\theta(k)$ and $\sigma(k)$ to one Markov chain. The following proposition can be easily accessible from Lemma 1 of [19].

Proposition 1 *The Markov chains $\theta(k)$ and $\sigma(k)$ of the system (7) can be mapped to the sequence $r(k) \in \mathcal{R} \triangleq \{1, 2, \cdots, n_y n_u\}$ by the following mapping $\Theta(\cdot,\cdot)$:*

$$r(k) = \Theta(\theta(k), \sigma(k)) \triangleq \theta(k) + (\sigma(k)-1)n_u, \tag{8}$$

Moreover, if $r(k)$ is given, the values of $\theta(k)$ and $\sigma(k)$ can be derived by $\phi_1(r(k))$ and $\phi_2(r(k))$:

$$\begin{cases} \sigma(k) = \phi_1(r(k)) \triangleq \left\lfloor \dfrac{r(k)-1}{n_u} \right\rfloor + 1, \\ \theta(k) = \phi_2(r(k)) \triangleq \mathrm{mod}(r(k)-1, n_u) + 1 \end{cases} \tag{9}$$

Obviously, there is a *one-to-one correspondence* between the variable $r(k)$ and the pair $(\theta(k), \sigma(k))$. According to the mapping described by (8), the transition probability matrix $\bar{\mathcal{P}} = \left[\bar{p}_{ij}(k)\right]_{n_u n_y \times n_u n_y}$ of the Markov chain $r(k)$ is obtained as follows:

$$\begin{aligned} \bar{p}_{ij}(k) &= \mathrm{Prob}(r(k+1) = j | i = r(k)) \\ &= \mathrm{Prob}\big(\sigma(k+1) = \phi_1(j) \big| \sigma(k) = \phi_1(i)\big) \\ &\quad \times \mathrm{Prob}\big(\theta(k+1) = \phi_2(j) \big| \theta(k) = \phi_2(i)\big) \\ &= p_1^{\phi_2(i)\phi_2(j)}(k) p_2^{\phi_1(i)\phi_1(j)}(k) \end{aligned} \tag{10}$$

where $p_1^{ij}(k)$ and $p_2^{ij}(k)$ have been defined in (2). Based on Proposition 1, the augmented system (7) can be rewritten as follows:

$$\begin{cases} \tilde{x}(k+1) = \vec{A}_{r(k)}(k)\tilde{x}(k) + \vec{B}_{r(k)}(k)u(k) \\ \qquad\qquad + \vec{D}_{r(k)}(k)\bar{\nu}(k) \\ \quad \bar{y}(k) = \vec{C}_{r(k)}(k)\tilde{x}(k) + \vec{E}_{r(k)}(k)\bar{\nu}(k) \\ \quad z(k) = \bar{M}(k)\tilde{x}(k) \end{cases} \tag{11}$$

where

$$\begin{aligned} &\vec{A}_{r(k)}(k) = \bar{A}_{\phi_1(r(k)),\phi_2(r(k))}(k), \quad \vec{B}_{r(k)}(k) = \bar{B}_{\phi_2(r(k))}(k), \\ &\vec{C}_{r(k)}(k) = \bar{C}_{\phi_1(r(k))}(k), \quad \vec{D}_{r(k)}(k) = \bar{D}_{\phi_1(r(k)),\phi_2(r(k))}(k), \\ &\vec{E}_{r(k)}(k) = \bar{E}_{\phi_1(r(k))}(k). \end{aligned}$$

2.3 Observer-based Controller

The observer-based control scheme for the system (11) is described by

$$\begin{cases} \hat{x}(k+1) = \vec{A}_{r(k)}(k)\hat{x}(k) + \vec{B}_{r(k)}(k)K_{r(k)}(k) \\ \qquad \times \hat{x}(k) + L_{r(k)}(k)\left(\bar{y}(k) - \vec{C}_{r(k)}(k)\hat{x}(k)\right) \\ \quad u(k) = K_{r(k)}(k)\hat{x}(k) \end{cases} \tag{12}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the state estimate of the system (11) with $n = n_x + n_y + n_u$, and the time-varying matrices $K_{r(k)}(k)$ and $L_{r(k)}(k)$ are controller parameters to be designed. Let the observer error be $e(k) = \tilde{x}(k) - \hat{x}(k)$. Then, the error dynamics can be obtained from (11) and (12) as follows:

$$\begin{aligned} e(k+1) &= \big(\vec{A}_{r(k)}(k) - L_{r(k)}(k)\vec{C}_{r(k)}(k)\big)e(k) \\ &\quad + \big(\vec{D}_{r(k)}(k) - L_{r(k)}(k)\vec{E}_{r(k)}(k)\big)\bar{\nu}(k) \end{aligned} \tag{13}$$

