Composite Control of Linear Quadratic Games in Delta Domain with Disturbance Observers

Yuan Yuan, Lei Guo*, and Zidong Wang

Abstract

In this paper, the disturbance-observer-based composite control problem is investigated for a class of delta domain linear quadratic game with both matched and unmatched disturbances. In the presence of the disturbances, the ϵ-Nash Equilibrium (ϵ-NE) is proposed to describe the outcome of the game. We aim to develop a composite control strategy integrating the disturbance-observer-based control and the feedback Nash strategies such that the matched disturbance is compensated and the individual cost function of each player is optimized. Sufficient conditions are given to ensure the existence of both the desired disturbance observer and the feedback Nash strategies in the delta domain, and then the explicit expressions of the observer gain and Nash strategies are provided. An upper bound for the ϵ-NE is given analytically which demonstrates the robustness of the Nash equilibrium. Finally, a simulation example on the two-area load frequency control problem is provided to illustrate the effectiveness of the proposed design procedure.

Index Terms

Linear quadratic (LQ) game, delta operator, disturbance observer, ϵ-Nash Equilibrium (ϵ-NE)

I. Introduction

It is well known that the game theory is arguably one of the most active research areas in operation research and control, and many different types of game theoretic methods have been developed in the literature, see e.g. [1]–[8] and the references therein. Among various types of games, the so-called linear quadratic (LQ) difference/differential games have gained particular attention since they are powerful in characterizing cooperation and confliction among different decision makers in a dynamic environment. On the other hand, the sampling rate of industrial systems is usually high due mainly to the rapid development of the sensing techniques, and the sampled-data issue becomes a major concern in the design of control systems. The delta operator theory is recognized as an effective tool in dealing with sampling issues in the sense that 1) it can overcome numerical illness resulting from fast sampling discrete-time system described by the z-operator; and 2) it can unify the analysis results in both continuous- and discrete-time settings [9]. To be more specific, it is essential to formulate the delta domain analogue of the traditional discrete system if the sampling rate is high. Owing to its theoretical significance and practical importance, the delta operator approach has been extensively exploited in studying the filtering and control problems, see e.g. [9]–[11]. Comparing with the vast literature with respect to the control of delta operator systems with a single decision maker, the corresponding results for control issue of dynamic games in the delta domain have been very few due

Y. Yuan is with the School of Automation Science and Electrical Engineering, Beihang University, 100191 China, and also with the Department of Computer Science, Brunel University London, Uxbridge, Middlesex, UB8 3PH, United Kingdom, Email: snowkey@aliyun.com.

Z. Wang is with the Department of Computer Science, Brunel University London, Uxbridge, Middlesex, UB8 3PH, United Kingdom. He is also with the Faculty of Engineering, King Abdulaziz University, Jeddah 21589, Saudi Arabia. Email: Zidong.Wang@brunel.ac.uk

L. Guo is with the School of Automation Science and Electrical Engineering, Beihang University, 100191 China. Email: lguo@buaa.edu.cn

probably to the technical difficulties, see [10]. It should be mentioned that the control of delta domain dynamic games requires the utilization of delta domain Riccati equations, which makes it different from most of the existing work on delta operator within the framework of linear matrix inequality (LMI), see e.g. [9], [12], [13].

Virtually almost all real-time control systems operate in the presence of certain types of disturbances due to a variety of reasons such as load variations and friction in electrical and mechanical systems, measurement noise, errors caused by actuators and sensors, network fluctuations and environmental disturbance, see e.g. [14]–[20]. In recent years, systems with simultaneous multi-source disturbances have stirred considerable research attention because of its practical significance, see e.g. [21]–[23]. Among various anti-disturbance control methods, the disturbance-observer-based control is capable of achieving fast dynamic response when dealing with disturbances since it provides a feedforward compensation patch directly counteracting the disturbance [24]. In most existing literature concerning the multi-player dynamic game, however, it has been implicitly assumed that no disturbance exists or all the disturbances can be fully estimated and compensated, which is impractical in most cases. For systems with immeasurable and unpredictable disturbances, pure Nash Equilibrium (NE) cannot be used as the final outcome of the game [25], [26] since it will also be affected by such disturbances. Very recently, the continuous LQ game affected by immeasurable disturbances has been considered in [27] and [28], where the adaptation and sliding mode mechanisms have been employed for the compensation. *Unfortunately, to the best of the author's knowledge, the composite control problem of LQ game using disturbance observer technique has not been adequately investigated, not to mention the case where the system is in the discrete or delta domain setting.* It is, therefore, the purpose of this paper to shorten such a gap.

In this paper, we aim to investigate the composite control problem of disturbance-observer-based control and LQ games in the delta domain. Note that the underlying dynamic model of the game is quite comprehensive to cover both matched and unmatched disturbances, thereby reflecting the reality closely. The problem addressed represents the first of few attempts to address the anti-disturbance control of delta domain LQ games with disturbances. The technical challenge lies in how to define the LQ games in the delta domain and how to describe the influences of disturbances on NE. To be more specific, the contributions of this paper are mainly threefold. *(1) The composite control for delta domain LQ game is proposed, which consists of disturbance-observer-based control and feedback Nash strategies. (2) The strategies for the composite control are developed such that the matched disturbance is compensated and individual cost function of each player is optimized. (3) The ϵ-NE is proposed to describe the dynamic coupling of the disturbance observer and LQ game, and furthermore, the epsilon level is estimated analytically.*

The rest of the paper is organised as follows. Section II provides the problem formulation and some basic assumptions made in this paper. In Section III, the delta domain conditions are provided to design the disturbance observer and develop the feedback Nash strategies for LQ games. The estimation for epsilon-level of the ϵ-NE is presented in IV. In Section V, a simulation example is given to demonstrate the effectiveness of the proposed approach. The conclusion is drawn in Section VI.

