

Non-Fragile State Estimation for Discrete Markovian Jumping Neural Networks

Nan Hou, Hongli Dong^{*}, Zidong Wang, Weijian Ren and Fuad E. Alsaadi

Abstract

In this paper, the non-fragile state estimation problem is investigated for a class of discrete-time neural networks subject to Markovian jumping parameters and time delays. In terms of a Markov chain, the mode switching phenomenon at different times is considered in both the parameters and the discrete delays of the neural networks. To account for the possible gain variations occurring in the implementation, the gain of the estimator is assumed to be perturbed by multiplicative norm-bounded uncertainties. We aim to design a non-fragile state estimator such that, in the presence of all admissible gain variations, the estimation error converges to zero asymptotically. By adopting the Lyapunov-Krasovskii functional and the stochastic analysis theory, sufficient conditions are established to ensure the existence of the desired state estimator that guarantees the stability of the overall estimation error dynamics. The explicit expression of such estimators is parameterized by solving a convex optimization problem via the semi-definite programming method. A numerical simulation example is provided to verify the usefulness of the proposed methods.

Index Terms

Non-fragile state estimation; estimator gain variations; Markovian jumping; time delays; nonlinearity

I. INTRODUCTION

Research area of theoretical investigation, algorithm development and practical application of recurrent neural networks (RNNs) has been growing in few recent decades. It is now clear that the internal states of the RNNs exhibit rich dynamic temporal behavior that can be ideally to be exploited to process arbitrary sequences of inputs. The popularity of RNNs in solving real-world problems (e.g. pattern recognition and dynamic optimization) places an increasing demand for dynamic analysis of RNNs. This demand generates new requirements for the stability synchronization analysis, state estimation and pinning control for RNNs, which have led to a rich body of literature, see e.g. [3]–[5], [8], [15], [21], [33], [34]. For example, in [21], the problem of globally asymptotic stability has been investigated for a kind of neural

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N. Hou, H. Dong and W. Ren are with the College of Electrical and Information Engineering, Northeast Petroleum University, Daqing 163318, China. (Email: shiningdhl@vip.126.com)

Z. Wang is with the Department of Computer Science, Brunel University London, Uxbridge, Middlesex, UB8 3PH, United Kingdom. He is also with the Faculty of Engineering, King Abdulaziz University, Jeddah 21589, Saudi Arabia. (Email: Zidong.Wang@brunel.ac.uk)

F. E. Alsaadi is with the Faculty of Engineering, King Abdulaziz University, Jeddah 21589, Saudi Arabia.

^{*} Corresponding author.

networks with discrete and distributed delays via the Lyapunov-Krasovskii stability theory and a linear matrix inequality (LMI) approach. The robust state estimation problem has been studied in [8] for a class of uncertain neural networks with time-varying delay.

In the context of neuron state estimation for RNNs, most results obtained so far have been based on the assumption that the parameter of the estimator can be realized accurately [3], [5], [8]. In fact, due to the complex and changeable environments (e.g. analogue-to-digital conversion, rounding errors, finite precision or internal noise), the parameter implemented is not necessarily the same as the ideal value. In other words, it is very often the case that the implemented parameters undergo certain drafts/variations/fluctuations which might give rise to the fragility (performance degradation or even instability) of the underlying systems. As such, the non-fragile issue has attracted a great deal of research attention in the past few years for dynamical systems, complex networks as well as neural networks, see [2], [10]–[12], [16], [23], [25] and the references therein. For example, a non-fragile H_∞ controller has been designed in [11] for a class of discrete system with randomly occurring gain variations, distributed delays and channel fadings. In [23], the non-fragile Kalman filter has been obtained in terms of the solutions to the algebraic Riccati equations.

As is well known, time delays occur inevitably in the neural networks, and the existence of time delays could lead to the undesired oscillation and even the instability of the neural network, see e.g. [9], [14], [20], [21], [37]. In this case, it is of great significance to research into the delayed neural networks and examine how the delays have an impact on the dynamical behaviors. In [14], mixed time-delays (discrete and distributed time-delays) have been considered in the neural network, and the problem of state estimation has been solved via the Lyapunov stability theory and the LMI technology. On the other hand, the nonlinear activation functions play an important role in the functioning of neural network. There are a variety of types of activation functions dependent on the nature of the problem to be solved, and a commonly used one is the sigmoid function. It is worth mentioning that, in [14], the activation functions have been allowed to be non-monotonic, which are more general than the traditional Lipschitz-type conditions. Because of the tighter bounds on the activation functions proposed in [14], less conservatism is expected in the stability analysis of the RNNs.

In practice, neural networks may be subject to network mode switching, which is regulated by a Markovian chain [13], [14], [19], [22], [24], [37]. For example, a Markov process governed continuous-time discrete-state homogeneous Markov process has been utilized in [22] to generate the jumping parameters in the discrete and finite state space, where the dynamics of the neural network can be stochastically exponentially stable in the mean square and independent of the time delays as long as certain conditions are met. In [13], both the stability and the synchronization problems have been analyzed for the discrete-time Markovian jumping neural networks with mixed mode-dependent time-delays, where the parameters of the neural network are changeable among modes in accordance with the Markovian chain and the discrete/distributed time-delays are also dependent on the Markovian jumping mode. Nevertheless, to the best of the authors' knowledge, the non-fragile state estimation problem for discrete-time Markovian jumping neural networks with time delays has not been adequately addressed in the literature yet, and the purpose of this paper is therefore to shorten such a gap.

In this paper, we deal with the non-fragile state estimation problem for a class of discrete-time neural

networks with Markovian jumping parameters and time-delays. To guarantee that the estimation error dynamic is asymptotically stable, the Lyapunov functional method and some matrix analysis techniques are employed to acquire the delay-dependent sufficient conditions. The results deduced are in terms of LMIs, which can be solved conveniently by the standard simulation software. Note that the gain uncertainties (multiplicative gain variations) are considered in the estimator of the neural networks with Markovian jumping parameters and mode-dependent delays. A numerical example is utilized to represent the usefulness of our research. The main contribution of this paper can be listed as follows. 1) The non-fragile state estimation problem is, for the first time, investigated for a class of discrete-time Markovian jumping neural networks with time delays. 2) Intensive stochastic analysis is conducted to obtain sufficient conditions that guarantee the convergence of the estimation errors against the gain variations as well as the nonlinear disturbances on the network outputs.

Notation: Throughout this paper, M^T means the transpose of M . \mathbb{R}^n means the n dimensional Euclidean space and $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices. The set of all non-positive integers is denoted by \mathbb{Z}^- . I and 0 denote the identity matrix and zero matrix, respectively. The notation $P > 0$ means that P is a real symmetric and positive definite matrix. $\mathbb{E}\{x\}$ and $\mathbb{E}\{x|y\}$ represent, respectively, the expectation of x and the expectation of x conditional on y . $\|x\|$ stands for the Euclidean norm of a vector x . In symmetric block matrices, the shorthand $\text{diag}\{A_1, A_2, \dots, A_n\}$ represents a block diagonal matrix with diagonal blocks being the matrices A_1, \dots, A_n , and the symbol $*$ denotes an ellipsis for terms induced by symmetry. If M is a symmetric matrix, $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ show the maximum eigenvalue and the minimum eigenvalue. The symbol \otimes denotes the Kronecker product. Matrices without explicitly stated dimensions are supposed to be compatible for matrix operations.