By defining variable $\eta(k) = \left[\tilde{x}^T(k)\ e^T(k)\right]^T$, we obtain the closed-loop system as follows:

$$\begin{cases} \eta(k+1) = \mathcal{A}_{r(k)}(k)\eta(k) + \mathcal{D}_{r(k)}(k)\bar{\nu}(k) \\ \quad z(k) = \mathcal{M}(k)\eta(k) \end{cases} \tag{14}$$

where

$$\begin{aligned} &\mathcal{A}_{r(k)}(k) = \begin{bmatrix} \mathscr{A}_{11}(k) & \mathscr{A}_{12}(k) \\ 0 & \mathscr{A}_{22}(k) \end{bmatrix}, \quad \mathcal{D}_{r(k)}(k) = \begin{bmatrix} \mathscr{D}_{11}(k) \\ \mathscr{D}_{21}(k) \end{bmatrix}, \\ &\mathscr{A}_{11}(k) = \vec{A}_{r(k)}(k) + \vec{B}_{r(k)}(k)K_{r(k)}(k), \\ &\mathcal{M}(k) = \left[\bar{M}(k)\ 0\right], \ \mathscr{A}_{12}(k) = -\vec{B}_{r(k)}(k)K_{r(k)}(k), \\ &\mathscr{A}_{22}(k) = \vec{A}_{r(k)}(k) - L_{r(k)}(k)\vec{C}_{r(k)}(k), \mathscr{D}_{11}(k) = \vec{D}_{r(k)}(k), \\ &\mathscr{D}_{21}(k) = \vec{D}_{r(k)}(k) - L_{r(k)}(k)\vec{E}_{r(k)}(k). \end{aligned}$$

We are now in the position to state the problem addressed in this paper as follows. We aim to design appropriate controller parameters $K_{r(k)}(k)$ and $L_{r(k)}(k)$ such that, for the given positive scalar γ, the closed-loop system (14) satisfies the following $\mathcal{H}_\infty$ performance requirement:

$$\begin{aligned} J &\triangleq \mathbb{E}\left\{\sum_{k=0}^{N-1}\left(\|z(k)\|^2 - \gamma^2\|\bar{\nu}(k)\|^2\right)\right\} \\ &\quad - \gamma^2\eta^T(0)W\eta(0) \le 0, \quad \forall(\bar{\nu}(k), \eta(0)) \ne 0 \end{aligned} \tag{15}$$

where W is a given positive definite matrix.

Remark 2 *In (15), an $\mathcal{H}_\infty$ performance index is proposed in the sense of mathematical expectation. Roughly speaking, such an index can better describe the weighted average $\mathcal{H}_\infty$ performance of the closed-loop system over the possible modes with individual weights (transition probabilities), and is therefore particularly suitable when the overall control performance becomes a concern. Also, such an index could facilitate the subsequent mathematical analysis and enhance the feasibility of the controller design problem. On the other hand, one could also ask for a certain $\mathcal{H}_\infty$ performance requirement to be met for all possible modes, and this might be useful for considering the worst-case performance at the cost of increasing the computational complexity.*

3 Main results

Lemma 1 *[11] Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be known nonzero matrices with appropriate dimensions. The solution X to $\min_X \|\mathcal{U}X\mathcal{W}-\mathcal{V}\|_F$ is $\mathcal{U}^\dagger\mathcal{V}\mathcal{W}^\dagger$.*

Lemma 2 *Consider the system (1) with the SCP constraints governed by the Markov chain $r(k)$ whose transition probability matrix is $\bar{\mathcal{P}}$ defined in (10). With the updating rule (6), let the controller parameters $\{K_i(k)\}_{0\le k\le N-1}$, $\{L_i(k)\}_{0\le k\le N-1}$, the disturbance attenuation level $\gamma>0$ and the positive definite matrix W be given. For any disturbance sequences $\{\omega(k)\}_{0\le k\le N-1}$ and $\{\nu(k)\}_{0\le k\le N-1}$, the augmented system (14) satisfies the $\mathcal{H}_\infty$ performance requirement if there exist a family of non-negative definite matrices $P_i(k)$ ($i\in\mathcal{R}$, $0\le k\le N-1$) (with final condition $P_i(N)=0$) satisfying the following backward recursive RDE:*

$$P_i(k)=\mathcal{A}_i^T(k)\bar{P}_i(k+1)\mathcal{A}_i(k)+\mathcal{M}^T(k)\mathcal{M}(k)+\mathcal{A}_i^T(k)\times\bar{P}_i(k+1)\mathcal{D}_i(k)\Delta_i^{-1}(k)\mathcal{D}_i^T(k)\bar{P}_i(k+1)\mathcal{A}_i(k) \tag{16}$$

subject to

$$\begin{cases}\Delta_i(k)=\gamma^2I-\mathcal{D}_i^T(k)\bar{P}_i(k+1)\mathcal{D}_i(k)>0\\ P_i(0)\le\gamma^2W\end{cases} \tag{17}$$

where

$$\bar{P}_i(k+1)=\sum_{j\in\mathcal{R}}\bar{p}_{ij}(k)P_j(k+1) \tag{18}$$

and the nonempty finite set $\mathcal{R}$ is defined in Proposition 1.

