Almost Sure H_∞ Sliding Mode Control for Nonlinear Stochastic Systems with Markovian Switching and Time-Delays

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Abstract

This paper investigates the almost sure H_∞ sliding mode control (SMC) problem for nonlinear stochastic systems with Markovian switching and time-delays. An integral sliding surface is first constructed for the addressed system. Then, by employing the stopping time method combined with martingale inequalities, sufficient conditions are established to ensure the almost surely exponential stability and the H_∞ performance of the system dynamics in the specified sliding surface. A SMC law is designed to guarantee the reachability of the specified sliding surface almost surely. Furthermore, the obtained results are applied to a class of special nonlinear stochastic systems with Markovian switching and time-delays, where the desired SMC law is obtained in terms of the solutions to a set of matrix inequalities. Finally, a numerical example is given to show the effectiveness of the proposed SMC scheme.

Index Terms

Nonlinear stochastic systems, Sliding mode control, Markovian switching, almost surely exponential stability.

I. Introduction

Since its inception in the early 1970s [30], the sliding mode control (SMC) (also known as variable structure control) has been a focus of research due to its advantage of strong robustness against model uncertainties, parameter variations and external disturbances. So far, the SMC methodologies have found successful applications in various engineering systems such as power systems, chemical processes, robot manipulators and aero-engineering, see [1], [13], [18] and the references therein. Generally speaking, the SMC approach can be briefly described as the design of a discontinuous control law to force the state trajectories onto a desired sliding surface and maintain there for all subsequent time.

In the past few decades, considerable research attention has been devoted to the theoretical research on SMC for different systems. These systems include, but are not limited to, uncertain systems [25], [35], stochastic systems [15], [21], [24], nonlinear systems [3] and fuzzy systems [36]. For example, In [25], a special integral-type switching function is constructed to guarantee that the system dynamics lies on the specified sliding surface. Very recently, the SMC problem of Markovian jump systems (MJSs) has gained particular research interests because of their practical applications in a variety of areas [5], [6], [19], [20], [28], [33]. The main reason could be that MJSs are very appropriate to model the dynamic systems whose structures are subject to random abrupt variation [26], [27].

This work was supported in part by the National Natural Science Foundation of China under Grants 61134009 and 61329301, the Royal Society of the U.K., and the Alexander von Humboldt Foundation of Germany.

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In particular, the sliding mode controllers have been designed in [6] for the systems with actuator degradation by estimating the loss of effectiveness of actuators, and in [19] for uncertain switched stochastic system with time-varying delays by utilizing the average dwell time method. However, so far, most available literature concerning the SMC problems for MJSs have been limited to the systems with some special nonlinearities, and the SMC problems for general nonlinear stochastic systems have not been paid adequate research attention despite its clear engineering significance.

On the other hand, time-delays are well known to be ubiquitous in practice. If not adequately taken into account in the analysis and synthesis, time-delays will inevitably degrade the system performances or even cause the instability. As a result, a great deal of work has been done in order to eliminate or compensate the effect caused by time-delays, see [2], [14], [17], [34], [35], [38] and the references therein. For example, the SMC problem has been studied in [38] for systems subject to both the Markovian jump parameters and time-delays. It should be pointed out that, the widely investigated mean-square stability implies that the variance of the state process is asymptotically bounded and, in this case, the system could be well-behaved on the average but the sample state trajectories could have a finite probability of being arbitrarily far from the system's equilibrium point. In other words, under certain circumstances, the commonly used mean-square measure might be crude to quantify the dynamic performance and is sometimes unacceptable for some real-world engineering with high reliability requirements. For instance, the precision requirement in the rocket control problem should be guaranteed with the probability 1. Therefore, the almost sure stability [22], [23], which describes the system performance from the viewpoint of system sample paths, has recently attracted considerable attention, see [4], [16], [17], [29] and the references therein. Up to now, the corresponding results mainly involve the performance analysis of input-to-state stability and stochastic stability in almost sure sense, and the controller structure is mostly dependent on state/output feedback. To the best of the authors' knowledge, *the SMC problem in almost sure sense* for nonlinear stochastic systems has not been properly investigated so far, not to mention the case when Markovian switching and time-delays are also involved. It is, therefore, the purpose of this paper to shorten such a gap.

Summarizing the above discussions, in this paper, we are motivated to study the SMC problem for nonlinear stochastic systems with Markovian switching and time-delays in almost sure sense. Two essential challenges are identified as follows: 1) how to design a sliding mode controller to guarantee the reachability of the specified sliding surface almost surely, and 2) how to examine the impact from both time-delays and exogenous disturbance on the control performance? To handle these two challenges, we first construct an integral-type sliding function and, following intensive stochastic analysis, the stability condition of the sliding mode dynamics is presented via Hamilton-Jacobi-Isaacs (HJI) inequalities. Furthermore, a sliding mode controller is designed to guarantee the trajectories of the system to be driven onto the sliding surface almost surely. The contribution of this paper is mainly threefold: *1) an integral-type sliding function is proposed in order to facilitate the SMC problem later; 2) the H_∞ performance combined with the almost sure exponential stability is utilized to evaluate the system performance; and 3) a design scheme for almost sure sliding mode controller is obtained by solving HJI inequalities for general nonlinear stochastic systems or matrix inequalities for special nonlinear stochastic systems.*

The rest of this paper is organized as follows. In Section II, a class of nonlinear stochastic systems with Markovian switching and time-delays are presented, and some preliminaries are briefly outlined. In Section III, the main results are established in the form of coupled HJI inequalities. These sufficient conditions are then applied to a class of special nonlinear stochastic systems with Markovian switching and time-delays in Section IV. Furthermore, a numerical example is proposed to demonstrate the effectiveness of the obtained results in Section V. Finally, conclusions are drawn in Section VI.

Notation The notation used here is fairly standard unless otherwise specified. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices, and $\mathbb{R}_+ = [0, +\infty)$. $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete probability space with a natural filtration $\{\mathscr{L}\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). M^T represents the transpose of the matrix M. $|\cdot|$ denotes the Euclidean norm. $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation. $\mathbb{P}\{\cdot\}$ means the probability. $C([-\tau, 0]; \mathbb{R}^n)$ denotes the family of all continuous $\mathbb{R}^n$-valued function φ on $[-\tau, 0]$ with the norm $|\varphi| = \sup\{|\varphi(\theta)| : -\tau \leq \theta \leq 0\}$. $C^b_{\mathcal{F}_0}([-\tau, 0); \mathbb{R}^n)$ is the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0); \mathbb{R}^n)$-value random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$. $L^1(\mathbb{R}_+; \mathbb{R}_+)$ denotes the family of functions $\lambda :\ \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty \lambda(t)dt < \infty$. $\mathcal{K}$ denotes a class of continuous (strictly) increasing functions μ from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $\mu(0) = 0$. $\mathcal{K}_\infty$ denotes a class of functions μ in $\mathcal{K}$ with $\mu(r) \to \infty$ as $r \to \infty$. $L_2(\mathbb{R}_+, \mathbb{R}^p)$ denotes the space of nonanticipative stochastic process $y(t) \in \mathbb{R}^p$ with respect to the filtration $\mathscr{F}_t$ satisfying $|y(t)|^2_{L_2} := \mathbb{E} \int_0^\infty |y(t)|^2 dt < \infty$.

II. Problem formulation

A. *The nonlinear stochastic systems with Markovian switching and time-delays*

Let $r(t)$ $(t \geq 0)$ be a right-continuous Markov chain taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if} \quad i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if} \quad i = j, \end{cases}$$

where $\triangle > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from mode i to mode j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$.