II. PROBLEM FORMULATION AND PRELIMINARIES

The Markov chain $\theta(k)$ ($k \geq 0$) takes values in a finite state space $S = \{1, 2, \dots, s\}$ with transition probability matrix $\Lambda = [\lambda_{ij}]_{s \times s}$ given by

$$\text{Prob}\{\theta(k+1) = j | \theta(k) = i\} = \lambda_{ij}, \quad \forall i, j \in S,$$

where $\lambda_{ij} \geq 0$ ($i, j \in S$) is the transition probability from i to j and $\sum_{j=1}^s \lambda_{ij} = 1$, $\forall i \in S$.

In this paper, we consider a discrete-time n -neuron Markovian jumping neural network described by the following dynamical equation:

$$\begin{aligned} x(k+1) &= A(\theta(k))x(k) + A_d(\theta(k))x(k - d_1(\theta(k))) + W(\theta(k))g(x(k)) \\ &\quad + W_d(\theta(k))g(x(k - d_2(\theta(k))))), \end{aligned} \quad (1)$$

$$y(k) = D(\theta(k))x(k) + E(\theta(k))h(x(k)), \quad (2)$$

$$x(k) = \psi(k), \quad \forall k \in \mathbb{Z}^-$$

where $x(k) = [x_1(k), x_2(k), \dots, x_n(k)]^T$ is the neural state vector; $g(x(k)) = [g_1(x_1(k)), g_2(x_2(k)), \dots, g_n(x_n(k))]^T$ represents the nonlinear activation function with the initial condition $g(0) = 0$; $d_1(\theta(k))$ and $d_2(\theta(k))$ denote the discrete time delays; $A(\theta(k)) = \text{diag}\{a_1(\theta(k)), a_2(\theta(k)), \dots, a_n(\theta(k))\}$ describes the rate with which the each neuron will reset its potential to the resting state in isolation when disconnected

from the networks and external inputs; $A_d(\theta(k)) = \text{diag} \{a_{d1}(\theta(k)), a_{d2}(\theta(k)), \dots, a_{dn}(\theta(k))\}$ is the parameter matrix of the state time delays; $W(\theta(k)) = [w_{ij}(\theta(k))]_{n \times n}$ is the connection weight matrix; $W_d(\theta(k)) = [w_{dij}(\theta(k))]_{n \times n}$ is the discretely delayed connection weight matrix; $y(k)$ is the output; $h(x(k))$ is the nonlinear disturbance on the output. $\psi(k)$ is a given initial sequence. The nonlinear vector-valued function $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $h(0) = 0$ is supposed to be continuous and satisfies the following sector-bounded condition

$$[h(x) - h(y) - \Phi(x - y)]^T [h(x) - h(y) - \Omega(x - y)] \leq 0 \quad (3)$$

for all $x, y \in \mathbb{R}^n$, where Φ and Ω are real matrices of appropriate dimensions.

The nonlinear vector-valued function $g(x(k))$ satisfies:

$$\|g(x(k) + \delta(k)) - g(x(k))\| \leq \|B\delta(k)\| \quad (4)$$

where, for all the system modes, $B = \text{diag} \{b_1, b_2, \dots, b_n\} > 0$ is a known matrix and $\delta(k)$ is a vector.

The set S contains s modes of equation (1)-(2), for $\theta(k) = i$, the system matrices of the i th mode are denoted by $A_i, A_{di}, W_i, W_{di}, D_i$, and the time delays of the i th mode are represented by d_{1i}, d_{2i} .

As mentioned in the introduction, the actual state estimator experiences gain variations with times. To cope with this problem, we consider a discrete-time state estimator in the following form:

$$\begin{aligned} \hat{x}(k+1) = & A_i \hat{x}(k) + A_{di} \hat{x}(k - d_{1i}) + W_i g(\hat{x}(k)) + W_{di} g(\hat{x}(k - d_{2i})) \\ & + (K_i + \Delta K_i)(y(k) - D_i \hat{x}(k) - E_i h(\hat{x}(k))) \end{aligned} \quad (5)$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the state of the estimator and K_i is the matrix to be designed. ΔK_i quantifies the gain variations corresponding to the following norm-bounded multiplicative form:

$$\Delta K_i = K_i H_k F(k) E_k, \quad (6)$$

where H_k, E_k are known matrices with appropriate dimensions and $F(k)$ is the unknown matrix satisfying $F^T(k)F(k) \leq I$.

Remark 1: Due to a variety of reasons such as rounding errors and finite precision, the parameters of the estimator might not be implemented accurately. In (5) and (6), ΔK_i ($i \in S$) is introduced in a multiplied form with hope to account the gain variations in a realistic way. We aim to design a state estimator that can achieve a satisfactory estimation performance even if the gains deviate from the expected values within an admissible bound.

Letting $e(k) = x(k) - \hat{x}(k)$ and combining (1)-(2) with (5), we can easily obtain the following error dynamics:

$$\begin{aligned} e(k+1) = & A_i e(k) + A_{di} e(k - d_{1i}) + W_i (g(x(k)) - g(\hat{x}(k))) + W_{di} (g(x(k - d_{2i})) - g(\hat{x}(k - d_{2i}))) \\ & - (K_i + \Delta K_i) D_i e(k) - (K_i + \Delta K_i) E_i (h(x(k)) - h(\hat{x}(k))). \end{aligned} \quad (7)$$

Furthermore, denote $\eta(k) = [x^T(k) \quad e^T(k)]^T$, $g(\eta(k)) = [g^T(x(k)) \quad g^T(x(k)) - g^T(\hat{x}(k))]^T$ and $h(\eta(k)) = [h^T(x(k)) \quad h^T(x(k)) - h^T(\hat{x}(k))]^T$. Combining the estimation error (7) with system (1)-(2), the augmented system model to be considered is given as follows:

$$\eta(k+1) = \mathcal{A}_i \eta(k) + \mathcal{A}_{di} \eta(k - d_{1i}) + \mathcal{W}_i g(\eta(k)) + \mathcal{W}_{di} g(\eta(k - d_{2i})) - \mathcal{K}_i \mathcal{D}_i \eta(k) - \mathcal{K}_i \mathcal{E}_i h(\eta(k)) \quad (8)$$

where

$$\begin{aligned} \mathcal{K}_i &= \text{diag}\{K_i + \Delta K_i, K_i + \Delta K_i\}, \quad \mathcal{A}_i = \text{diag}\{A_i, A_i\}, \quad \mathcal{A}_{di} = \text{diag}\{A_{di}, A_{di}\}, \\ \mathcal{W}_{di} &= \text{diag}\{W_{di}, W_{di}\}, \quad \mathcal{E}_i = \text{diag}\{0, E_i\}, \quad \mathcal{D}_i = \text{diag}\{0, D_i\}, \quad \mathcal{W}_i = \text{diag}\{W_i, W_i\}. \end{aligned}$$

Definition 1: The error dynamic (8) is exponentially stable in the mean square if there exist positive constants $\mu > 0$ and $0 < \alpha < 1$ satisfying

$$\mathbb{E}\{\|\eta(k)\|^2\} \leq \mu \alpha^k \sup_{i \in \mathbb{Z}^-} \mathbb{E}\{\|\psi(i)\|^2\}. \quad (9)$$

The objective of this paper is to design a non-fragile state estimator for neural network (1)-(2). More specifically, we are interested in finding an estimator described by (5) with allowable gain variations of the form (6) such that the resulting system (8) is asymptotically stable. By constructing new Lyapunov-Krasovskii functional, we will derive the sufficient conditions under which (5) becomes an asymptotic state estimator of neural network (1)-(2) and the gain matrices K_i ($i \in S$) will also be given explicitly.

