Hardware-efficient frequency offset and phase noise mitigation in coherent optical quadrature amplitude modulation systems

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A hardware-efficient adaptive algorithm for frequency offset and phase noise mitigation in coherent optical quadrature amplitude modulation systems is presented and analysed. Hardware efficiency for the mitigation of imperfections is achieved by computing directly the complex exponential, thus avoiding a CORDIC processor in the phase recovery loop. The design includes a square-root- and trigonometry-free update. Simulation results substantiate the theoretical findings.

Problem formulation: Consider an equalised and time-synchronised coherent optical quadrature amplitude modulation (CO-QAM) system. This Letter focuses on mitigation of laser phase noise (PN) and frequency offset (FO). Let

$$y_k = x_k \exp\left(i(\theta_k + k\Omega)\right) + w_k \tag{1}$$

denote the baud-rate samples of the output of the matched filter at the receiver, where $i = \sqrt{-1}$, k is the discrete index denoting the time kT, T is the symbol period, θ_k denotes the PN, x_k is an independent and identically distributed (i.i.d.) sequence, $\Omega = 2\pi T\Delta f$ represents normalised FO (where Δf is the frequency difference between the transmitter laser and the local oscillator (LO)), and w_k is a symmetric i.i.d. zero-mean Gaussian additive noise with independent real and imaginary parts. The PN, θ_k, follows a random walk model

$$\theta_k = \theta_{k-1} + q_k \tag{2}$$

where q_k is a zero-mean Gaussian distributed i.i.d. sequence with variance $\sigma_q^2 = 2\pi T\Delta\nu$, and $\Delta\nu$ is the combined linewidths of the transmitter laser and the LO [1]. Let $\widehat{\theta}_k$ be an estimate of θ_k available at the receiver, then an estimate of the signal x_k is given by $z_k = y_k \exp(-i\widehat{\theta}_k)$. One of the admissible methods for finding $\widehat{\theta}_k$ is to minimise the fourth-order statistics of the derotated sequence $\{z_k\}$ with respect to $\widehat{\theta}_k$ [2, 3]. The fourth-power cost is $J_4(z_k) = E[z_{k,\mathrm{R}}^4 + z_{k,\mathrm{I}}^4]$, which results in a globally convergent update

$$\widehat{\theta}_{k+1} = \widehat{\theta}_k - \mu(z_{k,\mathrm{R}}^2 - z_{k,\mathrm{I}}^2)z_{k,\mathrm{I}}z_{k,\mathrm{R}} \tag{3}$$

where μ is the loop gain (LG), and $z_{k,\mathrm{R}}$ and $z_{k,\mathrm{I}}$ are the in-phase and quadrature components of z_k, respectively. Once $\widehat{\theta}_k$ is known, the traditional way is to obtain the complex exponential $\exp(i\widehat{\theta}_k)$ using a CORDIC processor [4], which computes the required trigonometric quantities, i.e. $\cos(\widehat{\theta}_k)$ and $\sin(\widehat{\theta}_k)$. Note that a single-axis variant of (3) has been recently used in the vestigial sideband [5] and QAM-based CO-OFDM systems for adaptive PN mitigation [6].

In this Letter, we suggest minimising the cost function $J_4(z_k)$ with respect to $r_k := \exp(i\widehat{\theta}_k)$ to obtain a complex-valued update as follows:

$$r_{k+1} = r_k - \mu(z_{k,\mathrm{R}}^3 - i\, z_{k,\mathrm{I}}^3)y_k \tag{4}$$

where $z_k = r_k^* y_k$. Since $r_k := \exp(i\widehat{\theta}_k)$ is readily available, it avoids trigonometric operations. Note that update (4) requires normalisation to unit amplitude which may lead to needing a CORDIC processor to compute the required square-root. To avoid this, in this Letter r_{k+1} is normalised approximately by projecting it onto a unit polygon:

$$1r_{k+1} = \mathrm{proj}(r_k - \mu(z_{k,\mathrm{R}}^3 - iz_{k,\mathrm{I}}^3)y_k) \tag{5a}$$

$$\mathrm{proj}(g) := \begin{cases} g/(p_1|\Re[g]| + p_2|\Im[g]|), & \text{if } |\Re[g]| > |\Im[g]|, \\ g/(p_2|\Re[g]| + p_1|\Im[g]|), & \text{otherwise} \end{cases} \tag{5b}$$

where $\mathrm{proj}(\cdot)$ is performing a low-complexity normalisation, $\Re$ and $\Im$ denote real and imaginary operations, respectively, and the choice of $(p_1, p_2) = (15/16,\ 15/32)$ ensures a good polygonal fit (see Fig. 1).

Performance analysis: Below, we first analyse a generic synchroniser

$$\widehat{\theta}_{k+1} = \widehat{\theta}_k + \mu\mathcal{E}_k, \quad \text{where } \mathcal{E}_k = f(z_k) \tag{6}$$

and then extend the results to the update (5). Let $\psi_k = \theta_k + k\Omega - \widehat{\theta}_k$ be the residual, we obtain $z_k = y_k \exp(-i\widehat{\theta}_k) = x_k \exp(i\psi_k) + e_k$, where e_k has similar statistical properties as that of additive noise w_k.

Theorem 1: Consider any adaptive synchroniser of the form (6) and assume filter operation in steady state with PN but no FO. Assume further that $a_k = y_k \exp(-i\widehat{\theta}_{k,\mathrm{opt}})$, where $\widehat{\theta}_{k,\mathrm{opt}}$ (an optimal estimate of θ_k) varies according to the random-walk model $\widehat{\theta}_{k+1,\mathrm{opt}} = \widehat{\theta}_{k,\mathrm{opt}} + q_k$, and q_k (with the same statistics as defined in (2)) is independent of $\{a_m\}$ and $\{y_m, \widehat{\theta}_{0,\mathrm{opt}}\}$ for all $m < k$. Denoting $\psi_k = \widehat{\theta}_{k,\mathrm{opt}} - \widehat{\theta}_k$, the following variance relation holds:

$$2E\psi_k\mathcal{E}_k = \mu E\mathcal{E}_k^2 + \mu^{-1}\sigma_q^2 \tag{7}$$

Proof: See [7] for the energy conservation-based variance relation for an adaptive filter operating in a non-stationary environment.

Next, we mention the proposition of Moridi and Sari [8] in relation to the statistics involved in Theorem 1. □

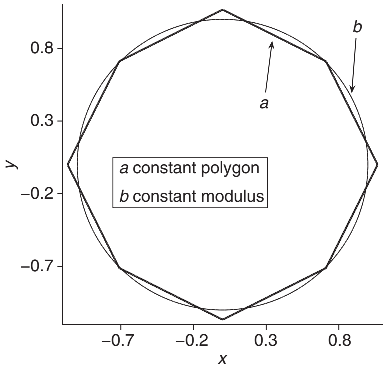

Fig. 1 *Polygon fit to constant modulus [9]. On xy-plain, the unit-radius polygon is expressed as (15/16) max(|x|, |y|) + (15/32) min(|x|, |y|) = 1*

Proposition 1: Consider any adaptive synchroniser of the form (6) and assume a filter operation in steady state, it is true that $E\psi_k\mathcal{E}_k = \alpha E\psi_k^2$, and $E\mathcal{E}_k^2 = \beta E\psi_k^2 + \gamma$, with appropriate α, β, and γ. The steady-state phase jitter variance (PJV) is $E\psi_\infty^2 = \mu\gamma/(2\alpha - \mu\beta)$, and the convergence speed is characterised by the time constant $-\log[1 - \mu(2\alpha - \mu\beta)]$.

By combining Theorem 1 and Proposition 1, we have an optimal LG for the mitigation of PN, as summarised in the following theorem.

