Detection, Isolation and Diagnosability Analysis of Intermittent Faults in Stochastic Systems

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***Abstract*—In this paper, the fault detection and isolation problem of intermittent faults (IFs) in a class of linear stochastic systems is investigated. For the detection and isolation of intermittent faults, it includes: i) to detect both the appearing time and the disappearing time of an IF; ii) to detect the appearing (disappearing) time of each IF before the subsequent disappearing (appearing) time; iii) to determine where the IFs happen. Based on the outputs of the observers we designed, a novel set of residuals is constructed by using the sliding time window technique, two hypothesis tests are proposed to detect all the appearing time and disappearing time of intermittent faults. The isolation problem of intermittent faults is also considered. Furthermore, within a statistical framework, the definition of the diagnosability of IFs is proposed, and a sufficient condition is brought forward for the diagnosability of intermittent faults. Quantitative performance analysis results for the false alarm rate and the missing detection rate are discussed, and the influences of some key parameters of the proposed scheme on performance indices such as the false alarm rate and the missing detection rate are analyzed rigorously. Effectiveness of the proposed scheme is illustrated via a simulation example of an unmanned helicopter longitudinal control system.**

***Index Terms*—Intermittent faults (IFs), fault detection and isolation, diagnosability, linear stochastic systems, hypothesis test.**

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Acronyms

DES	discrete event system
FDI	fault detection and isolation
IF	intermittent fault

IFAP -diagnosable	appearing time of the IF is probabilistically diagnosable
IFDP -diagnosable	disappearing time of the IF is probabilistically diagnosable
IFP -diagnosable	IF is probabilistically diagnosable
PF	permanent fault

NOTATION

$\mathbb{E}[\cdot]$	expectation
$P(\cdot)$	probability
$Var[\cdot]$	variance
$\Phi(\cdot)$	normal Gaussian distribution
b_i	ith fault signature of systems
$m_i(t)$	intermittent fault (IF) in b_i
$\mu_{i,q}$, $\nu_{i,q}$	qth unknown appearing, and disappearing time of the IF $m_i(t)$
$f_i(q)$	qth magnitude of the IF $m_i(t)$
ρ_i	lower bound of the magnitude of the IF $m_i(t)$
$\tau_{i,q}^{\text{dur}}$, $\tau_{i,q}^{\text{int}}$	qth duration time, and interval time of the IF $m_i(t)$
$\widetilde{\tau}_i^{\text{dur}}$, $\widetilde{\tau}_i^{\text{int}}$	minimal values of the duration time, and the interval time of the IF $m_i(t)$
$\mathcal{X}$	real vector space with typical elements of state vector $x(t)$
$\mathcal{S}$	invariant subspace, a A-invariant subspace is the subspace $\mathcal{S} \subseteq \mathcal{X}$ with the property $A\mathcal{S} \subseteq \mathcal{S}$
$\mathcal{B}_i$	image of b_i, $\mathcal{B}_i =$Im b_i
Ker C	kernel of C
$\underline{\mathcal{S}}(\cdot)$	set of all (C, A)-unobservability subspaces containing given subspaces
$\underline{Q_i}(\mathcal{W})$	set of all Q_i satisfying $(A + Q_iC)\mathcal{W} \subseteq \mathcal{W}$ for a given (C, A)-invariant subspace $\mathcal{W}$
dim$(\cdot)$	dimension of the given space
I_n	n-dimensional identity matrix
Δt_i	sliding time window for $m_i(t)$

$r_i(t, \Delta t_i)$	novel set of residuals
γ_{i_1}, γ_{i_2}	test sizes of hypothesis tests for the detection of appearing time, and disappearing time of $m_i(t)$
$\mu_{i,q}^d$, $\nu_{i,q}^d$	detection time of qth appearing time, and disappearing time of $m_i(t)$
Π_{i_1}, Π_{i_2}	detection thresholds for the appearing time, and the disappearing time of $m_i(t)$
ϑ_{i_1}, ϑ_{i_2}	confidence values of hypothesis tests for the detection of appearing time, and disappearing time of $m_i(t)$
$\widetilde{\mu}_{i,q}^d$, $\widetilde{\nu}_{i,q}^d$	worst allowable detection time of $\mu_{i,q}^d$, and $\nu_{i,q}^d$ for $m_i(t)$
δ_i^*	minimal length of the permissable sliding window for $m_i(t)$
R_{test}^i	hypothesis test result for the novel residual $r_i(t, \Delta t_i)$

I. INTRODUCTION

With the growing demand of the reliability and safety of complex industrial processes [32], [10], [5], [23], [20], the fault diagnosis problem of stochastic systems has received increasing attention in the past three decades [5], [9], [12], [14], [35], [37]. Compared with the hardware redundancy method, analytical redundancy based fault diagnosis schemes, such as model-based approaches [9], [17] and data-driven approaches [37], [12], are more appealing due to their low expense, high adaptabilities, and good performances. Consequently, a number of efficient analytical redundancy based fault diagnosis methods have been presented [2], [26], [21], [31].

With the development of digital circuit technologies and computer technologies, a large fraction of faults are actually IFs (also called transient faults) in most application fields, especially in electromechanical systems, power systems, military systems, and aerospace or aircraft systems [13], [8], [19], [27]. IFs are different from permanent faults [36]. In [30], IFs are defined as "Fault of an item for a limited period of time, following which the item recovers its ability to perform its required function without being subjected to any external corrective action" and "such faults are often recurrent" [8]. Owing to the fact that IFs can randomly appear and disappear with unknown magnitudes, they are one of the main factors resulting in false alarms and temporary failure in industrial processes [18]. According to [28], more than 50% of all pilot-reported operational faults in avionics are IFs. As illustrated in [1], the inspection and maintenance costs of IFs in large scale integrated circuits remain high while those for permanent faults (PFs) decrease. Moreover, many IFs are related to the degradation of systems or equipments [8]. Therefore, the FDI of IFs should be studied carefully to make corrective maintenance strategy and assure the reliability of systems or equipments.

Different from the FDI of permanent faults, the FDI of IFs requires to detect not only all the appearing time but also all the disappearance time of IFs. That is, we must detect the appearing (disappearing) time of IFs before the subsequent disappearing (appearing) time[6], [13]. Besides, the duration time and the interval time of an IF may be very short, so the FDI scheme for IFs must be fast enough [3]. Note that IFs can disappear without any corrective actions. The FDI of IFs must be accurate enough to avoid unnecessary reparation operations. Hence, it is hard to detect and isolate IFs at a fast speed as well as a high accuracy rate, which motivates us to investigate the FDI problem of IFs.

In the existing FDI results for IFs, many of them are based on the qualitative analysis methods and the data-driven methods for specific plants [20], [3], [34]. Most of them have focused on detecting whether an IF has happened in the system, but they have ignored the intermittent property of IFs and may be ineffective in detecting all the appearing time and disappearing time. In [15], the diagnosability of IFs based on discrete event systems (DES) was proposed which is of paramount significance. However, much prior information about system structures and IFs were required to build up the DES and no quantitative performance analysis was provided to assure the accuracy of the FDI of IFs. Based on the fact that IFs may change the mean of residuals, an effective approach was proposed in [6] to detect all the appearing and disappearing time of a scalar IF in linear stochastic systems with ideal measurements $y = x$ and a necessary and sufficient condition for the detectability of IFs was obtained. Since noises are inevitable in most practical systems, it makes more sense to study the FDI problem of IFs in linear stochastic systems with measurement noises, and analyze the diagnosability, the false alarm rate, and the missing detection rate, based on the proposed FDI scheme.

In this paper, a novel method is proposed to study the FDI problem of IFs in linear stochastic systems with measurement noises. The FDI problem of IFs includes: i) to detect both the appearing time and the disappearing time of an IF; ii) to detect the appearing (disappearing) time of each IF before the subsequent disappearing (appearing) time; iii) to determine where the IFs happen. In order to isolate IFs in different fault signatures, a set of observers sensitive to each fault signature is designed. Based on the outputs of observers, a novel set of residuals is constructed by introducing sliding time windows and two hypothesis tests are brought forward to detect all the appearing time and the disappearing time separately. The main contributions of this paper are as follows: 1) a novel scheme is provided to detect and isolate IFs in a class of linear stochastic systems with both process noises and measurement noises, where a novel set of residuals is constructed by using the sliding time window technique; 2) the diagnosability of IFs is defined within a statistical framework, and a sufficient condition is brought forward for the diagnosability of IFs; 3) some quantitative performance analysis results for the false alarm rate and the missing detection rate are presented and the influences of some key parameters of the proposed scheme on those performance indices are analyzed rigorously.

The rest of this paper is organized as follows. In Section II, the FDI problem of IFs in linear stochastic systems with measurement noises is mathematically formulated. A novel scheme is proposed to detect both the appearing time and the disappearing time of IFs and isolate them. A sufficient diagnosability condition for a class of IFs is provided in Section III. In Section IV, performance indices such as the detection time, the false alarm rate and the missing detection rate, are analyzed and the influence of key parameters of the proposed scheme on these performance indices are quantitatively analyzed. In Section V, simulation results are provided to illustrate the validity of the proposed scheme. Finally, we draw up the concluding remarks in Section VI.

II. PROBLEM FORMULATION

Consider a class of continuous stochastic linear time invariant (LTI) dynamic systems formulated as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + \sum_{i=1}^{l} b_i m_i(t) + Ew(t) \\ y(t) = Cx(t) + Dv(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^l$ and $y(t) \in \mathbb{R}^s$ are the state vector, input vector and output vector, respectively; $w(t) \in \mathbb{R}^p$ and $v(t) \in \mathbb{R}^d$ denote the system noises and measurement noises which are independent Gaussian white noises with known covariances R_w and R_v; $m_i(t) \in \mathbb{R}$ is the fault mode and $b_i \in \mathbb{R}^n$ is the fault signature, which is the ith column of B; the real constant system matrices A, B, E, C, D are of appropriate dimensions. Note that sensor faults and changes in system dynamics can be represented as pseudoactuator faults by modifying A, B, b_i, and C [21]. Hence, the following analysis will focus on the model with actuator faults.

In model (1), $m_i(t) : \mathbb{R}^+ \to \mathbb{R}$ stands for the IF signal which is in the following form [6]:

$$m_i(t) = \sum_{q=1}^{\infty}[\Gamma(t-\mu_{i,q}) - \Gamma(t-\nu_{i,q})] \cdot f_i(q), \quad (2)$$

where $\Gamma(t)$ is the step function; $\mu_{i,q}$ and $\nu_{i,q}$ stand for the qth unknown appearing and disappearing time of the IF in the fault signature b_i respectively satisfying $\mu_{i,q} < \nu_{i,q} < \mu_{i,q+1}$; the qth duration time of the IF is $\tau_{i,q}^{\text{dur}} = \nu_{i,q} - \mu_{i,q}$, and the qth interval time is $\tau_{i,q}^{\text{int}} = \mu_{i,q+1} - \nu_{i,q}$; $f_i(q) : \mathbb{N}^+ \to \mathbb{R}$ represents the unknown fault magnitude.

Assumption 1: There is no redundancy in different fault signatures, i.e. $\mathcal{B}_j \neq \mathcal{B}_k$, $\forall j \neq k$.

Assumption 2: i) Each IF $m_i(t)$ has a known lower bound represented by ρ_i satisfying $|f_i(q)| \geq \rho_i$. ii) The minimal values of $\tau_{i,q}^{\text{dur}}$ and $\tau_{i,q}^{\text{int}}$ of each IF are formulated as

$$\begin{cases} \widetilde{\tau}_i^{\text{dur}} \triangleq \inf\limits_{q \in \mathbb{N}^+} \tau_{i,q}^{\text{dur}}, \\ \widetilde{\tau}_i^{\text{int}} \triangleq \inf\limits_{q \in \mathbb{N}^+} \tau_{i,q}^{\text{int}}. \end{cases} \quad (3)$$

Let $\delta_i = \min\left\{\widetilde{\tau}_i^{\text{dur}}, \widetilde{\tau}_i^{\text{dur}}\right\}$ and assume δ_i is known.