III. MAIN RESULTS

Before stating our main results, we introduce the following lemmas:

Lemma 1: (Schur Complement) [1] Given constant matrices S_1, S_2, S_3 where $S_1 = S_1^T$ and $0 < S_2 = S_2^T$, then $S_1 + S_3^T S_2^{-1} S_3 < 0$ if and only if

$$\begin{bmatrix} S_1 & S_3^T \\ S_3 & -S_2 \end{bmatrix} < 0, \quad \begin{bmatrix} -S_2 & S_3 \\ S_3^T & S_1 \end{bmatrix} < 0. \quad (10)$$

Lemma 2: (S-procedure) [1] Let $L = L^T$, M and N be real matrices of appropriate dimensions with F satisfying $F^T F \leq I$, then

$$L + M F N + N^T F^T M^T < 0$$

if and only if there exists a positive scalar μ such that

$$L + \mu^{-1} M M^T + \mu N^T N < 0 \quad (11)$$

or, equivalently,

$$\Pi = \begin{bmatrix} L & M & \mu N^T \\ M^T & -\mu I & 0 \\ \mu N & 0 & -\mu I \end{bmatrix} < 0$$

For presentation convenience, we denote $\bar{d}_1 = \max\{d_{1i} | i \in S\}$, $\underline{d}_1 = \min\{d_{1i} | i \in S\}$, $\bar{d}_2 = \max\{d_{2i} | i \in S\}$, $\underline{d}_2 = \min\{d_{2i} | i \in S\}$, $\underline{\lambda} = \min\{\lambda_{ii} | i \in S\}$.

Firstly, we will derive a stability criterion for the discrete time Markovian jump neural network. The following theorem presents a sufficient condition on the asymptotic stability of (8).

Theorem 1: Let the parameters \mathcal{K}_i ($i \in S$) be known. The augmented system (8) is asymptotically stable if there exist a set of matrices $P_i > 0$ ($i \in S$), two matrices $Q > 0$, $R > 0$, and positive constant scalars κ_{1i} , κ_{2i} , κ_{3i} satisfying

$$\tilde{\Pi}_i = \begin{bmatrix} \tilde{\Pi}_{11_i} & 0 & 0 & 0 & 0 & \kappa_{3i} \frac{(I \otimes (\Phi + \Omega))^T}{2} & \mathcal{A}_i^T \bar{P}_i - \mathcal{D}_i^T \mathcal{K}_i^T \bar{P}_i \\ * & -Q & 0 & 0 & 0 & 0 & \mathcal{A}_{di}^T \bar{P}_i \\ * & * & \kappa_{2i} \bar{B}^T \bar{B} & 0 & 0 & 0 & 0 \\ * & * & * & \sigma_2 R - \kappa_{1i} I & 0 & 0 & \mathcal{W}_i^T \bar{P}_i \\ * & * & * & * & -R - \kappa_{2i} I & 0 & \mathcal{W}_{di}^T \bar{P}_i \\ * & * & * & * & * & -\kappa_{3i} I & -\mathcal{E}_i^T \mathcal{K}_i^T \bar{P}_i \\ * & * & * & * & * & * & -\bar{P}_i \end{bmatrix} < 0 \quad (12)$$

where

$$\tilde{\Pi}_{11_i} = -P_i + \sigma_1 Q + \kappa_{1i} \bar{B}^T \bar{B} - \kappa_{3i} \frac{I \otimes (\Phi \Omega + \Omega^T \Phi^T)}{2}, \quad P_i = \text{diag}\{P_{1i}, P_{1i}\}, \quad \bar{P}_i = \sum_{j=1}^s \lambda_{ij} P_j,$$

$$\xi(k, i) = \begin{bmatrix} \eta^T(k) & \eta^T(k - d_{1i}) & \eta^T(k - d_{2i}) & g^T(\eta(k)) & g^T(\eta(k - d_{2i})) & h^T(\eta(k)) \end{bmatrix}^T,$$

$$S_i = \begin{bmatrix} \mathcal{A}_i - \mathcal{K}_i \mathcal{D}_i & \mathcal{A}_{di} & 0 & \mathcal{W}_i & \mathcal{W}_{di} & -\mathcal{K}_i \mathcal{E}_i \end{bmatrix}, \quad \sigma_1 = (1 - \underline{\lambda})(\bar{d}_1 - \underline{d}_1) + 1,$$

$$\sigma_2 = (1 - \underline{\lambda})(\bar{d}_2 - \underline{d}_2) + 1.$$

Proof: : Define the following Lyapunov function:

$$V(\eta(k), k, \theta(k)) = V_1(\eta(k), k, \theta(k)) + V_2(\eta(k), k, \theta(k)) + V_3(\eta(k), k, \theta(k)) \quad (13)$$

where

$$V_1(\eta(k), k, \theta(k)) = \eta^T(k) P(\theta(k)) \eta(k),$$

$$V_2(\eta(k), k, \theta(k)) = \sum_{l=k-d_1(\theta(k))}^{k-1} \eta^T(l) Q \eta(l) + (1 - \underline{\lambda}) \sum_{m=\underline{d}_1}^{\bar{d}_1-1} \sum_{l=k-m}^{k-1} \eta^T(l) Q \eta(l),$$

$$V_3(\eta(k), k, \theta(k)) = \sum_{l=k-d_2(\theta(k))}^{k-1} g^T(\eta(l)) R g(\eta(l)) + (1 - \underline{\lambda}) \sum_{m=\underline{d}_2}^{\bar{d}_2-1} \sum_{l=k-m}^{k-1} g^T(\eta(l)) R g(\eta(l)),$$

For $i \in S$, we have

$$\begin{aligned} & \mathbb{E}\{V_1(\eta(k+1), k+1, \theta(k+1)) | \eta(k), \theta(k) = i\} - V_1(\eta(k), k, i) \\ & = \xi^T(k, i) S_i^T \bar{P}_i S_i \xi(k, i) - \eta^T(k) P_i \eta(k), \end{aligned} \quad (14)$$

$$\begin{aligned} & \mathbb{E}\{V_2(\eta(k+1), k+1, \theta(k+1)) | \eta(k), \theta(k) = i\} - V_2(\eta(k), k, i) \\ & = \sum_{j=1}^N \lambda_{ij} \sum_{l=k+1-d_{1j}}^k \eta^T(l) Q \eta(l) + (1 - \underline{\lambda}) \sum_{m=\underline{d}_1}^{\bar{d}_1-1} \sum_{l=k+1-m}^k \eta^T(l) Q \eta(l) \\ & \quad - \sum_{l=k-d_{1i}}^{k-1} \eta^T(l) Q \eta(l) - (1 - \underline{\lambda}) \sum_{m=\underline{d}_1}^{\bar{d}_1-1} \sum_{l=k-m}^{k-1} \eta^T(l) Q \eta(l) \end{aligned}$$

