A Note on the Path Formulation for $(\mathbb{O}(2), \mathbb{SO}(2))$-Forced Symmetry Breaking Formulation

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1 Introduction

In this note we deal with the liftability of vector fields over the projection onto the parameter space for $\mathbb{SO}(2)$-equivariant universal unfoldings (under contact equivalence) of $\mathbb{O}(2)$-equivariant problems of corank 2 and the application of those results to the path formulation for bifurcation problems with the forced symmetry breaking from $\mathbb{O}(2)$ to $\mathbb{SO}(2)$. In the non equivariant case it is well known that vector fields are liftable over the projection if and only if they are tangent to its discriminant and that they form a free module (cf. [14]). In [7] we discussed equivariant cases under finite groups where the two modules are different, although both are free. Here we have continuous Lie groups, with a simple action, but both modules are again equal and free.

Forced symmetry breaking occurs when the symmetry of the equations changes when a parameter is varied. In previous works we studied forced symmetry breaking in bifurcation equation from $\mathbb{O}(2)$ to some of its subgroups using a modification of the standard theory of [13] (no symmetry at all in [6], $\mathbb{D}_n$ in [5] and $\mathbb{SO}(2)$ in [8]). We use this last example to illustrate the use of path formulation to study bifurcation problems in the case of continuous Lie group actions. Bifurcation diagrams are identified with paths in the parameter space of the $\mathbb{SO}(2)$-universal unfolding F_0 of the cores $f_0(z) = f(z, 0)$. Equivalence between paths is given by diffeomorphisms preserving the discriminant Δ^{F_0} of the projection π of $F_0^{-1}(0)$ onto the unfolding parameter space of F_0. Without symmetry, the tangent space of the group of those diffeomorphisms is the module of liftable vector fields. In our case this module corresponds to the whole module of vector fields tangent to the discriminant so the group

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of liftable diffeomorphisms is the whole group of liftable diffeomorphisms preserving the discriminant.

In [10] we have shown how path formulation follows directly from the algebra of the parametrised contact equivalence theory, here in its version for forced symmetry breaking of [8]. This is the object of Chapter 2. Finally, we show in Chapter 3 that we get the module of liftable vector fields directly from the geometry of the situation. To exploit fully methods from algebraic geometry we complexify our situation and work in the holomorphic realm. As long as we consider real germs of finite codimension (hence finitely determined) our results are valid for real germs viewed as real slices of the complex objects.

1.1 Notation

We consider the actions of $\mathbb{C}^*$ and $\mathbb{Z}_2$ on the complex plane given by $t : (z_1, z_2) \mapsto (tz_1, t^{-1}z_2)$ and $\chi : (z_1, z_2) \mapsto (z_2, z_1)$. Together they form an action of the semi-direct product $\mathbb{O}^* = \mathbb{Z}_2 \ltimes \mathbb{C}^*$. This is the complexification of the usual orthogonal action of $\mathbb{O}(2)$ on $\mathbb{R}^2 \sim \mathbb{C}$ by setting $z_1 = z$ and $z_2 = \bar{z}$. We denote by $\mathrm{M}_2(\mathbb{C})$ the set of 2×2-matrices with complex coefficients and by $\mathrm{GL}_2(\mathbb{C})$ the subset of invertible 2×2-matrices. The identity in $\mathrm{GL}_2(\mathbb{C})$ is denoted by I_2.

The derivatives are denoted by subscripts, f_z for $\frac{\partial f}{\partial z}$ ..., and the superscript ^{o} denotes the value of any function at the origin, $f^o = f(0)$, $f^o_z = f_z(0)$ .... When clear from the context we still use z for $(z_1, z_2) \in \mathbb{C}^2$. For any variable, or set of variables, $a \in \mathbb{C}^n$, we denote by $\mathcal{O}_a$ the ring of analytic germs $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ and by $\mathcal{M}_a$ its maximal ideal. For $b \in \mathbb{C}^m$, let $\vec{\mathcal{O}}_{a,b}$ denote the $\mathcal{O}_a$-module of analytic germs $g : (\mathbb{C}^n, 0) \to \mathbb{C}^m$, and $\vec{\mathcal{M}}_{a,b}$ its submodule of germs vanishing at the origin. When b is clear from the context, we denote $\vec{\mathcal{O}}_{a,b}$ by $\vec{\mathcal{O}}_a$ and $\vec{\mathcal{M}}_{a,b}$ by $\vec{\mathcal{M}}_a$. In the real case we denote by $\mathcal{E}_a$ and by $\vec{\mathcal{E}}_{a,b}$ the corresponding ring and modules of smooth germs. We use the superscript Γ to indicate the rings of Γ-invariant or the module of Γ-equivariant germs. Let R be a ring, we denote by $< m_1 \ldots m_k >_R$ the R-module generated by the m_i's over R.