Theorem 2: Considering the variance relation (7) and Proposition 1, we have

$$E\psi_\infty^2 = \frac{\mu\gamma + \mu^{-1}\sigma_q^2}{2\alpha - \mu\beta} \tag{8}$$

The optimal LG that tracks the PN (2) with minimum PJV

$$(E\psi_\infty^2)_{\min} \approx \frac{2\sigma_q\gamma}{2\alpha\sqrt{\gamma} - \sigma_q\beta} \approx \frac{\sqrt{\gamma}\sigma_q}{\alpha} \tag{9}$$

is given by

$$\mu_* = \frac{\sqrt{4\gamma\alpha^2\sigma_q^2 + \beta^2\sigma_q^4} - \beta\sigma_q^2}{2\alpha\gamma} \approx \frac{\sigma_q}{\sqrt{\gamma}} \tag{10}$$

where approximations are based on the assumption that $4\gamma\alpha^2 \gg \beta^2\sigma_q^2$.

Proof is simple and thus skipped. Note from Theorem 2 that both PJV and LG are proportional to the standard deviation of the PN. The larger the deviation, the larger the LG to track. Next, we state a generic theorem for the evaluation of optimal LG for the system exhibiting a fixed FO.

Theorem 3: Consider any adaptive synchroniser of the form (6) with residual $\psi_k = \bar{\theta} + k\Omega - \widehat{\theta}_k$ (where $\bar{\theta}$ denotes a possible fixed phase offset), and assume a filter operation in steady state. The PJV is

$$E\psi_\infty^2 = \mu\gamma/(2\alpha - \mu\beta) + (E\psi_\infty)^2, \quad (E\psi_\infty = \Omega/(\mu\alpha)) \tag{11}$$

$E\psi_\infty$ is the tracking error and the optimal LG that minimises (11) is

$$\mu_* = \frac{\Omega^4\beta^4}{9\alpha^6\gamma^2\sqrt[3]{\delta}} - \frac{4\Omega^2\beta}{3\alpha^2\gamma\sqrt[3]{\delta}} + \sqrt[3]{\delta} + \frac{\Omega^2\beta^2}{3\alpha^3\gamma} \tag{12}$$

where

$$\delta = \sqrt{\left(\frac{2\Omega^2}{\alpha\gamma} - \frac{2\Omega^4\beta^3}{3\alpha^5\gamma^2} + \frac{\Omega^6\beta^6}{27\alpha^9\gamma^3}\right)^2 - \left(\frac{\Omega^4\beta^4}{9\alpha^6\gamma^2} - \frac{4\Omega^2\beta}{3\alpha^2\gamma}\right)^3} + \frac{2\Omega^2}{\alpha\gamma} - \frac{2\Omega^4\beta^3}{3\alpha^5\gamma^2} + \frac{\Omega^6\beta^6}{27\alpha^9\gamma^3}.$$

To mitigate both PN and FO, we have the following theorem.

Theorem 4: Consider any adaptive synchroniser of the form (6) with residual $\psi_k = \theta_k + k\Omega - \widehat{\theta}_k$, and assume a filter operation in steady state. The PJV is

$$E\psi_\infty^2 = \frac{\mu\gamma + \mu^{-1}\sigma_q^2}{2\alpha - \mu\beta} + \frac{\Omega^2}{\mu^2\alpha^2} \tag{13}$$

minimisation of which yields an optimal LG

$$\mu_* = \frac{\zeta^2}{\sqrt[3]{\delta}} - \frac{\eta}{\sqrt[3]{\delta}} + \sqrt[3]{\delta} + \zeta \tag{14}$$

where

$$\delta = \zeta^3 + \frac{2\Omega^2}{\alpha\gamma} - \frac{3}{2}\zeta\eta + \sqrt{\left(\zeta^3 + \frac{2\Omega^2}{\alpha\gamma} - \frac{3}{2}\zeta\eta\right)^2 - (\zeta^2 - \eta)^3},$$

$$\zeta = \frac{\Omega^2\beta^2 - \alpha^2\beta\sigma_q^2}{3\alpha^3\gamma} \quad \text{and} \quad \eta = \frac{4\Omega^2\alpha\beta - \alpha^3\sigma_q^2}{3\alpha^3\gamma}.$$

Finally, for the update (5), we have the following result.