Notation: Some standard notations are used throughout this paper. For a matrix M, M^T denotes its transpose. $M > 0$ (respectively, $M < 0$) means that M is positive definite (respectively, negative definite). $M \geq 0$ (respectively, $M \leq 0$) means that M is a non-negative (respectively, non-positive definite) matrix. $\lambda_{\max}\{M\}$ represents the largest eignvalue of the matrix M. The element in the ith row and jth column of matrix M would be denoted as M_{ij}. $\{M_i\}_{i=1}^{S}$ denotes the set of matrices from M_1 to M_S. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. For a vector $v(k)$, its Euclidean norm is denoted as $\|v(k)\| := \sqrt{v^T(k)v(k)}$ and $\|v(k)\|_2 := \sqrt{\sum_{k=1}^{K} \|v(k)\|^2} < \infty$. $l_2[0,\infty)$ is the space of square summable vectors. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. PROBLEM FORMULATION

The aim of this paper is to design the disturbance observer and develop the feedback Nash strategy for the delta domain LQ games with disturbances. The composite controller using the estimation provided by the disturbance observer can compensate for the matched disturbance and optimize the individual cost function of each player. In the presence of the unmatched disturbance and estimation error of the matched disturbance, the ϵ-NE is proposed to characterize the deviation from the pure NE.

A. Delta Operator System Model with Disturbance

Consider the following delta domain system with matched and unmatched disturbances:

$$(\Sigma):\ \delta x(k) = Ax(k) + \bar{B}(\bar{u}(k) + d_m(k)) + H_1 d_u(k), \tag{1}$$

where $x(k) \in \mathbb{R}^{n_1}$ is the state vector, $\bar{u}(k) = \begin{bmatrix} \bar{u}^{1T}(k) & \bar{u}^{2T}(k) & \cdots & \bar{u}^{ST}(k) \end{bmatrix}^T \in \mathbb{R}^{m_1}$ is the control input, and $d_m(k) = \begin{bmatrix} d_m^{1T}(k) & d_m^{2T}(k) & \cdots & d_m^{ST}(k) \end{bmatrix}^T \in \mathbb{R}^{m_1}$ is the matched disturbance, and $d_u(k) \in \mathbb{R}^{p_1}$ is the unmatched disturbance satisfying $\|d_u(k)\|^2 < \beta_1 < \infty$. As can be seen later, the inclusion of the disturbances will bring a deviation to NE and additional difficulty in the analysis. The matrix $\bar{B}$ can be partitioned as $\bar{B} = \begin{bmatrix} B^1 & B^2 & \cdots & B^S \end{bmatrix}$. A, $\bar{B}$ and H_1 in system (Σ) are all constant matrices with appropriate dimensions. As discussed in [15], [31], [32], we assume that the matched disturbance $d_m(k)$ can be described by an exogenous system with uncertainty as follows.

Assumption 1: The disturbance $d_m(k)$ can be formulated by the following exogenous system:

$$\begin{cases} \delta w(k) = Ww(k) + H_2\phi(k), \\ d_m(k) = Vw(k), \end{cases} \tag{2}$$

where $w_k \in \mathbb{R}^{q_1}$ is the internal state vector of the exogenous system, W, H_2 and V are known matrices with appropriate dimensions, and $\phi(k)$ is the additional bounded disturbance which belongs to $l_2[0, \infty)$.

Remark 1: As in [9], the delta operator is defined as follows:

$$\delta x(k) = \begin{cases} dx(t)/dt, & T_s = 0, \\ \frac{x(k+1)-x(k)}{T_s} & T_s \neq 0\ . \end{cases} \tag{3}$$

System (Σ) can be obtained from the following continuous model

$$\dot{x}(t) = A_s x(t) + \bar{B}_s \bar{u}(t) + \bar{B}_s d_m(t) + H_s d_u(t), \tag{4}$$

where A_s, $\bar{B}_s$ and H_s are matrices with appropriate dimensions in the continuous domain. It follows from [12] that the matrices in (1) can be obtained by $A = \frac{e^{A_s T_s} - I}{T_s}$, $\bar{B} = \frac{1}{T_s}\int_0^{T_s} e^{A_s(T_s-\tau)}\bar{B}_s d\tau$, and $H_1 = \frac{1}{T_s}\int_0^{T_s} e^{A_s(T_s-\tau)} H_s d\tau$.