$$\begin{aligned}
&= \lambda_{ii} \left(\sum_{l=k+1-d_{1i}}^k \eta^T(l)Q\eta(l) - \sum_{l=k-d_{1i}}^{k-1} \eta^T(l)Q\eta(l) \right) + (1-\underline{\lambda})(\bar{d}_1 - \underline{d}_1)\eta^T(k)Q\eta(k) \\
&\quad + \sum_{j=1, j \neq i}^N \lambda_{ij} \left(\sum_{l=k+1-d_{1j}}^k \eta^T(l)Q\eta(l) - \sum_{l=k-d_{1i}}^{k-1} \eta^T(l)Q\eta(l) \right) - (1-\underline{\lambda}) \sum_{l=k-\bar{d}_1+1}^{k-\underline{d}_1} \eta^T(l)Q\eta(l) \\
&= [(1-\underline{\lambda})(\bar{d}_1 - \underline{d}_1) + 1]\eta^T(k)Q\eta(k) - \eta^T(k-d_{1i})Q\eta(k-d_{1i}) - (1-\underline{\lambda}) \sum_{l=k-\bar{d}_1+1}^{k-\underline{d}_1} \eta^T(l)Q\eta(l) \\
&\quad + \sum_{j=1, j \neq i}^N \lambda_{ij} \left(\sum_{l=k+1-d_{1j}}^{k-1} \eta^T(l)Q\eta(l) - \sum_{l=k+1-d_{1i}}^{k-1} \eta^T(l)Q\eta(l) \right) \\
&\leq [(1-\underline{\lambda})(\bar{d}_1 - \underline{d}_1) + 1]\eta^T(k)Q\eta(k) - \eta^T(k-d_{1i})Q\eta(k-d_{1i}),
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&\mathbb{E}\{V_3(\eta(k+1), k+1, \theta(k+1)) | \eta(k), \theta(k) = i\} - V_3(\eta(k), k, i) \\
&= \sum_{j=1}^N \lambda_{ij} \sum_{l=k+1-d_{2j}}^k g^T(\eta(l))Rg(\eta(l)) + (1-\underline{\lambda}) \sum_{m=\underline{d}_2}^{\bar{d}_2-1} \sum_{l=k+1-m}^k g^T(\eta(l))Rg(\eta(l)) \\
&\quad - \sum_{l=k-d_{2i}}^{k-1} g^T(\eta(l))Rg(\eta(l)) - (1-\underline{\lambda}) \sum_{m=\underline{d}_2}^{\bar{d}_2-1} \sum_{l=k-m}^{k-1} g^T(\eta(l))Rg(\eta(l)) \\
&= \lambda_{ii} \left(\sum_{l=k+1-d_{2i}}^k g^T(\eta(l))Rg(\eta(l)) - \sum_{l=k-d_{2i}}^{k-1} g^T(\eta(l))Rg(\eta(l)) \right) \\
&\quad + (1-\underline{\lambda})(\bar{d}_2 - \underline{d}_2)g^T(\eta(k))Rg(\eta(k)) + \sum_{j=1, j \neq i}^N \lambda_{ij} \left(\sum_{l=k+1-d_{2j}}^k g^T(\eta(l))Rg(\eta(l)) \right. \\
&\quad \left. - \sum_{l=k-d_{2i}}^{k-1} g^T(\eta(l))Rg(\eta(l)) \right) - (1-\underline{\lambda}) \sum_{l=k-\bar{d}_2+1}^{k-\underline{d}_2} g^T(\eta(l))Rg(\eta(l)) \\
&= [(1-\underline{\lambda})(\bar{d}_2 - \underline{d}_2) + 1]g^T(\eta(k))Rg(\eta(k)) - g^T(\eta(k-d_{2i}))Rg(\eta(k-d_{2i})) \\
&\quad - (1-\underline{\lambda}) \sum_{l=k-\bar{d}_2+1}^{k-\underline{d}_2} g^T(\eta(l))Rg(\eta(l)) \\
&\quad + \sum_{j=1, j \neq i}^N \lambda_{ij} \left(\sum_{l=k+1-d_{2j}}^{k-1} g^T(\eta(l))Rg(\eta(l)) - \sum_{l=k+1-d_{2i}}^{k-1} g^T(\eta(l))Rg(\eta(l)) \right) \\
&\leq [(1-\underline{\lambda})(\bar{d}_2 - \underline{d}_2) + 1]g^T(\eta(k))Rg(\eta(k)) - g^T(\eta(k-d_{2i}))Rg(\eta(k-d_{2i})).
\end{aligned} \tag{16}$$

In terms of (14)-(16), we obtain

$$\begin{aligned}
&\mathbb{E}\{V(\eta(k+1), k+1, \theta(k+1)) | \eta(k), \theta(k) = i\} - V(\eta(k), k, i) \\
&\leq \xi^T(k, i)S_i^T \bar{P}_i S_i \xi(k, i) - \eta^T(k)P_i \eta(k) + \sigma_1 \eta^T(k)Q\eta(k) \\
&\quad - \eta^T(k-d_{1i})Q\eta(k-d_{1i}) + \sigma_2 g^T(\eta(k))Rg(\eta(k)) \\
&\quad - g^T(\eta(k-d_{2i}))Rg(\eta(k-d_{2i}))
\end{aligned} \tag{17}$$

Moreover, according to the constraint (4), it can be deduced that

$$\begin{aligned}\|g(\eta(k))\| &\leq \|\bar{B}\eta(k)\|, \\ \|g(\eta(k - d_{2i}))\| &\leq \|\bar{B}\eta(k - d_{2i})\|,\end{aligned}$$

with $\bar{B} = \text{diag}\{B, B\}$, and therefore,

$$\begin{aligned}\kappa_{1i}g^T(\eta(k))g(\eta(k)) - \kappa_{1i}\eta^T(k)\bar{B}^T\bar{B}\eta(k) &\leq 0 \\ \kappa_{2i}g^T(\eta(k - d_{2i}))g(\eta(k - d_{2i})) - \kappa_{2i}\eta^T(k - d_{2i})\bar{B}^T\bar{B}\eta(k - d_{2i}) &\leq 0.\end{aligned}\tag{18}$$

On the other hand, it follows from (3) that

$$\kappa_{3i}[h(\eta(k)) - (I \otimes \Phi)\eta(k)]^T[h(\eta(k)) - (I \otimes \Omega)\eta(k)] \leq 0.\tag{19}$$

Combination of (17), (18) and (19) results in

$$\begin{aligned}&\mathbb{E}\{V(\eta(k+1), k+1, \theta(k+1))|\eta(k), \theta(k) = i\} - V(\eta(k), k, i) \\ &\leq \xi^T(k, i)S_i^T\bar{P}_iS_i\xi(k, i) - \eta^T(k)P_i\eta(k) + \sigma_1\eta^T(k)Q\eta(k) \\ &\quad - \eta^T(k - d_{1i})Q\eta(k - d_{1i}) + \sigma_2g^T(\eta(k))Rg(\eta(k)) \\ &\quad - g^T(\eta(k - d_{2i}))Rg(\eta(k - d_{2i})) \\ &\quad - (\kappa_{1i}g^T(\eta(k))g(\eta(k)) - \kappa_{1i}\eta^T(k)\bar{B}^T\bar{B}\eta(k)) \\ &\quad - (\kappa_{2i}g^T(\eta(k - d_{2i}))g(\eta(k - d_{2i})) - \kappa_{2i}\eta^T(k - d_{2i})\bar{B}^T\bar{B}\eta(k - d_{2i})) \\ &\quad - [\kappa_{3i}h^T(\eta(k))h(\eta(k)) - \kappa_{3i}\eta^T(k)(I \otimes (\Phi + \Omega))^T h(\eta(k)) + \kappa_{3i}\eta^T(k)\frac{I \otimes (\Phi\Omega + \Omega^T\Phi^T)}{2}\eta(k)] \\ &= \xi^T(k, i)\tilde{\Pi}_i\xi(k, i)\end{aligned}\tag{20}$$

where $\tilde{\Pi}_i = S_i^T\bar{P}_iS_i + \Pi_i$ with

$$\Pi_i = \begin{bmatrix} \tilde{\Pi}_{11i} & 0 & 0 & 0 & 0 & \kappa_{3i}\frac{(I \otimes (\Phi + \Omega))^T}{2} \\ * & -Q & 0 & 0 & 0 & 0 \\ * & * & \kappa_{2i}\bar{B}^T\bar{B} & 0 & 0 & 0 \\ * & * & * & \sigma_2R - \kappa_{1i}I & 0 & 0 \\ * & * & * & * & -R - \kappa_{2i}I & 0 \\ * & * & * & * & * & -\kappa_{3i}I \end{bmatrix}.$$