2 Path Formulation for $(\mathbb{O}(2), \mathbb{SO}(2))$-Forced Symmetry Breaking Bifurcation

Following from the broad outline of [11] and the general framework of [3], we presented in [8] a general theory of unfoldings, finite determinacy and the recognition problem for forced symmetry breaking bifurcation problems of the type $f : (\mathbb{R}^2 \times \mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ where

$$f(z, \lambda) = f_1(z, \lambda) + \mu f_2(z, \lambda, \mu) \tag{2.1}$$

with f_1, f_2, $\mathbb{O}(2), \mathbb{SO}(2)$-equivariant, respectively, for their orthogonal actions on $\mathbb{R}^2$. We identify (as real vector spaces) $\mathbb{C}$ with $\mathbb{R}^2$ given by $z = x + iy$. We denote the bifurcation parameters by $\Lambda = (\lambda, \mu)$. In complex notation, the set of smooth equivariant bifurcation problems of type (2.1) is the module $\vec{\mathcal{E}}^{\mathbb{O}(2),\mathbb{SO}(2)}_{(z,\Lambda)} = < z >_{\mathcal{E}_{(u,\lambda)}} + < z, iz >_{\mathcal{E}(u,\Lambda)}$ where $u = z\bar{z}$ is the unique generator of the ring of $\mathbb{SO}(2)$-invariant germs.

2.1 Contact equivalence

Let $f, g \in \vec{\mathcal{E}}^{\mathbb{O}(2),\mathbb{SO}(2)}_{(z,\Lambda)}$ be two bifurcation maps like in (2.1), f is **contact equivalent** to g if

$$f(z,\Lambda) = T(z,\Lambda)\, g(X(z,\Lambda), L(\Lambda)) \tag{2.2}$$

where T is an equivariant matrix and (X, L) is an equivariant local change of coordinates around the origin in the (z,Λ)-space such that $\det T(0,0) > 0$, $X(0,0) = 0 = L(0)$, $\det X_z(0,0) > 0$, $\det L_\Lambda(0) > 0$, $L_2(\lambda, 0) = 0$ and (T, X, L) are $\mathbb{O}(2)$-equivariant when $\mu = 0$ and $\mathbb{SO}(2)$-equivariant when $\mu \neq 0$. The change of coordinates (2.2) means that the zero-sets $f^{-1}(0)$ and $g^{-1}(0)$ are diffeomorphic under the local diffeomorphism (X, L) which preserves the orientation of the (x,Λ)-space, the (λ,μ)-slice structure of the zero-set and its symmetries.

The set of contact equivalences (T, X, L) has a group structure of semi-direct product by composition. We denote by $\mathcal{K}^{\mathbb{O}(2),\mathbb{SO}(2)}_{\Lambda}$ the group of contact equivalences (T, X, L) acting on $\vec{\mathcal{E}}^{\mathbb{O}(2),\mathbb{SO}(2)}_{(z,\Lambda)}$ consisting of (T, X, L) that are $\mathbb{O}(2)$-equivariant when $\mu = 0$, but only $\mathbb{SO}(2)$-equivariant when $\mu \neq 0$. Nevertheless, $\mathcal{K}^{\mathbb{O}(2),\mathbb{SO}(2)}_{\Lambda}$ is a geometric subgroup of contact equivalences, hence it satisfies the abstract theorems of the general theory of [3]. We get thus the theories for universal unfoldings and determinacy with estimates of the higher order terms $\mathcal{P}(f)$, terms we can discard in each contact class of the normal forms. The following questions have been dealt with in [8].

1. To classify the bifurcation germs of topological codimension less or equal to 2 in relation to the change of coordinates (2.2).
2. To solve the recognition problem for these normal forms.
3. To describe the different bifurcation diagrams obtained by perturbing the normal form, that is, to study the universal unfolding of each normal form.

2.2 Path equivalence

Another approach is to associate with a bifurcation map f, a "path" in the space of deformation parameters of the **core** f_0 of f where $f_0(z) = f(z, 0)$.