Remark 2: The matched disturbances are defined as the disturbances existing in the control input channel, while the unmatched disturbances refer to disturbances which do not satisfy the 'matching condition' [33]. For instance, in [34], the lumped disturbances in magnetic leviation suspension system do not satisfy the 'matching' condition and can only be regarded as the unmatched disturbance. In [35], the parameter uncertainties which are seen as the unmatched disturbances do not enter the permanent magnet synchronous motor system via the control channel. It should be pointed out that, as we shall see subsequently, the proposed disturbance-observer-based composite control scheme could only compensate the unwanted impact of the matched disturbance. The influence of the unmatched disturbance will be quantified via the estimation of the upper bound of the ϵ-level.

B. Delta Domain Disturbance Observer

In this section, we design the following disturbance observer to estimate the matched disturbance $d_m(k)$

$$\begin{cases} \delta v(k) = (W + L\bar{B}V)\hat{w}(k) + L(Ax(k) + \bar{B}u(k)), \\ \hat{w}(k) = v(k) - Lx(k), \\ \hat{d}_m(k) = V\hat{w}(k), \end{cases} \tag{5}$$

where $v(k) \in \mathbb{R}^{m_2}$ is the state vector of the disturbance observer, $\hat{w}(k) \in \mathbb{R}^{m_2}$ is the estimation of $w(k)$, $\hat{d}_m(k) \in \mathbb{R}^{m_1}$ is the estimation of $d_m(k)$, and $L \in \mathbb{R}^{m_2 \times n_1}$ is the disturbance observer gain to be designed. Letting the estimation error be

$$e_w(k) := w(k) - \hat{w}(k), \tag{6}$$

it follows from (1), (2), (5) and (6) that

$$\delta e_w(k) = (W + L\bar{B}V)e_w(k) + H_2\phi(k) + LH_1 d_u(k). \tag{7}$$

Next, system (7) can be rewritten in a compact form as follows:

$$\delta e_w(k) = G_0 e_w(k) + Hd(k), \tag{8}$$

where $G_0 = W + L\bar{B}V$, $H = \begin{bmatrix} H_2 & LH_1 \end{bmatrix}$, and $d(k) = \begin{bmatrix} \phi^T(k) & d_u^T(k) \end{bmatrix}^T$. In the following, we introduce the definition of the robust stability of the error dynamics system (8).

Definition 1: For a given $\gamma > 0$, the error dynamics (8) is said to satisfy the H_∞ performance if the following inequality holds:

$$J = \sum\nolimits_{k=0}^{\infty} \{e_w^T(k)e_w(k) - \gamma^2 d^T(k)d(k)\} < 0, \tag{9}$$

under the zero initial condition.

C. Delta Domain Noncooperative Game

The design objective for system (Σ) is to develop a composite controller which can counteract the matched disturbance and minimize the individual cost function for each player. In this paper, we propose the following disturbance-observer-based composite control strategy

$$\bar{u}(k) = -\hat{d}_m + u(k), \tag{10}$$

where $\hat{d}_m$ is the feedforward compensation term, and $u(k) = [u^{1T}(k), u^{2T}(k), \ldots, u^{ST}(k)]^T$ is the feedback Nash strategy to be determined later. To facilitate understanding, we provide an example of the two-player LQ game with composite control strategy as shown in Fig. 1. From Fig. 1, we can see that the feedforward term compensates the matched disturbance based on the estimation provided by the disturbance observer. Meanwhile, the feedback loop of the composite control optimizes the objective of each player strategically and independently without any central coordination, which makes it a noncooperative game. Substituting $\bar{u}(k)$ and (6) into system (Σ) yields

$$(\Sigma_e): \ \delta x(k) = Ax(k) + \bar{B}u(k) + \bar{B}Ve_w(k) + H_1 d_u(k). \tag{11}$$

The system (Σ_e) is said to be the nominal system (Σ_n) if $e_w(k) \equiv 0$ and $d_u(k) \equiv 0$. The associated cost function of system (Σ_n) is

$$J_K^i(u^i, u^{-i}) = x^T(K)Q_\delta^K x(K) + \sum\nolimits_{k=0}^{K-1} \left\{ x^T(k)Q_\delta^i x(k) + \sum\nolimits_{j=1}^{S} u^{jT}(k)R_\delta^{ij} u^j(k) \right\}, \ k \in \mathbf{K}, \ i \in \mathbf{S} \tag{12}$$

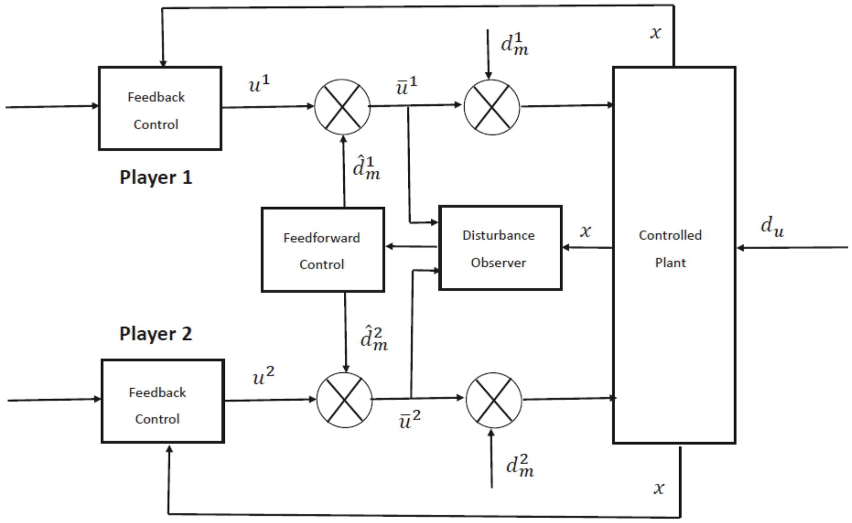


Fig. 1. The structure of the two-player LQ games with the proposed composite strategy.