By applying Lemma 1 to (12), we can deduce that $\tilde{\Pi}_i < 0$ ($i \in S$). For a sufficiently small scalar $\sigma_0 > 0$, one has

$$\mathbb{E}\{V(\eta(k+1), k+1, \theta(k+1))|\eta(k), \theta(k) = i\} - V(\eta(k), k, i) + \sigma_0\mathbb{E}\{\|\eta(k)\|^2\} \leq 0$$

According to the definition of $V(\eta(k), k, \theta(k))$, it is derived that

$$\begin{aligned}&\mathbb{E}\{V(\eta(k), k, \theta(k))\} \\ &\leq \left\{ \lambda_{\max}(P_i) + \lambda_{\max}(Q) \left[\bar{d}_1 + (1 - \lambda)\frac{(\bar{d}_1 - \underline{d}_1)(\bar{d}_1 + \underline{d}_1 - 1)}{2} \right] \right. \\ &\quad \left. + \lambda_{\max}(\bar{B}^TR\bar{B}) \left[\bar{d}_2 + (1 - \lambda)\frac{(\bar{d}_2 - \underline{d}_2)(\bar{d}_2 + \underline{d}_2 - 1)}{2} \right] \right\} \mathbb{E}\{\|\eta(k)\|^2\} = \rho_1\mathbb{E}\{\|\eta(k)\|^2\},\end{aligned}\tag{21}$$

and

$$\mathbb{E}\{V(\eta(0), 0, \theta(0))\} \leq \rho_1 \sup_{i \in \mathbb{Z}^-} \mathbb{E}\{\|\psi(i)\|^2\}. \quad (22)$$

For an arbitrary scalar $\mu > 1$, we can deduce that

$$\begin{aligned} & \mu^{k+1} \mathbb{E}\{V(\eta(k+1), k+1, \theta(k+1)) | \eta(k), \theta(k) = i\} - \mu^k V(\eta(k), k, i) \\ &= \mu^{k+1} \mathbb{E}\{V(\eta(k+1), k+1, \theta(k+1)) | \eta(k), \theta(k) = i\} - \mu^{k+1} V(\eta(k), k, i) \\ & \quad + \mu^k (\mu - 1) V(\eta(k), k, i) \\ & \leq \delta(\mu) \mu^k \mathbb{E}\{\|\eta(k)\|^2\}, \end{aligned} \quad (23)$$

with $\delta(\mu) = -\mu\sigma_0 + (\mu - 1)\rho_1$.

Summing the two sides of (23) from $k = 0$ to $k = N - 1$ leads to

$$\mu^N \mathbb{E}\{V(\eta(N), N, \theta(N))\} - \mathbb{E}\{V(\eta(0), 0, \theta(0))\} \leq \delta(\mu) \sum_{k=0}^{N-1} \mu^k \mathbb{E}\{\|\eta(k)\|^2\} \quad (24)$$

which is equivalent to

$$\mu^N \mathbb{E}\{V(\eta(N), N, \theta(N))\} \leq \mathbb{E}\{V(\eta(0), 0, \theta(0))\} + \delta(\mu) \sum_{k=0}^{N-1} \mu^k \mathbb{E}\{\|\eta(k)\|^2\} \quad (25)$$

Letting

$$\begin{aligned} \rho_0 = & \lambda_{\min}(P_i) + \lambda_{\min}(Q) \left[\underline{d}_1 + (1 - \underline{\lambda}) \frac{(\bar{d}_1 - \underline{d}_1)(\bar{d}_1 + \underline{d}_1 - 1)}{2} \right] \\ & + \lambda_{\min}(\bar{B}^T R \bar{B}) \left[\underline{d}_2 + (1 - \underline{\lambda}) \frac{(\bar{d}_2 - \underline{d}_2)(\bar{d}_2 + \underline{d}_2 - 1)}{2} \right], \end{aligned}$$

we have

$$\mathbb{E}\{V(\eta(N), N, \theta(N))\} \geq \rho_0 \mathbb{E}\{\|\eta(N)\|^2\}. \quad (26)$$

Furthermore, it is easy to prove that there exists $\mu_0 > 1$ so that $\delta(\mu_0) = -\mu_0\sigma_0 + (\mu_0 - 1)\rho_1 = 0$. On the basis of (22), (25) and (26), we have

$$\mathbb{E}\{\|\eta(N)\|^2\} \leq c_0 \left(\frac{1}{\mu_0}\right)^N \sup_{i \in \mathbb{Z}^-} \mathbb{E}\{\|\psi(i)\|^2\}, \quad c_0 = \frac{\rho_1}{\rho_0}, \quad (27)$$

which indicates the mean square exponential stability, and the proof is now complete. \blacksquare

Now we are in a position to deal with the design problem of the state estimator. The following result is derived from Theorem 1.

Theorem 2: There exists an asymptotic state estimator such that the augmented system (8) is asymptotically stable if there exist two sets of matrices $P_{1i} > 0$, X_i , positive constant scalars $\varphi_i (i \in S)$, two matrices $Q > 0$, $R > 0$, and positive constant scalars κ_{1i} , κ_{2i} , κ_{3i} satisfying

$$\begin{bmatrix} \tilde{\Pi}_{11i} & 0 & 0 & 0 & 0 & \kappa_{3i} \frac{(I \otimes (\Phi + \Omega))^T}{2} & \mathcal{A}_i^T \bar{P}_i - \mathcal{D}_i^T \mathcal{X}_i^T & 0 & \varphi_i \mathcal{D}_i^T \mathcal{E}_k^T \\ * & -Q & 0 & 0 & 0 & 0 & \mathcal{A}_{di}^T \bar{P}_i & 0 & 0 \\ * & * & \kappa_{2i} \bar{B}^T \bar{B} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \sigma_2 R - \kappa_{1i} I & 0 & 0 & \mathcal{W}_i^T \bar{P}_i & 0 & 0 \\ * & * & * & * & -R - \kappa_{2i} I & 0 & \mathcal{W}_{di}^T \bar{P}_i & 0 & 0 \\ * & * & * & * & * & -\kappa_{3i} I & -\mathcal{E}_i^T \mathcal{X}_i^T & 0 & \varphi_i \mathcal{E}_i^T \mathcal{E}_k^T \\ * & * & * & * & * & * & -\bar{P}_i & -\mathcal{X}_i \mathcal{H}_k & 0 \\ * & * & * & * & * & * & * & -\varphi_i I & 0 \\ * & * & * & * & * & * & * & * & -\varphi_i I \end{bmatrix} < 0 \quad (28)$$

where

$$\mathcal{M}_i = -\bar{P}_i \bar{\mathcal{K}}_i \mathcal{H}_k, \quad \mathcal{N}_i = \begin{bmatrix} \mathcal{E}_k \mathcal{D}_i & 0 & 0 & 0 & 0 & \mathcal{E}_k \mathcal{E}_i \end{bmatrix}, \quad \mathcal{X}_i = \text{diag}\{X_i, X_i\}.$$

Furthermore, the gain of the estimator is given by $K_i = P_{1_i}^{-1} X_i$ ($i \in S$).