Proposition 2: Considering the proposed synchroniser (5), and assuming $|r_k| \approx 1$, $r_k^* \exp(i\theta_k) = \exp(i\psi_k) \approx 1 + i\psi_k$, and a filter operation in steady state, it is true that $\alpha \approx E[6x_I^2x_R^2 - x_R^4 - x_I^4]$, $\beta \approx E[2x_R^8 + 70x_I^4x_R^4 - 56x_I^6x_R^2]$, and $\gamma \approx E[2x_I^2x_R^6 - 2x_I^4x_R^4 + 18\sigma_e^4x_R^4 + 12\sigma_e^8 + 2\sigma_e^2 x_R^6 + 48\sigma_e^6x_R^2 + 6\sigma_e^2x_I^2x_R^4 + 18\sigma_e^4x_I^2x_R^2]$.

Proof: On the basis of the assumption $|r_k| \approx 1$, we write (below $y_k = r_kz_k$)

$$\begin{aligned} r_{k+1} &= r_k - \mu(z_{k,R}^3 - i\,z_{k,I}^3)y_k = r_k(1 - \mu(z_{k,R}^3 - iz_{k,I}^3)z_k) \\ &= r_k(1 - \mu(z_{k,R}^4 + z_{k,I}^4 + i(z_{k,R}^2 - z_{k,I}^2)z_{k,R}z_{k,I})) \end{aligned}$$

Taking conjugate and multiplying with exp $(i\theta_k)$, we get $\exp(i\psi_{k+1}) \approx \exp(i\psi_k)(1 - \mu(z_{k,R}^4 + z_{k,I}^4 - i(z_{k,R}^2 - z_{k,I}^2)z_{k,R}z_{k,I}))$. Assuming $\exp(i\psi_k) \approx 1 + i\psi_k$, and comparing the coefficients, we get a real-valued update $\psi_{k+1} \approx \psi_k(1 - \mu(z_{k,R}^4 + z_{k,I}^4)) - \mu(z_{k,I}^2 - z_{k,R}^2)\, z_{k,R}z_{k,I}$. Further assuming $\mu(z_{k,R}^4 + z_{k,I}^4) \ll 1$, we obtain $\psi_{k+1} \approx \psi_k - \mu(z_{k,I}^2 - z_{k,R}^2)z_{k,R}z_{k,I}$. With $z_{k,R} = x_{k,R}\cos(\psi_k) - x_{k,I}\sin(\psi_k) + e_{k,R}$, $z_{k,I} = x_{k,I}\cos(\psi_k) + x_{k,R}\sin(\psi_k) + e_{k,I}$, and denoting $\mathcal{E}_k^2 := (z_{k,I}^2 - z_{k,R}^2)z_{k,R}z_{k,I}$, an exact expression for $E[\mathcal{E}_k^2|\psi_k]$ is evaluated but the result is too long to be included here. Exploiting Taylor's series-based simplifications $\sin(\psi) \approx \psi$ and $\cos(\psi) \approx 1 - 0.5\psi^2$, and statistics $Ee_R^2 = \sigma_e^2$, $Ee_R^4 = 3\sigma_e^4$, and $Ee_R^6 = 15\sigma_e^6$ for the quadrature components of additive Gaussian noise e_k, however, we obtain $E[\mathcal{E}_k^2|\psi_k] \approx (2Ex_R^8 + 70Ex_I^4x_R^4 - 56Ex_I^6x_R^2)\psi_k^2 + 12\sigma_e^8 + 2Ex_I^2x_R^6 - 2Ex_I^4 x_R^4 + 18\sigma_e^4Ex_E^4 + 2\sigma_e^2Ex_R^6 + 48\sigma_e^6Ex_R^2 + 6\sigma_e^2Ex_I^2x_R^4 + 18\sigma_e^4Ex_I^2x_R^2$. Similarly, we obtain $E[\mathcal{E}_k|\psi_k] \approx \psi_kE\,[6x_I^2x_R^2 - x_R^4 - x_I^4]$. □

