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Event-Based Input and State Estimation for Linear Discrete Time-Varying Systems

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In this paper, the joint input and state estimation problem is considered for linear discrete-time stochastic systems. An event-based transmission scheme is proposed with which the current measurement is released to the estimator only when the difference from the previously transmitted one is greater than a prescribed threshold. The purpose of this paper is to design an event-based recursive input and state estimator such that the estimation error covariances have guaranteed upper bounds at all times. The estimator gains are calculated by solving two constrained optimization problems and the upper bounds of the estimation error covariances are obtained in form of the solution to Riccati-like difference equations. Special efforts are made on the choices of appropriate scalar parameter sequences in order to reduce the upper bounds. In the special case of linear time-invariant system, sufficient conditions are acquired under which the upper bound of the error covariance of the state estimation is asymptomatically bounded. Numerical simulations are conducted to illustrate the effectiveness of the proposed estimation algorithm.

Keywords: Event-based estimation; input estimation; linear stochastic systems; recursive filter; time-varying systems.

1. Introduction

State estimation for stochastic systems is one of the fundamental problems in system and control theory. Among a variety of estimation schemes, the Kalman filtering and the H_∞ filtering algorithms are two most widely investigated ones that have been extensively applied in practice. The renowned Kalman filter provides optimal state estimates based on exact values of the system parameters and known input/output data Anderson & Moore (2005), and the popular H_∞ filter is capable of attenuating and rejecting the influence from the exogenous disturbances to the controlled output up to a given level. However, in practice, the exogenous inputs are usually unknown disturbances or unmodeled dynamics which may not be known *a prior*. In this case, both the traditional Kalman filter and the H_∞ filter Basin & Rodriguez-Ramirez (2012); Ding, Wang, Shen, & Shu (2012); Hu, Wang, Gao, & Stergioulas (2012); Wang, Shen, & Liu (2012) cannot yield an optimal state estimation for systems with unknown inputs. As such, it is desirable to design a new kind of filters capable of jointly estimating system states and unknown inputs.

The state and unknown input estimation problems have found wide applications in many areas such as fault detection and diagnosis Ding (2008), transportation management Yong, Zhu,& Frazzoli (2016) and geophysics Kitanidis (1987). So far, considerable research attention has been devoted to the problem of optimal filtering in the presence of unknown inputs and a rich body of literature has been available. In Kitanidis (1987), an optimal state estimator has been proposed

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for linear system under the assumption without requiring prior information of the unknown input. In Darouach & Zasadzinski (1997), a parameterized design method has been developed for the optimal state estimator, and both the stability and convergence conditions have been established for the designed estimator. Furthermore, the state estimation problem has been investigated in Cheng, Ye, Wang, & Zhou (2009); Darouach, Zasadzinski,& Boutayeb (2003) for linear system with direct feedthrough from the unknown input to the output. On the other hand, the *simultaneous* input and state estimation problem has been investigated in Gillijns & Moor (2007); Hsieh (2000).

It is worth pointing out that, up to now, almost all the results on optimal filtering problem with unknown inputs has implicitly adopted the time-based strategy whose main idea is to send the measurement data from the sensors to the filter at a *fixed* time interval. Due to its simplicity, the time-based strategy (or called periodic communication strategy) would facilitate the system analysis/design, and is acceptable for certain engineering systems where the energy supply of sensor and the bandwidth of the communication network are *not* a concern. However, in case of constrained resources, the time-based strategy might lead to unnecessary signal transmission and therefore cause a waste of energy consumption and bandwidth resource for sensors. For example, in networked control systems, the sensors are usually powered by batteries with limited capacity and the wireless communication networks are shared by many sensors with limited bandwidth Gungor & Hancke (2009); Gungor, Lu & Hancke (2010). As such, there is a need to develop more resource-efficient communication schemes.

In the past few years, the event-based strategy has become more and more popular for the sake of energy-saving because it provides the possibility of maintaining system performance under limited communication resources. With the event-based strategy, a sensor is triggered to send the measurement data if and only if certain events occur. Recently, the event-based state estimation problem has attracted considerable research interest and a number of research results have been reported Hu & Yue (2012); Liu, Wang, He, & Zhou (2014, 2015); Meng & Chen (2014); Shi, Chen, & Shi (2014); Sijs & Lazar (2012); Suh & Nguyen (2007); Wu, Jia, Johansson, & Shi (2013); Zhang & Han (2015). Nevertheless, the event-based transmission scheme does complicate the estimation problem considerably especially when no measurements are received by the estimator between two consecutive event-triggered instants. For this reason, a common assumption made in the literature is the Gaussian approximation which simplifies the estimator design. For example, in Suh & Nguyen (2007), a modified Kalman filter has been proposed for the discrete time-invariant system with a send-on-delta (SOD) event triggering mechanism. In Sijs & Lazar (2012), with a general description of the event-based strategy, an event-based estimator has been designed for the discrete time-invariant system using Gaussian sum approximations. Very recently, the maximum likelihood event-based estimation problem has been investigated in Shi et al. (2014). The assumption of Gaussian approximation, unfortunately, leads to the estimators with only approximate minimum mean square error, and one of the motivations of this paper is therefore to remove such an assumption by developing an efficient event-based estimator design algorithm in the presence of unknown inputs.

In the context of event-based estimation, by far, much research has been done for linear system, but the corresponding estimation problem *coupled with unknown input* has not yet received adequate research attention due mainly to the difficulty in handling the unknown input with no prior information. In addition, when the adoption of the event-based mechanism, the unbiasedness of both the input and the state estimate cannot be guaranteed in general, and the traditional time-based unbiased input/state estimator design methods are no longer applicable. As such, we are motivated to challenge the design problem of the joint input/state estimators according to the event-based strategy by employing a SOD concept Miskowicz (2006). Our aim is to obtain the joint input/state estimates that are precise within a known confidence interval even though only partial measurements at the event-triggered instants are accessible by the estimator.

The main contributions of this paper are highlighted as follows: *1) a joint input and state estimator is proposed for linear time-varying systems with unknown input based on a novel event-based* *transmission scheme; 2) the event indicator variable is introduced to reflect the triggering information and reduce the conservatism in the analysis of estimation performance; 3) upper bounds of the estimation error covariances are obtained recursively and then reduced by choosing proper scalar parameters and estimator gains according to a given procedure; and 4) the asymptotic boundedness of the obtained upper bounds is analysed for the linear time-invariant systems.*

Notations. The notations used throughout the paper are standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all $n \times m$ matrices, respectively. For a matrix $P \in \mathbb{R}^{n\times m}$, P^T, $\mathrm{Rk}\{P\}$, $P^+ \in \mathbb{R}^{n\times m}$ represent its transpose, rank and Moore-Penrose pseudo inverse, respectively. For square matrix A, A^{-1} (where A is invertible), $\mathrm{Tr}\{A\}$, and $\rho(A)$ represents its inverse, trace and spectral radius, respectively. I and 0 represent the identity matrix and a zero matrix with appropriate dimension, respectively. $\mathrm{diag}\{X_1, X_2, \ldots, X_n\}$ stands for a block-diagonal matrix with matrices $X_1, X_2, \ldots, X_n$ on the diagonal. $\mathbb{S}_+^n$ is the set of $n \times n$ positive semi-definite matrices. When $X \in \mathbb{S}_+^n$, we simply write $X \geq 0$. Similarly, $X \geq Y$ means $X - Y \geq 0$.

2. Problem Formulation And Preliminaries

Consider the following linear discrete time-varying system:

$$\begin{cases} x(k) = A(k-1)x(k-1) + G(k-1)d(k-1) + \omega(k-1) \\ y(k) = C(k)x(k) + \nu(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the system state, $d(k) \in \mathbb{R}^p$ is the unknown system input and $y(k) \in \mathbb{R}^m$ is the measurement output. The process noise $\omega(k) \in \mathbb{R}^n$ and the measurement noise $\nu(k) \in \mathbb{R}^m$ are assumed to be mutually uncorrelated, zero-mean random signals with known covariance matrices $W(k)$ and $R(k)$, respectively. The initial value $x(0)$ has mean $\bar{x}(0)$ and covariance $P(0|0)$. Without loss of generality, we follow Darouach & Zasadzinski (1997); Gillijns & Moor (2007); Kitanidis (1987), and assume that $m \geq p$ and $\mathrm{Rk}\{C(k)G(k-1)\} = \mathrm{Rk}\{G(k-1)\} = p$.