Proof: By defining

$$Y_{r(k)}(k)\triangleq\mathbb{E}\{\eta^T(k+1)P_{r(k+1)}(k+1)\eta(k+1)-\eta^T(k)P_i(k)\eta(k)|i=r(k)\} \tag{19}$$

and noticing (14), we have

$$\begin{aligned}&Y_{r(k)}(k)\\&=\mathbb{E}\Big\{\big(\mathcal{A}_i(k)\eta(k)+\mathcal{D}_i(k)\bar{\nu}(k)\big)^T\bar{P}_i(k+1)\big(\mathcal{A}_i(k)\eta(k)\\&\quad+\mathcal{D}_i(k)\bar{\nu}(k)\big)-\eta^T(k)P_i(k)\eta(k)|i=r(k)\Big\}\\&=\mathbb{E}\Big\{\eta^T(k)\left(\mathcal{A}_i^T(k)\bar{P}_i(k+1)\mathcal{A}_i(k)-P_i(k)\right)\eta(k)\\&\quad+2\eta^T(k)\mathcal{A}_i^T(k)\bar{P}_i(k+1)\mathcal{D}_i(k)\bar{\nu}(k)\\&\quad+\bar{\nu}^T(k)\mathcal{D}_i^T(k)\bar{P}_i(k+1)\mathcal{D}_i(k)\bar{\nu}(k)|i=r(k)\Big\}.\end{aligned} \tag{20}$$

Adding the following zero term

$$\|z(k)\|^2-\gamma^2\|\bar{\nu}(k)\|^2-\big(\|z(k)\|^2-\gamma^2\|\bar{\nu}(k)\|^2\big) \tag{21}$$

to both sides of (20) and then taking the mathematical expectation, we have

$$\begin{aligned}Y_{r(k)}(k)=&\,\mathbb{E}\Big\{\gamma^2\|\bar{\nu}(k)\|^2+\eta^T(k)\big(\mathcal{M}^T(k)\mathcal{M}(k)\\&+\mathcal{A}_i^T(k)\bar{P}_i(k+1)\mathcal{A}_i(k)-P_i(k)\big)\eta(k)-\|z(k)\|^2\\&+2\eta^T(k)\mathcal{A}_i^T(k)\bar{P}_i(k+1)\mathcal{D}_i(k)\bar{\nu}(k)-\bar{\nu}^T(k)\big(\gamma^2I\\&-\mathcal{D}_i^T(k)\bar{P}_i(k+1)\mathcal{D}_i(k)\big)\bar{\nu}(k)|i=r(k)\Big\}.\end{aligned} \tag{22}$$

Applying the completing squares method results in

$$\begin{aligned}Y_{r(k)}(k)=&\,\mathbb{E}\Big\{\eta^T(k)\big(\mathcal{A}_i^T(k)\bar{P}_i(k+1)\mathcal{A}_i(k)-P_i(k)\\&+\mathcal{M}^T(k)\mathcal{M}(k)\big)\eta(k)+(\bar{\nu}^*(k))^T\Delta_i(k)\bar{\nu}^*(k)\\&-\big(\bar{\nu}(k)-\bar{\nu}^*(k)\big)^T\Delta_i(k)\big(\bar{\nu}(k)-\bar{\nu}^*(k)\big)|i=r(k)\Big\}\\&-\mathbb{E}\Big\{\|z(k)\|^2-\gamma^2\|\bar{\nu}(k)\|^2\Big\}\end{aligned} \tag{23}$$

where

$$\bar{\nu}^*(k)=\Delta_{r(k)}^{-1}(k)\mathcal{D}_{r(k)}^T(k)\bar{P}_{r(k)}(k+1)\mathcal{A}_{r(k)}(k)\eta(k).$$

Taking the sum on both sides of (23) from 0 to $N-1$, we obtain

$$\begin{aligned}&\mathbb{E}\{\eta^T(N)P_{r(N)}(N)\eta(N)-\eta^T(0)P_{r(0)}(0)\eta(0)\}\\&=\mathbb{E}\Big\{-\sum_{k=0}^{N-1}\big(\bar{\nu}(k)-\bar{\nu}^*(k)\big)^T\Delta_{r(k)}(k)\big(\bar{\nu}(k)-\bar{\nu}^*(k)\big)\\&\quad-\sum_{k=0}^{N-1}\big(\|z(k)\|^2-\gamma^2\|\bar{\nu}(k)\|^2\big)\Big\}.\end{aligned} \tag{24}$$

Since $\Delta_i(k)>0$, $P_i(0)\le\gamma^2W$ and $P_i(N)=0$, we have

$$\begin{aligned}J=&\,\mathbb{E}\Big\{-\sum_{k=0}^{N-1}\big(\bar{\nu}(k)-\bar{\nu}^*(k)\big)^T\Delta_{r(k)}(k)\big(\bar{\nu}(k)-\bar{\nu}^*(k)\big)\\&+\eta^T(0)\left(P_i(0)-\gamma^2W\right)\eta(0)\Big\}\le0\end{aligned} \tag{25}$$

which means the pre-specified $\mathcal{H}_\infty$ performance is satisfied. The proof of is complete.