2.2.1 Paths

Consider the bifurcation map f as an unfolding of f_0 with parameter Λ. When $\Lambda = 0$, the group of contact equivalences $\mathcal{K}^{\mathbb{O}(2),\mathbb{SO}(2)}_{\Lambda}$ simplifies into $\mathcal{K}^{\mathbb{O}(2)}$, the classical group of $\mathbb{O}(2)$-equivariant contact equivalences without distinguished parameters. A germ f is said to be of **finite core** if f_0 is of finite $\mathcal{K}^{\mathbb{O}(2)}$-codimension. It is straightforward to see that, for each $n \geq 1$, there are exactly two cores of codimension n, namely $f_0(z) = \epsilon_n u^n z$ where $\epsilon_n^2 = 1$. We look at the $\mathbb{SO}(2)$-equivariant universal unfolding of f_0 because of the forced symmetry breaking problem. Compared with the other (finite) subgroups of $\mathbb{O}(2)$, a finite core is also of finite $\mathcal{K}^{\mathbb{SO}(2)}$-codimension, equal to $m = 2n$. We choose the basis of the $\mathcal{K}^{\mathbb{SO}(2)}$-normal space so that $F(z, \alpha, 0)$ is the $\mathcal{K}^{\mathbb{O}(2)}$-universal unfolding of f_0. Explicitly, the $\mathcal{K}^{\mathbb{SO}(2)}$-universal unfolding of f_0 with parameters $\gamma = (\alpha, \beta) \in \mathbb{R}^{2n}$ is $F_0(z, \alpha\,\beta) = \epsilon_n u^n z + \sum_{i=0}^{n-1} (\alpha_i + i\beta_i)\, u^i z$.

From the $\mathcal{K}^{\mathbb{SO}(2)}$-theory of unfoldings, there exists a mapping of unfoldings such that

$$f(z, \Lambda) = T(z, \Lambda)\, F_0(X(z, \Lambda), \bar{\gamma}(\Lambda)) \tag{2.3}$$

where $\bar{\gamma} : (\mathbb{R}^2, 0) \to (\mathbb{R}^m, 0)$ (that is, $\bar{\gamma}(0) = 0$) is the **path** associated with f. Note that T and X are invertible like in (2.2) but $\bar{\gamma}$ is usually obviously not. On the other hand, (2.3) means that f and $\bar{\gamma}^* F_0$ are contact equivalent with equivalence (T, X, I_2). In [9] we show that we can construct the path $\bar{\gamma}$ so that $\bar{\gamma}(\lambda, 0) = (\bar{\alpha}(\lambda), 0)$. The **space of paths** $\vec{\mathcal{P}}_m$ is defined as the set of paths $(p, 0) + \mu q$ where $p \in \vec{\mathcal{E}}_\lambda$ and $q \in \vec{\mathcal{E}}_\Lambda$. To be more precise, it is a finitely generated module over the system of rings $\{\mathcal{E}_\lambda, \mathcal{E}_\Lambda\}$, $\vec{\mathcal{P}}_m =< (1,0) >_{\mathcal{E}_\lambda} + < (1,0), (0,1) >_{\mathcal{E}_\Lambda}$ so

$$\bar{\gamma}(\Lambda) = (\bar{\alpha}^{(1)}(\lambda) + \mu\bar{\alpha}^{(2)}(\Lambda), \mu\bar{\beta}(\Lambda)).$$

2.2.2 Path equivalence

We can now define an equivalence between two paths with the same core. That is, $\bar{r}, \bar{s} : (\mathbb{R}^2, 0) \to (\mathbb{R}^m, 0)$ are **path equivalent** if

$$\bar{r}(\Lambda) = H(\Lambda, \bar{s}(L(\Lambda))) \tag{2.4}$$

where $L : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and the Λ-parametrised family $H : (\mathbb{R}^{2+m}, 0) \to (\mathbb{R}^m, 0)$ are orientation preserving local diffeomorphisms on $(\mathbb{R}^m, 0)$. Moreover, we ask that H preserves the discriminant Δ^{F_0} of F_0 in the sense that $H(\Lambda, \Delta^{F_0}) \subset \Delta^{F_0}$ for all $\Lambda \in (\mathbb{R}^2, 0)$ and lifts to diffeomorphisms of $(\mathbb{R}^{2+m}, 0)$ that preserves $F_0^{-1}(0)$. The **discriminant** Δ^{F_0} of F_0 is the set of values of γ where F_0 is singular. Because $\mathbb{O}(2)$ is continuous, this set corresponds to the whole of the projection of the zero set so $\Delta^{F_0} = \{\ \gamma \ \mid\ \exists z,\ F_0(z, \gamma) = 0\ \}$. For forced symmetry breaking we have to assume also that path equivalence preserve the section $\beta = 0$ of the discriminant.

For a fixed core f_0, the group of path equivalences is denoted by $\mathcal{K}_{\Delta^{F_0}}$. This group is a geometric subgroup which acts on the space of paths, hence the general theory of [3] applies. Remark that we cannot in general simplify H in (2.4) as a Λ-parametrised matrix like with the usual contact equivalence. An explicit description of the diffeomorphisms H is in general very hard, if not impossible. Nevertheless, the tangent space of $\bar{\gamma}$ can be determined explicitly. It involves the module $\mathrm{Derlog}(\Delta^{F_0})$ of vector fields tangent to the discriminant Δ^{F_0} and respecting the section $\beta = 0$. More explicitly it is the submodule of $\{\ \xi \in \vec{\mathcal{E}}_m \mid \xi(h) \in I(\Delta^{F_0}),\ \forall\, h \in \Delta^{F_0}\ \}$ respecting the section $\beta = 0$ of Δ^{F_0}. Therefore the extended tangent space to $\bar{\gamma} = (p, 0) + \mu q$ is

$$\mathcal{T}_e\mathcal{K}_\Delta(\bar{\gamma}) = \bar{\gamma}^* \mathrm{Derlog}(\Delta^{F_0})_{\{\mathcal{E}_\lambda, \mathcal{E}_\Lambda\}} + < (p_\lambda, 0) >_{\mathcal{E}_\lambda} + < \mu(p_\lambda, 0), q + \mu q_\mu >_{\mathcal{E}_\Lambda}.$$

The following results follow from [10].