We consider a class of nonlinear stochastic systems of the form

$$\begin{cases} dx(t) = \big[f(x(t), x(t-\tau), t, r(t)) + g(x(t), x(t-\tau), t, r(t))v(t)) \\ \qquad\qquad + B(r(t))(u(t) + \phi(x(t), x(t-\tau), t, r(t)))\big]dt \\ \qquad\qquad + G(r(t))[h(x(t), x(t-\tau), t, r(t)) + s(x(t), x(t-\tau), t, r(t))v(t)]d\omega(t), \\ y(t) = l(x(t), x(t-\tau), t, r(t)) + m(x(t), x(t-\tau), t, r(t))v(t), \end{cases} \tag{2.1}$$

with initial data $x(t) = x_0 \in C^b_{\mathcal{F}_0}([-\tau, 0); \mathbb{R}^n)$ and $r(0) = r_0 \in S$, where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$, $u(t) \in \mathbb{R}^p$ and $v(t) \in L_2([0\ \infty);\ \mathbb{R}^q)$ are the state vector, the controlled output, the control input and the exogenous disturbance input, respectively. The time-delay τ is a known positive integer. For a fixed mode $r(t)$, $B(r(t))$ and $G(r(t))$ are known constant matrices with appropriate dimension. Assume that $B(r(t))$ is a full column rank matrix, $\phi(x(t), x(t-\tau), t, r(t))$ is a matched nonlinear function representing the uncertainties, and $\omega(t)$ is a zero-mean one-dimensional Wiener process (Brownian Motion) satisfying $\mathbb{E}[d\omega(t)] = 0$ and $\mathbb{E}[d\omega^2(t)] = t$.

In this paper, f, g, h, s, l, m and ϕ are known smooth functions with $f(0, 0, t, i) = 0$, $g(0, 0, t, i) = 0$, $h(0, 0, t, i) = 0$, $s(0, 0, t, i) = 0$, $\phi(0, 0, t, i) = 0$, $l(0, 0, t, i) = 0$ and $m(0, 0, t, i) = 0$, and satisfy the following assumption.

Assumption 1: The measurable nonlinear function θ, which could be f, g, h, s, ϕ, l or m, satisfies the global Lipschitz condition, that is, there is a $L_\theta > 0$ such that

$$|\theta(x, z, t, i) - \theta(\bar{x}, \bar{z}, t, i)| \leq L_\theta(|x - \bar{x}| + |z - \bar{z}|), \tag{2.2}$$

for all $x, z, \bar{x}, \bar{z} \in \mathbb{R}^n,\ t \geq -\tau,\ i \in S$.

Remark 1: Under Assumption 1, it is easy to verify that the functions f and ϕ satisfy the local Lipschitz condition and the linear growth condition. Therefore, in terms of the well-known existence-and-unique theorem, one has that the system in (2.1) with $B(r(t)) = 0$ and $v(t) = 0$ has a unique solution which can be denoted by $x(t; x_0, r_0)$ for any initial data $x_0 \in C^b_{\mathcal{F}_0}([-\tau, 0); \mathbb{R}^n)$ and $t \geq -\tau$.

B. *The sliding mode controller*

Firstly, for the sake of notation simplification, denote $B(i) = B_i$ and $G(i) = G_i$ for each $r(t) = i \in S$. Also denote the function $\varphi(x(t), x(t-\tau), t, i)$ as φ which could be f, h, l, m or s when there is no ambiguity. Then, the following nonlinear integral sliding mode function is designed:

$$\sigma(t, i) = H_i x(t) - \int_0^t H_i[f(x(\theta), x(\theta - \tau), \theta, i) + B_i K_i x(\theta)]d\theta, \tag{2.3}$$

where K_i is a constant matrix to be determined. The constant matrix $H_i \in \mathbb{R}^{m\times n}$, which satisfies $H_iG_i = 0$ with H_iB_i being nonsingular, is also a parameter to be designed.

Remark 2: It should be mentioned that, since B_i is of full column rank, the nonsingularity of H_iB_i can be ensured by choosing $H_i = B_i^TX_i$ with $X_i > 0$, and the condition $H_iG_i = 0$ can be confirmed by solving $(B_i^TX_iG_i)^TB_i^TX_iG_i \leq \beta I$ for $\beta \geq 0$, which can be verified by using the Matlab LMI toolbox.

Now, integrating (2.1) from 0 to $t > 0$ and substituting $x(t)$ into (2.3) yield

$$\begin{aligned}\sigma(t,i) &= H_ix_0 + \int_0^t H_i[-B_iK_ix(\theta) + B_i(u(\theta) + \phi(x(\theta), x(\theta-\tau), \theta, i)) \\ &\quad + g(x(\theta), x(\theta-\tau), \theta, i)v(\theta)]d\theta. \end{aligned} \tag{2.4}$$

The surface $\sigma(t,i) = 0$ is called a sliding surface. It is well known that when $\dot{\sigma}(t,i) = 0$, the state trajectories of the system enter the sliding surface. In this case, the equivalent control law of the sliding mode is given by

$$u_{eq}(t) = K_i(x(t)) - \phi(x(t), x(t-\tau), t, i) - (H_iB_i)^{-1}H_ig(x(t), x(t-\tau), t, i))v(t). \tag{2.5}$$

By substituting (2.5) into (2.1), the resulting sliding mode dynamics can be written as:

$$\begin{cases} dx(t) = [f(x(t), x(t-\tau), t, i) + B_iK_ix(t) \\ \qquad\qquad + (I - B_i(H_iB_i)^{-1}H_i)g(x(t), x(t-\tau), t, i)v(t))]dt \\ \qquad\qquad + G_i[h(x(t), x(t-\tau), t, i) + s(x(t), x(t-\tau), t, i)v(t)]d\omega(t), \\ y(t) = l(x(t), x(t-\tau), t, i) + m(x(t), x(t-\tau), t, i)v(t). \end{cases} \tag{2.6}$$

For each $i \in S$, let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ denote the family of all nonnegative functions $V(x,t,i)$ on $\mathbb{R}^n \times \mathbb{R}_+ \times S$ that are twice continuously differentiable in x and once in t. If $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, define an infinitesimal generator $\mathcal{L}$ of (2.6) from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$ to $\mathbb{R}$ by

$$\begin{aligned}\mathcal{L}V(x,\xi,t,i) &= V_t(x,t,i) + V_x(x,t,i)[f + B_iK_ix + g_ev(t)] \\ &\quad + \frac{1}{2}[G_ih + G_isv]^TV_{xx}(x,t,i)[G_ih + G_isv] + \sum_{j=1}^N \gamma_{ij}V(x,t,j),\end{aligned}$$

where

$$\begin{aligned}V_t(x,t,i) &= \frac{\partial V(x,t,i)}{\partial t}, \quad V_x(x,t,i) = \left(\frac{\partial V(x,t,1)}{\partial x_i}, \cdots, \frac{\partial V(x,t,i)}{\partial x_n}\right), \\ V_{xx}(x,t,i) &= \left(\frac{\partial^2 V(x,t,i)}{\partial x_i \partial x_j}\right)_{n\times n}, \quad g_e = (I - B_i(H_iB_i)^{-1}H_i)g(x(t), x(t-\tau), t, i).\end{aligned}$$

Before formulating the problem to be dealt with in this work, the following definition is considered.

Definition 1: The systems (2.1) with $v(t) = 0$ is almost surely exponentially stable if there exists a scalar $\lambda > 0$ such that

$$\lim_{t\to\infty} \sup \frac{1}{t} \log(|x(t; x_0, r_0)|) \leq -\lambda \tag{2.7}$$

for any initial data $x_0 \in C^b_{\mathcal{F}_0}([-\tau, 0); \mathbb{R}^n)$.