So far, we have conducted the $\mathcal{H}_\infty$ performance analysis in terms of the solvability of a backward Riccati equation in Lemma 2. In the next stage, let us propose an approach for computing the appropriate controller parameters $K_{r(k)}(k)$ and $L_{r(k)}(k)$ in each step under the worst situation, i.e. $\bar{\nu}(k)=\bar{\nu}^*(k)=\Delta_{r(k)}^{-1}(k)\mathcal{D}_{r(k)}^T(k)\bar{P}_{r(k)}(k+1)\mathcal{A}_{r(k)}(k)\eta(k)$. For this purpose, we rewrite the augmented system (14) as follows:

$$\begin{aligned}\eta(k+1)=&\left(\bar{\mathcal{A}}_{r(k)}(k)+\mathcal{D}_{r(k)}(k)\mathcal{T}_{r(k)}(k)\right)\eta(k)\\&+\bar{\mathcal{B}}_{r(k)}(k)u(k)+\bar{\mathcal{I}}L_{r(k)}(k)\tilde{y}(k)\end{aligned} \tag{26}$$

where $\bar{\mathcal{A}}_{r(k)}(k)=\mathrm{diag}\{\vec{A}_{r(k)}(k),\vec{A}_{r(k)}(k)\}$, $\mathcal{T}_{r(k)}(k)=\Delta_{r(k)}^{-1}(k)\mathcal{D}_{r(k)}^T(k)\bar{P}_{r(k)}(k+1)\mathcal{A}_{r(k)}(k)$, $\bar{\mathcal{B}}_{r(k)}(k)=\big[\vec{B}_{r(k)}^T(k)\ 0\big]^T$, $\bar{\mathcal{I}}=\big[0\ -I\big]^T$, $\tilde{y}(k)=\vec{C}_{r(k)}(k)e(k)$ and $u(k)$ is defined in (12). Then, we define the following cost functional:

$$\bar{J}_{\bar{\nu}^*}\triangleq\mathbb{E}\Big\{\sum_{k=0}^{N-1}\big(\|z(k)\|^2+\varepsilon_1\|u(k)\|^2$$

$$+\varepsilon_2\|L_{r(k)}(k)\tilde{y}(k)\|^2)\Big\} \tag{27}$$

where ε_1 and ε_2 are known constants introduced for more flexibility in the controller parameter design.

Theorem 1 *Consider the system (1) with the SCP associating with (10) and the updating rule (6). Let the disturbance attenuation level $\gamma > 0$ and the positive definite matrix W be given. The augmented system (14) satisfies the $\mathcal{H}_\infty$ performance requirement if there exist solutions $\{(P_i(k),\ Q_i(k),\ K_i(k),\ L_i(k))\}_{0\le k\le N-1}$ ($i \in \mathcal{R}$) satisfying the recursive RDE (16) as well as the following recursive RDE:*

$$\begin{cases} Q_i(k) = \big(\bar{\mathcal{A}}_i(k)+\mathcal{D}_i(k)\mathcal{T}_i(k)\big)^T\bar{Q}_i(k+1)\big(\bar{\mathcal{A}}_i(k) \\ \qquad +\mathcal{D}_i(k)\mathcal{T}_i(k)\big)+\mathcal{M}^T(k)\mathcal{M}(k)+\bar{\Upsilon}(k) \\ \qquad +\tilde{\mathcal{I}}^TK_i^T(k)\bar{\mathcal{B}}_i^T(k)\bar{Q}_i(k+1)\bar{\mathcal{I}}L_i(k)\tilde{C}_i(k) \\ \qquad +\tilde{C}_i^T(k)L_i^T(k)\bar{\mathcal{I}}^T\bar{Q}_i(k+1)\bar{\mathcal{B}}_i(k)K_i(k)\tilde{\mathcal{I}} \\ Q_i(N)=0 \end{cases} \tag{28}$$

subject to

$$\begin{cases} P_i(0)\le\gamma^2W, P_i(k)\ge 0, P_i(N)=Q_i(N)=0 \\ \Delta_i(k)=\gamma^2I-\mathcal{D}_i^T(k)\bar{P}_i(k+1)\mathcal{D}_i(k)>0 \\ \bar{\Delta}_{1,i}(k)=\bar{\mathcal{B}}_i^T(k)\bar{Q}_i(k+1)\bar{\mathcal{B}}_i(k)+\varepsilon_1I>0 \\ \bar{\Delta}_{2,i}(k)=\bar{\mathcal{I}}^T\bar{Q}_i(k+1)\bar{\mathcal{I}}+\varepsilon_2I>0 \end{cases} \tag{29}$$

with the controller parameters given as follows:

$$K_i(k)=-\mathcal{P}_i(k)\tilde{\mathcal{I}}^\dagger,\quad L_i(k)=-\mathcal{Q}_i(k)\tilde{\mathcal{C}}_i^\dagger(k) \tag{30}$$