2.1 *Traditional unknown input and state estimator*

Up to now, lots of results have been developed with respect to the estimation problem with unknown input. The traditional unknown input and state estimator has the following general form:

$$\mathcal{E}_1 : \begin{cases} \hat{d}_t(k-1) = M_t(k)\big(y(k) - C(k)A(k-1)\hat{x}_t(k-1|k-1)\big) \\ \hat{x}_t(k|k-1) = A(k-1)\hat{x}_t(k-1|k-1) + G(k-1)\hat{d}_t(k-1) \\ \hat{x}_t(k|k) = \hat{x}_t(k|k-1) + K_t(k)\big(y(k) - C(k)\hat{x}_t(k|k-1)\big) \end{cases} \tag{2}$$

where $\hat{d}_t(k-1)$ is the estimate of the unknown input at time instant $k-1$, $\hat{x}_t(k|k-1)$ is the one-step prediction of $x(k)$ at time instant $k-1$, and $\hat{x}_t(k|k)$ is the estimate of $x(k)$ at time instant k with $\hat{x}_t(0|0) = \bar{x}(0)$. $M_t(k)$ and $K_t(k)$ are the estimator gain matrices to be determined at time instant k.

So far, to the best of the author's knowledge, almost all established results on unknown input and state estimation problem have been obtained according to the time-based mechanism whose idea is to send the measurements to the estimator at every time instant. Due to the resource limits on energy-consumption and communication bandwidth especially in wireless communication, the control system needs more energy-efficient and lower bitrate data transmission mechanisms than the time-based one. The event-based data transmission mechanism stands out as a promising solution to this issue because, with such a mechanism, only important measurements (rather than all measurements) are transmitted to accomplish the control/estimation tasks.

2.2 *Event-based unknown input and state estimator*

In order to reduce the energy consumption and communication burden, the measurement $y(k)$ is transmitted only when certain event generator is triggered. In this paper, the send-on-delta (SOD) triggering mechanism is adopted and characterized as follows.

Assume that the event triggering instants are $k_0, k_1, \dots,$ where $k_0 = 0$ is the initial time. Define $y_e(k) = y(k_j)$ for k $(k_j \le k \le k_{j+1})$ with the subscript "e" indicating event triggering. The sequence of event triggering instants $0 = k_0 \le k_1 \le \dots \le k_i \le \dots$ is determined iteratively by

$$k_{i+1} = \min\{k \in \mathbb{N} | k > k_i, \|y_e(k) - y(k)\|^2 > \sigma\} \tag{3}$$

where the threshold σ is a positive scalar.

Define $\delta(k) = y_e(k) - y(k)$. Under the event-based strategy, $\delta(k)$ will be reset to zero if the triggering condition is fulfilled. Consequently, the following inequality holds all the time:

$$\delta^T(k)\delta(k) \le \sigma. \tag{4}$$

With the event-based communication strategy, a recursive estimator for the system (1) is given as follows:

$$\mathcal{E}_2 : \begin{cases} \hat{d}_e(k-1) = M_e(k)\big(y_e(k) - C(k)A(k-1)\hat{x}_e(k-1|k-1)\big) \\ \hat{x}_e(k|k-1) = A(k-1)\hat{x}_e(k-1|k-1) + G(k-1)\hat{d}_e(k-1) \\ \hat{x}_e(k|k) = \hat{x}_e(k|k-1) + K_e(k)\big(y_e(k) - C(k)\hat{x}_e(k|k-1)\big) \end{cases} \tag{5}$$

where $\hat{d}_e(k-1)$ is the estimate of the unknown input at time instant $k-1$, $\hat{x}_e(k|k-1)$ is the one-step prediction of $x(k)$ at time instant $k-1$, and $\hat{x}_e(k|k)$ is the estimate of $x(k)$ at time instant k with $\hat{x}_e(0|0) = \bar{x}(0)$. $M_e(k)$ and $K_e(k)$ are the estimator gain matrices to be determined at time instant k.

In the event-based estimator $\mathcal{E}_2$, the input estimate $\hat{d}_e(k-1)$ is first obtained from $y_e(k)$ since $y_e(k)$ is the first event-triggered measurement that contains information about $d_e(k-1)$. Then, using both $\hat{d}_e(k-1)$ and the state estimate $\hat{x}_e(k-1|k-1)$, the a prior estimate $\hat{x}_e(k|k-1)$ is obtained. Finally, a posteriori estimate $\hat{x}_e(k|k)$ is obtained by updating $\hat{x}_e(k|k-1)$ with a correction term.

Substituting the first two equations into the last one in (5) leads to

$$\hat{x}_e(k|k) = A(k-1)\hat{x}_e(k-1|k-1) + L_e(k)\big(y_e(k) - C(k)A(k-1)\hat{x}_e(k-1|k-1)\big) \tag{6}$$

where

$$L_e(k) \triangleq K_e(k) + E(k)G(k-1)M_e(k), \quad E(k) \triangleq I - K_e(k)C(k). \tag{7}$$

Letting $\tilde{x}_e(k|k) = x(k) - \hat{x}_e(k|k)$, we have the following system that governs the estimation error dynamics:

$$\begin{aligned} \tilde{x}_e(k|k) =& A(k-1)x(k-1) + G(k-1)d(k-1) + \omega(k-1) \\ & - A(k-1)\hat{x}_e(k-1|k-1) - L_e(k)\big(y_e(k) - C(k)A(k-1)\hat{x}_e(k-1|k-1)\big). \end{aligned} \tag{8}$$

Noting that

$$y_e(k)-C(k)A(k-1)\hat{x}_e(k-1|k-1) = C(k)\big(A(k-1)\tilde{x}_e(k-1)+G(k-1)d(k-1)+\omega(k-1)\big)+\nu(k)+\delta(k), \tag{9}$$

(8) can be written as follows:

$$\begin{aligned}\tilde{x}_e(k|k) =&\big(I - L_e(k)C(k)\big)\big(A(k-1)\tilde{x}_e(k-1) + G(k-1)d(k-1) + \omega(k-1)\big)\\ &- L_e(k)(\nu(k) + \delta(k)).\end{aligned} \tag{10}$$

To eliminate the effect of the unknown input $d(k-1)$ on the state estimation error $\tilde{x}_e(k|k)$ in (10), the following lemma is needed.

Lemma 1: *For the designed event-based estimator $\mathcal{E}_2$ in (5), the estimation error $\tilde{x}_e(k|k)$ is unrelated to the unknown input $d(k)$ if the gain matrix $M_e(k)$ satisfies*

$$M_e(k)C(k)G(k-1) = I_p. \tag{11}$$

Proof. If the condition

$$(I - L_e(k)C(k))G(k-1) = 0 \tag{12}$$

holds, then (10) can be written as follows:

$$\tilde{x}_e(k|k) = \big(I - L_e(k)C(k)\big)\big(A(k-1)\tilde{x}_e(k-1) + \omega(k-1)\big) - L_e(k)(\nu(k) + \delta(k)), \tag{13}$$

which shows that the estimation error $\tilde{x}_e(k|k)$ is unrelated to the unknown input $d(k)$. In the following, we try to prove that if condition (11) holds, then (12) holds as well.

Noting $L_e(k) = K_e(k) + E(k)G(k-1)M_e(k)$ and $E(k) = I - K_e(k)C(k)$, it follows from (11) that

$$\begin{aligned}(I - L_e(k)C(k))G(k-1) &= G(k-1) - K_e(k)C(k)G(k-1) - E(k)G(k-1)\\ &= 0,\end{aligned} \tag{14}$$

and then the proof is complete. □

Remark 1: As pointed out in Gillijns & Moor (2007), in the traditional time-based estimator design, the estimator $\mathcal{E}_1$ is unbiased if and only if (11) is satisfied. It can be clearly seen that there exists at least one solution $M_e(k)$ to (11) under the assumption that $m \leq p$ and $m \geq p$ and $\mathrm{Rk}\{C(k)G(k-1)\} = \mathrm{Rk}\{G(k-1)\} = p$. On the other hand, in the event-based scenario, the state estimation error is not affected by the unknown input $d(k)$ when (11) holds.