III. THE FDI SCHEME FOR IFS

In this section, a novel scheme is proposed to study the FDI problem of IFs for system (1). An observer-type scheme is utilized to generate a novel set of residuals which can be used to isolate IFs in different fault signatures. But the residuals are subjecting to process noises and measurement noises. Then two hypothesis tests are provided to analyze the residuals to detect all the appearing time and the disappearing time of the IFs.

A. The Unidimensional Residual Design

For system (1), a set of observers governed by the following dynamics

$$\begin{cases} \dot{\omega}_i(t) = F_i\omega_i(t) - J_iy(t) + G_iu(t), \\ r_i(t) = M_i\omega_i(t) - H_iy(t) + K_iu(t), \end{cases} \quad (4)$$

is designed for each fault signature b_i such that the output of the observer $r_i(t)$ is decoupled from all the other IFs $m_j(t)$ $(j \neq i)$ but affected by $m_i(t)$ and $w(t)$ and $v(t)$. In (4), $\omega_i(t) \in \mathbb{R}^{\widetilde{n}}$ ($\widetilde{n}$ is defined below) is state of the ith observer; $u(t)$ and $y(t)$ are the input and output of system (1);

F_i, J_i, G_i, M_i, H_i and K_i are parameter matrices of appropriate dimensions to be designed.

Let $\mathcal{S}_i^* := \inf \underline{\mathcal{S}}(\sum_{j\neq i}\mathcal{B}_j)$. If $\mathcal{S}_i^* \cap \mathcal{B}_i = \mathbf{0}$ ($i = 1, \cdots, l$) is satisfied, the parameters of G_i, M_i, H_i and K_i in (4) can be calculated by using the following algorithm [21]: Let $Q_{i,0} \in \underline{Q_i}(\mathcal{S}_i^*)$. Let $P_i : \mathcal{X} \to \mathcal{X}/\mathcal{S}_i^*$ be the canonical projection with a right inverse P_i^{-r}, and let $A_{i,0}$ be the induced map on $\mathcal{X}/\mathcal{S}_i^*$, so we have $A_{i,0} = P_i(A + Q_{i,0}C)P_i^{-r}$. Construct H_i from $Ker\ H_iC = \mathcal{S}_i^* + Ker\ C$, and let $M_i = (H_iC)P_i^{-r}$, then the pair $(M_i, A_{i,0})$ is observable. So a $Q_{i,1}$ exists such that $\breve{\sigma}(F_i) = \Lambda_i$, where $F_i = A_{i,0} + Q_{i,1}M$, $\breve{\sigma}(F_i)$ is the spectrum of F_i and Λ_i is a given diagonal set. The rest of the coefficient matrices can be derived by $J_i = P_iQ_{i,0} + Q_{i,1}H_i$, $G_i = P_iB$ and $K_i = 0$. Let $\widetilde{n} = n - \dim(\mathcal{S}_i^*)$, and define $\varepsilon_i(t) = \omega_i(t) - P_ix(t)$, then we can obtain the dynamics of the $\widetilde{n}$ dimension observers as

$$
\begin{aligned}
\dot{\varepsilon}_i(t) =&F_i\omega_i(t) - J_i(Cx(t) + Dv(t)) \\
&+ G_iu(t) - P_iAx(t) - P_iBu(t) \\
&- P_ib_im_i(t) - P_iEw(t) \\
=&F_i\omega_i(t) - J_iCx(t) - J_iDv(t) \\
&- P_iAx(t) - P_ib_im_i(t) - P_iEw(t) \\
=&F_i\omega_i(t) - P_iQ_{i,0}Cx(t) - P_iAx(t) \\
&- P_ib_im_i(t) - J_iDv(t) - P_iEw(t) \\
=&F_i\varepsilon_i(t) - P_ib_im_i(t) - P_iEw(t) \\
&- J_iDv(t),
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
r_i(t) &= M_iw(t) - H_iy(t) \\
&= M_i\varepsilon_i(t) - H_iDv(t).
\end{aligned} \tag{6}
$$

From (5) and (6), we can conclude that the output of the ith observer $r_i(t)$ is decoupled from all the other IFs $m_j(t)$ ($j \neq i$) but affected by $m_i(t)$ and $w(t)$ and $v(t)$. Then, by introducing a sliding time window Δt_i ($0 < \Delta t_i \leq \delta_i$) [6], we can obtain the sliding-window based estimation error [22]

$$
\begin{aligned}
\varepsilon_i(t, \Delta t_i) &\triangleq \varepsilon_i(t) - e^{F_i\Delta t_i}\varepsilon_i(t - \Delta t_i) \\
&= \int_{t-\Delta t_i}^{t} e^{F_i(t-\tau)}[-P_ib_im_i(t) \\
&\quad - P_iEw(t) - J_iDv(t)]d\tau,
\end{aligned} \tag{7}
$$

In order to construct and analyze novel residuals, a convenient set of unidimensional observers governed by (4) should be designed for scalar IFs for the sake of conceptual and computational simplicity. Here, we propose the following two lemmas to design a set of unidimensional observers.

Lemma 1: For system (1) with the input matrix $B \in \mathbb{R}^{n\times l}(l \leq n)$, a set of unidimensional observers can be designed for each fault signature b_i if there exists a matrix $\widetilde{L}_i = [L_{i_1}\ L_{i_2}\ \cdots\ L_{i_z}] \in \mathbb{R}^{n\times z}(z \leq n-l)$ such that the following conditions can be satisfied

$$\dim(\mathcal{S}_i') = n - 1, \mathcal{S}_i' \cap \mathcal{B}_i = \mathbf{0},\ i = 1, \cdots, l, \tag{8}$$

where $\mathcal{S}_i' := \inf \underline{\mathcal{S}}(\sum_{j\neq i}^{l} \mathcal{B}_j + \sum_{k=1}^{z} \mathcal{L}_{i_k})$.

Proof: Note that $\mathcal{S}_i' \supset \underline{\mathcal{S}}_i^*$. If $\mathcal{S}_i' \cap \mathcal{B}_i = \mathbf{0}$ $(i = 1, \cdots, l)$ is satisfied, the parameter matrices of the ith observer, G_i, M_i, H_i and K_i, can be derived by using $\mathcal{S}_i'$ instead of $\underline{\mathcal{S}}_i^*$, and the output of the ith observer $r_i(t)$ is decoupled from all the other IFs $m_j(t)$ $(j \neq i)$ but affected by $m_i(t)$ and $w(t)$ and $v(t)$. Apparently, the order of the ith observer is $n - \dim(\mathcal{S}_i') = 1$. ∎

Lemma 2: For system (1) with overactuated actuators (i.e., $l > n$) [24], a set of unidimensional observers can be designed for each fault signature b_i if for each $(n-1)$ combination $b_{i_1}, \cdots, b_{i_{(n-1)}}$ of b_i's, there exist (C, A)-unobservable subspaces

$$\begin{aligned} &\mathcal{S}'_{i_1 i_2 \cdots i_{n-1}} \cap \mathcal{B}_k = \mathbf{0}, \ k \neq i_j, \\ &j = 1, \cdots, n-1, \end{aligned} \tag{9}$$

where $\mathcal{S}'_{i_1 i_2 \cdots i_{n-1}} := \inf \underline{\mathcal{S}}(\sum_{j=1}^{n-1} \mathcal{B}_{i_j})$.

Proof: It can be easily derived that $\dim(\mathcal{S}'_{i_1 i_2 \cdots i_{n-1}}) = n - 1$ under the assumption $\mathcal{B}_j \neq \mathcal{B}_k$, $\forall j \neq k$. Similar to the proof of Theorem 2.2 of [24] and Lemma 1, Lemma 2 can be easily obtained therefore the proof is omitted here. ∎

If the condition of Lemma 1 (for $l \leq n$) or Lemma 2 (for $l > n$) is satisfied, we can design a set of unidimensional observers ($F_i = \lambda_i$) governed by (4). According to (7), the following equation holds

$$\begin{aligned} M_i \varepsilon_i(t, \Delta t_i) &= M_i \varepsilon_i(t) - M_i e^{\lambda_i \Delta t_i} \varepsilon_i(t - \Delta t_i) \\ &= r_i(t) - e^{\lambda_i \Delta t_i} r_i(t - \Delta t_i) + H_i D v(t) \\ &\quad - e^{\lambda_i \Delta t_i} H_i D v(t - \Delta t_i), \end{aligned}$$

Define $r_i(t, \Delta t_i) \triangleq r_i(t) - e^{\lambda_i \Delta t_i} r_i(t - \Delta t_i)$, then we can calculate a novel set of residuals

$$\begin{aligned} r_i(t, \Delta t_i) = &\int_{t-\Delta t_i}^{t} e^{\lambda_i (t-\tau)} [-M_i P_i b_i m_i(\tau) \\ &- M_i P_i E_i w(\tau) - M_i P_i Q_{i,0} v(\tau) \\ &- \lambda_i H_i v(\tau) + M_i A_{i,0} M_i^{-r} H_i v(\tau)] d\tau \\ &- H_i D v(t) + e^{\lambda_i \Delta t_i} H_i D v(t - \Delta t_i), \end{aligned} \tag{10}$$

where λ_i is the only parameter to be determined.

Remark 1: In fact, based on a set of multidimensional observers, a novel set of residuals can be designed if we construct a special form of F_i in (4) as $F_i = \lambda_i \cdot I_{\widetilde{n}}$. As for the following two hypothesis tests for detecting all the appearing time and the disappearing time of IFs, a component by component strategy [4] or a Mahalanobis distance based strategy [11] can be utilized for the multidimensional residuals which we will consider in our future work.

B. Stochastic Properties of the Unidimensional Residuals

To analyze the sochastic properties of the residual $r_i(t,\Delta t_i)$, we rewrite $r_i(t,\Delta t_i)$ into five parts:

$$\begin{cases} p_{i0}(t,\Delta t_i) \triangleq -\int_{t-\Delta t_i}^{t} e^{\lambda_i(t-\tau)} M_i P_i b_i m_i(\tau)\mathrm{d}\tau, \\ p_{i1}(t,\Delta t_i) \triangleq -\int_{t-\Delta t_i}^{t} e^{\lambda_i(t-\tau)} M_i P_i E w(\tau)\mathrm{d}\tau, \\ p_{i2}(t,\Delta t_i) \triangleq -\int_{t-\Delta t_i}^{t} e^{\lambda_i(t-\tau)} (M_i P_i Q_{i,0} + \lambda_i H_i \\ \qquad\qquad - M_i A_{i,0} M_i^{-r} H_i) v(\tau)\mathrm{d}\tau, \\ p_{i3}(t,\Delta t_i) \triangleq H_i D v(t), \\ p_{i4}(t,\Delta t_i) \triangleq e^{\lambda_i \Delta t_i} H_i D v(t-\Delta t_i), \end{cases}$$

According to [7], $p_{i1}(t,\Delta t_i)$ and $p_{i2}(t,\Delta t_i)$ are both Gaussian distributed [33] with the mean $\mathbb{E}\left[p_{i1}(t,\Delta t_i)\right] = \mathbb{E}\left[p_{i2}(t,\Delta t_i)\right] = 0$. The variance of $p_{i2}(t,\Delta t_i)$ is derived by [6]

$$\begin{aligned} & Var\left[p_{i1}(t,\Delta t_i)\right] \\ =& \mathbb{E}\Big[\int_{t-\Delta t_i}^{t} e^{\lambda_i(t-\tau)} M_i P_i E w(\tau)\mathrm{d}\tau \cdot \int_{\mathrm{t}-\Delta \mathrm{t_i}}^{\mathrm{t}} \mathrm{e}^{\lambda_\mathrm{i}(\mathrm{t}-\tau)} \\ & \times M_i P_i E w(\tau)\mathrm{d}\tau\Big] \\ =& \int_{t-\Delta t_i}^{t}\int_{t-\Delta t_i}^{t} e^{\lambda_i(2t-\tau_1-\tau_2)} (M_i P_i E) \\ & \times \mathbb{E}[w(\tau_1)\, w(\tau_2)](M_i P_i E)^T \mathrm{d}\tau_1 \mathrm{d}\tau_2 \\ =& (M_i P_i E) R_w (M_i P_i E)^T \int_{t-\Delta t_i}^{t} e^{2\lambda_i(t-\tau)}\mathrm{d}\tau \\ =& \frac{1-e^{2\lambda_i \Delta t_i}}{-2\lambda_i}(M_i P_i E) R_w (M_i P_i E)^T. \end{aligned}$$