Theorem 2.1. (a) If $f \in \vec{\mathcal{E}}_{(z,\Lambda)}^{\mathbb{O}(2),\mathbb{SO}(2)}$ has a core of finite $\mathcal{K}^{\mathbb{O}(2)}$-codimension, there exists a path $\bar{\gamma}$ such that f is $\mathcal{K}_{\Lambda}^{\mathbb{O}(2),\mathbb{SO}(2)}$-equivalent to $\bar{\gamma}^* F_0$.
(b) $\mathrm{cod}_{\mathcal{K}_{\Delta F_0}} \bar{\gamma} < \infty$ if and only if $\mathrm{cod}_{\mathcal{K}_{\Lambda}^{\mathbb{O}(2),\mathbb{SO}(2)}} \bar{\gamma}^* F_0 < \infty$. In that case, a map G is a $\mathcal{K}_{\Delta F_0}$-universal unfolding of $\bar{\gamma}$ if and only if $G^* F_0$ is a $\mathcal{K}_{\Lambda}^{\mathbb{O}(2),\mathbb{SO}(2)}$-universal unfolding for $\bar{\gamma}^* F_0$.
(c) Let $\bar{\gamma}^1, \bar{\gamma}^2$ be two smooth paths in $\vec{\mathcal{P}}_m$. Then, $\bar{\gamma}^1$ is $\mathcal{K}_{\Delta F_0}$-equivalent to $\bar{\gamma}^2$ if and only if $\bar{\gamma}^{1*} F_0$ is $\mathcal{K}_{\Lambda}^{\mathbb{O}(2),\mathbb{SO}(2)}$-equivalent to $\bar{\gamma}^{2*} F_0$ for finite codimension problems.

In general, the group $\mathcal{K}_{\Lambda}^{\mathbb{O}(2),\mathbb{SO}(2)}$ induces an equivalence of theory for paths of finite codimension with the *subgroup* of $\mathcal{K}_{\Delta F_0}$ of diffeomorphisms that are 'liftable' over the projection π_{F_0} from the zero-set of F_0 onto the the space of parameters $\mathbb{R}^m$. It Section 3 we show that, as in many other cases, like the non equivariant case, this subgroup is actually the whole of $\mathcal{K}_{\Delta F_0}$.

2.3 Explicit Derlogs in Real Form

In this section we discuss the real form of $\mathrm{Derlog}(\Delta^{F_0})$ for the three cores we need for the classification of Theorem 2.2. In its liftable form, a vector field $\xi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ is in $\mathrm{Derlog}(\Delta^{F_0})$ if there exist germs X and T such that

$$\begin{aligned} (F_0)_z\, X(z,\gamma) + (F_0)_\gamma\, \xi(\gamma) &= T(z,\gamma)\, F(z,\gamma), \\ (F_0)_z(z,\alpha,0)\, X(z,\alpha,0) + (F_0)_\gamma(z,\alpha,0)\, \xi(\alpha,0) &= T(z,\alpha,0)\, F(z,\alpha,0). \end{aligned}$$

Note that, because of the $\mathbb{SO}(2)$-equivariance, the lifts are not unique, they are given modulo $(-y, x, 0, 0)$ which is in the kernel of $(F_0)_z$ modulo TF.

The generic case is $f_0(z) = \epsilon u z$ (core I). Its $\mathbb{SO}(2)$-unfolding is $(\epsilon u + \alpha) z + i\beta z$ and $\mathrm{Derlog}(\Delta^{F_0})$ is freely generated by

$$\xi_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} -\beta^2 \\ \alpha\beta \end{pmatrix} \tag{2.5}$$

where Δ^{F_0} is $\beta(\alpha^2 + \beta^2) = 0$. The lifts are $(z, 2\xi_1)$ and $(-(\epsilon u + \alpha) z, 2\xi_2)$, respectively.