Letting $\tilde{d}_e(k-1) = d(k-1) - \hat{d}_e(k-1)$, assume that condition (11) holds. Then, the input estimation error $\tilde{d}_e(k-1)$ is given as follows:

$$\tilde{d}_e(k-1) = -M_e(k)\big(C(k)A(k-1)\tilde{x}_e(k-1|k-1) + C(k)\omega(k-1) + \nu(k) + \delta(k)\big). \tag{15}$$

For presentation convenience, we denote

$$\begin{aligned}
\tilde{d}_e^u(k) &= \mathbb{E}\{\tilde{d}_e(k)\}, & \tilde{d}_e^s(k) &= \tilde{d}_e(k) - \tilde{d}_e^u(k),\\
\tilde{x}_e^u(k|k) &= \mathbb{E}\{\tilde{x}_e(k|k)\}, & \tilde{x}_e^s(k|k) &= \tilde{x}_e(k|k) - \tilde{x}_e^u(k|k),\\
\Sigma_e^u(k) &= \tilde{d}_e^u(k)(\tilde{d}_e^u(k))^T, & \Sigma_e^s(k) &= \mathbb{E}\{\tilde{d}_e^s(k)(\tilde{d}_e^s(k))^T\},\\
P_e(k|k) &= \mathbb{E}\{\tilde{x}_e(k|k)(\tilde{x}_e(k|k))^T\}, & \Sigma_e(k) &= \mathbb{E}\{\tilde{d}_e(k)(\tilde{d}_e(k))^T\},\\
P_e^u(k|k) &= \tilde{x}_e^u(k|k)(\tilde{x}_e^u(k|k))^T, & P_e^s(k|k) &= \mathbb{E}\{\tilde{x}_e^s(k|k)(\tilde{x}_e^s(k|k))^T\},
\end{aligned}$$

and then (13) and (15) can be rewritten in the following form:

$$\tilde{x}_e^s(k|k) = (I - L_e(k)C(k))(A(k-1)\tilde{x}_e^s(k-1|k-1) + \omega(k-1)) - L_e(k)\nu(k), \quad (16)$$

$$\tilde{x}_e^u(k|k) = (I - L_e(k)C(k))A(k-1)\tilde{x}_e^u(k-1|k-1) - L_e(k)\delta(k), \quad (17)$$

$$\tilde{d}_e^s(k-1) = -M_e(k)C(k)A(k-1)\tilde{x}_e^s(k-1|k-1) - M_e(k)C(k)\omega(k-1) - M_e(k)\nu(k), \quad (18)$$

$$\tilde{d}_e^u(k-1) = -M_e(k)C(k)A(k-1)\tilde{x}_e^u(k-1|k-1) - M_e(k)\delta(k) \quad (19)$$

where $\tilde{x}_e^s(k|k)$ and $\tilde{x}_e^u(k|k)$ represent the stochastic and deterministic parts of the state estimation error, respectively. Similarly, $\tilde{d}_e^s(k)$ and $\tilde{d}_e^u(k)$ represent the stochastic and deterministic parts of the input estimation error, respectively.

3. Estimator design

In this section, for the system (1) with the event-based estimator $\mathcal{E}_2$, we will first obtain the upper bounds of the error covariances of both the input and state estimates, and then look for appropriate gain matrices $M_e(k)$ and $K_e(k)$ such that the obtained upper bounds are minimized.

Before proceeding further, we introduce the following lemmas which will be used in subsequent developments.

Lemma 2: *Given two vectors $x, y \in \mathbb{R}^m$, the following inequality holds,*

$$(x+y)(x+y)^T \le (1+\varepsilon)xx^T + (1+\varepsilon^{-1})yy^T \quad (20)$$

where ε is an arbitrary positive scalar.

Proof. (20) follows from $(\sqrt{\varepsilon}x - \sqrt{\varepsilon^{-1}}y)(\sqrt{\varepsilon}x - \sqrt{\varepsilon^{-1}}y)^T \ge 0$ immediately. □

Lemma 3: *Define a matrix function $f : \mathbb{S}_+^n \mapsto \mathbb{R}$ as follows:*

$$f(X) = Tr\{(A^T X^{-1} A)^{-1}\}$$

where A is a given matrix of appropriate dimension and $A^T X^{-1} A$ is nonsingular. For two matrices $X_1, X_2 \in \mathbb{S}_+^n$, if $X_1 < X_2$, then $f(X_1) < f(X_2)$.

Proof. For two arbitrary positive definite matrices $X_1, X_2 \in \mathbb{S}_+^n$, assume that $X_1 < X_2$. Then, the following are true:

$$\begin{aligned}
0 < X_1 < X_2 \Rightarrow 0 < X_2^{-1} < X_1^{-1} \Rightarrow 0 < A^T X_2^{-1} A < A^T X_1^{-1} A\\
\Rightarrow 0 < (A^T X_1^{-1} A)^{-1} < (A^T X_2^{-1} A)^{-1} \Rightarrow f(X_1) < f(X_2).
\end{aligned}$$

The last deduction is based on the property that if matrix $A \le B$, then the inequality $\mathrm{Tr}\{A\} \le \mathrm{Tr}\{B\}$ holds as well. Now the proof is completed. □

Lemma 4: *[Anderson & Moore (2005)] Consider the following recursion equation*

$$P(k+1) = FP(k)F^T + Q$$

where matrix $P(k) \in \mathbb{R}^{n\times n}$*, and* F *and* Q *are known real matrices of appropriate dimensions. If* $|\lambda(F)| < 1$*, for arbitrary initial* $P(0)$*, we have* $\lim_{k\to\infty} P(k) = \bar{P}$ *where* $\bar{P}$ *is the solution to* $\bar{P} - F\bar{P}F^T = Q$.

Lemma 5: *Liu et al. (2015) For* $0 \le k \le N$*, suppose that* $X, Y \in \mathbb{R}^{n\times n}$, $X = X^T > 0$, $Y = Y^T > 0$, $\phi(X,k) = \phi^T(X,k) \in \mathbb{R}^{n\times n}$*. If*

$$\phi(X,k) \le \phi(Y,k), \quad \forall X \le Y, \tag{21}$$

then the solutions $M(k)$ *and* $N(k)$ *to the following difference equations*

$$M(k+1) \le \phi(M(k),k),\ N(k+1) = \phi(N(k),k),\ M(0) = N(0) > 0 \tag{22}$$

satisfy

$$M(k) \le N(k).$$

3.1 *Input estimation*

In this section, we consider the unknown input estimation problem. At time instant k, assume that $\hat{P}^u_e(k-1|k-1)$, the upper bound of estimation covariance matrix $P^u_e(k-1|k-1)$, is Known (the derivation of $\hat{P}^u_e(k-1|k-1)$ will be given in the next subsection). Given the event-based measurement $y_e(k)$, we aim to obtain the input estimate $\hat{d}_e(k)$ and an upper bound on the error covariance of the input estimate, and then we look for an appropriate estimation gain $M_e(k)$ which minimizes such an upper bound.

An upper bound on the error covariance of the input estimate is given in the following theorem.

Theorem 1: *Consider the linear system (1) and the event-based estimator* $\mathcal{E}_2$ *in* (5) *with event generator condition (3). Assume that the condition (11) is satisfied. For a given positive scalar sequence* $\{\varepsilon_1(k), k \in \mathbb{N}\}$*, an upper bound on the error covariance matrix of the input estimation* $\hat{\Sigma}_e(k-1)$ *is given by*

$$\hat{\Sigma}_e(k-1) = M_e(k)\Phi(k)M_e^T(k) \tag{23}$$

where

$$\Phi(k) = C(k)Q(k-1|k-1)C^T(k) + R(k) + (1+\varepsilon_1^{-1}(k))\sigma I,$$
$$Q(k-1|k-1) = A(k-1)\big(P^s_e(k-1|k-1) + (1+\varepsilon_1(k))\hat{P}^u_e(k-1)\big)A^T(k-1) + W(k-1).$$

Proof. First, let us derive the expression of $\Sigma_e(k-1)$. It follows from (18) and (19) that

$$\Sigma_e(k-1) = \Sigma_e^s(k-1) + \Sigma_e^u(k-1) \tag{24}$$

$$\Sigma_e^s(k-1) = M_e(k)\Big(C(k)(A(k-1)P_e^s(k-1|k-1)A^T(k-1) + W(k-1))C^T(k) + R(k)\Big)M_e^T(k) \tag{25}$$

$$\begin{aligned}\Sigma_e^u(k-1) = {} & M_e(k)\big(\delta(k)\delta^T(k) + C(k)A(k-1)P_e^u(k-1|k-1)A^T(k-1)C^T(k)\big)(M_e(k))^T \\ & +\big(M_e(k)C(k)A(k-1)\tilde{x}^u(k-1|k-1)\big)(M_e(k)\delta(k))^T \\ & +M_e(k)\delta(k)\big(M_e(k)C(k)A(k-1)\tilde{x}^u(k-1|k-1)\big)^T.\end{aligned} \tag{26}$$

Using Lemma 2, we obtain

$$\begin{aligned}&\big(M_e(k)C(k)A(k-1)\tilde{x}_e^u(k-1|k-1)\big)(M_e(k)\delta(k))^T + M_e(k)\delta(k)\big(M_e(k)C(k)A(k-1)\tilde{x}_e^u(k-1|k-1)\big)^T \\ \leq & M_e(k)\big(\varepsilon_1(k)C(k)A(k-1)P_e^u(k-1|k-1)A^T(k-1)C^T(k) + \varepsilon_1^{-1}(k)\delta(k)\delta^T(k)\big)M_e^T(k).\end{aligned} \tag{27}$$

Substituting (27) into (26) and noting that $P_e^u(k-1|k-1) \leq \hat{P}_e^u(k-1|k-1)$, we have

$$\Sigma_e(k-1) \leq \hat{\Sigma}_e(k-1),$$

where $\hat{\Sigma}_e(k-1)$ is given in (23). □

Now, we are ready to minimize the upper bound $\hat{\Sigma}_e(k-1)$ at each time instant by appropriately designing the estimator parameter $M_e(k)$.