where

$$\begin{aligned}
&\bar{\Upsilon}(k)=-\Upsilon_1(k)-\Upsilon_2(k)+\Upsilon_3(k)+\Upsilon_4(k),\\
&\bar{P}_i(k+1)=\sum_{j\in\mathcal{R}}\bar{p}_{ij}(k)P_j(k+1),\quad \tilde{C}_i(k)=\big[0\ \vec{C}_i(k)\big],\\
&\bar{Q}_i(k+1)=\sum_{j\in\mathcal{R}}\bar{p}_{ij}(k)Q_j(k+1),\quad \tilde{\mathcal{I}}=\big[I\ -I\big],\\
&\Upsilon_1(k)=\bar{\mathcal{A}}_i^T(k)\bar{Q}_i(k+1)\bar{\mathcal{B}}_i(k)\bar{\Delta}_{1,i}^{-1}(k)\bar{\mathcal{B}}_i^T(k)\\
&\qquad\times\bar{Q}_i(k+1)\bar{\mathcal{A}}_i(k),\\
&\Upsilon_2(k)=\bar{\mathcal{A}}_i^T(k)\bar{Q}_i(k+1)\bar{\mathcal{I}}\bar{\Delta}_{2,i}^{-1}(k)\bar{\mathcal{I}}^T\bar{Q}_i(k+1)\bar{\mathcal{A}}_i(k),\\
&\Upsilon_3(k)=\mathcal{T}_i^T(k)\mathcal{D}_i^T(k)\bar{Q}_i(k+1)\bar{\mathcal{B}}_i(k)K_i(k)\tilde{\mathcal{I}}\\
&\qquad+\tilde{\mathcal{I}}^TK_i^T(k)\bar{\mathcal{B}}_i^T(k)\bar{Q}_i(k+1)\mathcal{D}_i(k)\mathcal{T}_i(k),\\
&\Upsilon_4(k)=\mathcal{T}_i^T(k)\mathcal{D}_i^T(k)\bar{Q}_i(k+1)\bar{\mathcal{I}}L_i(k)\tilde{C}_i(k)\\
&\qquad+\tilde{C}_i^T(k)L_i^T(k)\bar{\mathcal{I}}^T\bar{Q}_i(k+1)\mathcal{D}_i(k)\mathcal{T}_i(k),\\
&\mathcal{P}_i(k)=\bar{\Delta}_{1,i}^{-1}(k)\bar{\mathcal{B}}_i^T(k)\bar{Q}_i(k+1)\bar{\mathcal{A}}_i(k),\\
&\mathcal{Q}_i(k)=\bar{\Delta}_{2,i}^{-1}(k)\bar{\mathcal{I}}^T\bar{Q}_i(k+1)\bar{\mathcal{A}}_i(k),
\end{aligned}$$

and the nonempty finite set $\mathcal{R}$ is defined in Proposition 1.

Proof: First, it follows from Lemma 2 that, if there exists solutions $P_i(k)$ to (28) such that $\Delta_i(k)>0$ and $P_i(0)<\gamma^2W$, then system (14) achieves the pre-specified $\mathcal{H}_\infty$ performance. In this case, the worst-case disturbance can be expressed as $\bar{\nu}^*(k)=\mathcal{T}_{r(k)}\eta(k)$. In what follows, by employing the worst-case disturbance, we aim to provide a design scheme of the controller parameters $K_i(k)$ and $L_i(k)$. For this purpose, we define

$$\begin{aligned}\bar{Y}_{r(k)}(k)\triangleq\mathbb{E}\{&\eta^T(k+1)Q_{r(k+1)}\eta(k+1)\\&-\eta^T(k)Q_i(k)\eta(k)|i=r(k)\}.\end{aligned} \tag{31}$$

Then, we have

$$\begin{aligned}
\bar{Y}_{r(k)}(k)=\mathbb{E}\Big\{&\eta^T(k)\big(\bar{\mathcal{A}}_i(k)+\mathcal{D}_i(k)\mathcal{T}_i(k)\big)^T\bar{Q}_i(k+1)\\
&\times\big(\bar{\mathcal{A}}_i(k)+\mathcal{D}_i(k)\mathcal{T}_i(k)\big)\eta(k)-\eta^T(k)Q_i(k)\eta(k)\\
&+2\eta^T(k)\big(\bar{\mathcal{A}}_i(k)+\mathcal{D}_i(k)\mathcal{T}_i(k)\big)^T\bar{Q}_i(k+1)\bar{\mathcal{B}}_i(k)u(k)\\
&+u^T(k)\bar{\mathcal{B}}_i^T(k)\bar{Q}_i(k+1)\bar{\mathcal{B}}_i(k)u(k)+2\eta^T(k)\big(\bar{\mathcal{A}}_i(k)\\
&+\mathcal{D}_i(k)\mathcal{T}_i(k)\big)^T\bar{Q}_i(k+1)\bar{\mathcal{I}}L_i(k)\tilde{y}(k)+\tilde{y}^T(k)L_i^T(k)\\
&\times\bar{\mathcal{I}}^T\bar{Q}_i(k+1)\bar{\mathcal{I}}L_i(k)\tilde{y}(k)+2u^T(k)\bar{\mathcal{B}}_i^T(k)\bar{Q}_i(k+1)\\
&\times\bar{\mathcal{I}}L_i(k)\tilde{y}(k)|i=r(k)\Big\}.
\end{aligned} \tag{32}$$

Since
$u(k)=K_{r(k)}(k)\tilde{\mathcal{I}}\eta(k), \tilde{y}(k)=\tilde{C}_{r(k)}(k)\eta(k), Q_i(N)=0,$
it follows from the cost function (27) that

$$\begin{aligned}\bar{J}_{\bar{\nu}^*}=\sum_{k=0}^{N-1}\Big(&\bar{Y}_{r(k)}(k)+\mathbb{E}\Big\{\|z(k)\|^2+\varepsilon_1\|u(k)\|^2\\&+\varepsilon_2\|L_{r(k)}(k)\tilde{y}(k)\|^2+\eta^T(0)Q_{r(0)}(0)\eta(0)\Big\}\Big).\end{aligned} \tag{33}$$