Proof: From Theorem 1, it is derived that

$$\tilde{\Pi}_i = \begin{bmatrix} \Pi_i & S_i^T \bar{P}_i \\ * & -\bar{P}_i \end{bmatrix} < 0 \quad (29)$$

Moreover, split $\tilde{\Pi}_i$ as follows

$$\tilde{\Pi}_i = \bar{\Pi}_i + \Delta \Pi_i \quad (30)$$

with

$$\begin{aligned} \bar{\Pi}_i &= \begin{bmatrix} \Pi_i & \bar{S}_i^T \bar{P}_i \\ * & -\bar{P}_i \end{bmatrix}, \quad \Delta \Pi_i = \begin{bmatrix} 0 & \tilde{S}_i^T \bar{P}_i \\ * & 0 \end{bmatrix}, \quad \bar{S}_i = \begin{bmatrix} \mathcal{A}_i - \bar{\mathcal{K}}_i \mathcal{D}_i & \mathcal{A}_{di} & 0 & \mathcal{W}_i & \mathcal{W}_{di} & -\bar{\mathcal{K}}_i \mathcal{E}_i \end{bmatrix}, \\ \tilde{S}_i &= \begin{bmatrix} -\tilde{\mathcal{K}}_i \mathcal{D}_i & 0 & 0 & 0 & 0 & -\tilde{\mathcal{K}}_i \mathcal{E}_i \end{bmatrix}, \quad \bar{\mathcal{K}}_i = \text{diag}\{K_i, K_i\}, \quad \tilde{\mathcal{K}}_i = \text{diag}\{\Delta K_i, \Delta K_i\}, \\ \mathcal{H}_k &= \text{diag}\{H_k, H_k\}, \quad \mathcal{E}_k = \text{diag}\{E_k, E_k\}, \quad \mathcal{F}(k) = \text{diag}\{F(k), F(k)\}, \quad \tilde{\mathcal{K}}_i = \bar{\mathcal{K}}_i \mathcal{H}_k \mathcal{F}(k) \mathcal{E}_k. \end{aligned}$$

Noting that $\bar{P}_i \tilde{S}_i = \mathcal{M}_i \mathcal{F}(k) \mathcal{N}_i$ and denoting $\tilde{\mathcal{M}}_i = [0 \ \mathcal{M}_i^T]^T$, $\tilde{\mathcal{N}}_i = [\mathcal{N}_i \ 0]$, we have

$$\tilde{\Pi}_i = \bar{\Pi}_i + \tilde{\mathcal{M}}_i \mathcal{F}(k) \tilde{\mathcal{N}}_i + \tilde{\mathcal{N}}_i^T \mathcal{F}^T(k) \tilde{\mathcal{M}}_i^T < 0. \quad (31)$$

According to Lemma 2, (31) is equivalent to

$$\bar{\Pi}_i + \varphi_i^{-1} \tilde{\mathcal{M}}_i \tilde{\mathcal{M}}_i^T + \varphi_i^{-1} (\varphi_i \tilde{\mathcal{N}}_i)^T (\varphi_i \tilde{\mathcal{N}}_i) < 0. \quad (32)$$

Combine (30) and (32) with the usage of Lemma 1, we can see that the inequality (29) is feasible if

$$\begin{bmatrix} \Pi_i & \bar{S}_i^T \bar{P}_i & 0 & \varphi_i \mathcal{N}_i^T \\ * & -\bar{P}_i & \mathcal{M}_i & 0 \\ * & * & -\varphi_i I & 0 \\ * & * & * & -\varphi_i I \end{bmatrix} < 0 \quad (33)$$

holds. Letting $\mathcal{X}_i = \bar{P}_i \bar{\mathcal{K}}_i$ ($i \in S$), it is easy to see that (33) is equivalent to (28) and the proof is now complete. \blacksquare

Remark 2: In Theorem 2, sufficient conditions are presented that ensure the dynamic system to be asymptotically stable. It is noted that all the system parameters, the delay information, the bounds on the gain variations as well as the jumping transition probabilities are all reflected in the main results. It is shown that the feasibility of the non-fragile state estimator design problem can be readily checked by the solvability of inequality (28). In the next section, a numerical simulation will be utilized to verify the proposed method.

IV. NUMERICAL EXAMPLE

In this section, we present an example to demonstrate the approach addressed. Consider a three-neuron neural network (1)-(2) with the following parameters:

$$\begin{aligned}
d_1(1) &= 2, \quad d_1(2) = 6, \quad d_2(1) = 1, \quad d_2(2) = 5, \quad A(1) = \text{diag}\{0.4, 0.3, 0.3\}, \quad A(2) = \text{diag}\{0.5, 0.2, -0.3\}, \\
A_d(1) &= \text{diag}\{0.05, 0.01, 0.04\}, \quad A_d(2) = \text{diag}\{0.03, 0.02, -0.01\}, \quad B = \text{diag}\{0.2, 0.3, 0.1\}, \\
W(1) &= \begin{bmatrix} 0.2 & 0.2 & -0.1 \\ 0 & 0.4 & 0.3 \\ -0.3 & 0 & 0.2 \end{bmatrix}, \quad W(2) = \begin{bmatrix} 0 & -0.2 & 0.1 \\ 0.1 & 0.2 & 0.3 \\ 0.1 & -0.1 & 0.2 \end{bmatrix}, \quad W_d(1) = \begin{bmatrix} 0.2 & 0 & 0.1 \\ 0.1 & 0.2 & 0 \\ 0.1 & 0 & 0.1 \end{bmatrix}, \\
W_d(2) &= \begin{bmatrix} 0.3 & 0.1 & -0.1 \\ 0.1 & 0.2 & 0 \\ 0.1 & 0.2 & 0.2 \end{bmatrix}, \quad D(1) = \begin{bmatrix} 1 & 0.8 & 0.7 \\ 0.5 & -0.7 & 0.9 \end{bmatrix}, \quad D(2) = \begin{bmatrix} -0.6 & 0.9 & 0.6 \\ 0.8 & 1 & 0.8 \end{bmatrix}, \\
E(1) &= \begin{bmatrix} 0.4 & 0.2 & 0.1 \\ 0 & 0.3 & 0.4 \end{bmatrix}, \quad E(2) = \begin{bmatrix} 0.2 & 0 & 0.1 \\ 0.4 & 0.4 & 0.3 \end{bmatrix}, \quad H_k = \text{diag}\{1, 1\}, \quad E_k = \begin{bmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{bmatrix}, \\
\Phi &= \begin{bmatrix} -0.29 & 0.29 \\ 0 & 0.6 \end{bmatrix}, \quad \Omega = \begin{bmatrix} -0.6 & 0.29 \\ 0 & 0.4 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}.
\end{aligned}$$

Take the activation functions as follows:

$$g_1(x_1(k)) = -0.2 \tanh(x_1(k)), \quad g_2(x_2(k)) = 0.3 \tanh(x_2(k)), \quad g_3(x_3(k)) = 0.1 \tanh(x_3(k)).$$

in which $x_l(k)$ ($l = 1, 2, 3$) represents the l -th element of the system state $x(k)$. Meanwhile, the nonlinear vector-valued function $h(x(k))$ is chosen as

$$h(x(k)) = \begin{bmatrix} -0.6x_1(k) + \tanh(0.3x_2(k)) + 0.3x_3(k) \\ 0.6x_2(k) - \tanh(0.2x_3(k)) \\ \tanh(0.3x_1(k)) + 0.3x_2(k) \end{bmatrix},$$