Theorem 2: *Consider the linear system (1) and the event-based estimator $\mathcal{E}_2$ in* (5) *with event generator condition (3). If the parameter $M_e(k)$ is chosen as*

$$M_e(k) = \Pi^{-1}(k)G^T(k-1)C^T(k)\Phi^{-1}(k), \tag{28}$$

then 1) the condition (11) is satisfied; 2) The upper bound $\hat{\Sigma}_e(k-1)$ (given in (23)*) on the error covariance of the input estimation is minimized and the minimized upper bound is given by*

$$\hat{\Sigma}_e(k-1) = \Pi^{-1}(k) \tag{29}$$

where

$$\Pi(k) = G^T(k-1)C^T(k)\Phi^{-1}(k)C(k)G(k-1),$$

$\Phi(k)$ and $Q(k)$ are defined in Theorem 1.

Proof. We need to search for an appropriate gain matrix $M_e(k)$ which minimizes the upper bound matrix $\hat{\Sigma}_e(k-1)$, and the corresponding problem can be equivalently written as the following constrained optimization problem:

$$\begin{aligned}&\min_{M_e(k)} \hat{\Sigma}_e(k-1), \\ &\text{subject to } M_e(k)C(k)G(k-1) = I_p.\end{aligned} \tag{30}$$

Using the completion-of-squares method, $\hat{\Sigma}_e(k-1)$ can be rearranged as follows:

$$\begin{aligned}\hat{\Sigma}_e(k-1) =& \big(M_e(k) - \Pi^{-1}(k)G^T(k-1)C^T(k)\Phi^{-1}(k)\big)\Phi(k) \\ &\times \big(M_e(k) - \Pi^{-1}(k)G^T(k-1)C^T(k)\Phi^{-1}(k)\big)^T + \Pi^{-1}(k)\end{aligned} \tag{31}$$

By choosing

$$M_e(k) = \Pi^{-1}(k)G^T(k-1)C^T(k)\Phi^{-1}(k),$$

it can be easily found that the equality constraint in (30) is satisfied and $\hat{\Sigma}_e(k-1)$ is minimized as

$$\hat{\Sigma}_e(k-1) = \Pi^{-1}(k).$$

This completes the proof. □

3.2 *State estimation*

In this section, we consider the estimation problem of the system state. We are interested in finding an appropriate gain matrix $K_e(k)$ for the event-based estimator $\mathcal{E}_2$ such that the upper bound on the error covariance of the state estimation is minimized. First, an upper bound on the error covariance of the state estimation is given in the following theorem.

Theorem 3: *Consider the linear system (1) and the event-based estimator $\mathcal{E}_2$ in* (5) *with event generator condition (3). Let the condition* (11) *be satisfied. Assume that, for a given positive scalar sequence $\{\varepsilon_2(k), k \in \mathbb{N}\}$, there exist two sets of real-valued matrices $\hat{P}_e^u(k|k)$ and $L_e(k)$ satisfying the following Riccati-like difference equation with the initial condition $\hat{P}_e^u(0|0) = 0$:*

$$\hat{P}_e^u(k|k) = \phi(\hat{P}_e^u(k-1|k-1), k-1), \tag{32}$$

where

$$\phi(\hat{P}_e^u(k-1|k-1), k-1) = (1+\varepsilon_2(k))\bar{A}_e(k-1)\hat{P}_e^u(k-1|k-1)\bar{A}_e^T(k-1) + (1+\varepsilon_2^{-1}(k))\sigma L_e(k)L_e^T(k),$$

$$\bar{A}_e(k-1) = (I-L_e(k)C(k))A(k-1), \quad L_e(k) = K_e(k)+E(k)G(k-1)M_e(k), \quad E(k) = I-K_e(k)C(k).$$

Then, we have $\hat{P}_e^u(k|k) \geq P_e^u(k|k)$. Accordingly, an upper bound $\hat{P}_e(k|k)$ on the estimation error covariance matrix $P_e(k|k)$ is given as follows:

$$\hat{P}_e(k|k) = P_e^s(k|k) + \hat{P}_e^u(k|k) \tag{33}$$

where

$$\begin{aligned}P_e^s(k|k) =& \bar{A}(k-1)P_e^s(k-1|k-1)\bar{A}^T(k-1) + L_e(k)R(k)L_e^T(k) \\ &+ (I-L_e(k)C(k))W(k-1)(I-L_e(k)C(k))^T, \quad P_e^s(0|0) = P(0|0).\end{aligned}$$

Proof. From (16) and (17), it is straightforward to obtain that

$$
\begin{aligned}
P_e^s(k|k) &= \bar{A}_e(k-1)P_e^s(k-1|k-1)\bar{A}_e^T(k-1) + L_e(k)R(k)L_e^T(k) && (34)\\
&\quad +(I-L_e(k)C(k))W(k-1)(I-L_e(k)C(k))^T, && (35)\\
P_e^u(k|k) &= \bar{A}_e(k-1)P_e^u(k-1|k-1)\bar{A}_e^T(k-1) + L_e(k)\delta(k)\delta^T(k)L_e^T(k)\\
&\quad +\bar{A}_e(k-1)\tilde{x}_e^u(k-1|k-1)(L_e(k)\delta(k))^T + L_e(k)\delta(k)(\bar{A}_e(k-1)\tilde{x}_e^u(k-1|k-1))^T && (36)
\end{aligned}
$$

For an arbitrary positive scalar $\varepsilon_2(k)$, it follows from Lemma 2 that

$$
\begin{aligned}
&\bar{A}_e(k-1)\tilde{x}_e^u(k-1|k-1)(L_e(k)\delta(k))^T + L_e(k)\delta(k)(\bar{A}(k-1)\tilde{x}_e^u(k-1|k-1))^T\\
\le&\varepsilon_2(k)\bar{A}_e(k-1)P_e^u(k-1|k-1)\bar{A}_e^T(k-1) + \varepsilon_2^{-1}(k)\sigma L_e(k)L_e^T(k)
\end{aligned}
$$

which, together with (36), indicates that $\phi(P_e^u(k-1|k-1),k-1) \ge P_e^u(k|k)$. As $\hat{P}_e^u(0|0) = P_e^u(0|0) = 0$, and $\hat{P}_e^u(k|k)$ can be calculated iteratively by the Riccati-like difference equation $\hat{P}_e^u(k|k) = \phi(\hat{P}_e^u(k-1|k-1),k-1)$. It follows from Lemma 5 that

$$
\hat{P}_e^u(k|k) \ge P_e^u(k|k), \forall k > 0, \tag{37}
$$

and, furthermore, we can easily obtain from (33) and (37) that

$$
\hat{P}_e(k|k) \ge P_e(k|k), \forall k > 0
$$

and the proof is now complete. □

Before we design the estimator, we denote

$$
\begin{aligned}
\Omega(k) \;\triangleq\; &\Big\{\Big[\tilde{P}_e(k-1|k-1) + \Lambda(k)G^T(k-1)\Big]C^T(k)\Xi^{-1}(k) - G(k-1)M_e(k)\Big\}\\
&\times\Big\{I - \big[I-C(k)G(k-1)M_e(k)\big]^+\big[I-C(k)G(k-1)M_e(k)\big]\Big\}
\end{aligned}
$$

In the following theorem, the upper bound matrix $\hat{P}_e(k|k)$ at each time instant is minimized by appropriately designing the estimator parameter $K_e(k)$.