Completing the square with respect to $u(k)$ and $\tilde{y}(k)$, it can be derived from (29) that

$$\begin{aligned}\bar{J}_{\bar{\nu}^*}=&\mathbb{E}\big\{\eta^T(0)Q_{r(0)}(0)\eta(0)\big\}+\sum_{k=0}^{N-1}\mathbb{E}\Big\{\big(u(k)+u_i^*(k)\big)^T\\&\times\bar{\Delta}_{1,i}(k)\big(u(k)+u_i^*(k)\big)+\big(L_i(k)\tilde{y}(k)+\tilde{y}_i^*(k)\big)^T\\&\times\bar{\Delta}_{2,i}(k)\big(L_i(k)\tilde{y}(k)+\tilde{y}_i^*(k)\big)\big|i=r(k)\Big\}\end{aligned} \tag{34}$$

where

$$\begin{aligned}&u_i^*(k)=\bar{\Delta}_{1,i}^{-1}(k)\bar{\mathcal{B}}_i^T(k)\bar{Q}_i(k+1)\bar{\mathcal{A}}_i(k)\eta(k)=\mathcal{P}_i(k)\eta(k),\\&\tilde{y}_i^*(k)=\bar{\Delta}_{2,i}^{-1}(k)\bar{\mathcal{I}}^T\bar{Q}_i(k+1)\bar{\mathcal{A}}_i(k)\eta(k)=\mathcal{Q}_i(k)\eta(k).\end{aligned}$$

Consider the following inequality:

$$\begin{aligned}\bar{J}_{\bar{\nu}^*}=&\mathbb{E}\big\{\eta^T(0)Q_{r(0)}(0)\eta(0)\big\}+\sum_{k=0}^{N-1}\mathbb{E}\Big\{\big(u(k)\\&+u_i^*(k)\big)^T\bar{\Delta}_{1,i}(k)\big(u(k)+u_i^*(k)\big)+\big(L_i(k)\tilde{y}(k)\\&+\tilde{y}_i^*(k)\big)\bar{\Delta}_{2,i}(k)\big(L_i(k)\tilde{y}(k)+\tilde{y}_i^*(k)\big)\big|i=r(k)\Big\}\\
\le&\sum_{k=0}^{N-1}\mathbb{E}\Big\{\|K_i(k)\tilde{\mathcal{I}}+\mathcal{P}_i(k)\|_F^2\|\bar{\Delta}_{1,i}(k)\|_F\|\eta(k)\|^2\\&+\|L_i(k)\tilde{\mathcal{C}}_i(k)+\mathcal{Q}_i(k)\|_F^2\|\bar{\Delta}_{2,i}(k)\|_F\|\eta(k)\|^2\Big\}\\&+\mathbb{E}\big\{\eta^T(0)Q_{r(0)}(0)\eta(0)\big\}.\end{aligned} \tag{35}$$

For the purpose of suppressing the cost function (27), the controller parameters $K_i(k)$ and $L_i(k)$ can be selected in each iteration backward as follows:

$$\begin{cases} K_i(k)=\arg\min\limits_{K_i^*(k)}\Big\|K_i^*(k)\tilde{\mathcal{I}}+\mathcal{P}_i(k)\Big\|_F \\ L_i(k)=\arg\min\limits_{L_i^*(k)}\Big\|L_i^*(k)\tilde{\mathcal{C}}_i(k)+\mathcal{Q}_i(k)\Big\|_F \end{cases} \tag{36}$$

Then, it follows from Lemma 1 that (30) is the solution of the optimization problem (36). The proof is complete.

Next, we proceed to examine the impact from the SCP constraints on the design of the control parameter with hope to simplify the computational algorithm. To be specific, we now study the "special" structure of the matrix $K_i(k)$ $(i \in \mathcal{R})$ according to the controller parameters given by (30). Firstly, let us focus our attention on the structure of $\bar{\mathcal{B}}_i(k)$. It is easy to obtain that $\bar{\mathcal{B}}_i(k) = \begin{bmatrix}\bar{B}^T_{\phi_2(i)}(k) & 0\end{bmatrix}^T = \mathscr{B}(k)\Phi^u_{\phi_2(i)}$ where $\mathscr{B}(k) = \begin{bmatrix}B^T(k) & 0 & I & 0\end{bmatrix}^T$. Therefore, we have

$$\begin{aligned}\bar{\Delta}_{1,i}(k) &= \bar{\mathcal{B}}_i^T(k)\bar{Q}_i(k+1)\bar{\mathcal{B}}_i(k) + \varepsilon_1 I \\ &= \Phi^u_{\phi_2(i)}\mathscr{B}^T(k)\bar{Q}_i(k+1)\mathscr{B}(k)\Phi^u_{\phi_2(i)} + \varepsilon_1 I. \end{aligned} \tag{37}$$