where $\mathbf{K} := \{1, 2, \ldots, K\}$ is the set of time indexes and $\mathbf{S} := \{1, 2, \ldots, S\}$ is the set of players. As in [5], we assume that $Q_\delta^K > 0$, $Q_\delta^i > 0$ and $R_\delta^{ij} > 0$ for all $i, j \in \mathbf{S}$. Let u^{-i} denote the collection of actions of all players except player i, i.e., $u^{-i} := \{u^1, , \ldots, u^{i-1}, u^{i+1}, \ldots, u^S\}$. In system (Σ_n), the pure NE can be employed to describe the final outcome of the noncooperative game, which is $J_K^{i*} := J_K^i(u^{i*}, u^{-i*}) \leq J_K^i(u^i, u^{-i*})$ [**?**], [36]. However, it should be pointed out that in system (Σ_e), there exist estimation error $e_w(k)$ and unmatched disturbance $d_u(k)$, which can lead to a deviation from the pure NE. To deal with system (Σ_e), we are now in a position to introduce the concept of ϵ-NE as follows:

$$\hat{J}_K^{i*} := \hat{J}_K^i(u^{i*}, u^{-i*}) \leq \hat{J}_K^i(u^i, u^{-i*}) + \epsilon_K^i, \quad i = 1, 2, \ldots, S, \tag{13}$$

where $\hat{J}_K^i$ describes the cost function with the same structure as J_K^i but contaminated with perturbations $e_w(k)$ and $d_u(k)$, and the scalar $\epsilon_K^i \geq 0$ is a parameter characterizing the deviation from the pure NE ($\epsilon_K^i \equiv 0$). To proceed, let U_{admis}^i be a class of admissible control actions $u^i(k)$, which contains all non-stationary feedback controllers satisfying $\left\|u^i(k)\right\| \leq \bar{\delta}^i$.

In this paper, our main objective is to design the composite controller (10) for system (11) such that, for matched and unmatched disturbances, the ϵ-NE is obtained and the epsilon level can be estimated analytically. In other words, we aim to design a composite controller such that the following design objectives O1), O2) and O3) are achieved simultaneously.

O1) Design the disturbance observer gain L such that the error dynamics (8) is asymptotically stable while satisfying the H_∞ performance.

O2) Design the feedback Nash strategy $u(k)$ for system Σ_n such that the pure NE can be obtained, i.e. $J_K^i(u^{i*}, u^{-i*}) \leq J_K^i(u^i, u^{-i*})$.

O3) With the proposed composite control strategy (10), estimate an upper bound for the scalar ϵ_K^i such that (13) holds with $u^i(k) \in U_{admis}^i$.

Remark 3: For the purpose of describing the situation where central coordination is infeasible, the NE has been used as the design objective instead of the social optimal solution. It should be noticed that performance obtained by NE solution cannot be better than that by the social optimal solution. However, while dealing with the large scaled systems, it is more practical to use NE as the control performance index since it is often the case that each controller/decision maker only optimizes its own cost function in a noncooperative setting. As such, the NE has been used in a number of control problems, such as the coordinated control problem of multi-agent system [29], the power control problem [30], and the control problem of the heating, ventilation and air conditioning system [6].

III. Disturbance-observer-based Controller Design

In this section, the delta domain conditions are provided to design the disturbance observer gain and the feedback Nash strategies, respectively.

To proceed, we introduce the following lemmas which will be used in the proofs of our main results.

Lemma 1: [9] For any $x(k)$ and $y(k)$, the following property of delta operator holds

$$\delta(x(k)y(k)) = y(k)\delta x(k) + x(k)\delta y(k) + T_s\delta x(k)\delta y(k),$$

where T_s is the sampling interval.

Lemma 2: [28] Let x, y be any n_1-dimensional real vectors, and let Π be an $n_1 \times n_1$ symmetric positive semi-definite matrix. Then, we have $x^Ty + y^Tx \leq x^T\Pi x + y^T\Pi^{-1}y$.

A. Disturbance Observer Design

In the following theorem, a sufficient condition is given to design the disturbance observer gain in terms of the LMI method.

Theorem 1: For given scalars $\gamma > 0$ and $T_s > 0$, assume that there exist the matrix $P > 0$ and matrix Γ satisfying

$$\begin{bmatrix} I - \frac{1}{T_s}P & 0 & 0 & T_sW^TP + T_sV^T\bar{B}^T\Gamma^T + P \\ 0 & -\gamma^2 I & 0 & T_sH_2^TP \\ 0 & 0 & -\gamma^2 I & T_sH_1^T\Gamma^T \\ T_sPW + T_s\Gamma\bar{B}V + P & T_sPH_2 & T_s\Gamma H_1 & -T_sP \end{bmatrix} < 0, \tag{14}$$

then the error system (8) is robustly stable. Moreover, the disturbance observer gain is given by $L = P^{-1}\Gamma$.