Theorem 4: *Consider the linear system (1) and the event-based estimator $\mathcal{E}_2$ (5) with event generator condition (3). Assume that the estimation parameter $M_e(k)$ is chosen as in (28). The matrix $\hat{P}_e(k|k)$ given in (33), which is an upper bound on the error covariance $P_e(k|k)$ of the state estimation, can be minimized at the iteration when $\Omega(k) = 0$ with the parameter $K_e(k)$ given by*

$$
K_e(k) = \Big(\big(\tilde{P}_e(k-1|k-1)+\Lambda(k)G^T(k-1)\big)C^T(k)\Xi^{-1}(k)-G(k-1)M_e(k)\Big)\big(I-C(k)G(k-1)M_e(k)\big)^+ \tag{38}
$$

and the minimum given by

$$
\begin{aligned}
\hat{P}_e(k|k) =&\Lambda(k)G^T(k-1)C^T(k)\Xi^{-1}(k)C(k)G(k-1)\Lambda^T(k) + \tilde{P}_e(k-1|k-1)\\
&-\tilde{P}_e(k-1|k-1)C^T(k)\Xi^{-1}(k)C(k)\tilde{P}_e(k-1|k-1)
\end{aligned} \tag{39}
$$

where

$$
\begin{aligned}
&\tilde{P}_e(k-1|k-1) = A(k-1)\big(P_e^s(k-1|k-1) + (1+\varepsilon_2(k))\hat{P}_e^u(k-1|k-1)\big)A^T(k-1) + W(k-1),\\
&\Xi(k) = C(k)\tilde{P}_e(k-1|k-1)C^T(k) + R(k) + (1+\varepsilon_2^{-1}(k))\sigma I,\\
&\Lambda(k) = \big(G(k-1) - \tilde{P}_e(k-1|k-1)C^T(k)\Xi^{-1}(k)C(k)G(k-1)\big)\big(G^T(k-1)C^T(k)\Xi^{-1}(k)C(k)G(k-1)\big)^{-1}.
\end{aligned}
\tag{40}
$$

In the special case that the two sets of positive scalar sequences are identical, that is, $\varepsilon_2(k) = \varepsilon_1(k), \forall k \in \mathbb{N}$, *the expression of* $K_e(k)$ *reduces to the following equation,*

$$
K_e(k) = \tilde{P}_e(k-1|k-1)C^T(k)\Xi^{-1}(k). \tag{41}
$$

Proof. For locally minimum-variance estimation, we first look for $L_e(k)$ which minimizes $\hat{P}_e(k|k)$ subject to the constraint $L_e(k)C(k)G(k-1) = G(k-1)$. Using the completion-of-squares method, $\hat{P}_e(k|k)$ can be rewritten as follows:

$$
\begin{aligned}
\hat{P}_e(k|k) =&\Big(L_e(k)\Xi(k) - \tilde{P}_e(k-1|k-1)C^T(k) - \Lambda(k)G^T(k-1)C^T(k)\Big)\Xi^{-1}(k)\\
&\times\Big(L_e(k)\Xi(k) - \tilde{P}_e(k-1|k-1)C^T(k) - \Lambda(k)G^T(k-1)C^T(k)\Big)^T\\
&+ \tilde{P}_e(k-1|k-1) + \Lambda(k)G^T(k-1)C^T(k)\Xi^{-1}(k)C(k)G(k-1)\Lambda^T(k)\\
&- \tilde{P}_e(k-1|k-1)C^T(k)\Xi^{-1}(k)C(k)\tilde{P}_e(k-1|k-1)
\end{aligned}
\tag{42}
$$

where $\tilde{P}_e(k-1|k-1)$, $\Lambda(k)$, $\Xi(k)$ are defined in (40).

By choosing

$$
L_e(k) = \big(\tilde{P}_e(k-1|k-1) + \Lambda(k)G^T(k-1)\big)C^T(k)\Xi^{-1}(k), \tag{43}
$$

it can be found that $\hat{P}_e(k|k)$ is minimized and the minimum of $\hat{P}_e(k|k)$ is given by

$$
\begin{aligned}
\hat{P}_e(k|k) =&\Lambda(k)G^T(k-1)C^T(k)\Xi^{-1}(k)C(k)G(k-1)\Lambda^T(k) + \tilde{P}_e(k-1|k-1)\\
&- \tilde{P}_e(k-1|k-1)C^T(k)\Xi^{-1}(k)C(k)\tilde{P}_e(k-1|k-1).
\end{aligned}
$$

Note that $L_e(k) = K_e(k) + \big(I - K_e(k)C(k)\big)G(k-1)M_e(k)$ and $\Omega(k) = 0$, it is easy to see that the minimum-norm solution $K_e(k)$ to

$$
\big(\tilde{P}_e(k-1|k-1) + \Lambda(k)G^T(k-1)\big)C^T(k)\Xi^{-1}(k) = K_e(k) + \big(I - K_e(k)C(k)\big)G(k-1)M_e(k) \tag{44}
$$

exists and is given by

$$
K_e(k) = \Big(\big(\tilde{P}_e(k-1|k-1) + \Lambda(k)G^T(k-1)\big)C^T(k)\Xi^{-1}(k) - G(k-1)M_e(k)\Big)\big(I - C(k)G(k-1)M_e(k)\big)^+.
$$

When $\varepsilon_2(k) = \varepsilon_1(k)$, we have $\Phi(k) = \Xi(k)$. Accordingly, $K_e(k)$ is obtained from (44) as follows:

$$
K_e(k) = \tilde{P}_e(k-1|k-1)C^T(k)\Xi^{-1}(k).
$$

This completes the proof. $\square$

Remark 2: In case that $\Omega(k) \neq 0$, the estimator gain (38) would lead to a practical (not necessarily minimum-variance) solution with guaranteed upper bound $\hat{P}_e(k|k)$. On the other hand, if the threshold of event-triggering σ is set to be zero, then the event-based mechanism reduces to the traditional time-based mechanism and, accordingly, our proposed estimator reduces to the optimal time-based estimator proposed in Gillijns & Moor (2007).

3.3 *Discussion on choosing the scalar parameters*

From Theorems 1 and 3, it is clear that the estimation performance at time instant k depends on system data and the scalar sequences $\varepsilon_i(0), \varepsilon_i(1), \ldots, \varepsilon_i(k), i = 1, 2$. It means that, to compute an optimal event-based estimator at time k, the scalar sequences $\varepsilon_i(0), \varepsilon_i(1), \ldots, \varepsilon_i(k-1), i = 1, 2$, need to be re-computed, and so do the corresponding estimator gain matrices (28) and (38). The optimization over the scalar sequences becomes numerically intractable as the time instant k tends to $+\infty$.

To reduce the computation complexity, instead of optimizing the performance over all the k scalar parameters, a practical way is to optimize the trace of matrices $\hat{\Sigma}_e(k)$ and $\hat{P}_e(k|k)$ over a fixed length of scalar parameters $\varepsilon_i(k+1-T), \varepsilon_i(k+2-T), \ldots, \varepsilon_i(k), i = 1, 2$. For the special case that the length T is equal to 1, an optimal and suboptimal algorithms on how to choose the scalar parameters are given below respectively.

Proposition 1: *For the event-based estimator $\mathcal{E}_2$ in* (5) *with the parameters $M_e(k)$ and $K_e(k)$ given in (28) and (38), respectively, $Tr\{\hat{\Sigma}_e(k)\}$ and $Tr\{\hat{P}_e(k|k)\}$ are minimized if the scalars $\varepsilon_1(k)$ and $\varepsilon_2(k)$ are given as follows:*

$$\varepsilon_1(k) = \arg\min_{\varepsilon_1(k)} Tr\big\{\Pi^{-1}(k)\big\} \tag{45}$$

$$\begin{aligned}\varepsilon_2(k) = &\arg\min_{\varepsilon_2(k)} Tr\Big\{\Lambda(k)G^T(k-1)C^T(k)\Xi^{-1}(k)C(k)G(k-1)\Lambda^T(k) + \tilde{P}_e(k-1|k-1)\\ &- \tilde{P}_e(k-1|k-1)C^T(k)\Xi^{-1}(k)C(k)\tilde{P}_e(k-1|k-1)\Big\}.\end{aligned} \tag{46}$$

An analytical suboptimal scalar $\varepsilon_1(k)$ can be chosen as follows:

$$\varepsilon_1(k) = \begin{cases} \sqrt{\frac{\sigma}{\bar{\rho}(k)}}, & \text{if } \Phi\big(\sqrt{\frac{\sigma}{\bar{\rho}(k)}}, k\big) < \Phi\big(\sqrt{\frac{\sigma}{\underline{\rho}(k)}}, k\big) \\ \sqrt{\frac{\sigma}{\underline{\rho}(k)}}, & \text{otherwise.} \end{cases} \tag{47}$$

where $\bar{\rho}(k), \underline{\rho}(k)$ are the minimum and the maximum eigenvalues of $C(k)A(k-1)\hat{P}_e(k-1|k-1)A^T(k-1)C^T(k)$, respectively.