Denoting $\bar{\Omega}(k) \triangleq \mathscr{B}^T(k)\bar{Q}_i(k+1)\mathscr{B}(k) \triangleq \begin{bmatrix}\Omega_{ij}(k)\end{bmatrix}_{n_u \times n_u}$, it follows from (37) that

$$\bar{\Delta}_{1,i}(k) = \mathrm{diag}\Big\{ \underbrace{\varepsilon_1, \varepsilon_1, \cdots, \varepsilon_1}_{\phi_2(i)-1}, \bar{\varepsilon}_1, \underbrace{\varepsilon_1, \varepsilon_1, \cdots, \varepsilon_1}_{n_u-\phi_2(i)} \Big\} \tag{38}$$

where $\bar{\varepsilon}_1 = \varepsilon_1 + \Omega_{\phi_2(i)\phi_2(i)}(k)$. Then, $K_{r(k)}(k)$ can be rewritten as follows:

$$\begin{aligned} & K_{r(k)}(k) = -\mathcal{P}_{r(k)}(k)\tilde{\mathcal{I}}^\dagger \\ &= -\bar{\Delta}^{-1}_{1,r(k)}(k)\bar{\mathcal{B}}^T_{r(k)}(k)\bar{Q}_{r(k)}(k+1)\bar{\mathcal{A}}_{r(k)}(k)\tilde{\mathcal{I}}^\dagger \\ &= -\bar{\Delta}^{-1}_{1,r(k)}(k)\Phi^u_{\phi_2(r(k))}\mathscr{B}^T(k)\bar{Q}_{r(k)}(k+1)\bar{\mathcal{A}}_{r(k)}(k)\tilde{\mathcal{I}}^\dagger \\ &= -\Phi^u_{\theta(k)}\bar{\Delta}^{-1}_{1,r(k)}(k)\mathscr{B}^T(k)\bar{Q}_{r(k)}(k+1)\bar{\mathcal{A}}_{r(k)}(k)\tilde{\mathcal{I}}^\dagger \end{aligned} \tag{39}$$

where $\theta(k)$ is derived by (9). Subsequently, denoting

$$\begin{aligned}\mathcal{K}(k) &\triangleq \bar{\Delta}^{-1}_{1,r(k)}(k)\mathscr{B}^T(k)\bar{Q}_{r(k)}(k+1)\bar{\mathcal{A}}_{r(k)}(k)\tilde{\mathcal{I}}^\dagger \\ &\triangleq \begin{bmatrix}\mathcal{K}_1^T(k) & \mathcal{K}_2^T(k) & \cdots & \mathcal{K}^T_{n_u}(k)\end{bmatrix}^T,\end{aligned}$$

one has the special structure of $K_{r(k)}(k)$ as follows:

$$K_{r(k)}(k) = \Big[\underbrace{0\ 0\ \cdots\ 0}_{\theta(k)-1} \quad \mathcal{K}^T_{\theta(k)}(k) \quad \underbrace{0\ 0\ \cdots\ 0}_{n_u-\theta(k)}\Big]^T \tag{40}$$

where the zero entries are the reflection of the adoption of the scheduled SCP.

Remark 3 *It is worth mentioning that the special structure of $K_{r(k)}(k)$ in (40) is generated mainly due to the matrix $\phi^u_{\theta(k)}$ indicating the selected actuator obtaining access to the C/A network at time instant k. Under the scheduled SCP, only $u_{\theta(k)}(k)$ is transmitted to the actuator and others (i.e. $u_i(k)$ $(i \neq \theta(k))$) have no effect on the update of $\bar{u}(k)$ at time instant k. In other words, only the calculation of $u_{\theta(k)}(k) = \mathcal{K}_{\theta(k)}(k)\hat{x}(k)$ is "necessary" for the controller design purpose of the time-varying system (1) at time instant k. Therefore, the structure of the matrix $K_{r(k)}(k)$ is reasonable for suppressing the cost function (27) and the zero entries in (40) would simplify the design algorithm to a great extent.*

By means of Theorem 1, we can summarize the Finite-Horizon Observer-Based $\mathcal{H}_\infty$ Controller Design (FHOBCD) algorithm as follows:

Remark 4 *In this paper, the finite-horizon observer-based $\mathcal{H}_\infty$ controller is designed for a time-varying systems with the SCP by solving coupled backward recursive RDEs. Note that Lemma 2 and Theorem 1 are proved mainly by the "completing the square" technique which results in very little conservatism. It can be observed from Algorithm FHOBCD that, in the controller design procedure, all the important factors contributing to the system complexity have been reflected which include 1) the time-varying system parameters; 2) the transition probabilities of the SCP; and 3) the prescribed disturbance attenuation level.*

Algorithm FHOBCD:

Step 1. Set $k = N-1$, $i = r(N-1) \in \mathcal{R}$. Then $P_i(N) = 0$ and $Q_i(N) = 0$ are available.

Step 2. Calculate the matrices $\bar{\Delta}_{1,i}(k)$ and $\bar{\Delta}_{2,i}(k)$ by, (29) respectively. If $\bar{\Delta}_{1,i}(k)$ and $\bar{\Delta}_{2,i}(k)$ are all positive definite, then the controller parameters $K_i(k)$ and $L_i(k)$ can be solved by (30), and go to the next step, else jump to *Step 6*.