Proof: Let us construct the Lyapunov function as $V(e_w(k)) = e_w^T(k)Pe_w(k)$. For $e_w(0) = 0$, it follows from Lemma 1 that

$$\begin{aligned} J &\leq \sum\nolimits_{k=0}^{\infty} \{e_w^T(k)e_w(k) - \gamma^2 d^T(k)d(k) + \delta V(e_w(k))\} \\ &= \sum\nolimits_{k=0}^{\infty} \{e_w^T(k)e_w(k) - \gamma^2 d^T(k)d(k) + \{\delta e_w^T(k)\}Pe_w(k) + e_w^T(k)P\{\delta e_w(k)\} + \{\delta e_w^T(k)\}P\{\delta e_w(k)\}\} \\ &= \sum\nolimits_{k=0}^{\infty} \Phi(k), \end{aligned}$$

where

$$\begin{aligned} \Phi(k) \;=\; & \sum\nolimits_{k=0}^{\infty} \{e_w^T(k)e_w(k) - \gamma^2 d^T(k)d(k) + (G_0e_w(k) + Hd(k))^TPe_w(k) \\ & + e_w^T(k)P(G_0e_w(k) + Hd(k)) + T_s(G_0e_w(k) + Hd(k))^TP(G_0e_w(k) + Hd(k))\}. \end{aligned}$$

The matrix $\Phi(k)$ can be further rewritten as $\Phi(k) = \begin{bmatrix} e_w^T(k) & d^T(k) \end{bmatrix} M_1 \begin{bmatrix} e_w(k) \\ d(k) \end{bmatrix}$, where

$$M_1 = \begin{bmatrix} I + G_0^TP + PG_0 + T_sG_0^TPG_0 & H^TP + T_sH^TPG_0 \\ PH + T_sG_0^TPH & -\gamma^2 I + T_sH^TPH \end{bmatrix}.$$

It is clear that $M_1 < 0$ implies $G_0^TP + PG_0 + T_sG_0^TPG_0 < 0$ which indicates that system (8) is asymptotically stable [9]. On the other hand, by employing the Schur Complement, $M_1 < 0$ implies that

$$M_2 = \begin{bmatrix} I - \frac{1}{T_s}P & 0 & T_sG_0^TP + P \\ 0 & -\gamma^2 I & T_sH^TP \\ T_sPG_0 + P & T_sPH & -T_sP \end{bmatrix} < 0. \tag{15}$$

Substituting the coefficient matrices of (8) into (15) and denoting $\Gamma = PL$ yield

$$M_3 = \begin{bmatrix} I - \frac{1}{T_s}P & 0 & 0 & T_sW^TP + T_sV^T\bar{B}^T\Gamma^T + P \\ 0 & -\gamma^2 I & 0 & T_sH_2^TP \\ 0 & 0 & -\gamma^2 I & T_sH_1^T\Gamma^T \\ T_sPW + T_s\Gamma\bar{B}V + P & T_sPH_2 & T_s\Gamma H_1 & -T_sP \end{bmatrix} < 0, \tag{16}$$

It is easy to conclude that $J < 0$ if $M_3 < 0$. Therefore, according to Definition 1, it can be verified that system (8) is robustly stable. The proof is complete now. ■

It should be pointed out that the sampling rate is explicitly expressed in the sufficient condition in Theorem 1, which differs from the results of the traditional discrete systems [15]. The delta domain results can be transformed into the continuous (respectively, discrete) domain analogue if $T_s \to 0$ (respectively, $T_s = 1$). As a by-product, we point out that an optimization problem can be formulated as follows:

$$\begin{aligned} &\gamma^* := \min_{P>0,\Gamma} \gamma, \\ &\text{subject to (14).} \end{aligned} \tag{17}$$

B. Strategy Design for LQ Game

In this section, the conditions are given to develop the feedback Nash strategies for system Σ_n with the cost function (12). In the following theorem, the existence and uniqueness conditions of NE for delta domain game are provided, and the explicit expression of feedback Nash strategies is given.

Theorem 2: Consider the system Σ_n with the cost function (12). There exists a unique feedback NE solution if the following conditions are simultaneously satisfied:

1) The matrix $\Theta(k) = \begin{pmatrix} \Theta_{ij}(k) \end{pmatrix}$ is invertible with

$$\begin{aligned} \Theta_{ii}(k) &= R_\delta^{ii} + T_sB^{iT}Z^i(k+1)B^i, \\ \Theta_{ij}(k) &= T_sB^{iT}Z^i(k+1)B^j. \end{aligned}$$

2) The following inequalities hold

$$R_\delta^{ii} + T_sB^{iT}Z^i(k+1)B^i > 0,$$

and $Z^i(k)$ obeys the following backward recursion

$$\begin{cases} -\delta Z^i(k) = Q_\delta^i + \sum_{j=1}^{S} P^{jT}(k)R_\delta^{ij}P^j(k) + F^T(k)Z^i(k+1) \\ \qquad + Z^i(k+1)F(k) + T_sF^T(k)Z^i(k+1)F(k), \\ Z^i(k) = Z^i(k+1) - T_s\delta Z^i(k), \quad Z^i(K) = Q_\delta^K \end{cases} \tag{18}$$

and $F(k) = A - \sum_{j=1}^{S} B^jP^j(k)$. The controller takes the form of $u^{i*}(k) = -P^i(k)x(k)$ with the parameters obeying

$$\bar{P}(k) = \Theta^{-1}(k)\Xi(k) \tag{19}$$

where

$$\begin{cases} \Xi_{ii}(k) = B^{iT}Z^i(k+1)(T_sA + I), \\ \Xi_{ij}(k) = 0, \\ P(k) = \begin{bmatrix} P^{1T}(k) & P^{2T}(k) & \ldots & P^{ST}(k) \end{bmatrix}^T. \end{cases}$$

Moreover, the corresponding value of NE is $J_K^{i*} = x^T(0)Z^i(0)x(0)(i \in \mathbf{S})$ with $x(0)$ being the initial value.