For the next case, $f_0(z) = \epsilon_2 u^2 z$ (case II), of $\mathbb{SO}(2)$-unfolding $(\epsilon_2 u^2 + \alpha_1 + \alpha_2 u, \beta_1 + \beta_2 u)$, the discriminant is given by

$$\beta_2(\alpha_1^2 + \beta_1^2)(\epsilon_2 \beta_1^2 + \alpha_1 \beta_2 - \alpha_2 \beta_1 \beta_2) = 0,$$

with $\mathrm{Derlog}(\Delta^{F_0})$ freely generated by

$$\xi_1 = \begin{bmatrix} 2\alpha_1 \\ \alpha_2 \\ 2\beta_1 \\ \beta_2 \end{bmatrix}, \xi_2 = \begin{bmatrix} \alpha_1\alpha_2 \\ 2\epsilon\alpha_1 \\ \alpha_2\beta_1 \\ \epsilon_2\beta_1 \end{bmatrix}, \xi_3 = \begin{bmatrix} \beta_1\beta_2 \\ \beta_2^2 \\ -\alpha_1\beta_2 \\ \epsilon_2\beta_1 - \alpha_2\beta_2 \end{bmatrix}, \xi_4 = \begin{bmatrix} \alpha_1\beta_2 \\ \epsilon_2\beta_1 \\ \beta_1\beta_2 \\ 0 \end{bmatrix}.$$

The lifts are $(z, 2\xi_1)$, $((\epsilon_2 u + \alpha_2) z, 2\xi_2)$, $(0, \xi_3)$ and $(\beta_2 z, 2\xi_4)$, respectively.

Finally, we shall need $f_0(z) = \epsilon_3 u^3 z$ (case III), of $\mathbb{SO}(2)$-unfolding $(\epsilon_2 u^2 + \alpha_1 + \alpha_2 u + \alpha_3 u^2, \beta_1 + \beta_2 u + \beta_3 u^2)$. Its Derlog is freely generated by

$$
\begin{aligned}
\xi_1(\alpha,\beta) &= (3\alpha_1, 2\alpha_2, \alpha_3, 3\beta_1, 2\beta_2, \beta_3),\\
\xi_2(\alpha,\beta) &= (3\epsilon_3\alpha_1\alpha_3, 3\alpha_1 + \epsilon_3\alpha_2\alpha_3, 3\alpha_2, 2\epsilon_3\alpha_3\beta_1, 2\beta_1 + \epsilon_3\alpha_3\beta_2, \beta_2),\\
\xi_3(\alpha,\beta) &= (\epsilon_3\alpha_1\alpha_2 + \alpha_1\alpha_3^2 - \alpha_1\beta_3^2 + 2\epsilon_3\beta_1\beta_2, -\epsilon_3\beta_1\beta_3 + 2\epsilon_3\alpha_1\alpha_3 + 2\alpha_2\alpha_3^2 + 2\epsilon_3\beta_2^2, 3\alpha_1 + \epsilon_3\alpha_2\alpha_3 + \alpha_3\beta_3^2,\\
&\qquad \epsilon_3\alpha_2\beta_1 - 2\epsilon_3\alpha_1\beta_2 - \beta_1\beta_3^2, 3\epsilon_3\alpha_3\beta_1 - \epsilon_3\alpha_1\beta_3 - 2\epsilon_3\alpha_2\beta_2, 3\beta_1 - 2\epsilon_3\alpha_2\beta_3 + \beta_3^3),\\
\xi_4(\alpha,\beta) &= (2\alpha_1\beta_3, \epsilon_3\beta_1 + \alpha_2\beta_3, \epsilon_3\beta_2, 2\beta_1\beta_3, \beta_2\beta_3, 0),\\
\xi_5(\alpha,\beta) &= (2\alpha_1\alpha_2, 2\alpha_1\beta_3 + \alpha_2\beta_2, \epsilon_3\beta_1 + \alpha_2\beta_3, 2\beta_1\beta_2, \beta_2^2, \epsilon_3\beta_2\beta_3),\\
\xi_6(\alpha,\beta) &= (\beta_1\beta_3, \beta_2\beta_3, \beta_3^2, -\alpha_1\beta_3, \epsilon_3\beta_1 - \alpha_2\beta_3, \epsilon_3\beta_2 - \alpha_3\beta_3).
\end{aligned}
$$

The lifts are $(z, 2\xi_1)$, $((u+\epsilon_3\alpha_3)z, 2\xi_2)$, $((u^2 + \epsilon_3\alpha_3 + 5\epsilon_3\alpha_2 - \beta_3^2)z, 2\xi_3)$, $(\beta_3 z, 2\xi_4)$, $((\beta_3 u - \beta_2)z, 2\xi_5)$ and $(0, \xi_6)$, respectively.

Remarks. 1. Note that the real discriminants are to be seen as the real slices of the discriminants of the complexification of Section 3 where the calculations are sketched.

2. Note that $\mathrm{Derlog}(\Delta^{F_0}) = <\xi_1 \ldots \xi_n>_{\mathcal{O}_\alpha} \oplus <\xi_{n+1} \ldots \xi_m>_{\mathcal{O}_\alpha}$ where $\xi_i(\alpha, 0) = 0$, $n+1 \leq i \leq m$.