With the above parameters, by using the Matlab LMI Toolbox, we solve the LMIs (28) and obtain the feasible solution as follows:

$$\begin{aligned}
P_{1_1} &= \begin{bmatrix} 2.2901 & 0.2480 & 0.3279 \\ 0.2480 & 2.1991 & 0.4568 \\ 0.3279 & 0.4568 & 0.7215 \end{bmatrix}, \quad P_{1_2} = \begin{bmatrix} 0.5058 & 0.6416 & -0.1061 \\ 0.6416 & 3.4805 & 0.5923 \\ -0.1061 & 0.5923 & 2.5076 \end{bmatrix}, \\
Q &= \begin{bmatrix} 0.3130 & 0.8064 & 0.1840 & -0.0020 & -0.0003 & -0.0010 \\ 0.8064 & 4.1512 & 0.8091 & -0.0040 & -0.0018 & -0.0012 \\ 0.1840 & 0.8091 & 1.2084 & 0.0355 & 0.0115 & 0.0075 \\ -0.0020 & -0.0040 & 0.0355 & 0.3631 & 0.5922 & 0.1459 \\ -0.0003 & -0.0018 & 0.0115 & 0.5922 & 4.1258 & 1.1988 \\ -0.0010 & -0.0012 & 0.0075 & 0.1459 & 1.1988 & 0.7853 \end{bmatrix},
\end{aligned}$$

$$R = \begin{bmatrix} 1.1090 & 0.1122 & -0.1198 & -0.0014 & -0.0007 & -0.0012 \\ 0.1122 & 1.0793 & 0.1618 & 0.0024 & 0.0029 & 0.0012 \\ -0.1198 & 0.1618 & 0.6710 & -0.0006 & 0.0004 & -0.0009 \\ -0.0014 & 0.0024 & -0.0006 & 1.0969 & 0.0705 & -0.0692 \\ -0.0007 & 0.0029 & 0.0004 & 0.0705 & 1.0937 & 0.0892 \\ -0.0012 & 0.0012 & -0.0009 & -0.0692 & 0.0892 & 0.8489 \end{bmatrix},$$

$$X_1 = \begin{bmatrix} 3.5552 & 5.4237 \\ 3.1330 & 3.3232 \\ 3.7760 & 4.4454 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -0.1555 & 0.5270 \\ -0.1802 & 0.7601 \\ -1.2481 & -0.3058 \end{bmatrix},$$

$$\varphi_1 = 1.9261, \quad \varphi_2 = 4.6476, \quad \kappa_{11} = 4.7167, \quad \kappa_{21} = 7.4847, \quad \kappa_{31} = 5.4295,$$

$$\kappa_{12} = 4.1119, \quad \kappa_{22} = 7.5078, \quad \kappa_{32} = 2.7628.$$

It follows from Theorem 2 that (5) becomes an asymptotic state estimator of the neural network (1)-(2) with the given parameters, and the estimator gain matrices are calculated as

$$K_1 = \begin{bmatrix} 0.0085 & 0.0159 \\ 0.0037 & 0.0023 \\ 0.0461 & 0.0529 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.0628 & 0.0945 \\ 0.0160 & 0.0061 \\ -0.0562 & -0.0096 \end{bmatrix}.$$

Next, a simulation is given to further verify the stability of neural network (1)-(2) and the performance of the estimator (5). Figure 1 represents the switching of the system modes. Figure 2 depicts the estimation error with the initial condition $x(k) = [0.26 \ -0.2 \ 0.1]^T$ ($k \in [-5, 0]$). Figure 3 - Figure 5 show the states of the neural network $x_1(k)$, $x_2(k)$, $x_3(k)$ and the states of the estimator $\hat{x}_1(k)$, $\hat{x}_2(k)$, $\hat{x}_3(k)$, respectively, from which we can see that the non-fragile estimator is effective and has a relatively good estimation performance.

V. CONCLUSIONS

In this paper, the non-fragile state estimation problem has been investigated for a class of discrete-time neural networks with Markovian parameters and mode-dependent time-delays. Considering that the dynamics of the estimation error needs to be globally stable in the mean square, an asymptotic state estimator has been derived to estimate the neuron states with available output measurements. With the Lyapunov-Krasovskii functional, the LMI based sufficient conditions have been established to ensure the existence of the asymptotic state estimator. The gain of the estimator has been designed in terms of the solution to an LMI. A simulation example has been utilized to represent the effectiveness of the derived method. It should be pointed out that the main results **shown** in this paper can be extended to the filter design and the control applications for other discrete-time delayed systems (e.g. genetic regulatory networks). **In addition**, the methods here could be further employed to the non-fragile state estimation problems for discrete neural networks with more complicated network-induced phenomena such as fading measurements [26], [29], [31], [32], randomly occurring faults [27] **and randomly occurring incomplete measurements** [6], [7], [17], [18], [28], [30], [35], [36], [38].

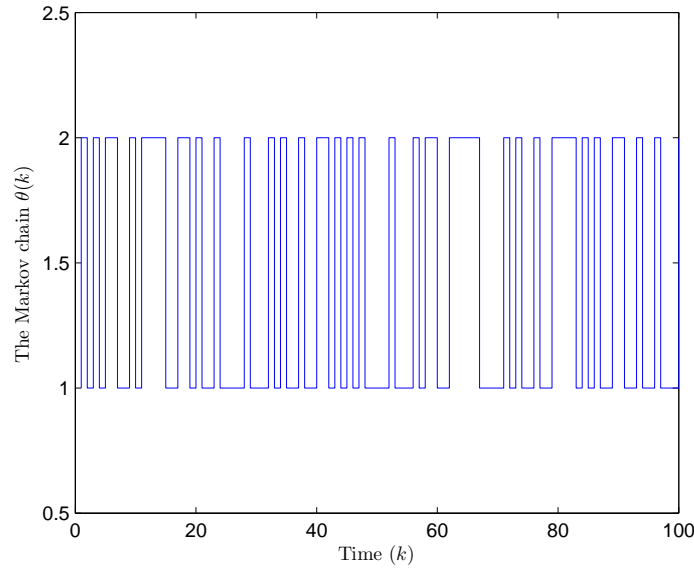


Fig. 1: Modes evolution

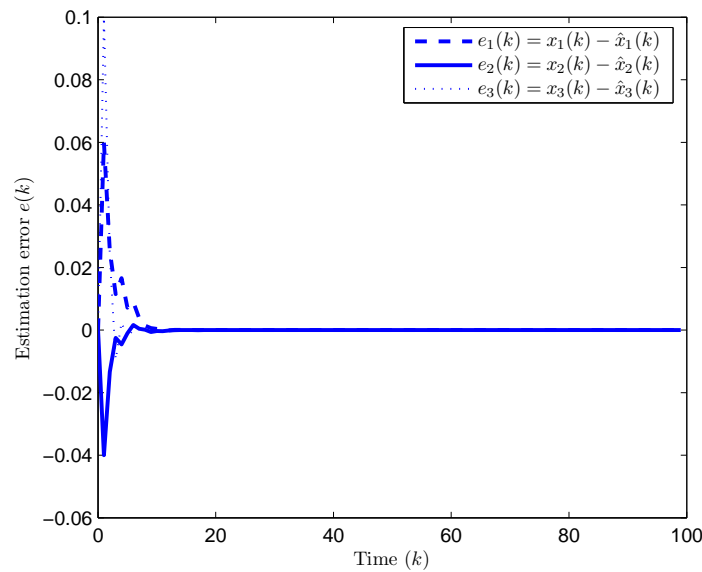


Fig. 2: Estimation errors

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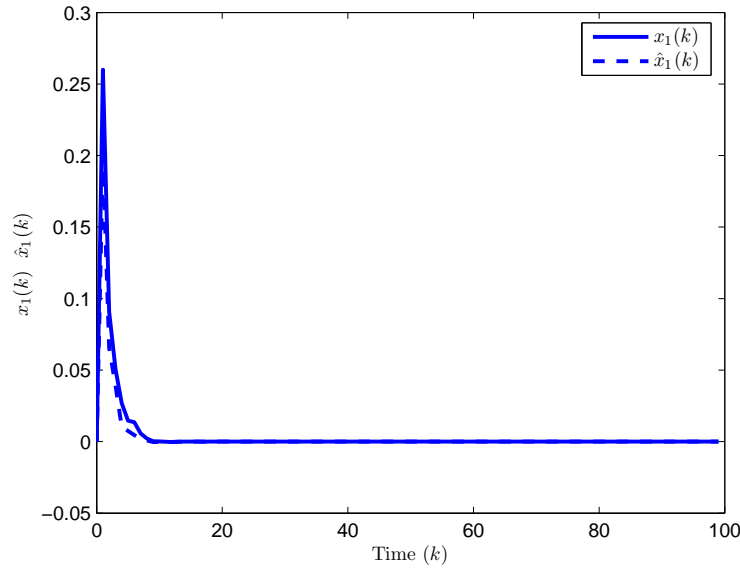