Proof: Let us choose the delta domain cost function as $V^i(x(k+1)) = \frac{1}{2T_s}x^T(k+1)Z^i(k+1)x(k+1)$. It follows from Lemma 1 that

$$\begin{aligned} V^i(x(k+1)) &= \frac{1}{2}\delta(x^T(k)Z^i(k)x(k)) + \frac{1}{2T_s}x^T(k)Z^i(k)x(k) \\ &= \frac{1}{2}\delta x^T(k)Z^i(k+1)x(k) + \frac{1}{2}x^T(k)Z^i(k+1)\delta x(k) \\ &\quad + \frac{1}{2}T_s\delta x^T(k)Z^i(k+1)\delta x(k) + \frac{1}{2T_s}x^T(k)Z^i(k+1)x(k). \end{aligned} \tag{20}$$

By the principle of dynamic programming as in [36], we have

$$\begin{aligned} V^i(x(k)) &= \min_{u^i(k)}\left\{\frac{1}{2}x^T(k)Q^i_\delta x(k) + \frac{1}{2}\sum\nolimits_{j=1}^{S} u^{jT}(k)R^{ij}_\delta u^j(k) + V^i(x(k+1))\right\} \\ &= \min_{u^i(k)}\left\{\frac{1}{2}x^T(k)Q^i_\delta x(k) + \frac{1}{2}(\sum\nolimits_{j=1}^{S} u^{jT}(k)R^{ij}_\delta u^j(k)\right. \\ &\quad + \frac{1}{2}(Ax(k) + \sum\nolimits_{i=1}^{S} B^i u^i(k))^T Z^i(k+1)x(k) \\ &\quad + \frac{1}{2}x^T(k)Z^i(k+1)(Ax(k) + \sum\nolimits_{i=1}^{S} B^i u^i(k)) \\ &\quad + \frac{1}{2}T_s(Ax(k) + \sum\nolimits_{i=1}^{S} B^i u^i(k))^T Z^i(k+1)(Ax(k) \\ &\quad \left. + \sum\nolimits_{i=1}^{S} B^i u^i(k)) + \frac{1}{2T_s}x^T(k)Z^i(k+1)x(k)\right\}. \end{aligned} \tag{21}$$

Since the second derivative of the right-hand side of (20) yields $R^{ii}_\delta + T_s B^{iT}Z^i(k+1)B^i > 0$, it is easy to conclude that the function $V^i(x(k))$ is strictly convex with respect to $u^i(k)$. The minimum value can be obtained by setting $\partial V^i(x(k))/\partial u^i(k) = 0$, i.e.,

$$\begin{aligned} 0 &= R^{ii}_\delta u^i(k) + B^{iT}Z^i(k+1)x(k) + T_s B^{iT}Z^i(k+1)Ax(k) \\ &\quad + T_s B^{iT}Z^i(k+1)\sum\nolimits_{i=1}^{S} B^{iT}u^i(k). \end{aligned} \tag{22}$$

Then, the optimal control strategies $u^i(k) = -P^i(k)x(k)$ can be obtained by

$$B^{iT}Z^i(k+1)(T_sA+I) = (R^{ii}_\delta + T_sB^{iT}Z^i(k+1)B^i)P^i(k) + T_sB^{iT}Z^i(k+1)\sum\nolimits_{j=1,j\neq i}^{S} B^jP^j(k). \tag{23}$$

It follows from (23) that there exists a unique NE if $\Theta(k)$ is invertible with $\Theta(k)\bar{P}(k) = \Xi(k)$. Substituting $u^i(k) = -P^i(k)x(k)$ into (20) yields

$$\begin{aligned} 0 &= Q^i_\delta + \sum\nolimits_{j=1}^{S} P^{jT}(k)R^{ij}_\delta P^j(k) + F^T(k)Z^i(k+1) \\ &\quad + Z(k{+}1)F(k) + T_sF^T(k)Z^i(k+1)F(k) + \delta Z^i(k) \end{aligned} \tag{24}$$

with $F(k) = A - \sum_{j=1}^{S} B^jP^j(k)$, which completes the proof of Theorem 2. ■

Remark 4: It is worth mentioning that the results in Theorem 2 can be readily extended to the time-varying case where the state matrices are $A(k)$ and $B^i(k)$, $i \in \mathbf{S}$, $k \in \mathbf{K}$. Moreover, the sampling interval is explicitly reflected in Theorem 2. If the sampling period T_s in (18) and (19) is replaced by a time-varying sampling period, the delta domain LQ game theoretic control strategy in Theorem 2 is still applicable.