The purpose of this paper is to design a SMC law such that the following requirements are met simultaneously:

a) the state trajectory of system (2.6) is globally driven onto the sliding mode surface $\sigma(t,i) = 0$ almost surely and, in a subsequent time, the sliding mode dynamics (2.6) with $v(t) \equiv 0$ is almost surely exponentially stable;

b) under the zero initial condition, for a given disturbance attenuation level $\gamma > 0$ and all nonzero $v \in L_2(\mathbb{R}_+;\ \mathbb{R}^q)$, the controlled output y satisfies

$$|y(t)|^2_{L_2} \leq \gamma^2|v(t)|^2_{L_2}. \tag{2.8}$$

III. Main Results

In this section, we aim to establish a unified framework to solve the addressed problem of almost sure H_∞ SMC for nonlinear stochastic systems with Markovian switching and time-delays. A sufficient condition is presented to ensure that the sliding mode dynamics (2.6) is almost surely stable and the desired H_∞ performance is achieved. Then, a design scheme of the sliding mode controller is proposed to guarantee the reachability of the specified sliding surface almost surely.

A. Performance analysis

Theorem 3.1: Consider the nonlinear stochastic system (2.6) and the sliding mode surface (2.3). For any $i \in S$, let the disturbance attenuation level $\gamma > 0$ be given. The sliding mode dynamics (2.6) is almost surely exponentially stable and the H_∞ performance is achieved if there exist positive scalars c_1, c_2, λ_1, λ_2 with $\lambda_1 > \lambda_2$, the function $V(\eta, t, i) \in C^{2,1}(\mathbb{R}^n \times [-\tau, \infty) \times S; \mathbb{R}_+)$ with $V(0, t, i) = 0$ and matrices $K_i \in \mathbb{R}^m$, $H_i \in \mathbb{R}^{m\times n}$ such that

$$c_1|x|^2 \leq V(x, t, i) \leq c_2|x|^2, \tag{3.1}$$

$$V_t + V_x(f + B_iK_i) + \frac{1}{2}h^T G_i^T V_{xx} h G_i + \sum_{j=1}^{N} \gamma_{ij} V_j + \Xi + \lambda_1 x^2 - \lambda_2 \xi^2 + \frac{1}{2} l^2 < 0, \tag{3.2}$$

$$\gamma^2 I - s^T G_i^T V_{\eta\eta} s G_i - m^T m > 0, \tag{3.3}$$

$$H_i G_i = 0, \tag{3.4}$$

hold for all $(x, t, i) \in \mathbb{R}^n \times [-\tau, \infty) \times S$, where H_iB_i is nonsingular and

$$\Xi = \frac{1}{2}\left(g_e^T V_x + s^T G_i^T V_{xx} G_i h + m^T l\right)^T \left(\gamma^2 I - s^T G_i^T V_{xx} G_i s - m^T m\right)^{-1} \left(g_e^T V_x + s^T G_i^T V_{xx} G_i h + m^T l\right).$$

Proof: Firstly, let us prove that the sliding mode dynamics (2.6) with $v(t) = 0$ is almost surely exponentially stable. For this purpose, denote $x(t-\tau) = \xi$, $V(x, t, i) = V_i$ ($i \in S$) and

$$f_e(x(t), x(t-\tau), t, i) = f(x(t), x(t-\tau), t, i) + B_iK_ix(t),$$
$$h_e(x(t), x(t-\tau), t, i) = G_i h(x(t), x(t-\tau), t, i).$$

By using (3.2) and (3.3), the infinitesimal generator $\mathcal{L}V(x, \xi, t, i)$ with respect to (2.6) is given as

$$\begin{aligned} \mathcal{L}V(x, \xi, t, i) &= V_t + V_x(f + B_iK_i) + \frac{1}{2}h^T G_i^T V_{xx} h G_i + \sum_{j=1}^{N} \gamma_{ij} V_j \\ &< -\lambda_1 x^2 + \lambda_2 \xi^2 - \frac{1}{2} l^2 \\ &\leq -\lambda_1 x^2 + \lambda_2 \xi^2. \end{aligned} \tag{3.5}$$

Then, by utilizing Itô' formula, it follows from (3.1) and (3.5) that

$$\begin{aligned} &\mathbb{E}[e^{\lambda t} V(x(t), t, i)] \\ = \ &\mathbb{E}V(x_0, 0, i) + \mathbb{E}\int_0^t e^{\lambda s}[\lambda V(x(s), s, i) + \mathcal{L}V(x(s), x(s-\tau), s, i)]ds \\ \leq \ &c_2|x_0|^2 + \mathbb{E}\int_0^t e^{\lambda s}[\lambda c_2|x(s)|^2 - \lambda_1|x(s)|^2 + \lambda_2|x(s-\tau)|^2]ds \\ \leq \ &c_2|x_0|^2 + \lambda_2\tau e^{\lambda\tau}\sup|x_0|^2 + \mathbb{E}\int_0^t e^{\lambda s}[\lambda c_2|x(s)|^2 - \lambda_1|x(s)|^2 + \lambda_2 e^{\lambda\tau}|x(s)|^2]ds. \end{aligned} \tag{3.6}$$

Because there is an unique root $\lambda > 0$ for the equation $\lambda c_2 - \lambda_1 + \lambda_2 e^{\lambda\tau} = 0$, one has

$$\mathbb{E}[e^{\lambda t} V(x(t), t, i)] \leq (c_2 + \lambda_2\tau e^{\lambda\tau})\sup|x_0|^2. \tag{3.7}$$

Letting $M = \frac{c_2+\lambda_2\tau e^{\lambda\tau}}{c_1}\sup|x_0|^2$, by means of the condition (3.1), we can obtain

$$\mathbb{E}|x(t)|^2 \le Me^{-\lambda t}, \quad t \ge 0. \tag{3.8}$$

Furthermore, there exists an arbitrary $\varepsilon \in (0, \lambda/2)$ such that

$$\mathbb{E}|x(t)|^2 \le Me^{-(\lambda-\varepsilon)t} \tag{3.9}$$

holds.

On the other hand, for any given $\delta > 0$, there exists an integer k_0 such that $(k_0-1)\delta \ge \tau$. Denoting the sequence $k = k_0, k_0+1, \cdots$, we can get that

$$\begin{aligned}\mathbb{E}\left[\sup_{(k-1)\delta\le t\le k\delta}|x(t)|^2\right] &\le 3^2\mathbb{E}|x((k-1)\delta)|^2 + 3^2\mathbb{E}\left(\int_{(k-1)\delta}^{k\delta}\left|f_e(x(s),x(s-\tau),s,i)\right|ds\right)^2\\ &\quad + 3^2\mathbb{E}\left[\sup_{(k-1)\delta\le t\le k\delta}\left|\int_{(k-1)\delta}^{t}h_e(x(s),x(s-\tau),s,i)dw(s)\right|^2\right].\end{aligned} \tag{3.10}$$

For the second part of the right-hand side of (3.10), it is not difficult to see that

$$\begin{aligned}\mathbb{E}\left(\int_{(k-1)\delta}^{k\delta}|f_e(x(s),x(s-\tau),s,i)|ds\right)^2 &\le \mathbb{E}\left(\delta\sup_{(k-1)\delta\le s\le k\delta}|f_e(x(s),x(s-\tau),s,i)|\right)^2\\ &\le 2(L\delta)^2\mathbb{E}\left[\sup_{(k-1)\delta\le s\le k\delta}(|x(s)|^2+|x(s-\tau(s))|^2)\right].\end{aligned} \tag{3.11}$$