2.4 Classification

We recover then the classification obtained by direct methods using $\mathcal{K}_\Lambda^{\mathbb{O}(2),\mathbb{SO}(2)}$ in [8].

Theorem 2.2. The paths in $\vec{\mathcal{P}}_m$ of topological $\mathcal{K}_{\Delta^{F_0}}$-codimension up to 2 are listed in the following table. The symbols I_n^i, II^i and III_n^i represent the paths, of $\mathbb{O}(2)$-codimension n, of topological codimension i corresponding to the normal forms for the cores I, II and III, respectively. The normal forms correspond to paths $(\bar{\alpha}, \mu\bar{\beta})$. Note that the elements of the normal space do not include the terms in the modal parameters of the normal forms.

CASE	NORMAL FORM	NORMAL SPACE	top-cod	cod
I_1^0	$(\delta\lambda, \epsilon_0)$	0	0	0
I_1^1	$(\delta\lambda, \kappa_1\lambda + \kappa_2\mu)$	$(0,1)$	1	1
I_1^{2a}	$(\delta\lambda, \kappa_1\lambda + \kappa_5\mu^2)$	$(0,1),(0,\mu)$	2	2
I_1^{2b}	$(\delta\lambda, \kappa_2\mu + \kappa_3\lambda^2)$	$(0,1),(0,\lambda)$	2	2
I_2^2	$(\delta_2\lambda^2 + \kappa_0\mu, \kappa_1\lambda)$	$(1,0),(0,1)$	2	2
II_1^1	$(\delta\lambda, 0, \epsilon_0, 0)$	$(0,1,0,0)$	1	1
II_1^2	$(\delta\lambda, m_1\mu, m_2\mu + \kappa_5\mu^2, \kappa_6)$	$(0,1,0,0),(0,0,1,0)$	2	4
II_3^2	$(\delta_2\lambda^2, m_3\lambda, \epsilon_0, 0)$	$(1,0,0,0),(\lambda,0,0,0)$	2	3
III_2^2	$(\delta\lambda, 0, 0, \epsilon_0, 0, 0)$	$(0,1,0,0,0,0),(0,0,1,0,0,0)$	2	2

The modal parameters are m_1, m_2, m_3 with conditions $m_1 m_2(\epsilon_2 m_2 - \kappa_6 m_1) \neq 0$ and $m_3^2 \neq 4\epsilon_2\delta_2$. The coefficients $\epsilon, \epsilon_2, \delta, \delta_2, \kappa, \kappa_i$, $i = 0, 1, 2, 3, 5, 6$, are all non zero, normalised to ± 1.

Proof. We use the calculations for $\mathcal{K}_{\Delta^{F_0}}$. We need to calculate the pullback by $\bar{\gamma}$ of $\mathrm{Derlog}(\Delta^{F_0})$. For the core I_1, $\mathrm{Derlog}(\Delta^{F_0})_{\{\mathcal{E}_\lambda, \mathcal{E}_\Lambda\}} = < \xi_1 >_{\mathcal{E}_\lambda} + < \mu\xi_1, \xi_2 >_{\mathcal{E}_\Lambda}$. For the core II, $\mathrm{Derlog}(\Delta^{F_0})_{\{\mathcal{E}_\lambda, \mathcal{E}_\Lambda\}} = < \xi_1, \xi_2 >_{\mathcal{E}_\lambda} + < \mu\xi_1, \mu\xi_2, \xi_3, \xi_4 >_{\mathcal{E}_\Lambda}$, etc. For instance for I_1^i, $i = 1, 2a, 2b$ and 2, the paths are given by $\bar{\alpha}^{(1)}(\lambda) = \delta\lambda$, $\bar{\alpha}^{(2)}(\Lambda) = 0$ and we have different $\bar{\beta}$'s. After some calculations using (2.5), we find that the extended tangent space is equal to

$$< \begin{bmatrix} \delta \\ \mu\bar{\beta}_\lambda \end{bmatrix} >_{\mathcal{E}_\lambda} + \mu < \begin{bmatrix} \delta \\ \mu\bar{\beta}_\lambda \end{bmatrix}, \begin{bmatrix} 0 \\ \bar{\beta} - \bar{\beta}_\lambda \end{bmatrix}, \begin{bmatrix} 0 \\ \bar{\beta} + \mu\bar{\beta}_\mu - \delta\lambda^2\mu\bar{\beta}_\lambda \end{bmatrix}, \begin{bmatrix} 0 \\ \lambda\bar{\beta}_\lambda + \mu^2\bar{\beta}^2\bar{\beta}_\lambda \end{bmatrix} >_{\mathcal{E}_\Lambda}.$$

Replacing by the different expressions for $\bar{\beta}$ and evaluating the normal spaces we get all the results. $\square$

2.5 Comments on the relationship between the two approaches

Although the two theories coincide for finite codimension problems, we can make the following remarks.