Remark 5: The delta domain results in Theorem 2 can be regarded as a unified form of the results in both discrete and continuous domains. It is easy to transform the delta domain results in Theorem 2 to its discrete-domain analogue by using $A_z = T_sA + I$ and $B^i_z = T_sB^i$. For such a transformation to the continuous domain, one has

$$\begin{cases} P^i(t) = \underset{T_s\to 0}{lim} P^i_\delta = \left\{R^{ii}_\delta\right\}^{-1} B^{iT}Z(t)\ , \\ \lim\limits_{T_s\to 0} \delta Z^i = \dot{Z}^i. \end{cases} \tag{25}$$

Substituting (25) into the backward recursion (18) yields

$$Q_\delta^i+\sum\nolimits_{j=1}^{S} Z^{iT}(t)B^i\{R_\delta^{jj}\}^{-1}R_\delta^{ij}\{R_\delta^{jj}\}^{-1}B^{iT}Z^i(t)+\tilde{F}^T(t)(k)Z^i(t)+Z^i(t)\tilde{F}(t)+\dot{Z}^i(t)=0, \tag{26}$$

where $\tilde{F}(t)=A-\sum_{i=1}^{S}B^i\{R_\delta^{ii}\}^{-1}B^{iT}Z^i(t)$. Note that (26) is consistent with the Riccati recursions in the continuous domain [36], which verifies the fact that Theorem 2 can unify the results in both discrete and continuous domains.

IV. Robustness Analysis of ϵ-Equilibrium

It should be noticed that the feedback Nash gain $P^i(k)$ $(i\in\mathbf{S})$ has been obtained for system Σ_n. In the presence of $e_w(k)$ and $d_u(k)$, a certain deviation from the pure NE will be caused. In this subsection, we will consider the system Σ_e and provide the estimation for the epsilon level of ϵ-NE.

Firstly, let us introduce the following notations.

1) $x^*(k)$ is the state vector for the system (Σ_n) if each player uses NE strategies.
2) $x(k)$ is the state vector for the system (Σ_e) if each player uses NE strategies.
3) $\tilde{x}(k|i)$ is the state vector for the system (Σ_n) if each player uses NE strategies except player i. Note that we still have $u^i(\tilde{x}(k))\in U^i_{admis}$.
4) $\breve{x}(k|i)$ is the state vector for the system (Σ_e) if each player uses NE except player i.

To facilitate further developments, set

$$\begin{cases}\|e_w(k)\|^2\le\beta_2,\\ V_1(k)=z^T(k)S_1(k)z(k),\;\; z(k)=x(k)-x^*(k),\\ V_2^i(k)=\tilde{z}^T(k)S_2^i(k)\tilde{z}(k),\;\; \tilde{z}(k)=\tilde{x}(k|i)-\breve{x}(k|i),\\ \tilde{S}_1:=\{\tilde{S}_1-S_1(k)>0,\forall k\in[0,K]\},\;\; \lambda_1=\lambda_{\max}\{T_sV^T\bar{B}^T(\tilde{S}_1+\tilde{S}_1\Lambda_1^{-1}\tilde{S}_1+\Lambda_3)\bar{B}V\},\\ \lambda_2=\lambda_{\max}\{T_sH_1^T(\tilde{S}_1+\tilde{S}_1\Lambda_2^{-1}\tilde{S}_1+\tilde{S}_1\Lambda_3^{-1}\tilde{S}_1)H_1\},\\ \tilde{S}_2^i:=\{\tilde{S}_2^i-S_2^i(k)>0,\forall k\in[0,K]\},\\ \kappa_{1,i}=4T_sB^{iT}(\tilde{S}_2^i+\tilde{S}_2^i\tilde{\Lambda}_1^{-1}\tilde{S}_2^i+\tilde{\Lambda}_4+\tilde{\Lambda}_5)B^i,\\ \lambda_{3,i}=\lambda_{\max}\left\{T_sV^T\bar{B}^T(\tilde{S}_2^i+\tilde{S}_2^i\tilde{\Lambda}_2^{-1}\tilde{S}_2^i+\tilde{S}_2^i\tilde{\Lambda}_4^{-1}S_2^i+\tilde{\Lambda}_6)\bar{B}V\right\},\\ \lambda_{4,i}=\lambda_{\max}\left\{T_sH_1^T(\tilde{S}_2^i+\tilde{S}_2^i\tilde{\Lambda}_3^{-1}\tilde{S}_2^i+\tilde{S}_2^i\tilde{\Lambda}_5^{-1}\tilde{S}_2^i+\tilde{S}_2^i\tilde{\Lambda}_6^{-1}\tilde{S}_2^i)H_1\right\},\\ \varsigma_1=\lambda_1\beta_2+\lambda_2\beta_1,\;\; \varsigma_{2,i}=\lambda_{3,i}\beta_2+\lambda_{4,i}\beta_1+\kappa_{1,i}\bar{\delta}^{i2},\\ \hat{Q}_\delta^i=Q_\delta^i+\sum\nolimits_{j=1}^{S}P^{jT}R_\delta^{ij}P^j,\; \tilde{Q}_\delta^i=Q_\delta^i+\sum\nolimits_{j\ne i}^{S}P^{jT}R_\delta^{ij}P^j.\end{cases} \tag{27}$$

Based on the dynamics of the nominal system, we can assume that

$$\sum\nolimits_{k=0}^{K}\|x^*(k)\|^2\le\beta_3<\infty,\qquad \sum\nolimits_{k=0}^{K}\|\breve{x}(k)\|^2\le\beta_4<\infty. \tag{28}$$

Now, we are in the position to provide the estimation for the epsilon level of ϵ-NE.