For the third part of the right-hand side of (3.10), by applying the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}&\mathbb{E}\left[\sup_{(k-1)\delta\le s\le k\delta}\left|\int_{(k-1)\delta}^{t}|h_e(x(s),x(s-\tau),s,i)dw(s)\right|^2\right]\\ &\le 4\mathbb{E}\left(\int_{(k-1)\delta}^{k\delta}|h_e(x(s),x(s-\tau(s)),s,i)|^2ds\right)\\ &\le 4\mathbb{E}\left(\delta\sup_{(k-1)\delta\le s\le k\delta}|h_e(x(s),x(s-\tau),s,i)|^2\right)\\ &\le 8L^2\delta\mathbb{E}\left[\sup_{(k-1)\delta\le s\le k\delta}(|x(s)|^2+|x(s-\tau)|^2)\right].\end{aligned} \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10) yields

$$\begin{aligned}&\mathbb{E}\left[\sup_{(k-1)\delta\le t\le k\delta}|x(t)|^2\right] \le 3^2Me^{-(\lambda-\varepsilon)(k-1)\delta}\\ &+(3L)^2(2\delta^2+8\delta)\mathbb{E}\left[\sup_{(k-1)\delta\le s\le k\delta}|x(s)|^2\right]+(3L)^2(2\delta^2+8\delta)\mathbb{E}\left[\sup_{(k-1)\delta\le s\le k\delta}|x(s-\tau)|^2\right].\end{aligned} \tag{3.13}$$

For the second part of the right-hand side of above inequality, letting δ be sufficiently small to satisfy $(3L)^2(\delta^2+4\delta) < \frac{1}{4}$, and considering (3.9) and (3.13), we have

$$\mathbb{E}\left[\sup_{(k-1)\delta\le t\le k\delta}|x(t)|^2\right] \le 19Me^{-(\lambda-\varepsilon)((k-1)\delta-\tau)}. \tag{3.14}$$

For the third part of the right-hand side of (3.13), it follows from the fact $(k-1)\delta \ge \tau$ and (3.8) that

$$\mathbb{E}\left[\sup_{(k-1)\delta\le s\le k\delta}|x(s-\tau)|^2\right] \le Me^{-(\lambda-\varepsilon)((k-1)\delta-\tau)}. \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13) and using the well-known Chebyshev's inequality, we have

$$\begin{aligned}&\mathbb{P}\left\{\omega : \sup_{(k-1)\delta\leq t\leq k\delta}|x(t)| > e^{-(\lambda-2\varepsilon)((k-1)\delta-\tau)/2}\right\}\\ &\leq \frac{\mathbb{E}\left[\sup_{(k-1)\delta\leq t\leq k\delta}|x(t)|^2\right]}{e^{-(\lambda-2\varepsilon)((k-1)\delta-\tau)}}\\ &\leq 19Me^{-\varepsilon((k-1)\delta-\tau)}.\end{aligned}\tag{3.16}$$

Applying the well-known Borel-Cantelli lemma, we can get that for almost all $\omega\in\Omega$,

$$\sup_{(k-1)\delta\leq t\leq k\delta}|x(t)|\leq e^{-(\lambda-2\varepsilon)((k-1)\delta-\tau)/2}\tag{3.17}$$

holds for all but finitely many k. Hence there exists a $k_1(\omega)$, for all $\omega\in\Omega$ excluding a $\mathbb{P}$-null set such that (3.17) holds whenever $k\geq\max\{k_0,k_1\}$. Consequently, for almost all $\omega\in\Omega$, when $(k-1)\delta\leq t\leq k\delta$ and $k\geq\max\{k_0,k_1\}$, one has

$$\frac{1}{t}\log(|x(t)|)\leq -\frac{(\lambda-2\varepsilon)((k-1)\delta-\tau)/2}{2t}\leq -\frac{(\lambda-2\varepsilon)((k-1)\delta-\tau)/2}{2k\delta},$$

which implies

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(|x(t)|)\leq -\frac{\lambda-2\varepsilon}{2}.\tag{3.18}$$

Therefore, the required (2.7) follows by letting $\varepsilon\to 0$, which means that the system (2.6) with $v(t)\equiv 0$ is almost surely exponentially stable.

Now, let us consider the H_∞ performance of the sliding mode dynamics (2.6). Firstly, the infinitesimal generator $\mathcal{L}V(x,\xi,t,i)$ associated with the system (2.6) is obtained as

$$\begin{aligned}&\mathcal{L}V(x,\xi,t,i)\\ =\;& V_t(x,t,i)+V_x(x,t,i)[f+B_iK_i]\\ &+\sum_{j=1}^{N}\gamma_{ij}V_j+V_x(x,t,i)(I-B_i(H_iB_i)^{-1}H_i)g(x(t),x(t-\tau),t,i)v(t)\\ &+\frac{1}{2}[G_ih+G_isv(t)]^TV_{xx}(x,t,i)[G_ih+G_isv(t)]\\ =\;& -\left[v(t)-\left(\gamma^2I-s^TG_i^TV_{xx}G_is-m^Tm\right)^{-1}\left(g_e^TV_x+s^TG_i^TV_{xx}G_ih+m^Tl\right)\right]^T\\ &\times\left(\gamma^2I-s^TG_i^TV_{xx}G_is-m^Tm\right)\\ &\times\left[v(t)-\left(\gamma^2I-s^TG_i^TV_{xx}G_is-m^Tm\right)^{-1}\left(g_e^TV_x+s^TG_i^TV_{xx}G_ih+m^Tl\right)\right]\\ &+\frac{1}{2}\left(g_e^TV_x+s^TG_i^TV_{xx}G_ih+m^Tl\right)^T\left(\gamma^2I-s^TG_i^TV_{xx}G_is-m^Tm\right)^{-1}\\ &\times\left(g_e^TV_x+s^TG_i^TV_{xx}G_ih+m^Tl\right)+V_t(x,t,i)+V_x(x,t,i)[f+B_iK_i]\\ &+\sum_{j=1}^{N}\gamma_{ij}V(x,t,j)+\frac{1}{2}\gamma^2v^T(t)v(t)-l^Tmv-\frac{1}{2}v^T(t)m^Tmv(t).\end{aligned}\tag{3.19}$$

Thus, it follows from (3.2) and (3.3) that

$$\begin{aligned}\mathcal{L}V(x,\xi,t,i)\leq\;& -\lambda_1\eta^2+\lambda_2\xi^2+\frac{1}{2}\gamma^2v^T(t)v(t)-l^Tmv(t)-\frac{1}{2}v^T(t)m^Tmv(t)-\frac{1}{2}l^Tl\\ \leq\;& -\lambda_1\eta^2+\lambda_2\xi^2+\frac{1}{2}\gamma^2v^T(t)v(t)-\frac{1}{2}y^T(t)y(t).\end{aligned}\tag{3.20}$$

Because of $\lambda_1>\lambda_2$, integrating both sides of (3.20) from 0 to $T>0$ and taking expectation result in

$$\mathbb{E}V(x,t,i)\;=\;\mathbb{E}\int_0^T\mathcal{L}V(x(t),x(t-\tau),t,i)dt$$

$$\leq \quad \mathbb{E}\int_0^T \left(\frac{1}{2}\gamma^2 v^T(t)v(t) - \frac{1}{2}y^T(t)y(t)\right)dt. \tag{3.21}$$

Finally, noting the fact that $V(x(T),T,i) \geq 0$ and $V(0,t,i) = 0$, we have

$$\begin{aligned} 0 \quad &\leq \quad \frac{1}{2}\mathbb{E}\int_0^T |y(t)|^2 dt \\ &\leq \quad \frac{1}{2}\int_0^T \gamma^2 |v(t)|^2 dt - \mathbb{E}V(x,T,i) \\ &\leq \quad \frac{1}{2}\int_0^T \gamma^2 |v(t)|^2 dt, \end{aligned} \tag{3.22}$$

for all $i \in S$ and $T > 0$. Therefore, the conclusion (2.8) follows immediately from (3.22) by letting $T \to \infty$. The proof is complete. ■

Remark 3: Generally speaking, it is not easy to design a controller $u(t) = K_i x(t)$ ($i \in S$) for general nonlinear stochastic systems by Theorem 3.1 in form of HJI inequalities with equality constraints. So, in the subsequent section, for a class of special nonlinear stochastic systems, the aforementioned SMC problem can be converted into solving a set of linear matrix inequalities (LMIs).