- The set-up for path equivalence is independent of the number of parameters and their structure. If $\lambda \in \mathbb{R}^k$, the paths are maps $\bar{r} : (\mathbb{R}^k, 0) \to (\mathbb{R}^m, 0)$ and the contribution of Derlog Δ^{F_0} in the tangent spaces does not depend on k. The path formulation makes explicit which singular behaviour is attributable to the core, which to the paths.

- To establish path equivalence we actually need to complexify the situation to establish the result we use about the Derlogs, but nothing will be lost because we are only interested in germs of finite codimension, that is, equivalent to polynomials.

- Solving the recognition problem using the group action of $\mathcal{K}_\Lambda^{\mathbb{O}(2),\mathbb{SO}(2)}$ is easier for explicit simplification to the normal forms.

3 Liftable Vector Fields

3.1 $\mathbb{C}^*$-Theory in $\vec{\mathcal{O}}_z^{\mathbb{C}^*}$

The rings of $\mathbb{C}^*$-invariant germs is generated by $\bar{u}(z) = z_1 z_2$ which is also χ-invariant. The module $\vec{\mathcal{O}}_{(z,a)}^{\mathbb{C}^*}$ of $\mathbb{C}^*$-equivariant maps with parameter $a \in (\mathbb{C}^m, 0)$ is generated by $Z_1(z) = (z_1, z_2)$, $Z_2(z) = (iz_1, -iz_2)$ and the module $\vec{\mathcal{O}}_{(z,a)}^{\mathbb{C}^*}$ of $\mathbb{O}^*$-equivariant maps is generated by Z_1. We choose our basis so that the algebra in the complexified situation equals the algebra in the real case. We use the standard equivariant contact equivalences $\mathcal{K}^\Gamma$ with groups $\Gamma = \mathbb{C}^*$ or $\mathbb{O}^*$. To calculate the tangent space for $\mathcal{K}^{\mathbb{C}^*}$ we need the module of $\mathbb{C}^*$-equivariant matrices acting on $\mathbb{C}^2$. It is generated by $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $M_2(z) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $M_3(z) = \begin{pmatrix} 0 & z_1^2 \\ z_2^2 & 0 \end{pmatrix}$

and $M_4(z) = \begin{pmatrix} 0 & iz_1^2 \\ -iz_2^2 & 0 \end{pmatrix}$. The extended tangent space of $f(z) = r\, Z_1 + s\, Z_2$, identified with (r,s), is generated by $(r,s), (-s,r), (us,0), (0,us)$ and $(r+2ur_u, s+2us_u)$.

To construct the space of paths, we need to get the $\mathbb{C}^*$-universal unfoldings of the $\mathbb{O}^*$-equivariant cores. Before complexifying the situation, recall that the cores f_0 of finite $\mathcal{K}^{\mathbb{O}(2)}$-codimension are given by $f_0(z) = \epsilon_n u^n z$ for some $n \in \mathbb{N}$. Thus $T_e\mathcal{K}^{\mathbb{SO}(2)}(f) =< (u^n, 0), (0, u^n) >_{\mathcal{E}_u} = \vec{\mathcal{M}}^n$, so f_0 is always of finite $\mathcal{K}^{\mathbb{SO}(2)}$-codimension and its $\mathbb{SO}(2)$-universal unfolding is equal to $(\epsilon_n u^n + \sum_{j=1}^n \alpha_j u^{j-1})\, z + (\sum_{j=1}^n \beta_j u^{j-1})\, iz$. Let $F(z,a) = (R(u,a) + iS(u,a))\, z$ be a germ in $\vec{\mathcal{E}}^{\mathbb{SO}(2)}_{(z,a)}$, and so its complexification is $(F, \bar{F}) = R(u,a)\, Z_1 + S(u,a)\, Z_2$. We keep track of the sign of u^k to be able to go back to the real case.

The discriminant of F_0 is formed of 3 varieties: $R(0,a,b) = 0$, $S(0,a,b)$ and $P(u,a,b) = Q(u,a,b) = 0$ that correspond to the projection of the zero-set of F_0.

3.2 $\mathbb{C}^*$-Derlog

For our 3 cores we can use directly the following result of Saito's to show that the liftable vector fields and the vector fields tangent to the discriminant form the same free module.

Theorem 3.1. (Saito [16]) If the vector fields $\{\xi_i\}_{i=1}^m$ are in Derlog(Δ^{F_0}) and the determinant they form $|\xi_1 \ldots \xi_m|$ is a reduced defining equation for Δ^{F_0} then they generate freely Derlog(Δ^{F_0}).