Proof. With the obtained optimal gain matrices $M_e(k)$ and $K_e(k)$, we search for the optimal/suboptimal scalar parameters $\varepsilon_1(k)$ and $\varepsilon_2(k)$. From (29) and (42), it is straightforward to derive the optimal $\varepsilon_1(k)$ and $\varepsilon_2(k)$, which are given in (45) and (46), respectively. However, since it is numerical intractable to compute the analytical solution for the optimal $\varepsilon_1(k)$ from (45), we would like to look for a suboptimal $\varepsilon_1(k)$. Instead of searching for the optimal $\varepsilon_1(k)$ from the interval $(0, +\infty)$, in the following, a suboptimal $\varepsilon_1(k)$ belonging to the interval $\Big(0, \sqrt{\frac{\sigma}{\bar{\rho}(k)}}\Big] \bigcup \Big[\sqrt{\frac{\sigma}{\underline{\rho}(k)}}, +\infty\Big)$, is derived in the analytical form.

Choosing two arbitrary scalar variables $\varepsilon_2(k) > \tilde{\varepsilon}_1(k) > 0$, we have

$$\begin{aligned}\Phi(\tilde{\varepsilon}_1(k), k) - \Phi(\varepsilon_1(k), k) &= (\tilde{\varepsilon}_1(k) - \varepsilon_1(k))C(k)A(k-1)\hat{P}_e(k|k)A^T(k-1)C^T(k) + \sigma(\tilde{\varepsilon}_1^{-1}(k) - \varepsilon_1^{-1}(k))I_m \\ &= (\tilde{\varepsilon}_1(k) - \varepsilon_1(k))\big(C(k)A(k-1)\hat{P}_e(k|k)A^T(k-1)C^T(k) - \frac{\sigma}{\varepsilon_1(k)\tilde{\varepsilon}_1(k)}I_m\big),\end{aligned}$$

from which we conclude the following:

(i) if $\varepsilon_1(k), \tilde{\varepsilon}_1(k) \in \left(0, \sqrt{\frac{\sigma}{\bar{\rho}(k)}}\right]$, then $\Phi(\tilde{\varepsilon}_1(k), k) < \Phi(\varepsilon_1(k), k)$;

(ii) if $\varepsilon_1(k), \tilde{\varepsilon}_1(k) \in \left[\sqrt{\frac{\sigma}{\underline{\rho}(k)}}, +\infty\right)$, then $\Phi(\tilde{\varepsilon}_1(k), k) > \Phi(\varepsilon_1(k), k)$;

(iii) if $\varepsilon_1(k), \tilde{\varepsilon}_1(k) \in \left[\sqrt{\frac{\sigma}{\bar{\rho}(k)}}, \sqrt{\frac{\sigma}{\underline{\rho}(k)}}\right]$, then $\Phi(\tilde{\varepsilon}_1(k), k)$ and $\Phi(\varepsilon_1(k), k)$ are not dominated by each other.

On the other hand, it follows from Lemma 3 that $\mathrm{Tr}\{\Sigma_e(k)\}$ is a strictly increasing function of $\Phi(k)$. Hence, it is known that, for $\varepsilon_1(k) \in \left(0, \sqrt{\frac{\sigma}{\bar{\rho}(k)}}\right] \bigcup \left[\sqrt{\frac{\sigma}{\underline{\rho}(k)}}, +\infty\right)$, $\mathrm{Tr}\{\Sigma_e(k)\}$ attains the minimum when

$$\varepsilon_1(k) = \begin{cases} \sqrt{\frac{\sigma}{\bar{\rho}(k)}}, & \text{if } \Phi(\sqrt{\frac{\sigma}{\bar{\rho}(k)}}, k) < \Phi(\sqrt{\frac{\sigma}{\underline{\rho}(k)}}, k) \\ \sqrt{\frac{\sigma}{\underline{\rho}(k)}}, & \text{otherwise.} \end{cases}$$

This completes the proof. □

The complete procedure of our proposed estimation algorithm is described in Algorithm 1.

Remark 3: Different from the traditional time-based estimation problem, in the event-based estimation one, the exact values of the measurements at the time instants when no transmission is triggered cannot be obtained by the estimator. Instead, only the inequality form of the measurement information in (4) is known. In our algorithm, using the information of measurement (expressed in (4)) and the system parameters, the input and state estimates are obtained with a guaranteed upper bound on estimation covariances.

4. Boundedness Analysis

In the section, we investigate the asymptotic boundedness properties of the upper bound $\hat{P}_e(k|k)$ for the time-invariant system. Without notation confusion, when referring to the time invariant system (1), it is explicitly assumed that the parameter matrices are fixed as constant matrices, that is, $A(k) = A$, $G(k) = G$, $C(k) = C$, $W(k) = W$, and $R(k) = R$.

To facilitate our analysis, existing results on time-based estimation problems for time-invariant systems are summarized in the following lemma.

Lemma 6: *Darouach & Zasadzinski (1997) Consider the linear time-invariant system with unknown input (1) and the time-based estimator $\mathcal{E}_1$ in (2). The corresponding error covariance matrix $P_t(k|k)$ of the state estimation converges to a unique fixed positive semi-definite matrix $\bar{P}_t$ for any given initial condition $P_t(0|0)$ if and only if the following two equations hold,*

$$\mathrm{Rk}\left\{\begin{bmatrix} zI_n - A & G \\ C & 0 \end{bmatrix}\right\} = n + p, \forall z \in \mathbb{C}, |z| \geq 1. \tag{48}$$

Algorithm 1 Event-based Simultaneous Input and State Estimation (ESISE)

Initialize:
  $k = 0, \hat{P}_e^u(0) = 0, P_e^s(0) = P(0|0)$;
while $k \geq 1$ **do**
  if opt="optimal" **then**
    Choose the scalar $\varepsilon_1(k)$ via (45) ;
    Calculate the input estimate gain $M_e(k)$ via (28);
    Calculate the upper bound of the input estimation $\hat{\Sigma}_e(k-1)$ via (29);
    Choose the scalar $\varepsilon_2(k)$ via (46) ;
    Calculate the state estimate gain $K_e(k)$ via (38) ;
    Calculate $\hat{P}_e(k|k)$ via (39) ;
  else if opt="sub-optimal" **then**
    Choose the scalar $\varepsilon_1(k)$ via (47) ;
    Calculate the input estimate gain $M_e(k)$ via (28);
    Calculate the upper bound of the input estimation $\hat{\Sigma}_e(k-1)$ via (29);
    Set the scalar $\varepsilon_2(k) = \varepsilon_1(k)$;
    Calculate the state estimate gain $K_e(k)$ via (41) ;
    Calculate $\hat{P}_e(k|k)$ via (39) ;
  end if
  Input and state estimate $\hat{d}_e(k)$, $\hat{x}_e(k|k)$ via (5);
  $k = k + 1$;
end while

$$\mathrm{Rk}\left\{\begin{bmatrix} A - e^{j\omega}I & G & W^{\frac{1}{2}} & 0 \\ e^{j\omega}C & 0 & 0 & R^{\frac{1}{2}} \end{bmatrix}\right\} = n + m, \forall \omega \in [0, 2\pi]. \tag{49}$$

Moreover, with the associate limiting gain matrices $K_t := \lim_{k\to\infty} K_t(k)$, $M_t := \lim_{k\to\infty} M_t(k)$, the time-invariant estimator is stable as well, i.e., the spectral radius $\rho(\bar{A}_t)$ of matrix $\bar{A}_t := (I - L_t C)A$ is less than 1, where $L_t = K_t + (I - K_t C)GM_t$.