B. Reachability analysis

In this part, we consider the reachability of the sliding mode surface $\sigma(t,i) = 0$. In the following theorem, a SMC law is provided to guarantee that the state trajectories of system (2.6) is globally driven onto the sliding mode surface $\sigma(t,i) = 0$ almost surely.

Theorem 3.2: Consider the system (2.6) with the sliding mode function $\sigma(t,i)$ in (2.4). If the SMC law is chosen as follows

$$u(t,i) = \begin{cases} K_i x(t) - \rho(t)\frac{(H_iB_i)^T\sigma(t,i)}{|(H_iB_i)^T\sigma(t,i)|}, & \text{if} \quad \sigma(t,i) \neq 0, \\ K_i x(t), & \text{if} \quad \sigma(t,i) = 0 \end{cases} \tag{3.23}$$

where

$$\rho(t) \geq \frac{2\alpha}{|(H_iB_i)^T|\sqrt{|\sigma(t,i)|}} + L_\phi|x(t)| + L_\phi|x(t-\tau)| + L_g|(H_iB_i)^{-1}H_i|(|x(t)v(t)| + |x(t-\tau)v(t)|)$$

with a small constant $\alpha > 0$, then the state trajectory of system (2.6) is globally driven onto the sliding mode surface $\sigma(t,i) = 0$ almost surely.

Proof: Firstly, due to the condition $H_iG_i = 0$, differentiating sliding mode function $\sigma(t,i)$ in (2.4) yields

$$\dot{\sigma}(t,i) = H_i[-B_iK_ix(t) + B_i(u(t) + \phi(x(t),x(t-\tau),t,i)) + g(x(t),x(t-\tau),t,i)v(t)]. \tag{3.24}$$

Choose the Lyapunov function candidate as

$$V(t,i) = \frac{1}{2}\sigma(t,i)^T\sigma(t,i). \tag{3.25}$$

It follows from (3.23) and (3.24) that

$$\begin{aligned} \dot{V}(t,i) &= \sigma(t,i)^T\dot{\sigma}(t,i) \\ &= \sigma(t,i)^T H_i[-B_iK_ix(t) + B_i(u(t) + \phi(x(t),x(t-\tau),t,i)) \\ &\quad + g(x(t),x(t-\tau),t,i)v(t)] \\ &= \sigma(t,i)^T H_i[-B_iK_ix(t) + B_i(K_i(x) - \rho(t)\frac{(H_iB_i)^T\sigma(t,i)}{|(H_iB_i)^T\sigma(t,i)|} \\ &\quad + \phi(x(t),x(t-\tau),t,i)) + g(x(t),x(t-\tau),t,i)v(t)]. \end{aligned} \tag{3.26}$$

Furthermore, it follows from (3.24), (3.26) and Assumption 1 that

$$\dot{V}(t,i) \leq -2\alpha\sqrt{|\sigma(t,i)|}. \tag{3.27}$$

Therefore, we can get that

$$\frac{d|\sigma(t,i)|}{dt} = \frac{d\sqrt{V(t,i)}}{dt} \leq -\alpha, \tag{3.28}$$

which implies $|\sigma(t,i)|$ converges to zero almost surely. The proof is complete. ■

IV. SMC FOR A SPECIAL CASE

To demonstrate that Theorem 3.1 serves as a theoretic basis for the problem of almost sure H_∞ SMC for nonlinear stochastic systems with Markovian switching and time-delays, in this section, we aim to show that Theorem 3.1 can be specialized to a special case, that is, the linear Markovian switching systems with Lipschitz-type nonlinearities. The specialized result is described in terms of LMIs, which can be solved by using the Matlab LMI toolbox.

Firstly, denote the matrix associated with the ith mode by $\Gamma_i \triangleq \Gamma(r(t)=i)$ where the matrix Γ could be A, A_d, B, C, C_d, D, D_1, D_2, E, E_d, G, H, K, X, Y or P.

Consider the following special nonlinear stochastic systems with Markovian switching and time-delays

$$\begin{cases} dx(t) = [A_i x(t) + A_{di}x(t-\tau) + f(x(t), x(t-\tau), t, r(t)) + B_i(u(t) + \phi(x(t), x(t-\tau), t, r(t))) \\ \qquad\qquad + D_{1i}v(t)]dt + G_i[E_i x(t) + E_{di}x(t-\tau) + D_{2i}v(t)]d\omega(t), \\ y(t) = C_i x(t) + C_{di}x(t-\tau) + D_i v(t), \end{cases} \tag{4.1}$$

where $A_i, A_{di}, B_i, C_i, D_i, D_{1i}, D_{2i}, E_i, E_{di}, G_i$ are known constant matrices of appropriate dimensions, and matrix B_i has full column rank. $f(x(t), x(t-\tau), t, i)$ and $h(x(t), x(t-\tau), t, i)$ are known nonlinear functions, and $\phi(x(t), x(t-\tau), t, i)$ is an unknown nonlinear uncertainty. Furthermore, it is assumed that $f(x(t), x(t-\tau), t, i)$ and $\phi(x(t), x(t-\tau), t, i)$ satisfy Assumption 1.

Now, we choose the integral sliding mode function as follows:

$$\sigma(t,i) = H_i x(t) - \int_0^t H_i[A_i x(\theta) + A_{di}x(\theta-\tau) + f(x(\theta), x(\theta-\tau), \theta, i) + B_i K_i x(\theta)]d\theta. \tag{4.2}$$

Then, by following a similar line as in (2.4)-(2.6), the sliding mode dynamics can be obtained as

$$\begin{cases} dx(t) = [A_i x(t) + A_{di}x(t-\tau) + f(x(t), x(t-\tau), t, i) + B_i K_i x(t) \\ \qquad\qquad + (I - B_i(H_i B_i)^{-1}H_i)D_{1i}v(t))]dt + G_i[E_i x(t) + E_{di}x(t-\tau) + D_{2i}v(t)]d\omega(t), \\ y(t) = C_i x(t) + C_{di}x(t-\tau) + D_i v(t). \end{cases} \tag{4.3}$$

Following the same lines as in Theorem 3.2 and Theorem 3.1, we have the following two results where only the sketches of the proofs are given in order to ensure the conciseness.

Theorem 4.1: Consider the system (4.3). If there exist positive matrices $P_i > 0$, matrices Y_i and positive real scalars λ_1, λ_2, λ_3 with $\lambda_1 > \lambda_2$ satisfying the following LMIs

$$\begin{bmatrix} \Omega_i & A_{di}^T & D_{1i}^T P_i & \Lambda_i & E_i^T G_i^T P_i & \Theta_{1i} & 0 & C_i^T & B_i P_i \\ * & -\lambda_2 I & 0 & 0 & E_{di}^T G_i^T P_i & 0 & \sqrt{2}P_i L_f & C_{di}^T & 0 \\ * & * & -\gamma^2 I & 0 & D_{2i}^T G_i^T P_i & 0 & 0 & D_i^T & 0 \\ * & * & * & -J_i & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -P_i & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\Delta_i & 0 & 0 & 0 \\ * & * & * & * & * & * & -P_i & 0 & 0 \\ * & * & * & * & * & * & 0 & -I & 0 \\ * & * & * & * & * & * & 0 & 0 & -\lambda_3 I \end{bmatrix} < 0, \quad \forall i \in S \tag{4.4}$$

with the constraint

$$\lambda_3 I \leq P_i B_i^T P_i^{-1} B_i P_i \tag{4.5}$$

where

$$\begin{aligned}
\Omega_i &= A_iP_i + P_iA_i^T + B_iY_i + Y_i^TB_i^T + \gamma_{ii}P_i, \\
\Lambda_i &= [P_1, \cdots, P_{i-1}, P_{i+1}, \cdots, P_N], \\
J_i &= \text{diag}\{\gamma_{i1}^{-1}P_1, \cdots, \gamma_{ii-1}^{-1}P_{i-1}, \gamma_{ii+1}^{-1}P_{i+1}, \cdots, \gamma_{iN}^{-1}P_N\}, \\
\Theta_{1i} &= [\sqrt{2}L_fP_i \quad P_i], \quad \Delta_i = \text{diag}\{-P_i, \ -\lambda_1 I\},
\end{aligned}$$

then the closed-loop system (4.3) with $v(t) = 0$ is almost surely exponentially stable and, meanwhile, the H_∞ performance is achieved. In this case, the controller gain matrices are $K_i = Y_iP_i^{-1}$ and $H_i = B_i^TP_i^{-1}$ for $i \in S$.