Owing to our analytical findings, we can evaluate the steady-state performance of the proposed update (5) in the presence of PN or FO or both for any given LG, QAM size, and SNR values. Since the analysis assumes an open-eye condition, therefore, we are not considering other optical interferences like polarisation mode and chromatic dispersions.

Simulation results: We consider a 2.5 Gbit/s, Gray-encoded, 16-QAM signalling (i.e. two uncorrelated 625 Mbaud quaternary data streams) under four different simulation scenarios as depicted in the legend of Fig. 2, compute the simulated PJV (for one of the polarisations), and compare it with analytical PJV. We also consider differential encoding to resolve the four-fold phase ambiguity. As the proposed update attempts to calculate directly the complex exponential, the steady-state value of PJV (for the given LG, μ) is estimated as

$$\begin{aligned} E\psi_\infty^2(\mu) &= \lim_{k\to\infty} E(\theta_k + k\Omega - \widehat{\theta}_k)^2 \\ &\approx \frac{1}{c}\sum_{j=1}^{N_{\text{runs}}}\sum_{k=1}^{N_{\text{iter}}} \frac{(\sin(\theta_{k,j} + k\Omega) - \Im[r_{k,j}])^2}{N_{\text{runs}}N_{\text{iter}}} \end{aligned} \tag{15}$$

where $c = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\cos(t)^2\,dt = (2+\pi)/(2\pi) = 0.8183$ is obtained by exploiting the mean-value theorem and Lipschitz continuity of $\sin(t)$, i.e. by realising the fact that $E(t_1 - t_2)^2 \geq (1/c)E(\sin(t_1) - \sin(t_2))^2$, for t_1, $t_2 \in [-\pi/4, \pi/4)$, and $c > 0$. Moreover, $\theta_{k,j}$ and $r_{k,j}$, respectively, denote the realisation of PN and the estimated complex-exponential in the kth iteration of the jth run, where $N_{\text{runs}} = 500$ and $N_{\text{iter}} = 5000$. Each simulation point is thus obtained as an average of N_{runs} runs each having N_{iter} iterations with independent and random realisation of PN, additive noise, and signal points. Our analytical findings are in close agreement with the simulation results, as seen in Fig. 2.

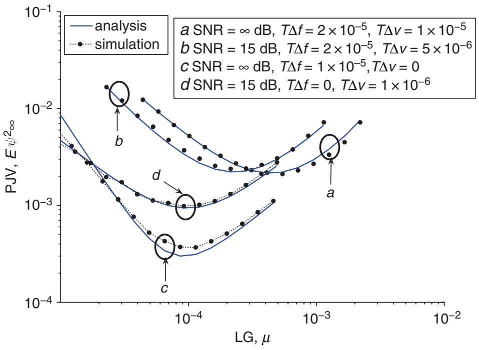

Fig. 2 *Plots of PJV against LG for 16-QAM signalling*

Conclusion: A hardware-efficient adaptive synchroniser for CO-QAM systems is proposed for the first time, is analysed and demonstrated to mitigate laser PN and FO. Compared with the traditional synchroniser, the proposed update computes the complex exponential directly without requiring any trigonometric and square-root operations. Results for the PJV are obtained for a wide range of LG values under different simulation scenarios, where the analytically obtained optimal LGs are found to be in close agreement with the simulated ones. Our proposed technique is proved as a substitute for the traditionally used ones.

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One or more of the Figures in this Letter are available in colour online.

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