Similarly, the variance of $p_{i2}(t,\Delta t_i)$ can be obtained. Note that $w(t)$ and $v(t)$ are independent white Gaussian noises. Based on the definition and properties of mean R-S integral [33], [16], $p_{ij}(t,\Delta t_i)$, $j \in \{1,2,3,4\}$, are mutually independent distributions [29] by direct calculation of mutual covariances (where $\forall j \neq l, j, l \in \{1,2,3,4\}$, $\mathbb{E}\left[p_{i_j}(t,\Delta t_i) p_{i_l}^T(t,\Delta t_i)\right] = 0$). Let $p_i(t,\Delta t_i) = \sum_{j=1}^{4} p_{ij}(t,\Delta t_i)$. Then, the mean of $p_i(t,\Delta t_i)$ is $\mathbb{E}\left[p_i(t,\Delta t_i)\right] = 0$. Moreover, we are able to derive the variance of $p_i(t,\Delta t_i)$ as

$$\begin{aligned} & Var[p_i(t,\Delta t_i)] \\ =& -\frac{1-e^{2\lambda_i \Delta t_i}}{2\lambda_i}\Bigg(\left(M_i P_i Q_{i,0} - M_i A_{i,0} M_i^{-r} H_i\right) \\ & \times R_v \left(M_i P_i Q_{i,0} - M_i A_{i,0} M_i^{-r} H_i\right)^T \\ & + M_i P_i E R_w \left(M_i P_i E\right)^T \Bigg) + \frac{1-e^{2\lambda_i \Delta t_i}}{2} \\ & \times \Bigg(\left(M_i P_i Q_{i,0} - M_i A_{i,0} M_i^{-r} H_i\right) R_v H_i^T \\ & + H_i R_v \left(M_i P_i Q_{i,0} - M_i A_{i,0} M_i^{-r} H_i\right)^T \Bigg) \\ & - \frac{\lambda_i\left(1-e^{2\lambda_i \Delta t_i}\right)}{2} H_i R_v H_i^T \\ & + \left(1+e^{2\lambda_i \Delta t_i}\right)\left(H_i D\right) R_v \left(H_i D\right)^T. \end{aligned} \tag{11}$$

Let $\sigma(\Delta t_i) = \sqrt{Var[p_i(t,\Delta t_i)]}$, and therefore $p_i(t,\Delta t_i) \sim \Phi(0,\sigma^2(\Delta t_i))$, where $\Phi(\cdot)$ is a Normal Gaussian Distribution .

C. The Detection of Appearing Time for IFs

Note that $p_{i0}(t,\Delta t_i) = 0$, when $\mu_{i,q} < \nu_{i,q} < t-\Delta t_i \leq t$. According to [4], [6], we introduce the following hypothesis test to detect the appearing time of the IF $m_i(t)$ i.e. $\mu_{i,q}$:

$$\begin{cases} H_{i,0}^A : \mathbb{E}[r_i(t,\Delta t_i)] = 0, \\ H_{i,1}^A : \mathbb{E}[r_i(t,\Delta t_i)] \neq 0. \end{cases} \tag{12}$$

Using the Mahalanobis distance $d(\cdot,\cdot)$ [11], we can obtain the test acceptance region for a given small value of the test size γ_{i_1} in the form of

$$\begin{aligned} B_{\gamma_{i_1}}^A(\Delta t_i) &= \big(L_{\gamma_{i_1}}^A(\Delta t_i), U_{\gamma_{i_1}}^A(\Delta t_i)\big] \\ &= \big(-h_{\frac{\gamma_{i_1}}{2}}\sigma(\Delta t_i), h_{\frac{\gamma_{i_1}}{2}}\sigma(\Delta t_i)\big], \end{aligned} \tag{13}$$

where $h_{\frac{\gamma_{i_1}}{2}}$ is the value of a Normal Gaussian Distribution $\Phi(\cdot)$ satisfying $1-\Phi\big(h_{\frac{\gamma_{i_1}}{2}}\big) = \frac{\gamma_{i_1}}{2}$. The time instant, when the qth appearing time of $m_i(t)$ is detected, can be defined as a random variable

$$\mu_{i,q}^d = \inf\left\{ t > \mu_{i,q} : r_i(t,\Delta t_i) \notin B_{\gamma_{i_1}}^A(\Delta t_i) \right\}. \tag{14}$$

We can summarize the rules for the detection of appearing time $\mu_{i,q}$ using (12) for the IF $m_i(t)$ as follows:

1) $\mu_{i,q} < \mu_{i,q}^d < \nu_{i,q}$;
2) $d\left(r_i(t,\Delta t_i), 0\right) \le h_{\frac{\gamma_{i_1}}{2}}$, $\forall t \in (\mu_{i,q}, \mu_{i,q}^d)$;
3) $d\left(r_i(\mu_{i,q}^d,\Delta t_i), 0\right) > h_{\frac{\gamma_{i_1}}{2}}$.

Here, $\Pi_{i_1} = \pm h_{\frac{\gamma_{i_1}}{2}}\sigma(\Delta t_i)$ is the detection threshold for the appearing time of $m_i(t)$ for given Δt_i and γ_{i_1}.

One of the most important challenges in detecting the qth appearing time of $m_i(t)$ is to determine $\mu_{i,q}^d$ before the qth disappearing time of $m_i(t)$. Therefore, we require that $\mu_{i,q}^d < \nu_{i,q}$ which can be used to formalize the diagnosability of the appearing time for IFs in a probabilistic sense.

As is demonstrated in [4], we can construct a confidence region $\widetilde{B}_{\vartheta_{i_1}}^A(\Delta t_i) \subset \mathbb{R}$ at each time t ($t \ge \mu_{i,q} + \Delta t_i$) such that $P\big(r_i(\Delta t_i) \in \widetilde{B}_{\vartheta_{i_1}}^A(t,\Delta t_i) \big| H_{i,1}^A\big) = 1-\vartheta_{i_1}$ by using the scheme (12). Owing to $p_{i0}(t,\Delta t_i) = \frac{M_i P_i b_i f_i(q)}{\lambda_i}(1-e^{\lambda_i \Delta t_i})$, when $\mu_{i,q} < t-\Delta t_i \le t \le \nu_{i,q}$, we can calculate the confidence region $\widetilde{B}_{\vartheta_{i_1}}^A(t,\Delta t_i)$ as

$$\begin{aligned} \widetilde{B}_{\vartheta_{i_1}}^A(\Delta t_i) &= \big(\widetilde{L}_{\vartheta_{i_1}}^A(\Delta t_i), \widetilde{U}_{\vartheta_{i_1}}^A(\Delta t_i)\big] \\ &= \big(\kappa_1(\Delta t_i) - h_{\frac{\vartheta_{i_1}}{2}}\sigma(\Delta t_i), \\ &\qquad \kappa_1(\Delta t_i) + h_{\frac{\vartheta_{i_1}}{2}}\sigma(\Delta t_i)\big], \end{aligned} \tag{15}$$

where $\kappa_1(\Delta t_i) = \frac{M_i P_i b_i f_i(q)}{\lambda_i}\left(1-e^{\lambda_i \Delta t_i}\right)$ [11]. Note that Δt_i is not only the length of the sliding window of $r_i(t,\Delta t_i)$ but also the allowable maximal detection delay for the appearing time of $m_i(t)$. Letting $\widetilde{\delta}_i^A \triangleq \inf\Big\{\Delta t_i > 0 : \widetilde{B}_{\vartheta_{i_1}}^A(\Delta t_i) \cap B_{\gamma_{i_1}}^A(\Delta t_i) = \varnothing\Big\}$ and $\widetilde{\mu}_{i,q}^d \triangleq \mu_{i,q} + \widetilde{\delta}_i^A$, then $\widetilde{\mu}_{i,q}^d$ is the worst allowable detection time of $\mu_{i,q}$ when $\Delta t_i = \widetilde{\delta}_i^A$ [6]. If $0 < \widetilde{\delta}_i^A \le \widetilde{\tau}_i^{\text{dur}}$, we can derive that $\widetilde{\mu}_{i,q}^d \le \nu_{i,q}$ which implies that the qth ($q \in \mathbb{N}^+$) appearing time of $m_i(t)$ can be detected before the qth IF disappearance time $\nu_{i,q}$ in the worst case. Consequently, we give the following definition, inspired by [4], [6].

Definition 1: Under Assumption 2, for given γ_{i_1} and ϑ_{i_1}, if $0 < \widetilde{\delta}_i^A \leq \widetilde{\tau}_i^{\text{dur}}$, then we say the appearing time of the IF $m_i(t)$ is probabilistically diagnosable or IFAP-diagnosable using the proposed scheme (4) and (12).

Inspired by [4], we can postulate the following theorem.

Theorem 1: If an IF $m_i(t)$ is IFAP-diagnosable, then

$$P(\mu_{i,q}^d \leq \overline{\mu}_{i,q}^d \leq \nu_{i,q}|H_{i,1}^A) \geq 1-\vartheta_{i_1}, \qquad (16)$$

where $\overline{\mu}_{i,q}^d = \mu_{i,q} + \Delta t_i$.

Proof: Apparantly, we have $\overline{\mu}_{i,q}^d \leq \nu_{i,q}$. By construction of the confidence region $\widetilde{B}_{\vartheta_{i_1}}^A(\Delta t_i)$ and the definition of $\overline{\mu}_{i,q}^d$, we can derive that

$$P\left(r_i(\overline{\mu}_{i,q}^d, \Delta t_i) \notin B_{\gamma_{i_1}}^A(\Delta t_i)|H_{i,1}^A\right)$$
$$\geq P\left(r_i(\overline{\mu}_{i,q}^d, \Delta t_i) \in \widetilde{B}_{\vartheta_{i_1}}^A(\Delta t_i)|H_{i,1}^A\right) = 1-\vartheta_{i_1}.$$

Thus, $P(\mu_{i,q}^d \leq \overline{\mu}_{i,q}^d \leq \nu_{i,q}|H_{i,1}^A) \geq 1-\vartheta_{i_1}$ is obtained. ■

Remark 2: In fact, (16) is related to the detection speed (or detection time) of the appearing time for $m_i(t)$ in a probabilistic sense. Note that $\mu_{i,q}^d$ is related to the given test size γ_{i_1} and ϑ_{i_1}. Then, by settling λ_i, γ_{i_1} and ϑ_{i_1} appropriately in (4) and (12), we are able to realize the expected detection speed of the appearing time for $m_i(t)$.

Based on Definition 1, we give the following theorem.