Fig. 3: $x_1(k)$ and its estimate $\hat{x}_1(k)$

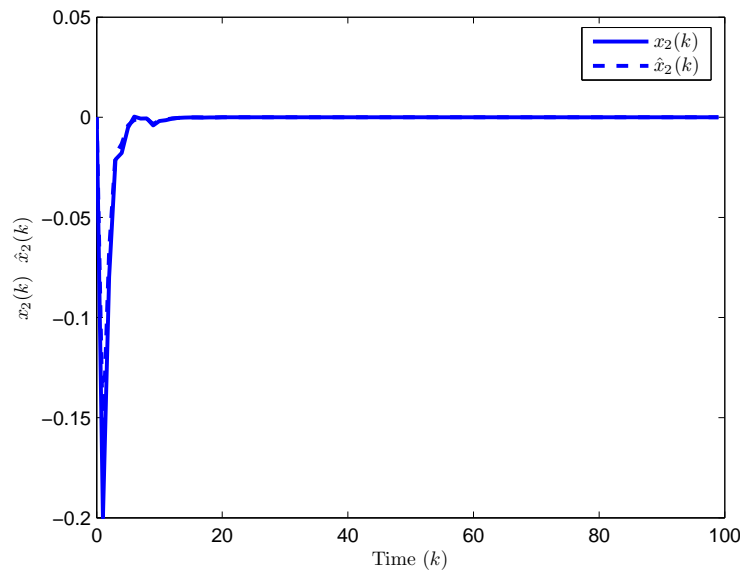


Fig. 4: $x_2(k)$ and its estimate $\hat{x}_2(k)$

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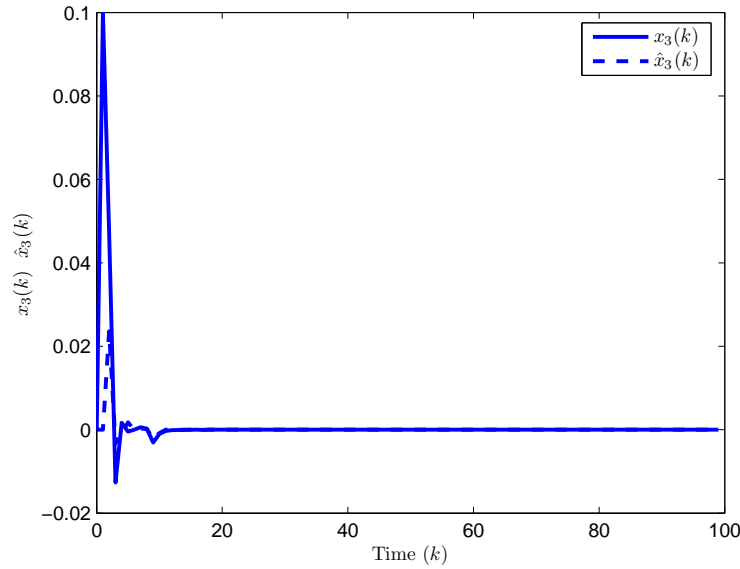
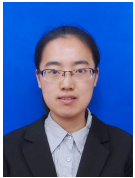


Fig. 5: $x_3(k)$ and its estimate $\hat{x}_3(k)$

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Nan Hou received the M.Eng. degree in Automation in 2015 from Northeast Petroleum University, Daqing, China. She is currently studying for the Ph.D. degree in Petroleum and Natural Gas Engineering at Northeast Petroleum University, Daqing, China. Her research interests include networked control systems, oil gas information and control engineering.



Hongli Dong received the Ph.D. degree in Control Science and Engineering in 2012 from Harbin Institute of Technology, Harbin, China. From July 2009 to January 2010, she was a Research Assistant in the Department of Applied Mathematics, the City University of Hong Kong. From October 2010 to January 2011, she was a Research Assistant in the Department of Mechanical Engineering, the University of Hong Kong. From January 2011 to January 2012, she was a Visiting Scholar in the Department of Information Systems and Computing, Brunel University, London, U.K. From November 2012 to October 2014, she was an Alexander von Humboldt research fellow at the University of Duisburg-Essen, Duisburg, Germany. She is currently a professor with the College of Electrical and Information

Engineering, Northeast Petroleum University, Daqing, China.

Dr. Dong's current research interests include robust control and networked control systems. She is a very active reviewer for many international journals.



Zidong Wang was born in Jiangsu, China, in 1966. He received the B.Sc. degree in mathematics in 1986 from Suzhou University, Suzhou, China, and the M.Sc. degree in applied mathematics in 1990 and the Ph.D. degree in electrical engineering in 1994, both from Nanjing University of Science and Technology, Nanjing, China.

He is currently Professor of Dynamical Systems and Computing in the Department of Computer Science, Brunel University London, U.K. From 1990 to 2002, he held teaching and research appointments in universities in China, Germany and the UK. Prof. Wang's research interests include dynamical systems, signal processing, bioinformatics, control theory and applications. He has published more than 300 papers in refereed international journals. He is a holder of the Alexander von Humboldt Research Fellowship of Germany, the JSPS Research Fellowship of Japan, William Mong Visiting Research Fellowship of Hong Kong.

Prof. Wang is a Fellow of the IEEE. He is serving or has served as an Associate Editor for 12 international journals, including IEEE Transactions on Automatic Control, IEEE Transactions on Control Systems Technology, IEEE Transactions on Neural Networks, IEEE Transactions on Signal Processing, and IEEE Transactions on Systems, Man, and Cybernetics - Systems. He is also a Fellow of the Royal Statistical Society and a member of program committee for many international conferences.



Weijian Ren received the M.S. degree in Control Theory and Control Engineering in 1990 and Ph.D. degree in Oil Storage and Transport Engineering in 2006, both from Northeast Petroleum University, Daqing, China. She is currently a professor at Northeast Petroleum University, Daqing, China. Her research interests include modelling and control of complex systems, fault diagnosis and simulation.



Fuad E. Alsaadi received the B.S. and M.Sc. degrees in electronic and communication from King AbdulAziz University, Jeddah, Saudi Arabia, in 1996 and 2002. He then received the Ph.D. degree in Optical Wireless Communication Systems from the University of Leeds, Leeds, UK, in 2011. Between 1996 and 2005, he worked in Jeddah as a communication instructor in the College of Electronics and Communication. He is currently an assistant professor of the Electrical and Computer Engineering Department within the Faculty of Engineering, King Abdulaziz University, Jeddah, Saudi Arabia. He published widely in the top IEEE communications conferences and journals and has received the Carter award, University of Leeds for the best PhD. He has research interests in optical systems and networks, signal processing, synchronization and systems design.