Theorem 5: *Consider the linear time-invariant system with unknown input (1) and event generator condition (3). Assume that both (48) and (49) are satisfied and an event-based estimator is designed according to Algorithm 1. With an arbitrarily chosen constant scalar $\varepsilon_2 \in (0, \bar{\varepsilon})$, where $\bar{\varepsilon} = \frac{1}{\rho^2(\bar{A}_t)} - 1$, the state error covariance matrix $P_e(k|k)$ is bounded and the upper bound $\hat{P}_e(k|k)$ is asymptotically convergent.*

Proof. 1). First, we prove that, for the event-based state estimator, when the filter gain $K_e(k)$ is set to be equivalent to the optimal gain $K_t(k)$ obtained in the time-based scenario, then the state estimation error covariance is bounded.

When the filter parameters are chosen as $K_e(k) = K_t(k)$, $L_e(k) = L_t(k)$, then $\bar{A}_e = \bar{A}_t$. From Lemma 6, it is known that $\bar{A}_e$ is a stable matrix and $\lim_{k\to\infty} P_t(k|k) = \bar{P}$. Moreover, it is easily found that $P_e^s(k|k)$ coincides with $P_t(k|k)$ and hence $P_e^s(k|k)$ converges to matrix $\bar{P}_t$. As $\bar{A}_e$ is a stable matrix and $\varepsilon \in (0, \bar{\varepsilon})$, then $\tilde{A}_e := \sqrt{1+\varepsilon}\bar{A}_e$ is a stable matrix as well.

Noting that $\hat{P}_e^u(k|k)$ satisfies

$$\hat{P}_e^u(k|k) = \tilde{A}_e \hat{P}_e^u(k-1|k-1)\tilde{A}_e^T + (1+\varepsilon^{-1})\sigma L_e L_e^T,$$

it follows from Lemma 4 that $\hat{P}_e^u(k|k) \to \bar{P}_e^u$ when $k \to \infty$, where $\bar{P}_e^u = \tilde{A}_e \bar{P}_e^u \tilde{A}_e^T + (1+\varepsilon^{-1})\sigma L_e L_e^T$. Furthermore, by noticing the fact that $\hat{P}_e(k|k) = \hat{P}_e^s(k|k) + \hat{P}_e^u(k|k)$, we have $\lim_{k\to\infty} P_e(k|k) = \bar{P}_t + \bar{P}_e^u$.

2). Next, through the induction approach, we aim to prove that, with the proposed optimal filter parameters $K_e(k)$ and $M_e(k)$, the upper bound matrix $\hat{P}^u(k|k)$ is always less than the one with the gain $K_t(k)$. That is, we would like to show that

$$\hat{P}_e\Big(k|k, K_e(k), \hat{P}_e\big(k-1|k-1, K_e(k-1)\big)\Big) \leq \hat{P}_e\Big(k|k, K_t(k), \hat{P}_e\big(k-1|k-1, K_t(k-1)\big)\Big). \quad (50)$$

When $k = 0$, $P_e^s(0|0) = P(0|0)$, and $P_e^u(0|0) = 0$, it is easy to find that $\hat{P}_e(0|0, K_e(0)) = \hat{P}_e(0|0, K_t(0))$. Suppose that, when $k = i-1$, $\hat{P}_e(i-1|i-1, K_e(i-1)) \leq \hat{P}_e(i-1|i-1, K_t(i-1))$ and we like to prove that (50) holds for $k = i$. In this case, since $K_e(i)$ minimizes $\hat{P}_e(i|i)$ given $\hat{P}_e(i-1|i-1)$, we can see that

$$\hat{P}_e\Big(i|i, K_e(i), \hat{P}_e\big(i-1|i-1, K_e(i-1)\big)\Big) \leq \hat{P}_e\Big(i|i, K_t(i), \hat{P}_e\big(i-1|i-1, K_e(i-1)\big)\Big). \quad (51)$$

On the other hand, it can be found that

$$\begin{aligned}
&\hat{P}_e\Big(i|i, K_t(i), \hat{P}_e\big(i-1|i-1, K_t(i-1)\big)\Big) - \hat{P}_e\Big(i|i, K_t(i), \hat{P}_e\big(i-1|i-1, K_e(i-1)\big)\Big) \\
&= \bar{A}_e(i-1)\Big((1+\varepsilon)\big(\hat{P}_e^u(i-1|i-1, K_t(i-1)) - \hat{P}_e^u(i-1|i-1, K_e(i-1))\big) \\
&- \big(P_e^s(i-1|i-1, K_e(i-1)) - P_e^s(i-1|i-1, K_t(i-1))\big)\Big)\bar{A}_e^T(i-1).
\end{aligned} \quad (52)$$

Noting that $P_e^s(i-1|i-1, K_t(i-1)) \leq P_e^s(i-1|i-1, K_e(i-1))$, and $\hat{P}_e(i-1|i-1, K_e(i-1)) \leq \hat{P}_e(i-1|i-1, K_t(i-1))$, it can be inferred that

$$\begin{aligned}
0 &\leq P_e^s\big(i-1|i-1, K_e(i-1)\big) - P_e^s\big(i-1|i-1, K_t(i-1)\big) \\
&\leq \hat{P}_e^u\big(i-1|i-1, K_t(i-1)\big) - \hat{P}_e^u\big(i-1|i-1, K_e(i-1)\big).
\end{aligned}$$

Substituting the above inequalities into (52), one obtains

$$\hat{P}_e\Big(i|i, K_t(i), \hat{P}_e\big(i-1|i-1, K_e(i-1)\big)\Big) \leq \hat{P}_e\Big(i|i, K_t(i), \hat{P}_e\big(i-1|i-1, K_t(i-1)\big)\Big). \quad (53)$$

Combining the inequalities (51) and (53) leads to (50).

3). In this step, we aim to prove that the upper bound is asymptotically bounded. Noting that

$$\lim_{k\to\infty} \hat{P}_e\Big(k|k, K_t(k), \hat{P}_e\big(k-1|k-1, K_t(k-1)\big)\Big) = \hat{P},$$

it follows from (53) that

$$\lim_{k\to\infty} \hat{P}_e\Big(k|k, K_e(k), \hat{P}_e\big(i-1|i-1, K_e(i-1)\big)\Big) \leq \hat{P}.$$

This completes the proof. $\square$

5. Numerical example

The simulation example proposed in Hsieh (2000) is used here to demonstrate the effectiveness of our proposed estimator, where the parameters for the linear system equations (1) are given by

$$\begin{aligned} A(k) &= \begin{bmatrix} 0.1 & 0.5 & 0.08 \\ 0.6 & 0.01 & 0.04 \\ 0.1 & 0.7 & 0.05 \end{bmatrix}, \ W(k) = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}, \\ C(k) &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \ G(k) = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \ R(k) = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}. \end{aligned}$$

The initial state is $x(0) = [1\ 1\ 1]^T$. The initial estimate of the state is assumed to be zero and its covariance is given by $P(0|0) = \text{diag}\{10, 10, 10\}$. The unknown input is given by

$$d(k) = 50u_s(k) - 100u_s(k-20)$$

where $u_s(k)$ is the unit-step function. The simulation time is 40 time steps. The threshold of the event-generator is set as $\sigma = 40$.

In order to ensure the generality of the experimental results, 100 Monte-Carlo simulations are run. The notion of mean square error (MSE) is adopted to evaluate the estimation accuracy. Let $\text{MSE}_{i,k}$ denotes MSE for the kth-run for the estimate of the ith state. The estimation for the accuracy on the ith state is $\text{MSE}_i = \frac{1}{100}\sum_{k=1}^{100}(x_{i,k} - \hat{x}_{i,k})^2$, and the average estimation performance of all states (AMSE) is defined as $\text{AMSE} := \frac{1}{n}\sum_{j=1}^{n}\text{MSE}_j$, where n is the number of the state variables.