Theorem 2: Considering IFs satisfying Assumption 2, for given γ_{i_1}, ϑ_{i_1}, a sufficient condition to guarantee $\widetilde{\mu}_{i,q}^d \leq \nu_{i,q}$ for the IF $m_i(t)$ such that the appearing time of $m_i(t)$ is IFAP-diagnosable using the proposed scheme (4) and (12), is

i) Lemma 1 for ($l \leq n$) or Lemma 2 for ($l > n$) is satisfied;

ii) $\rho_i^2 \geq \xi_i^2 \pi_i$;

iii) $0 < \delta_i^A \leq \Delta t_i \leq \widetilde{\tau}_i^{\text{dur}}$;

where

$$\begin{cases} \xi_i = \frac{h_{\frac{\gamma_{i_1}}{2}} + h_{\frac{\vartheta_{i_1}}{2}}}{M_i P_i b_i}, \pi_i = \max\left\{\pi_{i_1}, \pi_{i_2}\right\}, \\ \delta_i^A = \frac{1}{\lambda_i} \ln \frac{4\epsilon_i - \sqrt{16\epsilon_i \eta_i \lambda_i^2 - 4{\eta_i}^2 \lambda_i^4 + 4\lambda^2(\alpha_i - \lambda_i\beta_i + \lambda_i^2\zeta_i)^2}}{4\epsilon_i - 2\lambda_i^2\eta_i - 2\lambda_i(\alpha_i - \lambda_i\beta_i + \lambda_i^2\zeta_i)}, \end{cases} \qquad (17)$$

and the relative parameters in (17) are defined as follows:

$$\begin{cases} \alpha_i = (M_i P_i Q_{0,i} - M_i A_{0,i} M_i^{-r} H_i) R_v \\ \qquad \times (M_i P_i Q_{0,i} - M_i A_{0,i} M_i^{-r} H_i)^T \\ \qquad + (M_i P_i E) R_w (M_i P_i E)^T, \\ \beta_i = (M_i P_i Q_{0,i} - M_i A_{0,i} M_i^{-r} H_i) R_v H_i^T \\ \qquad + H_i R_v (M_i P_i Q_{0,i} - M_i A_{0,i} M_i^{-r} H_i)^T, \\ \zeta_i = H_i R_v H_i^T, \eta_i = (H_i D) R_v (H_i D)^T, \\ \epsilon_i = \frac{\rho_i^2}{\xi_i^2}, \widetilde{\epsilon}_i = \frac{f_i^2(q)}{\xi_i^2}, \\ \pi_{i_1} = \frac{-\zeta_i \lambda_i^3 + (\beta_i + \eta_i)\lambda_i^2 - \alpha_i \lambda_i}{2}, \\ \pi_{i_2} = \frac{-{\zeta_i}^2 \lambda_i^4 + 2\beta_i \zeta_i \lambda_i^3 - ({\beta_i}^2 + 2\alpha_i \zeta_i - {\eta_i}^2)\lambda_i^2 + 2\alpha_i \beta_i \lambda_i - {\alpha_i}^2}{4\eta_i}. \end{cases} \qquad (18)$$

Proof: If Lemma 1 (for $l \leq n$) or Lemma 2 (for $l > n$) is satisfied, we can construct the novel residual $r_i(t, \Delta t_i)$ for each fault signature b_i such that $r_i(t, \Delta t_i)$ is decoupled from all the other IFs $m_j(t)$ ($j \neq i$) but affected by $m_i(t)$ and $w(t)$ and $v(t)$, according to (10).

In the case that $-M_iP_ib_if_i(q) > 0$, we have

$$\begin{aligned}&\mathbb{E}[r_i(t,\Delta t_i)|H^A_{i,1}] - \mathbb{E}[r_i(t,\Delta t_i)|H^A_{i,0}]\\ =&\int_{t-\Delta t}^{t} e^{\lambda_i(t-\tau)}[-M_iP_ib_im_i(\tau)]\mathrm{d}\tau \\ =&\kappa_1(\Delta t_i),\end{aligned} \tag{19}$$

according to (12) [6]. Consider the function

$$\begin{aligned}g(e^{\lambda_i\Delta t_i}) =&[2\epsilon_i - \lambda_i^2\eta_i - \lambda_i(\alpha_i - \lambda_i\beta_i + \lambda_i^2\zeta_i)]\\ &\times e^{2\lambda_i\Delta t_i} - 4\epsilon_i e^{\lambda_i\Delta t_i} + [2\epsilon_i - \lambda_i^2\eta_i\\ &+ \lambda_i(\alpha_i - \lambda_i\beta_i + \lambda_i^2\zeta_i)].\end{aligned} \tag{20}$$

Apparently, $(\alpha_i - \lambda_i\beta_i + \lambda_i^2\zeta_i)$ is positive because $\alpha_i - \lambda_i\beta_i + \lambda_i^2\zeta_i = (M_iP_iQ_{i,0} - M_iA_{i,0}M_i^{-r}H_i - \lambda_iH_i)R_v(M_iP_iQ_{i,0} - M_iA_{i,0}M_i^{-r}H_i - \lambda_iH_i)^T + (M_iP_iE_i)R_w(M_iP_iE_i)^T > 0$. According to ii) of Theorem 2, we can get

$$\begin{aligned}&[2\epsilon_i - \lambda_i^2\eta_i - \lambda_i(\alpha_i - \lambda_i\beta_i + \lambda_i^2\zeta_i)] >\\ &[2\epsilon_i - \lambda_i^2\eta_i + \lambda_i(\alpha_i - \lambda_i\beta_i + \lambda_i^2\zeta_i)] > 0,\end{aligned} \tag{21}$$

For $\Delta t_i \geq \delta_i^A$, it is easy to derive $e^{\lambda_i\Delta t_i} \leq e^{\lambda_i\delta_i^A}$. From (17), we have $g(e^{\lambda_i\delta_i^A}) = 0$. According to iii) of Theorem 2, we obtain $g(e^{\lambda_i\Delta t_i}) > 0$ for $e^{\lambda_i\Delta t_i} \leq e^{\lambda_i\delta_i^A}$, which implies $\forall\ \Delta t_i \geq \delta_i^A$, $g(e^{\lambda_i\Delta t_i}) = [2\epsilon_i - \lambda_i^2\eta_i - \lambda_i(\alpha_i - \lambda_i\beta_i + \lambda_i^2\zeta_i)]e^{2\lambda_i\Delta t_i} - 4\epsilon_i e^{\lambda_i\Delta t_i} + [2\epsilon_i - \lambda_i^2\eta_i + \lambda_i(\alpha_i - \lambda_i\beta_i + \lambda_i^2\zeta_i)] \geq 0$, i.e.

$$\begin{aligned}\epsilon_i \geq &-\frac{\lambda_i(1+e^{\lambda_i\Delta t_i})}{2(1-e^{\lambda_i\Delta t_i})}\alpha_i + \frac{\lambda_i^2(1+e^{\lambda_i\Delta t_i})}{2(1-e^{\lambda_i\Delta t_i})}\beta_i\\ &-\frac{\lambda_i^3(1+e^{\lambda_i\Delta t_i})}{2(1-e^{\lambda_i\Delta t_i})}\zeta_i + \frac{\lambda_i^2(1+e^{2\lambda_i\Delta t_i})}{(1-e^{\lambda_i\Delta t_i})^2}\eta_i.\end{aligned} \tag{22}$$

Combined with (11), it can be obtained that

$$\widetilde{\epsilon_i} \geq \epsilon_i \geq \frac{\lambda_i^2}{(1-e^{\lambda_i\Delta t_i})^2}\sigma^2(\Delta t_i). \tag{23}$$

Owing to that $-M_iP_ib_if_i(q) > 0$, (23) can be transformed to

$$\begin{aligned}&\frac{-M_iP_ib_if_i(q)}{-\lambda_i}(1-e^{\lambda_i\Delta t_i})\\ &\geq \left(h_{\frac{\gamma_i}{2}}\sigma(\Delta t_i) + h_{\frac{\vartheta_i}{2}}\sigma(\Delta t_i)\right).\end{aligned} \tag{24}$$

From (12), (17) and (24), we can derive that $\exists\delta_i^A \in (0, \widetilde{\tau}_i^{\mathrm{dur}}]$ such that $\forall\ \delta_i^A \leq \Delta t_i \leq \widetilde{\tau}_i^{\mathrm{dur}}$,

$$\begin{aligned}&\mathbb{E}[r_i(t,\Delta t_i)|H^A_{i,1}] - \mathbb{E}[r_i(t,\Delta t_i)|H^A_{i,0}]\\ &\geq h_{\frac{\gamma_i}{2}}\sigma(\Delta t_i) + h_{\frac{\vartheta_i}{2}}\sigma(\Delta t_i).\end{aligned} \tag{25}$$

Then,

$$\begin{aligned}&\left\{\Delta t_i : \widetilde{L}^A_{\vartheta_{i_1}}(\Delta t_i) \geq U^A_{\gamma_{i_1}}(\Delta t_i)\right\}\\ =&\left\{\Delta t_i : \mathbb{E}[r_i(t,\Delta t_i)|H^A_{i,1}] - h_{\frac{\gamma_i}{2}}\sigma(\Delta t_i) \geq \right.\\ &\left.\mathbb{E}[r_i(t,\Delta t_i)|H^A_{i,0}] + h_{\frac{\vartheta_i}{2}}\sigma(\Delta t_i)\right\}\\ =&\left\{\Delta t_i : \mathbb{E}[r_i(t,\Delta t_i)|H^A_{i,1}] - \mathbb{E}[r_i(t,\Delta t_i)|H^A_{i,0}]\right.\\ &\left.\geq h_{\frac{\gamma_i}{2}}\sigma(\Delta t_i) + h_{\frac{\vartheta_i}{2}}\sigma(\Delta t_i)\right\}\\ &\supseteq [\delta_i^A, \widetilde{\tau}_i^{\mathrm{dur}}] \neq \varnothing.\end{aligned} \tag{26}$$

Consequently, we obtain that $0 < \widetilde{\delta}_i^A \leq \delta_i^A \leq \widetilde{\tau}_i^{\text{dur}}$ implying $\widetilde{\mu}_{i,q}^d \leq \mu_{i,q}^d + \delta_i^A \leq \mu_{i,q} + \widetilde{\tau}_i^{\text{dur}} \leq \mu_{i,q}^d + \tau_q^{\text{dur}} \leq \nu_{i,q}$. We can get the same conclusion in a similar way in the case that $-M_i P_i b_i f_i(q) < 0$.

To sum up, we have verified that IFs $m_i(t)$ $(i = 1, \cdots, l)$ satisfying Theorem 2 are IFAP-diagnosable by using the proposed scheme (4) and (12). ■

Theorem 3: If IFs are IFAP-diagnosable using the proposed scheme (4) and (12), the minimal length of the allowable sliding window (also the minimal value of the allowable maximal detection time) for the detection of appearing time for the IF $m_i(t)$ is

$$\inf \Psi_i = \delta_i^A, \tag{27}$$

where δ_i^A is defined in (17), and

$$\Psi_i = \Big\{ \Delta t_i > 0 : \widetilde{B}_{\vartheta_{i_1}}^A(\Delta t_i) \cap B_{\gamma_{i_1}}^A(\Delta t_i) = \varnothing, \quad 0 < \Delta t_i \leq \widetilde{\tau}_i^{\text{dur}} \Big\}.$$

Proof: Apparently, the set of allowable Δt_i for the detection of appearing time for $m_i(t)$ is Ψ_i. In the case that $-M_i P_i b_i f_i(q) > 0$, we derive that

$$\Psi_i = \Big\{ \Delta t_i : \widetilde{L}_{\vartheta_{i_1}}^A(\Delta t_i) \geq U_{\gamma_{i_1}}^A(\Delta t_i), \quad 0 < \Delta t_i \leq \widetilde{\tau}_i^{\text{dur}} \Big\}. \tag{28}$$