Proof: Firstly, choose the Lyapunov function candidate as

$$V(x, t, i) = x^T(t)X_ix(t). \tag{4.6}$$

Then, in light of Assumption 2, choose $H_i = B_i^TX_i$. The infinitesimal generator $\mathcal{L}V(x(t), x(t-\tau), t, i)$ associated with the system (4.3) is given as

$$\begin{aligned}
&\mathcal{L}V(x(t), x(t-\tau), t, i) \\
=\ & 2x^T(t)X_i[A_ix(t) + A_{di}x(t-\tau) + f(x(t), x(t-\tau), t, i) + B_iK_ix(t) \\
&+ (I - B_i(B_i^TX_iB_i)^{-1}B_i^TX_i)D_{1i}v(t))] + \sum_{j=1}^{N}\gamma_{ij}x^T(t)X_jx(t) \\
&+ [G_i(E_ix(t) + E_{di}x(t-\tau) + D_{2i}v(t))]^TX_i[G_i(E_ix(t) + E_{di}x(t-\tau) + D_{2i}v(t))].
\end{aligned} \tag{4.7}$$

In terms of Assumption 1, it is not difficult to obtain that

$$\begin{aligned}
&2x^T(t)X_if(x(t), x(t-\tau), t, i) \\
&\le x^T(t)X_i^2x(t) + f(x(t), x(t-\tau), t, i)^Tf(x(t), x(t-\tau), t, i) \\
&\le x^T(t)X_i^2x(t) + L_f^2(|x(t)| + |x(t-\tau)|)^2 \\
&\le x^T(t)X_i^2x(t) + 2L_f^2x^T(t)x(t) + 2L_f^2x^T(t-\tau)x(t-\tau).
\end{aligned} \tag{4.8}$$

Therefore, one has

$$\begin{aligned}
&\mathcal{L}V(x(t), x(t-\tau), t, i) \\
\le\ & x^T(t)\Big[X_iA_i + A_i^TX_i + X_iB_iK_i + K_i^TB_i^TX_i + X_i^2 + 2L_f^2I + \sum_{j=1}^{N}\gamma_{ij}X_j + X_iB_i(B_i^TX_iB_i)^{-1}B_i^TX_i\Big]x(t) \\
&+ 2x^T(t)A_{di}x(t-\tau) + 2L_f^2x^T(t-\tau)x(t-\tau) + 2x^T(t)X_iD_{1i}v(t) + v^T(t)D_{1i}^TX_iD_{1i}v(t) \\
&+ [G_iE_ix(t) + G_iE_{di}x(t-\tau) + G_iD_{2i}v(t)]^TX_i[G_iE_ix(t) + G_iE_{di}x(t-\tau) + G_iD_{2i}v(t)].
\end{aligned} \tag{4.9}$$

Furthermore, for $v(t) = 0$, substituting (4.8) into (4.9) results in

$$\mathcal{L}V(x(t), x(t-\tau), t, i) + \lambda_1^{-1}x^2 - \lambda_2x^2(t-\tau) \le [x^T(t) \quad x^T(t-\tau)]\Pi_i \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}, \tag{4.10}$$

where

$$\begin{aligned}
\Psi_i &= X_iA_i + A_i^TX_i + X_iB_iK_i + K_i^TB_i^TX_i + \sum_{j=1}^{N}\gamma_{ij}X_j + X_i^2 + 2L_f^2I + \lambda_1^{-1}I, \\
\Pi_i &= \begin{bmatrix} \Psi_i & X_iA_{di} \\ A_{di}^TX_i & 2L_f^2I - \lambda_2I \end{bmatrix} + \begin{bmatrix} E_i^TG_i^T \\ E_{di}^TG_i^T \end{bmatrix} X_i[G_iE_i \quad G_iE_{di}].
\end{aligned} \tag{4.11}$$

By using the Schur Complement Lemma, $\Pi_i < 0$ ($\forall i \in S$) is equivalent to

$$\begin{bmatrix} \bar{\Psi}_i & X_i A_{di} & \bar{\Lambda}_i & E_i^T G_i^T X_i & \bar{\Theta}_{1i} & 0 \\ * & -\lambda_2 I & 0 & E_{di}^T G_i^T X_i & 0 & \sqrt{2} L_f X_i \\ * & * & -\tilde{J}_i & 0 & 0 & 0 \\ * & * & * & -X_i & 0 & 0 \\ * & * & * & * & \bar{\Delta}_i & 0 \\ * & * & * & * & * & -X_i \end{bmatrix} < 0, \quad \forall i \in S, \tag{4.12}$$

where

$$\begin{aligned} \bar{\Psi}_i &= X_i A_i + A_i^T X_i + X_i B_i K_i + K_i^T B_i^T X_i + \gamma_{ii} X_i, \\ \bar{\Lambda}_i &= [X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_N]_{n \times (N-1)n}, \\ \tilde{J}_i &= \text{diag}\{\gamma_{i1}^{-1} X_1, \cdots, \gamma_{ii-1}^{-1} X_{i-1}, \gamma_{ii+1}^{-1} X_{i+1}, \cdots, \gamma_{iN}^{-1} X_N\}, \\ \bar{\Theta}_{1i} &= [\sqrt{2} L_f X_i \quad I], \quad \bar{\Delta}_i = \text{diag}\{-X_i, \ -\lambda_1 I\}. \end{aligned}$$

In what follows, let $P_i = X_i^{-1}$ and define

$$\bar{J}_{1i} = \text{diag}\{P_1, \cdots, P_{i-1}, P_{i+1}, \cdots, P_N\}, \quad \bar{J}_{2i} = \text{diag}\{P_i, P_i\}.$$

Pre- and post-multiplying (4.12) by $\text{diag}\{P_i, I, \bar{J}_{1i}, P_i, \bar{J}_{2i}, P_i\}$ yield

$$\begin{bmatrix} \Omega_i & A_{di}^T & \Lambda_i & E_i^T G_i^T P_i & \Theta_{1i} & 0 \\ * & -\lambda_2 I & 0 & E_{di}^T G_i^T P_i & 0 & \sqrt{2} P_i L_f \\ * & * & -J_i & 0 & 0 & 0 \\ * & * & * & -P_i & 0 & 0 \\ * & * & * & * & -\Delta_i & 0 \\ * & * & * & * & * & -P_i \end{bmatrix} < 0, \quad \forall i \in S. \tag{4.13}$$

Therefore, it follows from (4.4) and (4.13) that, for all $[x^T(t) \quad x^T(t-\tau)]^T \neq 0$

$$\mathcal{L}V(x(t), x(t-\tau), t, i) \leq -\lambda_1^{-1} x^2 + \lambda_2 x^2(t-\tau), \tag{4.14}$$

which means that the sliding mode dynamics (2.6) with $v(t) \equiv 0$ is almost surely exponentially stable.