The general fact that the $\mathbb{C}^*$-Derlogs are always free modules and correspond to the liftable vector fields follows from the same type of arguments as in the general case, but we need to check some conditions explicitly. Note first that, from the Malgrange Preparation Theorem, the normal space $\mathcal{N}_e\mathcal{K}^{\mathbb{C}^*}(F_0) = \vec{\mathcal{O}}^{\mathbb{C}^*}_{(z,a)}/\mathcal{T}_e\mathcal{K}^{\mathbb{C}^*}(F)$ is freely generated as an $\mathcal{O}_{\bar{a}}$-module by $\{h_i\}_{i=1}^m$, say. The following formula

$$\varphi(\bar{\alpha}) = \sum_{i=1}^m \bar{\alpha}_i(a)\, h_i(z)$$

defines a linear epimorphism:

$$\mathcal{O}_a^m \xrightarrow{\varphi} \mathcal{N}_e\mathcal{K}^{\mathbb{C}^*}(F_0) \to 0.$$

The kernel of φ is the $\mathbb{C}^*$-Derlog of liftable vector fields of F_0 (in [10] we called this kernel the **algebraic Derlog** of Δ^{F_0}). That it is free follows from the method of Teissier ([17]) as adapted by Damon ([4]) because those modules are still Cohen-Macaulay (cf. [2, 15]). The support of the kernel of φ is the projection of the zero-set of F_0 onto the parameter space.

To see that vector fields tangent to the discriminant lift over the projection we use the necessary and sufficient conditions for liftability in [1]. Note that the orbit space of $\mathbb{C}^2$ by the $\mathbb{C}^*$-action is smooth (cf. [15]).

References

[1] J.W. Bruce, A. du Plessis, L.C. Wilson. *Discriminants and Liftable Vector Fields.* Jour. Alg. Geom. 3 (1994), 725-753.

[2] J.W. Bruce, R.M. Roberts. *Critical Points of Functions on Analytic Varieties.* Topology 27 (1988), 57-90.

[3] J. Damon. *The Unfolding and Determinacy Theorems for Subgroups of $\mathcal{A}$ and $\mathcal{K}$*, Mem. Amer. Math. Soc. **306**, Providence RI 1984.

[4] J. Damon. *On the Legacy of Free Divisors: Discriminants and Morse Type Singularities.* Amer. J. Math. 120 (1998), 453-492.

[5] J.E. Furter, A.M. Sitta. $\mathbb{D}_n$*-Forced Symmetry Breaking in* $\mathbb{O}(2)$*-Equivariant Equations*, Proceedings of the Royal Society Edinburgh 132A, 2002, 1185-1218.

[6] J.E. Furter, M.A. Ruas and A.M. Sitta. *Singularity theory and Forced Symmetry Breaking in Equations.* Preprint.

[7] J.E. Furter, A.M. Sitta. *Geometric Theory of Bifurcation Problems: Path Formulation.* Preprint.

[8] J.E. Furter, A.M. Sitta. *Breaking the Reflection in* $\mathbb{O}(2)$*-Equivariant Bifurcation Problems.* Preprint.

[9] J.E. Furter, A.M. Sitta and I. Stewart. *Singularity Theory and Equivariant Bifurcation Problems with Parameter Symmetry.* Proc. Cam. Math. Soc. 120 (1996), 547-578.

[10] J.E. Furter, A.M. Sitta, I. Stewart. *Algebraic path formulation for equivariant bifurcation problems.* Math. Proc. Camb. Phil. Soc. 124 (1998), 275-304.

[11] M. Golubitsky and D.G. Schaeffer. *A discussion of symmetry and symmetry breaking*, Proc. Symp. Pure Math. **40** (I), AMS, Providence RI 1983.

[12] M. Golubitsky and D.G. Schaeffer. *Singularities and Groups in Bifurcation Theory I*, Applied Math. Sci. **51**, Springer-Verlag, New York 1985.

[13] M. Golubitsky, I.N. Stewart and D.G. Schaeffer. *Singularities and Groups in Bifurcation Theory II*, Applied Math. Sci. **69**, Springer-Verlag, New York 1988.

[14] E.J.N. Looijenga. *Isolated Singular Points of Complete Intersections.* LMS Lecture Notes 17, Cambridge University Press, 1984.

[15] J. Montaldi, D. van Straten. *Quotient spaces and critical points of invariant functions for* $\mathbb{C}^*$*-actions.* Pub. IRMA 16, Lille 26 1991.

[16] K. Saito. *Theory of logarithmic differential forms and logarithmic vector fields.* J.Fac.Sci. Univ. Tokyo Sec 1A 27 (1980), 265-291.

[17] B. Tessier. *The hunting of invariants in the geometry of the discriminant.* in: P. Holm, ed., Real and Complex Singularities, Oslo 1976 (Sijthoff and Noordhoff, Alphen aan den Rijn 1977), 565-677.