By calculation of $\widetilde{L}_{\vartheta_{i_1}}^A(\Delta t_i)$ and $U_{\gamma_{i_1}}^A(\Delta t_i)$, we have $\widetilde{L}_{\vartheta_{i_1}}^A(\Delta t_i) \geq U_{\gamma_{i_1}}^A(\Delta t_i)$, which implies $\kappa_1(\Delta t_i) - h_{\frac{\vartheta_{i_1}}{2}} \sigma(\Delta t_i) \geq h_{\frac{\gamma_{i_1}}{2}} \sigma(\Delta t_i)$. Then, we can derive that $\frac{-M_i P_i b_i f_i(q)}{(h_{\frac{\gamma_{i_1}}{2}} + h_{\frac{\vartheta_{i_1}}{2}})} \geq \frac{-\lambda_i \sigma(\Delta t_i)}{(1-e^{\lambda_i \Delta t_i})} > 0$, i.e. $\widetilde{\epsilon}_i \geq \frac{\lambda_i^2 \sigma^2(\Delta t_i)}{(1-e^{\lambda_i \Delta t_i})^2}$. Note that $\widetilde{\epsilon}_i \geq \frac{\lambda_i^2 \sigma^2(\Delta t_i)}{(1-e^{\lambda_i \Delta t_i})^2}$ should be satisfied for all $f_i(q)$ $(q \in \mathbb{N}^+)$. That is, (22) must be satisfied. Considering $e^{\lambda_i \Delta t_i}$ $(0 < e^{\lambda_i \Delta t_i} < 1)$ as a whole variable, we can get that (22) is equal to $g(e^{\lambda_i \Delta t_i}) \geq 0$, when Theorem 2 is satisfied. For simplicity, defining

$$\begin{cases} \widetilde{\mathcal{D}}_i &= \alpha_i - \lambda_i \beta_i + \lambda_i^2 \zeta_i, \\ \widetilde{\mathcal{A}}_i &= 2\epsilon_i - \lambda_i^2 \eta_i - \lambda_i(\alpha_i - \lambda_i \beta_i + \lambda_i^2 \zeta_i), \\ \widetilde{\mathcal{B}}_i &= -4\epsilon_i, \\ \widetilde{\mathcal{C}}_i &= 2\epsilon_i - \lambda_i^2 \eta_i + \lambda_i(\alpha_i - \lambda_i \beta_i + \lambda_i^2 \zeta_i), \end{cases}$$

then we can get that $\widetilde{\mathcal{B}}_i < 0 < \widetilde{\mathcal{D}}_i$, $\widetilde{\mathcal{C}}_i < \widetilde{\mathcal{A}}_i$. Therefore, we arrive at

$$\Psi_i = \Big\{ \Delta t_i > 0 : g(e^{\lambda_i \Delta t_i}) = \widetilde{\mathcal{A}}_i e^{2\lambda_i \Delta t_i} + \widetilde{\mathcal{B}}_i e^{\lambda_i \Delta t_i} + \widetilde{\mathcal{C}}_i \geq 0, \quad 0 < \Delta t_i \leq \widetilde{\tau}_i^{\text{dur}} \Big\}. \tag{29}$$

Only in the case that Ψ_i is a nonempty set. Thus, we can derive that

$$\Psi_i = \left\{ \Delta t_i > 0 : e^{\lambda_i \Delta t_i} \leq \phi_i \right\}, \tag{30}$$

where $\phi_i = \frac{4\epsilon_i - \sqrt{16\epsilon_i \eta_i \lambda_i^2 - 4{\eta_i}^2 \lambda_i^4 + 4\lambda^2(\alpha_i - \lambda_i \beta_i + \lambda_i^2 \zeta_i)^2}}{4\epsilon_i - 2\lambda_i^2 \eta_i - 2\lambda_i(\alpha_i - \lambda_i \beta_i + \lambda_i^2 \zeta_i)}$.

In conclusion, we obtain that

$$\inf \Psi_i = \inf\{\Delta t_i : \delta_i^A \leq \Delta t_i \leq \widetilde{\tau}_i^{\text{dur}}\} = \delta_i^A. \tag{31}$$

In the case that $-M_i P_i b_i f_i(q) < 0$, we can get the same conclusion. Consequently, the minimal length of the allowable sliding window (also the minimal value of the allowable maximal detection time) for the detection of the qth appearing time for the IF $m_i(t)$ is δ_i^A. ■

D. The Detection of Disappearing Time for IFs

Note that $|f_i(q)| \geq \rho_i$. We can get that

$$|\mathbb{E}[r_i(t, \Delta t)]| \geq \frac{|-M_i P_i b_i| \, \rho_i}{-\lambda_i}(1 - e^{\lambda_i \Delta t_i}), \quad (32)$$

for $\mu_{i,q} \leq t - \Delta t_i \leq t \leq \nu_{i,q}$. Let $\kappa_0(\Delta t_i) = \frac{|-M_i P_i b_i| \rho_i}{-\lambda_i}(1 - e^{\lambda_i \Delta t_i})$. The hypothesis test for the detection of disappearing time $\nu_{i,q}$ for $m_i(t)$ is proposed as

$$\begin{cases} H_{i,0}^D : |\mathbb{E}[r_i(t, \Delta t)]| \geq \kappa_0(\Delta t_i), \\ H_{i,1}^D : |\mathbb{E}[r_i(t, \Delta t)]| < \kappa_0(\Delta t_i). \end{cases} \quad (33)$$

Similar to the analysis on the detection of appearing time for $m_i(t)$, for a given γ_{i_2}, we can calculate the test acceptance region of (33) as

$$\begin{aligned} & B_{\gamma_{i_2}}^D(\Delta t_i) \\ & = \big(-\infty, U_{\gamma_{i_2}}^D(\Delta t_i) \big] \cup \big[L_{\gamma_{i_2}}^D(\Delta t_i), +\infty \big) \\ & = \big(-\infty, -\kappa_0(\Delta t_i) + h_{\gamma_{i_2}} \sigma(\Delta t_i) \big] \\ & \quad \cup \big[\kappa_0(\Delta t_i) - h_{\gamma_{i_2}} \sigma(\Delta t_i), +\infty \big) . \end{aligned} \quad (34)$$

The time instant when the disappearing time of IFs is detected can be defined as the random variable $\nu_{i,q}^d = \inf \Big\{ t > \nu_{i,q} : r_i(t, \Delta t_i) \notin B_{\gamma_{i_2}}^D(\Delta t_i) \Big\}$. The hypothesis test decision for the detection of disappearing time $\nu_{i,q}$ is

i) $\nu_{i,q} < \nu_{i,q}^d < \mu_{i,q+1}$;

ii) $r_i(t, \Delta t_i) \in B_{\gamma_{i_2}}^D(\Delta t_i)$, $\forall t \in (\nu_{i,q}, \nu_{i,q}^d)$;

iii) $r_i(\nu_{i,q}^d, \Delta t_i) \notin B_{\gamma_{i_2}}^D(\Delta t_i)$.

Moreover, $\Pi_{i_2} = \pm \big(\kappa_0(\Delta t_i) - h_{\gamma_{i_2}} \sigma(\Delta t_i) \big)$ is the detection threshold for the disappearing time of $m_i(t)$ for given Δt_i and γ_{i_2}. To formalize the detection requirement $\nu_{i,q}^d < \mu_{i,q+1}$, we can also construct a $((1 - \vartheta_{i_2}) \times 100)\,\%$ confidence region $\widetilde{B}_{\vartheta_{i_2}}^D(\Delta t_i) \subset \mathbb{R}$ at time t

$$\begin{aligned} & \widetilde{B}_{\vartheta_{i_2}}^D(\Delta t) \\ & = \Big(\widetilde{L}_{\vartheta_{i_2}}^D(\Delta t_i), \widetilde{U}_{\vartheta_{i_2}}^D(\Delta t_i) \Big) \\ & = \big(\mathbb{E}(r_i(t, \Delta t) | H_{0,1}^D) - h_{\vartheta_{i_2}} \sigma(\Delta t_i), \\ & \qquad \mathbb{E}(r_i(t, \Delta t) | H_{0,1}^D) + h_{\vartheta_{i_2}} \sigma(\Delta t_i) \big) \\ & = \big(-h_{\vartheta_{i_2}} \sigma(\Delta t_i), h_{\vartheta_{i_2}} \sigma(\Delta t_i) \big) , \end{aligned} \quad (35)$$

such that $P(r_i(t, \Delta t_i) \in \widetilde{B}_{\vartheta_{i_2}}^D(\Delta t_i) | H_{i1}^D) < 1 - \vartheta_{i_2}$ for $\mu_{i,q} \leq t - \Delta t_i \leq t \leq \nu_{i,q}$.

Similar to Definition 1, we can give the following diagnosablility definition for the detection of disappearing time for $m_i(t)$ in a probabilistic sense, based on $\widetilde{\delta}_i^D = \inf\{\Delta t_i > 0 : \widetilde{B}_{\vartheta_{i_2}}^D(\Delta t_i) \cap B_{\gamma_{i_2}}^D(\Delta t_i) = \varnothing\}$ and $\widetilde{\nu}_{i,q}^d \triangleq \nu_{i,q} + \widetilde{\delta}_i^D$.

Definition 2: Under Assumption 2, for given γ_{i_2} and ϑ_{i_2}, if $0 < \widetilde{\delta}_i^D \leq \widetilde{\tau}_i^{\text{int}}$, then we say the disappearing time of the IF $m_i(t)$ is probabilistically diagnosable (IFDP-diagnosable) by using the proposed scheme (4) and (33).

Corollary 1: If an IF $m_i(t)$ is IFDP-diagnosable, then

$$P(\nu_{i,q}^d \leq \overline{\nu}_{i,q}^d \leq \mu_{i,q+1} | H_{i,1}^D) \geq 1 - \vartheta_{i_2}, \tag{36}$$

where $\overline{\nu}_{i,q}^d = \nu_{i,q} + \Delta t_i$.

Proof: The proof of Corollary 1 is similar to Theorem 1, which is omitted here. ■

Corollary 2: Considering IFs satisfying Assumption 2, for given γ_{i_2}, ϑ_{i_2}, a sufficient condition to assure $\widetilde{\nu}_{i,q}^d \leq \mu_{i,q}$ for the IF $m_i(t)$ such that the disappearing time of $m_i(t)$ is IFDP-diagnosable using the proposed scheme (4) and (33), is

i) Lemma 1 for ($l \leq n$) or Lemma 2 for ($l > n$) is satisfied;

ii) $\rho_i^2 > \hat{\xi}_i^2 \pi_i$;

iii) $0 < \delta_i^D \leq \Delta t_i \leq \widetilde{\tau}_i^{\text{int}}$;

where

$$\begin{cases} \hat{\xi}_i = \frac{h_{\gamma_{i_2}} + h_{\vartheta_{i_2}}}{M_i P_i b_i}, \hat{\epsilon}_i = \frac{\rho_i^2}{\hat{\xi}_i^2}, \\ \delta_i^D = \frac{1}{\lambda_i} \ln \frac{4\hat{\epsilon}_i - \sqrt{16\hat{\epsilon}_i \eta_i \lambda_i^2 - 4\eta_i{}^2 \lambda_i^4 + 4\lambda^2(\alpha_i - \lambda_i \beta_i + \lambda_i^2 \zeta_i)^2}}{4\hat{\epsilon}_i - 2\lambda_i^2 \eta_i - 2\lambda_i(\alpha_i - \lambda_i \beta_i + \lambda_i^2 \zeta_i)}, \end{cases} \tag{37}$$

and the relative parameters of α_i, β_i, ζ_i, η_i, and π_i are illustrated in (17).