Similarly, it follows from (4.4), (4.8) and (4.9) that, for all $[x^T(t) \quad x^T(t-\tau) \quad v^T(t)]^T \neq 0$, the inequality

$$\mathcal{L}V(x(t), x(t-\tau), t, i) \leq -\lambda_1^{-1} x^2(t) + \lambda_2 x^2(t-\tau) + \gamma^2 v^T v - y^T y \tag{4.15}$$

is true. Therefore, the required H_∞ performance now follows from Theorem 3.1 by the similar line. The proof is complete. ∎

Now we are ready to give the design technique of the SMC controller.

Theorem 4.2: Consider the system (4.1) with the sliding mode function $\sigma(t, i)$ in (4.2). The state trajectories of the sliding mode dynamics (4.3) are globally driven onto the sliding mode surface $\sigma(t, i) = 0$ almost surely by adopting the SMC law (3.23) in Theorem 3.2 where the gain matrices K_i and H_i satisfy the conditions in Theorem 4.1.

Remark 4: The main results in Theorems 3.2-4.2 are described in terms of the feasibility to a few LMIs. Note that, for the standard LMI system, the algorithm has a polynomial-time complexity. That is, the number $\mathcal{N}(\varepsilon)$ of flops needed to compute an ε-accurate solution is bounded by $O(\mathcal{M}\mathcal{N}^3 \log(\mathcal{V}/\varepsilon))$, where $\mathcal{M}$ is the total row size of the LMI system, $\mathcal{N}$ is the total number of scalar decision variables, $\mathcal{V}$ is a data-dependent scaling factor, and ε is relative accuracy set for algorithm. Obviously, the computational complexity of the LMI-based algorithms depends polynomially on the network size and the variable dimensions. In order to reduce the computation burden, a possible way is to obtain the estimator gains in a node-by-node way. Fortunately, research on LMI optimization is a very active area in the applied mathematics, optimization and the operations research community, and substantial speed-ups can be expected in the future.

V. An Illustrative Example

In this section, we present an illustrative example to demonstrate the effectiveness of the proposed design scheme of sliding mode controllers for nonlinear stochastic systems with Markovian switching and time-delays in the form of (4.1).

Let the generator Γ be

$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} = \begin{bmatrix} 0.6 & -0.6 \\ -0.2 & 0.2 \end{bmatrix}$$

Suppose that the system involves two modes, and the system data are given as follows:

Mode 1:

$$A_1 = \begin{bmatrix} -1.2 & 3 & 0.2 \\ 0.3 & -2.5 & 0.4 \\ -0.8 & 0.4 & -0.3 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -0.2 & 0.3 & 0.1 \\ 0.1 & -0.5 & 0.2 \\ 0.3 & -0.6 & -0.25 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.7 & -1.3 \\ -0.3 & 0.8 \\ 0.6 & -1 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 0.12 \\ 0.2 \\ 0.1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0.1 & -0.05 \\ 0.3 & 0.1 \\ 0.2 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.4 & 0.1 & 0.6 \\ 0.3 & 0.5 & 1.0 \end{bmatrix}, \quad E_{d1} = \begin{bmatrix} 0.1 & 0.12 & 0.2 \\ 0.2 & 0.4 & 0.1 \end{bmatrix},$$

$$D_{21} = \begin{bmatrix} 0.15 \\ 0.22 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ 0.25 & 0.3 & 0.4 \end{bmatrix}, \quad C_{d1} = \begin{bmatrix} 0.2 & 0.5 & -0.4 \\ 0.5 & -0.3 & 0.25 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.2 \\ 0.15 \end{bmatrix}.$$

The nonlinear function f and ϕ are as follows

$$f(x(t), x(t-\tau), t, 1) = \left[0.14\sqrt{x_1^2(t) + x_2^2(t)}\sin x_3(t) \quad 0.13\sqrt{|x_2(t)x_3(t-\tau)|} \quad 0.15\sqrt{x_1^2(t) + x_3^2(t)}\right]^T,$$
$$\phi(x(t), x(t-\tau), t, 1) = [0.17\sin x_1(t-\tau) \quad 0.25\sin x_2(t) \quad 0.26\sin x_3(t)]^T.$$

Mode 2:

$$A_2 = \begin{bmatrix} -1.5 & 0.2 & 0.3 \\ -0.1 & -0.5 & 0.4 \\ -0.6 & 0.2 & -0.5 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.2 & -0.3 & 0.12 \\ 0.1 & 0.4 & -0.5 \\ -0.4 & -0.3 & 0.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.8 & -1.3 \\ -0.3 & 0.7 \\ 0.5 & -0.8 \end{bmatrix},$$

$$D_{12} = \begin{bmatrix} 0.2 \\ 0.1 \\ 0.13 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.2 & 0.05 \\ 0.1 & 0 \\ 0.12 & 0.1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.2 & 0.2 & 0.3 \\ 0.1 & 0.25 & 0.5 \end{bmatrix}, \quad E_{d2} = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.25 & 0.3 & 0.15 \end{bmatrix},$$

$$D_{22} = \begin{bmatrix} 0.1 \\ 0.14 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -0.5 & 0.3 & 0.1 \\ 0.2 & -0.4 & 0.25 \end{bmatrix}, \quad C_{d2} = \begin{bmatrix} 0.4 & 0.2 & -0.3 \\ -0.6 & 0.4 & 0.2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}.$$

The nonlinear function f and ϕ are as follows

$$f(x(t), x(t-\tau), t, 2) = \left[0.2\sqrt{x_1^2(t) + x_2^2(t)} \quad 0.14\sqrt{|x_2(t)x_3(t-\tau)|}\sin x_1(t) \quad 0.1\sqrt{x_1^2(t) + x_3^2(t)}\right]^T,$$
$$\phi(x(t), x(t-\tau), t, 2) = [0.35\sin x_1(t-\tau) \quad 0.25\sin x_2(t) \quad 0.3\sin x_3(t)]^T.$$

It is easy to see that Assumption 1 can be met with $L_f = 0.25$ and $L_\phi = 0.39$. By using Matlab LMIs toolbox to solve the LMIs (4.4) with constraint (4.5), we can get the permitted minimum γ is 0.4432 and the feasible solution $\lambda_1 = 0.2$, $\lambda_2 = 0.04$, $\lambda_3 = 0.005$ and

$$P_1 = \begin{bmatrix} 0.0337 & 0.0165 & -0.0068 \\ 0.0165 & 0.0379 & -0.0140 \\ -0.0068 & -0.0140 & 0.0274 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.0243 & 0.0055 & -0.0045 \\ 0.0055 & 0.0231 & -0.0062 \\ -0.0045 & -0.0062 & 0.0239 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} -118.9170 & -930.8616 & -520.7245 \\ -82.0009 & -462.5317 & -229.0995 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} -155.0931 & -602.6083 & -248.9477 \\ -93.7367 & -360.1865 & -129.0207 \end{bmatrix}.$$

Then, the desired parameters of sliding mode controller can be designed as

$$K_1=\begin{bmatrix}-15.8220 & -29.9526 & -0.3882\\ -8.8347 & -15.6761 & 0.7730\end{bmatrix},\quad K_2=\begin{bmatrix}-5.9825 & -13.2598 & -1.5402\\ -3.6908 & -8.0531 & -0.4427\end{bmatrix}.$$

$$H_1=\begin{bmatrix}0.0024 & 0.0002 & -0.0020\\ -0.0072 & -0.0007 & 0.0065\end{bmatrix},\quad H_2=\begin{bmatrix}-0.0029 & -0.0005 & 0.0056\\ 0.0050 & 0.0009 & -0.0095\end{bmatrix}.$$

In the simulation, the exogenous disturbance input is selected as $v(t)=\frac{\exp(1/t)}{1+t}$, the step size $\triangle$ and the time-delay τ are 0.001 and 0.2, respectively. Simulation results are shown in Figs. 1$\sim$4, where Fig. 1 and Figs. 2 depict, respectively, the trajectories of state $x(t)$ without $v(t)$ and the trajectories of state $x_2(t)$ with $v(t)$, Figs. 3 plots the control signals $u_1(t)$ and Figs. 4 shows the trajectories of $\sigma_1(t)$ with $v(t)$, which verifies the design scheme of sliding mode controller proposed in this paper.