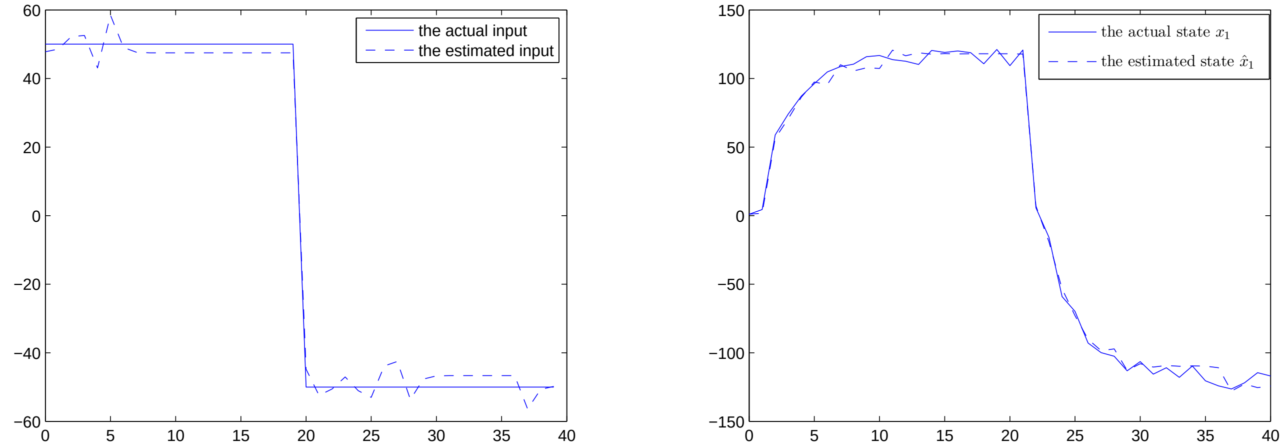
Figs. 1-2 show the actual and the estimated values of the unknown input and the system states. It can be seen that the proposed event-based estimator can estimate the input and the system state accurately. Fig. 3 shows the AMSE of the input and the states, which confirms that the AMSEs stay below their upper bounds. Moreover, it can be seen that the upper bounds converge to constant values, which confirms the asymptotic boundedness property of upper bounds proposed in Theorem 5.

To illustrate the effect of the parameter sequence $\varepsilon_1(k)$ on the estimation performance, a comparison experiment is implemented. The two estimator are of the same structure, and in one estimator, the suboptimal parameter sequences $\varepsilon_1(k)$ is calculated based on the equation (47), while in the other estimator, the parameter sequences are chosen arbitrarily as $\varepsilon_1(k) = 0.5$ for all k. From Fig. 3-b, it can be found that the estimator with the suboptimal parameter sequence $\varepsilon_1(k)$ yields a tighter bound on the estimation error covariance than the one with an arbitrarily chosen parameter sequence $\varepsilon_1(k)$.

Fig. 4 shows the triggering events during the simulation period. Compared with the time-based mechanism, it can be found that the transmission times are significantly reduced, which clearly shows the superiority of the proposed event-based mechanism.

6. Conclusion

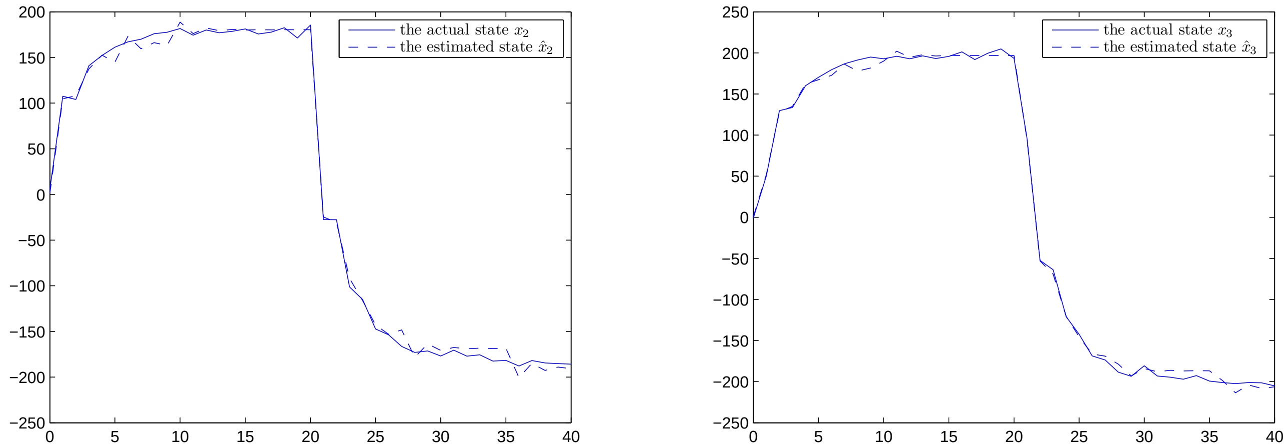
In this paper, an event-based joint input and state estimator has been proposed for the sake of reducing the sensor data transmission rate and the energy consumption. Based on an SOD concept, the sensors transmit the measurements when the prescribed conditioned is violated. By using the inductive method and intensive analysis on the estimation error, upper bounds of the estimation error covariances are obtained recursively. Subsequently, by choosing some scalar parameters properly, such upper bounds are reduced. In addition, for linear time-invariant system, the upper bounds



(a) The input.

(b) The state x_1.

Figure 1. The actual and estimated states x_1

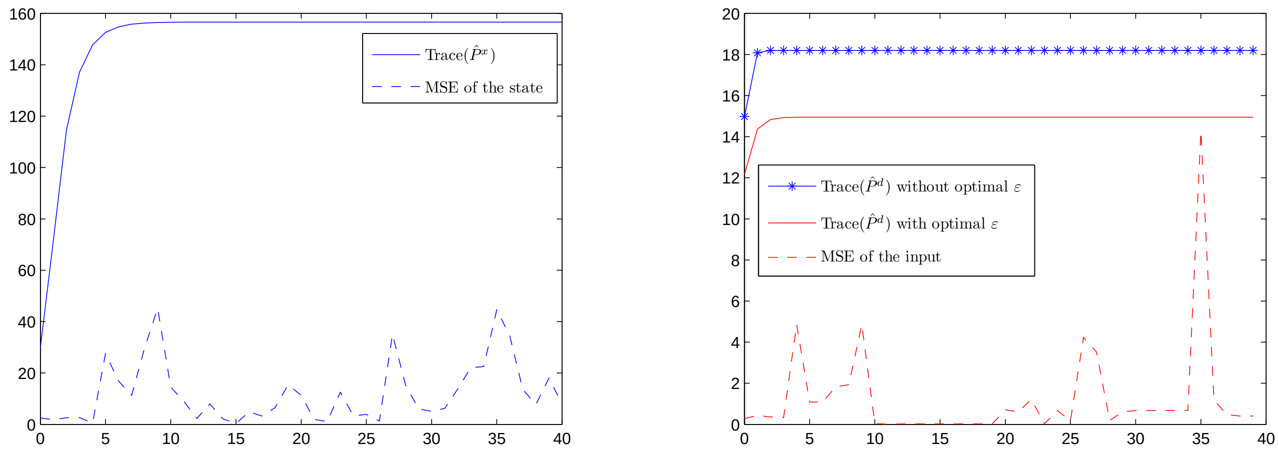


(a) The state x_2.

(b) The state x_3.

Figure 2. The actual and estimated states x_1 and x_2

are proved to be asymptotically bounded under certain conditions. Finally, through a numerical simulation, we have demonstrated that the proposed event-based estimator yields acceptable estimation performance while reduces the number of transmission greatly. Our future research topic would be the extension of the main results of this paper to more complex systems, see e.g. Chen, Liang, & Wang (2016); Li, Shen, Liu, & Alsaadi (2016); Liu, Liu, & Alsaadi (2016); Liu, Wei, Song, & Liu (2016a,b); Li, Wei, Han, & Liu (2016); Liu, Liu, Obaid, & Abbas (2016); Shu, Zhang, Shen, & Liu (2016); Wen, Cai, Liu, & Wen (2016); Zeng, Wang, & Zhang (2016); Zhang, Wang, Liu, Ding, & Alsaadi (2017); Zhang, Ma, & Liu (2016).



(a) The state. (b) The input.

Figure 3. The AMSEs and their upper bounds

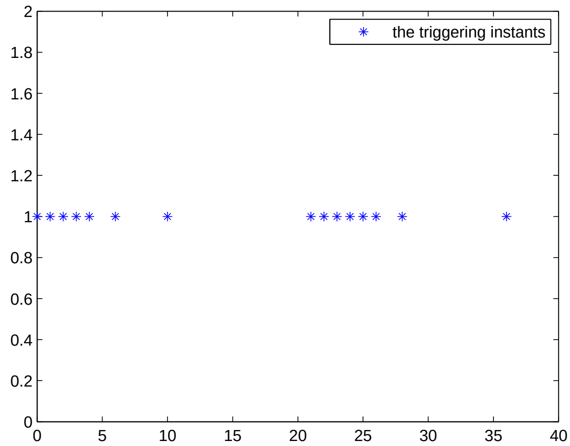


Figure 4. The triggering sequence

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