Proof: Apparently, in the case that $-M_i P_i b_i f_i(q) > 0$, we have

$$\begin{aligned} \widetilde{\delta}_i^D &= \inf \left\{ \Delta t_i > 0 : \widetilde{B}_{\vartheta_{i_2}}^D(\Delta t_i) \cap B_{\gamma_{i_1}}^D(\Delta t_i) = \varnothing \right\} \\ &= \inf \left\{ \Delta t_i > 0 : \left(h_{\vartheta_{i_2}} + h_{\gamma_{i_2}} \right) \sigma(\Delta t_i) \right. \\ &\qquad \left. \leq \kappa_0(\Delta t_i) \right\}. \end{aligned}$$

As is shown in the proof of Theorem 2, we can derive that $\exists \delta_i^D \in \left(0, \widetilde{\tau}_i^{\text{int}}\right]$ such that

$$\begin{aligned} \widetilde{\delta}_i^D &= \inf \left\{ \Delta t_i > 0 : \widetilde{B}_{\vartheta_{i_2}}^D(\Delta t_i) \cap B_{\gamma_{i_1}}^D(\Delta t_i) = \varnothing \right\} \\ &\supseteq \left[\delta_i^D, \widetilde{\tau}_i^{\text{int}} \right] \neq \varnothing, \end{aligned}$$

if ii) and iii) are satisfied. That is, $0 < \widetilde{\delta}_i^D \leq \widetilde{\tau}_i^{\text{int}}$ is satisfied such that $\widetilde{\nu}_{i,q}^d < \mu_{i,q+1}$ be true. The same conclusion can be drawn up in the case that $-M_i P_i b_i f_i(q) > 0$. So the fault $m_i(t)$ are IFDP-diagnosable using the proposed scheme (4) and (33). ■

Corollary 3: If IFs are IFDP-diagnosable using the proposed scheme (4) and (33), the minimal length of the allowable sliding window (also the minimal value of the allowable maximal detection time) for the detection of disappearing time for $m_i(t)$ is

$$\begin{aligned} \inf \Big\{ & \Delta t_i > 0 : \widetilde{B}_{\vartheta_{i_2}}^D(\Delta t_i) \cap B_{\gamma_{i_2}}^D(\Delta t_i) = \varnothing, \\ & 0 < \Delta t_i \leq \widetilde{\tau}_i^{\text{int}} \Big\} = \delta_i^D, \end{aligned} \tag{38}$$

where δ_i^D is defined in (37).

Proof: The proof is similar to Theorem 3 and omitted here for space limitation. ■

E. Diagnosability of IFs

The detection and isolation of IFs means: i) to detect both the appearing time and the disappearing time of an IF; ii) to detect the appearing (disappearing) time of each IF before the subsequent disappearing (appearing) time; iii) to determine where the IFs happen. Therefore, combined the hypothesis test (12) with (33), we can detect all the appearing time and the disappearing time of IFs and determine where they happen using (4). Here, we define the diagnosability of IFs as follows.

Definition 3: Under Assumption 2, for given γ_{i_1}, ϑ_{i_1}, γ_{i_2} and ϑ_{i_2}, the IF $m_i(t)$ is probabilistically diagnosable (IFP-diagnosable) by using the proposed scheme (4), (12) and (33) if the IF $m_i(t)$ is IFAP-diagnosable and IFDP-diagnosable.

According to the constructions of $B^A_{\gamma_{i_1}}(\Delta t_i)$ and $B^D_{\gamma_{i_2}}(\Delta t_i)$, we can infer that $B^A_{\gamma_{i_1}}(\Delta t_i) \cap B^D_{\gamma_{i_2}}(\Delta t_i) = \varnothing$ should be satisfied for a settled sliding time window Δt_i [6]. Then, we can easily derive that the following sufficient diagnosability condition for IFs.

Theorem 4: Consider IFs satisfying Assumptions 1 and 2, for given γ_{i_1}, ϑ_{i_1}, γ_{i_2} and ϑ_{i_2}, a sufficient condition to assure the IFs are IFP-diagnosable using the scheme (4), (12) and (33), is

1) Lemma 1 for ($l \leq n$) or Lemma 2 for ($l > n$) is satisfied;

2) $\rho_i^2 \geq \pi_i \max\left\{\xi_i^2, \hat{\xi}_i^2, \breve{\xi}_i^2\right\}$;

3) $0 < \delta_i^* \leq \Delta t_i \leq \min\{\widetilde{\tau}_i^{\text{int}}, \widetilde{\tau}_i^{\text{dur}}\}$;

where

$$\begin{cases} \breve{\xi}_i = \dfrac{\frac{h_{\gamma_{i_1}} + h_{\gamma_{i_2}}}{2}}{M_i P_i b_i}, \breve{\epsilon}_i = \dfrac{\rho_i^2}{\breve{\xi}_i^2}, \\ \delta_i^C = \dfrac{1}{\lambda_i} \ln \dfrac{4\breve{\epsilon}_i - \sqrt{16\breve{\epsilon}_i\eta_i\lambda_i^2 - 4\eta_i{}^2\lambda_i^4 + 4\lambda^2(\alpha_i - \lambda_i\beta_i + \lambda_i^2\zeta_i)^2}}{4\breve{\epsilon}_i - 2\lambda_i^2\eta_i - 2\lambda_i(\alpha_i - \lambda_i\beta_i + \lambda_i^2\zeta_i)}, \\ \delta_i^* = \max\left\{\delta_i^A, \delta_i^D, \delta_i^C\right\}. \end{cases} \tag{39}$$

and the relative parameters of α_i, β_i, ζ_i, η_i, ϵ_i, and $\breve{\epsilon}_{r,i}$ are defined in (17), (37).

Proof: Similar to the proof of Theorem 2, we can easily calculate $\breve{\epsilon}_i$ and δ_i^C based on $\left\{B^A_{\gamma_{i_1}}(\Delta t_i) \cap B^D_{\gamma_{i_2}}(\Delta t_i) = \varnothing\right\}$. Combining Theorem 2 with Corollary 2, Theorem 4 can be proved apparently which is omitted here due to space limitation. ■

IV. Performance Analysis

There are three basic performance indices for each FDI method: the detection time, the false alarm rate and the missing detection rate.

A. The Detection Time

For an IFAP-diagnosable IF $m_i(t)$, we can infer that at least $((1-\vartheta_{i_1})\%)$ of the appearing time of $m_i(t)$ can be detected before the time instant $\overline{\mu}^d_{i,q}$ according to Theorem 1. Besides, in a probabilistic framework, the minimal value of the permissable maximal detection time Δt_i (illustrated as $\widetilde{\delta}_i^A$) for the appearing time of $m_i(t)$ can be lowered down by choosing λ_i. Similar conclusions can be drawn up for an IFDP-diagnosable IF $m_i(t)$.

B. False Alarm Rates and Missing Detection Rates

Based on the fact that it is impossible to always make a right decision to retain or reject null hypothesis, false alarm and missing detection are inevitable using our method. Moreover, these performance indices play an important role in setting some key parameters in the proposed scheme. Here, we will give an upper bound of each rate. Let R^i_{test} be the hypothesis test result for $r_i(t, \Delta t_i)$ at time t. Then, we can derive the following results.

1) The detection of appearing time for an IFAP-diagnosable IF at time t:

a) The false alarm of the detection of appearing time for an IFAP-diagnosable IF means the case in which we have detected the appearing time of the IF when there is no IF or the IF has disappeared actually. That is, we reject the null hypothesis $H^A_{i,0}$ when it is true. So the *false alarm rate of the detection of appearing time for an IFAP-diagnosable IF* by using the proposed method is

$$\begin{aligned} &P\left(R^i_{test} = H^A_{i,1} \mid H^A_{i,0}\right) \\ &= 1 - P\left(R^i_{test} = H^A_{i,0} \mid H^A_{i,0}\right) \\ &= \gamma_{i_1}. \end{aligned}$$

b) The missing detection of the detection of appearing time for an IFAP-diagnosable IF at time t is the event $\left\{\mu^d_{i,q} > \nu_{i,q} | t \geq \mu_{i,q}\right\}$ which contains the case $\mu^d_{i,q} = +\infty$ (implying that the qth appearing time has never been detected) and the case $\nu_{i,q} < \mu^d_{i,q} < +\infty$ (implying that the qth appearing time has been detected after the IF disappears). Thus, we have

$$\begin{aligned} P\left(\mu^d_{i,q} > \nu_{i,q} \middle| H^A_{i,1}\right) &\leq P\left(\mu^d_{i,q} > \widetilde{\mu}^d_{i,q} \middle| H^A_{i,1}\right) \\ &= P\left(r_i(\widetilde{\mu}^d_{i,q}, \Delta t_i) \in B^A_{\gamma_{i_1}}(\Delta t_i) \middle| H^A_{i_1}\right). \end{aligned}$$

In the case that $[-M_i P_i b_i f_i(q)] > 0$, we have $\widetilde{L}^A_{\vartheta_{i_1}}(\Delta t_i) > U^A_{\gamma_{i_1}}(\Delta t_i)$. Hence,

$$\begin{aligned} &P\left(r_i(\widetilde{\mu}^d_{i,q}, \Delta t_i) \in B^A_{\gamma_{i_1}}(\Delta t_i) \middle| H^A_{i_1}\right) \\ &= P\left(L^A_{\vartheta_{i_1}}(\Delta t_i) < r_i(\widetilde{\mu}^d_{i,q}, \Delta t_i) \leq U^A_{\gamma_{i_1}}(\Delta t_i) \middle| H^A_{i,1}\right) \\ &< P\left(r(\widetilde{\mu}^d_{i,q}, \Delta t_i) \leq \widetilde{L}^A_{\vartheta_{i_1}}(\Delta t_i) \middle| H^A_{i,1}\right) \\ &= \frac{\vartheta_{i_1}}{2}. \end{aligned}$$

Therefore, *the missing detection rate of the detection of appearing time for an IFAP-diagnosable IF at time t* is $P\left(\mu^d_{i,q} > \nu_{i,q} \middle| H^A_{i,1}\right) < \frac{\vartheta_{i_1}}{2}$.

2) The detection of disappearing time for an IFDP-diagnosable IF at time t:

a) The false alarm of the detection of disappearing time for an IFDP-diagnosable IF at time t is the case where we have detected the disappearing time of an IF when it has not disappeared. Thus, the *false alarm rate of the detection of disappearing time for an IFDP-diagnosable IF at* *time* t is

$$P\left(R^i_{\textit{test}} = H^D_{i,1} \mid H^D_{i,0}\right)$$
$$= 1 - P\left(r_i(t, \Delta t_i) \notin B^D_{\gamma_{i_1}}(\Delta t_i) | H^D_{i,0}\right)$$
$$< \gamma_{i_2}.$$

b) Similarly, the missing detection rate of the detection of disappearing time for an IFDP-diagnosable IF at time t is the probability of the event $\left\{\nu^d_{i,q} > \mu_{i,q+1} | t \geq \nu_{i,q}\right\}$. In the case that $[-M_i P_i b_i f_i(q)] > 0$, we have $\widetilde{L}^A_{\vartheta_{i_1}}(\Delta t_i) > U^A_{\gamma_{i_1}}(\Delta t_i)$. Therefore, we can derive that

$$P\left(\nu^d_{i,q} > \mu_{i,q+1} \middle| H^D_{i,1}\right) \leq P\left(\nu^d_{i,q} > \widetilde{\nu}^d_{i,q} \middle| H^D_{i,1}\right)$$
$$= P\left(r(\widetilde{\nu}^d_{i,q}, \Delta t_i) \geq L^D_{\gamma_{i_1}}(\Delta t_i) \middle| H^D_{i,1}\right)$$
$$\leq P\left(r(\widetilde{\nu}^d_{i,q}, \Delta t_i) \geq \widetilde{U}^D_{\vartheta_{i_2}}(\Delta t_i) \middle| H^D_{i,1}\right)$$
$$= \vartheta_{i_2}.$$

Consequently, the *missing detection rate of the detection of disappearing time for an IFDP-diagnosable IF at time* t is $P(\nu^d_{i,q} > \mu_{i,q+1} | H^D_{i,1}) \leq \vartheta_{i_2}$.

C. The Influences of Some Key Parameters on Performance Indices

In this section, we will quantitatively analyze the influences of some key parameters of the proposed scheme on performance indices such as the false alarm rate and the missing detection rate.

Some key parameters of our proposed method, γ_{i_1}, ϑ_{i_1}, γ_{i_2}, ϑ_{i_2}, λ_i and Δt_i, can be settled according to practical detection demands. As illustrated in [4], these parameters can be selected depending on whether a false alarm or a quick fault detection is of more importance. According to J. Neyman and E. S. Pearson's criterion [25], it is necessary to choose a small value of γ_{i_1} and γ_{i_2} to avoid the primary Type I error (the fault alarm rate), while we try to lessen the Type II error (the missing detection rate) as possible as we can. Consequently, we can propose the following two theorems to analyze the influences of some key parameters of the proposed scheme on these performance indices.