VI. CONCLUSIONS

In this paper, we have investigated the almost sure H_∞ SMC problems for a class of nonlinear stochastic systems with Markovian switching and time-delays. An integral-type sliding function has been constructed and a SMC law has been designed such that the closed-loop system reaches the specified sliding mode surface almost surely. By employing the stopping time method combined with martingale inequalities, some sufficient conditions have been proposed to guarantee that the sliding dynamics is almost surely exponentially stable and the H_∞ norm from the external inputs to the controlled outputs is less than a given level γ. Furthermore, based on the proposed results, for a class of special nonlinear stochastic systems with Markovian switching and time-delays, the desired SMC law has been designed by solving a set of LMIs. Finally, a numerical simulation example has been provided to demonstrate the effectiveness and applicability of the proposed design approach. Future research topics would include the extension of the main results to more complex systems such as networked control systems with fading measurements [7], [8], [10] and randomly occurring phenomena [11], [12], [32], [37].

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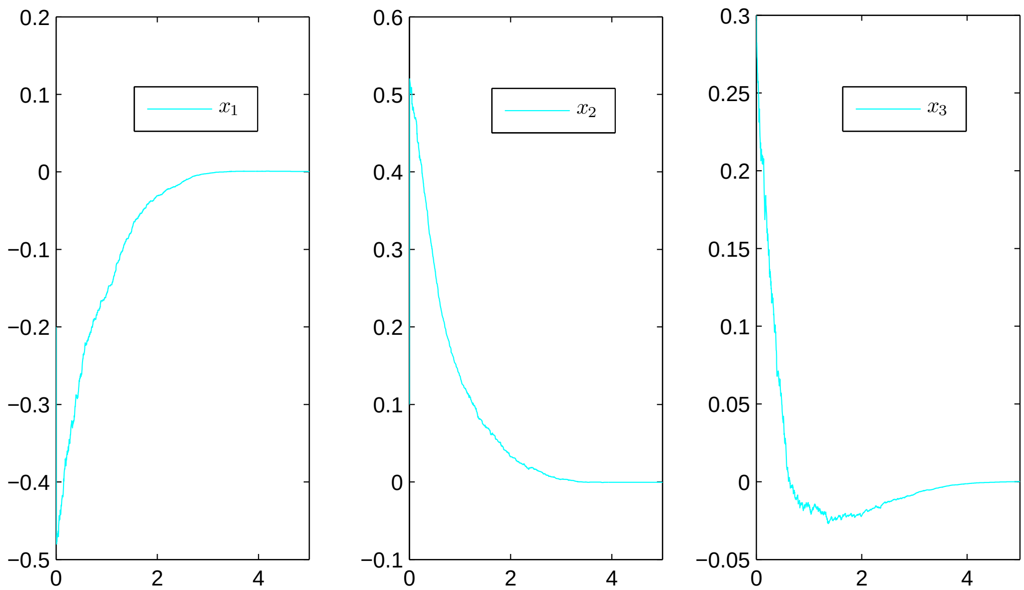

Fig. 1. Trajectories of state $x(t)$ without $v(t)$.

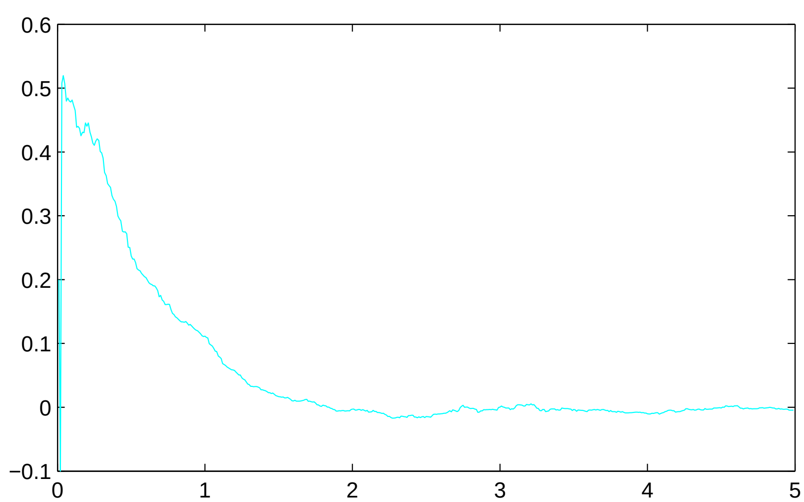

Fig. 2. Trajectories of state $x_2(t)$ with $v(t)$.

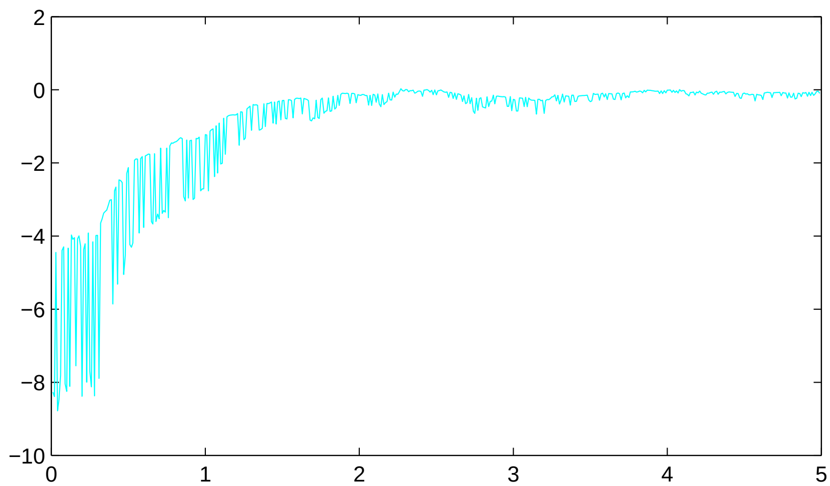

Fig. 3. Control signals $u_1(t)$ with $v(t)$.

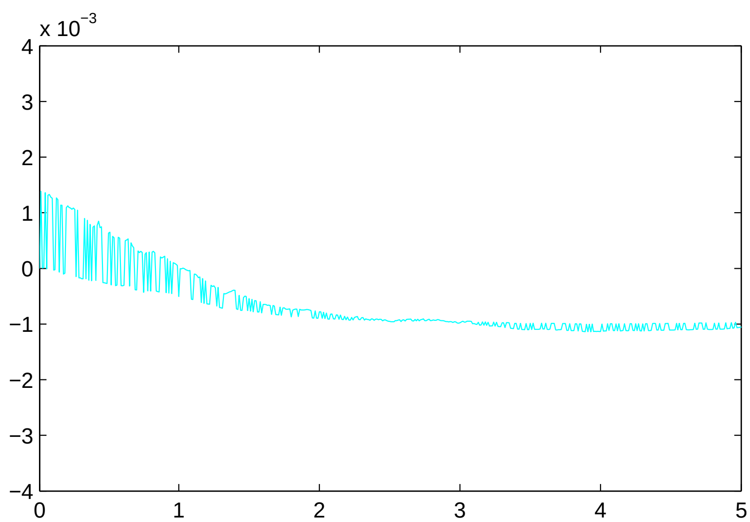

Fig. 4. Trajectories of $\sigma_1(t)$ with $v(t)$.

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