Step 3. Compute the matrix $\Delta_i(k)$. If $\Delta_i(k)$ is positive definite, then step to the next procedure, else jump to *Step 6*.

Step 4. Solve the backward RDEs of (16) and (28) to get $P_i(k)$ and $Q_i(k)$, respectively.

Step 5. If $k \neq 0$, set $k = k-1$, $i = r(k) \in \mathcal{R}$ and go back to *Step 2*, else turn to the next step.

Step 6. If the condition $\{\bar{\Delta}_{r,i}(k) > 0\ (r = 1,2),\ \Delta_i(k) > 0,\ P_i(0) \leq \gamma^2 W\}$ is not satisfied, this algorithm is infeasible. Stop.

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4 AN ILLUSTRATIVE EXAMPLE

In this section, we present an example to illustrate the effectiveness of the proposed $\mathcal{H}_\infty$ controller design scheme. Consider system (1) with the following parameters:

$$A(k) = \begin{bmatrix} 0.45 + \frac{\sin(2k)}{10} & -0.40 & 0.65 \\ -0.50 & 0.65 + \frac{\cos(2k)}{10} & 0.50 \\ 0.60 & -0.30 & -0.4 - 0.3e^{-\frac{k}{10}} \end{bmatrix},$$

$$B(k) = \begin{bmatrix} 0.64 & 0.65 \\ 0.58 & 0.52 \\ 0.3 & 0.25 \end{bmatrix}, \quad C(k) = \begin{bmatrix} 0.65 & -0.30 & -0.40 \\ 0.55 & 0.40 & -0.45 \end{bmatrix},$$

$$D(k) = \begin{bmatrix} 0.04 \\ -0.06 \\ 0.05 \end{bmatrix}, E(k) = \begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix}, M(k) = \begin{bmatrix} 0.2 & -0.1 & 0.2 \end{bmatrix}.$$

The matrices $\mathcal{P}_1(k)$ and $\mathcal{P}_2(k)$ are taken to be

$$\mathcal{P}_1(k) = \begin{bmatrix} 0.6 + 0.1(-1)^k & 0.4 - 0.1(-1)^k \\ 0.55 & 0.45 \end{bmatrix},$$

$$\mathcal{P}_2(k) = \begin{bmatrix} 0.6 + 0.1(-1)^k & 0.4 - 0.1(-1)^k \\ 0.55 & 0.45 \end{bmatrix}.$$

The matrices $E_1(k)$ and $E_2(k)$ are defined as follows:

$$E_1(k) = \begin{bmatrix} 0.3 & -0.2 \end{bmatrix}^T, \quad E_2(k) = \begin{bmatrix} 0.2 & -0.3 \end{bmatrix}^T.$$

In this example, the $\mathcal{H}_\infty$ performance level γ, positive definite matrix W and time-horizon N are selected as 0.99, $2.2I$ and 80, respectively. The scalars ε_1 and ε_2 are selected as $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.01$, respectively. The exogenous disturbance inputs are selected as

$$\nu(k) = 0.8\sin(1.2k), \quad \omega(k) = 0.6\cos(k).$$

Based on the given algorithm, the set of solutions to recursive RDEs in Theorem 1 are obtained and the simulation result is shown in Figs. 2, where Fig. 2 depicts the controlled output trajectories of the open-loop and closed-loop system.

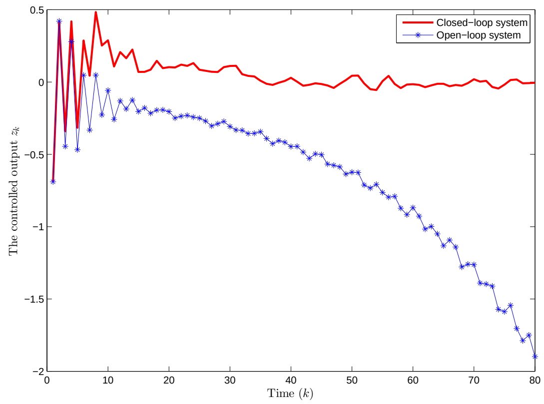

Fig. 2. The output trajectories of the open-loop and closed-loop system

5 CONCLUSION

In this paper, the observer-based $\mathcal{H}_\infty$ control problem has been investigated for a class of discrete time-varying systems with the SCP over a given finite horizon. The signal transmission between the plant and remote controller has been implemented via two communication networks where the SCP has been applied to determine which sensor (or actuator) obtains access to the network. The Markov chains are employed to characterize the random nature of the SCP. An observer-based controller has been designed to construct the control law. By employing the completing squares method and the stochastic analysis techniques, the sufficient conditions have been derived to guarantee the $\mathcal{H}_\infty$ performance of the closed-loop system. Moreover, the desired controller parameters have been achieved by solving two coupled recursive RDEs. Then, the corresponding analysis on the structure of the controller parameters has been conducted subject to the SCP scheduling. Finally, an illustrative example has been given to highlight the effectiveness of the proposed design strategy.

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