Theorem 5: For given γ_{i_1}, ρ_i and δ_i, the minimal value of feasible ϑ_{i_1} satisfies

$$\begin{aligned}\vartheta_{i_1\min} > 2\Bigg[&\Phi\left(h_{\frac{\gamma_{i_1}}{2}} - \frac{\kappa_0(\delta_i)}{\sigma_i(\delta_i)}\right)\\ &+\Phi\left(h_{\frac{\gamma_{i_1}}{2}} + \frac{\kappa_0(\delta_i)}{\sigma_i(\delta_i)}\right) - 1\Bigg],\end{aligned} \tag{40}$$

where $\kappa_0(\delta_i) = \frac{|M_i P_i b_i| \rho_i}{-\lambda_i}\left(1 - e^{\lambda_i \delta_i}\right)$, and $\sigma_i(\delta_i) = \sqrt{Var[r_i(t, \delta_i)]}$.

Proof: When $t - \Delta t_i \leq \mu_{i,q} < t \leq \nu_{i,q}$, let $\Delta t^{(1)}_i = t - \mu_{i,q}$. It is apparent to derive that $\mathbb{E}\left(r_i(t, \Delta t_i) \middle| H^A_{i,0}\right) = 0$ and $\mathbb{E}\left(r_i(t, \Delta t_i) \middle| H^A_{i,1}\right) = \frac{-M_i P_i b_i f_i(q)}{-\lambda_i}(1 - e^{\lambda_i \Delta t^{(1)}_i})$. Define

$$\begin{aligned}\varrho_{i_1}(\Delta t^{(1)}_i, \Delta t_i) \triangleq \Big|&\mathbb{E}\left(r_i(t, \Delta t_i) \middle| H^A_{i,1}\right)\\ &- \mathbb{E}\left(r_i(t, \Delta t_i) \middle| H^A_{i,0}\right)\Big|\\ &= \frac{|M_i P_i b_i| f_i(q)}{-\lambda_i}(1 - e^{\lambda_i \Delta t^{(1)}_i}).\end{aligned}$$

Letting $P_{i_1}(\Delta t_i, \Delta t_i^{(1)})$ be the probability of the Type II error, then, we get $P_{i_1}(\Delta t_i, \Delta t_i^{(1)}) = P\left(R_{test}^i = H_{i,0}^A | H_{i,1}^A\right) < \frac{\vartheta_{i_1}}{2}$. According to [25], it can be derived that $P_{i_1}(\Delta t_i, \Delta t_i^1)$ is becoming smaller as $\varrho_{i_1}(\Delta t_i^{(1)}, \Delta t_i)$ is getting bigger. Apparently, considering $|f_i(q)| \geq \rho_i$, we have $\varrho_{i_1}(\Delta t_i^{(1)}, \Delta t_i)_{\max} = \kappa_0(\delta_i)$, when $\Delta t_i^{(1)} = \Delta t_i = \delta_i$ for all $f_i(q)$ ($q \in \mathbb{N}^+$). In this case, we can calculate $P_{i_1}(\Delta t_i, \Delta t_i^{(1)})_{\min}$ as follows.

$$
\begin{aligned}
P_{i_1}(\delta_i) &\triangleq P_{i_1}(\Delta t_i, \Delta t_i^{(1)})_{\min} \\
&= P\left(r_i(t,\delta_i) \in B_{\gamma_{i_1}}^A(\delta_i) \mid H_{i_1}^A\right) \\
&= P\left(-h_{\frac{\gamma_{i_1}}{2}}\sigma_i(\delta_i) < r(t,\delta_i)\right. \\
&\qquad\qquad \left.\leq h_{\frac{\gamma_{i_1}}{2}}\sigma_i(\delta_i) \middle| H_{i,1}^A\right).
\end{aligned} \tag{41}
$$

Let $\Phi(\cdot)$ be the standard normal distribution. Then, we have

$$
\begin{aligned}
P_{i_1}(\delta_i) = &\Phi\left(h_{\frac{\gamma_{i_1}}{2}} - \frac{\kappa_0(\delta_i)}{\sigma_i(\delta_i)}\right) \\
&- \Phi\left(-h_{\frac{\gamma_{i_1}}{2}} - \frac{\kappa_0(\delta_i)}{\sigma_i(\delta_i)}\right) \\
= &\Phi\left(h_{\frac{\gamma_{i_1}}{2}} - \frac{\kappa_0(\delta_i)}{\sigma_i(\delta_i)}\right) \\
&+ \Phi\left(h_{\frac{\gamma_{i_1}}{2}} + \frac{\kappa_0(\delta_i)}{\sigma_i(\delta_i)}\right) - 1.
\end{aligned} \tag{42}
$$

Consequently, we can derive that $\vartheta_{i_1\min} > 2P_{i_1}(\delta_i)$, i.e.

$$
\begin{aligned}
\vartheta_{i_1\min} > 2\Bigg[&\Phi\left(h_{\frac{\gamma_{i_1}}{2}} - \frac{\kappa_0(\delta_i)}{\sigma_i(\delta_i)}\right) \\
&+\Phi\left(h_{\frac{\gamma_{i_1}}{2}} + \frac{\kappa_0(\delta_i)}{\sigma_i(\delta_i)}\right) - 1\Bigg].
\end{aligned} \tag{43}
$$

■

Similarly, we have

Corollary 4: For given γ_{i_2}, ρ_i and δ_i, the minimal value of feasible ϑ_{i_2} satisfies

$$
\vartheta_{i_2\min} > \Phi\left(h_{\gamma_{i_2}} - \frac{\kappa_0(\delta_i)}{\sigma_i(\delta_i)}\right). \tag{44}
$$

Proof: The proof is similar to Theorem 6 and is omitted here. ■

V. Simulation Example

In this section, our proposed FDI scheme for IFs is applied to an unmanned helicopter longitudinal control system. This closed-loop control system consists of two main rotors rotated by two servo mechanisms. Here we consider the FDI problem of IFs for each actuator of this system.

$$
\begin{cases}
\dot{x}(t) &= Ax(t) + Bu(t) \\
&\quad + \sum_i^2 b_i m_i(t) + w(t), \\
y(t) &= Cx(t) + v(t),
\end{cases} \tag{45}
$$

where $x(t) = \begin{bmatrix} h_s(t) & v_s(t) & p_s(t) & \theta_d(t) \end{bmatrix}^T$ is the system state with initial condition $x(0) = \begin{bmatrix} 0.1 & 0.2 & 0 & 0.1 \end{bmatrix}^T$ consisting of the horizontal velocity $h_s(t)$ (km/s), the vertical velocity $v_s(t)$ (km/s), the pitch velocity $p_s(t)$ (deg/s), and the pitching angle $\theta_d(t)$ (deg). $u(t) = \begin{bmatrix} \delta_c(t) & \delta_e(t) \end{bmatrix}$ is the input vector where $\delta_c(t)$ is the collective pitch (deg) and $\delta_e(t)$ is the longitudinal cyclic pitch (deg); $w(t)$ and $v(t)$ are the Gauss distributed process noises and measurement noises with covariances, R_w and R_v; b_i is the actuator fault signature which is the corresponding column of the input matrix B; $m_i(t)$ is the IF mode where $m_1(t)$ is the IF signal of the collective pitch and $m_2(t)$ is the IF signal of the longitudinal cycle variable pitch. The constant system matrices A, B, C are shown as follows:

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.707 & 1.420 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0.0000 & 0.0000 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

$$R_w = 0.1 \times \begin{bmatrix} 1 & 0.5 & 0.2 & 0.1 \\ 0.5 & 1 & 0.2 & 0.1 \\ 0.2 & 0.1 & 1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 1 \end{bmatrix},$$

$$R_v = 0.01 \times \begin{bmatrix} 1 & 0.3 & 0.1 & 0.2 \\ 0.3 & 1 & 0.2 & 0.1 \\ 0.1 & 0.2 & 1 & 0.2 \\ 0.2 & 0.1 & 0.2 & 1 \end{bmatrix}.$$

To achieve precise trajectory tracking of the unmanned helicopter, we design the following state feedback tracking control law $u(t) = Kx(t) - K_r y_r(t)$ according to the target flight path $y_r(t)$. The matrices of the controller are

$$K = \begin{bmatrix} -0.1455 & 0.0148 & 0.2435 & 0.4478 \\ 0.0864 & 0.0575 & -0.1292 & -0.3763 \end{bmatrix},$$

$$K_r = \begin{bmatrix} 0.1990 & -0.0233 & 0 & -0.0268 \\ -0.0584 & -0.1100 & 0 & -0.1037 \end{bmatrix}.$$

The input signal and the output signal of system (45) are illustrated in Fig. 1 and Fig. 2.

According to Lemma 1, it is easy to verify that $\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$ are feasible solutions of (8). So a set of two observers that is decoupled from either actuator fault signature can be designed. In fact, by introducing the virtual fault signatures $L_{i_1} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$ and $L_{i_2} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$ $(i = 1, 2)$, the order of either observer is reduced from 3 to 1, which makes it convenient for us to analyze their outputs. And some of the parameters, G_i, H_i, M_i and K_i, are shown as follows:

$$G_1 = \begin{bmatrix} 0.5243 & 0.0000 \end{bmatrix}, G_2 = \begin{bmatrix} -0.0000 & 1.1146 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0.9997 & 0.0232 & -0.0000 & -0.0000 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0.9923 & -0.1238 & 0.0000 & 0.0000 \end{bmatrix},$$

$$M_1 = M_2 = 1, K_1 = K_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

We set $\gamma_{1,1} = \gamma_{1,2} = \gamma_{2,1} = \gamma_{2,2} = 0.05$, $\vartheta_{1,1} = \vartheta_{1,2} = \vartheta_{2,1} = \vartheta_{2,2} = 0.1$, respectively. Based on Theorem 4, we can give a feasible IFP-diagnosable region of IFs for system (45) (shown in Fig. 4) once λ_i is determined. We can conclude that the there is a minimal value of IF magnitudes ρ_i for each IFP-diagnosable IF, which is the primary diagnosable condition. Based on a given ρ_i, we can detect IFs with duration (interval) time longer than δ_i^*. For example, we can detect the practical IFs of $\delta_1^* \geq 2.037$s when $\rho_1 = 1.5$, and $\delta_1^* \geq 1.065$s when $\rho_1 = 1$, from Fig. 4. Besides, it is noted that λ_i can be designed for different FDI requirements according to Theorem 5 and Corollary 4. Here, we set $\lambda_1 = -0.3$ and $\lambda_2 = -0.5$ for instance. It is easy to verify that the IF $m_1(t)$ ($\rho_1 = 1.6$, $\delta_1 = 2.2$s) and the IF $m_2(t)$ ($\rho_2 = 1$, $\delta_1 = 1.4$s) fulfil the conditions in Theorem 4 by choosing $\Delta t_1 = 2.0$s and $\Delta t_2 = 1.2$s.

The detection results using the proposed scheme are illustrated in Fig. 5 $\sim$ Fig. 8. Comparing the novel residual $r_i(t, \Delta t_i)$ with the IF signal $m_i(t)$, we can find that $r_1(t, \Delta t_1)$ is robust to $m_2(t)$ and $r_2(t, \Delta t_1)$ robust to $m_1(t)$. So we can easily isolate IFs in different actuator fault signatures. Meanwhile, we can find that by introducing an appropriate sliding time window Δt_i, the novel residual $r_i(t, \Delta t_i)$ is more effective to be used to detect the IF $m_i(t)$ than the output of the

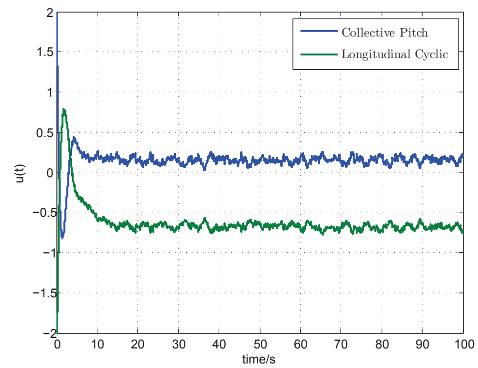

Fig. 1. The state feedback tracking control law.

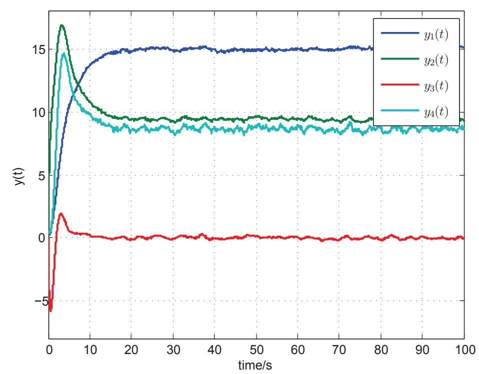

Fig. 2. The output of the normal system.

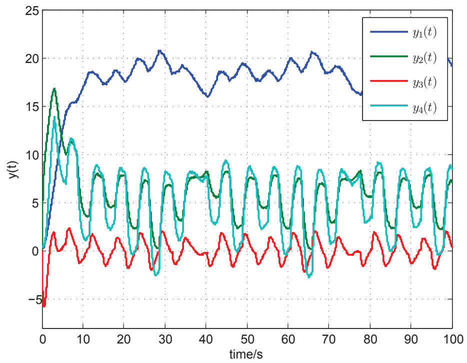

Fig. 3. The system output when IFs happens.

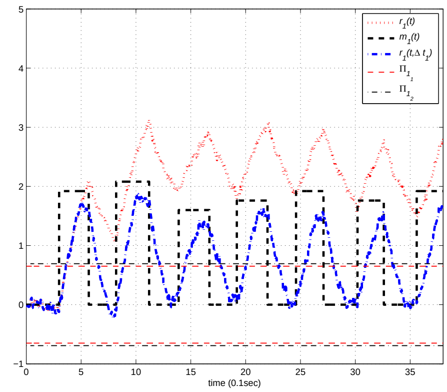

Fig. 5. The residuals of $r_1(t, \Delta t_1)$ when an IF happens in the fault signature b_1.

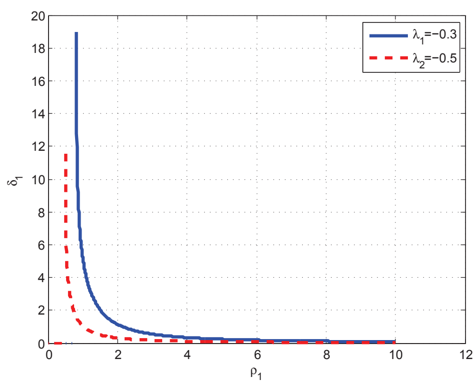

Fig. 4. The diagnosability analysis of IFs by using the proposed scheme.

observer $r_i(t)$. Take the detection results of $m_2(t)$ as example. As illustrated in Fig. 8, both $r_2(t, \Delta t_2)$ and $r_2(t)$ exceed the detection threshold Π_{2_1} so fast that the qth appearing time of $m_2(t)$ ($\mu_{2,q}$) can be detected before $\nu_{2,q}$. But after $\nu_{2,q}$, $r_2(t)$ takes so much time to reach below the detection threshold of Π_{2_2} that $r_2(t)$ is still beyond Π_{2_2} when $t = \mu_{2,q+1}$. But $r_2(t, \Delta t_2)$ decays below the detection threshold of Π_{2_2} before $\overline{\nu}^d_{i,q}$ ($\overline{\nu}^d_{i,q} \leq \nu^d_{i,q}$). As a result, all the appearing (disappearing) time of $m_i(t)$ can be detected before the subsequent disappearing (appearing) time. Besides, from the detection results, it should be noted that there is somehow a contradiction between the detection of appearing time and the detection of disappearing time: an IF with a larger magnitude is easier to detect its appearing time but harder to detect its disappearing time before the subsequent appearing time. That is one of the reasons why traditional FDI methods are not effective to solve the FDI problem of IFs in stochastic systems. The related detection results are also illustrated in Table I, where I, $\overline{\mu}^d_{i,q}$ and $\overline{\nu}^d_{i,q}$ are upper bounds of $\mu^d_{i,q}$ and $\nu^d_{i,q}$, respectively.

In order to show the effectiveness of the pro-

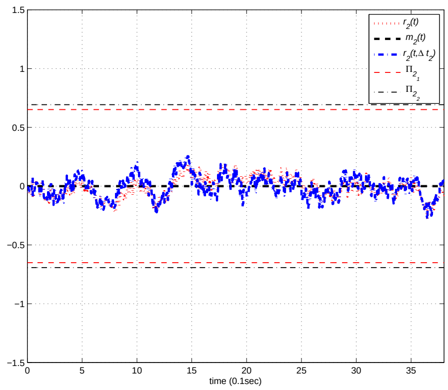

Fig. 6. The residuals of $r_2(t, \Delta t_2)$ when an IF happens in the fault signature b_1.

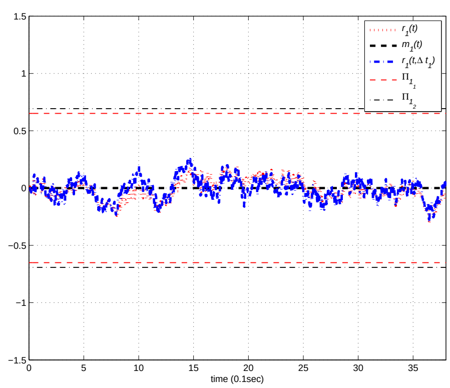

Fig. 7. The residuals of $r_1(t, \Delta t_1)$ when an IF happens in the fault signature b_2.

posed scheme, we also utilize the classical Kalman filter to solve the FDI problem of IFs in fault signatures, b_1 and b_2. The residuals are generated by $r_i(k) = y(k) - \hat{y}(k)$ where the process noises and measurement noises are much smaller than those in system (1). Considering that the Kalman

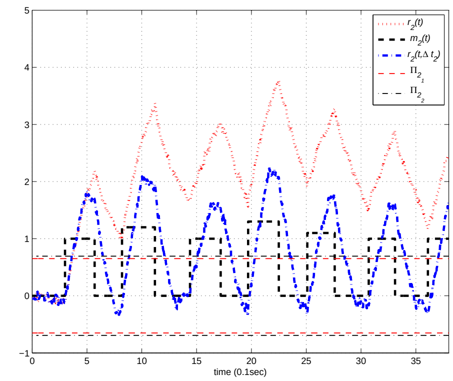

Fig. 8. The residuals of $r_2(t, \Delta t_2)$ when an IF happens in the fault signature b_2.

TABLE I

DETECTION RESULTS USING THE PROPOSED DETECTION SCHEME.

q_{th}	$\mu_{1,q}$	$\mu^d_{1,q}$	$\overline{\mu}^d_{1,q}$	$\nu_{1,q}$	$\nu^d_{1,q}$	$\overline{\nu}^d_{1,q}$
1	3.00	3.59	5.20	5.70	5.48	7.90
2	8.20	8.87	10.40	11.20	12.13	13.40
3	13.90	14.45	16.10	16.70	17.66	18.90
4	19.20	20.06	21.40	22.00	22.86	24.20
5	24.60	25.36	26.80	27.10	28.00	29.30
6	30.20	90.91	32.40	32.60	33.37	34.80
q_{th}	$\mu_{2,q}$	$\mu^d_{2,q}$	$\overline{\mu}^d_{2,q}$	$\nu_{2,q}$	$\nu^d_{2,q}$	$\overline{\nu}^d_{2,q}$
1	3.00	3.60	4.40	5.70	6.43	7.10
2	8.20	8.82	9.60	11.20	12.06	12.60
3	14.40	15.10	15.80	17.20	18.15	18.60
4	19.70	20.28	21.10	22.50	23.50	23.90
5	25.10	25.90	26.50	27.60	28.38	29.00
6	30.70	31.45	32.10	33.10	34.01	34.50

filter based fault detection method can not perform the isolation of IFs in different fault signatures. Here, we only discuss the effectiveness of detecting the appearing (disappearing) time of IFs. For the same IFs, $m_1(t)$ and $m_2(t)$, the detection results are shown in Fig. 9 and Fig. 10. Fig. 9 shows that the classical Kalman filter based fault detection method can not detect both the appearing time and the disappearing time of $m_1(t)$ while the appearing (disappearing) time of $m_2(t)$ can be detected at a high false alarm rate and missing detection rate in Fig. 10. Note that the residual $r_i(t, \Delta t_i)$ can decay fast to zero which makes it possible to detect the appearing (disappearing) time of an IF before the subsequent disappearing (appearing) time. Consequently, we can conclude that the proposed method is more effective to detect all the appearing (disappearing) time.

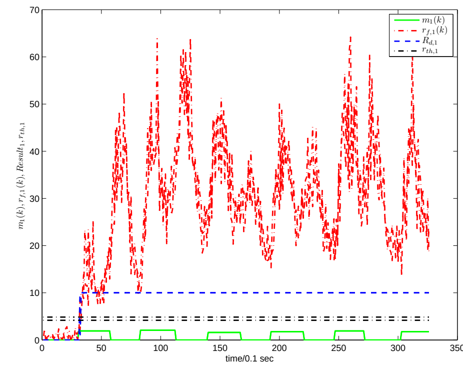

Fig. 9. The detection results of an IF in the fault signature b_1 by using the Kalman filter.

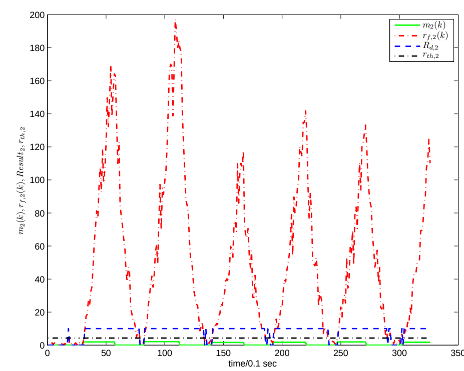

Fig. 10. The detection results of an IFs in the fault signature b_2 by using the Kalman filter.

VI. Conclusions

In this paper, we have addressed the FDI problem for IFs in linear stochastic systems with measurement noises. A novel scheme has been proposed to detect all the appearing (disappearing) time before the subsequent disappearing (appearing) time of IFs and isolate them. In order to isolate IFs, a novel set of residuals has been constructed based on the outputs of observers. Then two hypothesis tests have been provided to detect the appearing time and the disappearing time respectively. Within a probabilistic framework, the diagnosability of IFs has been defined, based on which a sufficient condition has been obtained for a class of IFs. Furthermore, performance indices such as the detection speed, the false alarm rate and the missing detection rate have been analyzed quantitatively and the effects of some key parameters of the given scheme on these performance indices are analyzed rigorously. Finally, a simulation example of an unmanned helicopter longitudinal control system has been illustrated to show how to perform the proposed scheme and demonstrate the validity of the scheme.

Since IFs are very common in modern industrial processes, the problem of diagnosing IFs is appealing to us both in theory and practice. The research in this area is emerging and has attracted more and more research attentions. The following topics towards the IFs are promising in future: 1) the FDI of vector IFs in complex systems is an interesting problem since the IFs considered in this paper is assumed to be the scalar type; 2) the diagnosis of IFs with small magnitudes is worthy to investigate. From Theorem 4, it can be seen that incipient IFs are harder to detect and isolate compared with the IFs with large magnitudes; 3) the investigation of diagnosis for IFs in nonlinear systems; 4) the derivation of fast diagnosis methodology for IFs.

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