Theorem 3: Consider the system Σ_e with the cost function (12). For given NE strategies u^{i*} and positive definite matrices $\{\Lambda_j\}_{j=1}^3$, L_1, $\tilde{L}_2^i$, $\{\tilde{\Lambda}_j^i\}_{j=1}^6$ ($i \in \mathbf{S}$), if there exist positive definite matrices S_1 and S_{2i} such that the following delta domain Lyapunov equations hold

$$\begin{cases} -\delta S_1(k) = T_s\tilde{A}^T S_1(k+1)\tilde{A} + \tilde{A}^T S_1(k+1) + S_1(k+1)\tilde{A} \\ \qquad + \dfrac{1}{T_s}\left\{L_1 + (T_s\tilde{A}+I)^T\Lambda_1(T_s\tilde{A}+I) + (T_s\tilde{A}+I)^T\Lambda_2(T_s\tilde{A}+I)\right\}, \\ S_1(k) = S_1(k+1) - T_s\delta S_1(k), \quad S_1(K) = 0, \\ -\delta S_2^i(k) = T_s\tilde{A}^{iT} S_2^i(k+1)\tilde{A}^i + \tilde{A}^{iT} S_2^i(k+1) + S_2^i(k+1)\tilde{A}^i \\ \qquad + \dfrac{1}{T_s}\left\{(T_s\tilde{A}^i+I)^T(\tilde{\Lambda}_1+\tilde{\Lambda}_2+\tilde{\Lambda}_3)(T_s\tilde{A}^i+I) + L_2^i\right\}, \\ S_2^i(k) = S_2^i(k+1) - T_s\delta S_2^i(k), \quad S_2^i(K) = 0, \end{cases} \tag{29}$$

with $\tilde{A} = A - \sum_{i=1}^S B^i P^i(k)$ and $\tilde{A}^i = A - \sum_{j\neq i}^S B^j P^j(k)$, then the NE strategies in Theorem 2 provide an ϵ-Nash equilibrium, i.e.,

$$\hat{J}_K^{i*} = \hat{J}_K^i(u^{i*}, u^{-i*}) \leq \hat{J}_K^i(u^i, u^{-i*}) + \epsilon_K^i, \tag{30}$$

where

$$\begin{aligned} \epsilon_K^i \;=\; & \lambda_{\max}\{\hat{Q}_\delta^i\}\{T_s\lambda_{\min}\{L_1\}^{-1}K\varsigma_1 + \lambda_{\min}\{L_1\}^{-1}V_1(0)\} \\ & +2\sqrt{\beta_3\{T_s\lambda_{\min}\{L_1\}^{-1}K\varsigma_1 + \lambda_{\min}\{L_1\}^{-1}V_1(0)\}\}} \\ & +\lambda_{\max}\{\tilde{Q}_\delta^i\}\{T_s\lambda_{\min}\{L_2^i\}^{-1}K\varsigma_{2,i} + \lambda_{\min}\{L_2^i\}^{-1}V_2^i(0) \\ & +2\sqrt{\beta_4\{T_s\lambda_{\min}\{L_2^i\}^{-1}K\varsigma_{2,i} + \lambda_{\min}\{L_2^i\}^{-1}V_2^i(0)\}\}} + 4\lambda_{\max}\{R_\delta^{ii}\}K\bar{\delta}^{i2}. \end{aligned} \tag{31}$$

Proof: According to the definition of NE, we have

$$J_K^i(u^{i*}, u^{-i*}) \leq J_K^i(u^i, u^{-i*}). \tag{32}$$

It follows that

$$\hat{J}_K^i(u^{i*}, u^{-i*}) \leq \hat{J}_K^i(u^i, u^{-i*}) + \underbrace{\Delta J_{K1}^i + \Delta J_{K2}^i}_{\epsilon_K^i}, \tag{33}$$

where $\Delta J_{K1}^i := \hat{J}_K^i(u^{i*}, u^{-i*}) - J_K^i(u^{i*}, u^{-i*})$ and $\Delta J_{K2}^i := J_K^i(u^i, u^{-i*}) - \hat{J}_K^i(u^i, u^{-i*})$. In the following, the terms ΔJ_{K1}^i and ΔJ_{K2}^i will be estimated, respectively.

Firstly, let us estimate the term ΔJ_{K1}^i. According to the NE strategies without noise, we obtain

$$\hat{J}_K^i(u^{i*}, u^{-i*}) = \sum\nolimits_{k=0}^K x^T(k)\hat{Q}_\delta^i(k)x(k), \tag{34}$$

where $\hat{Q}_\delta^i = Q_\delta^i + \sum_{j=1}^S P^{jT}R_\delta^{ij}P^j$. When the noise exists, one has

$$J_K^i(u^{i*}, u^{-i*}) = \sum\nolimits_{k=0}^K x^{*T}(k)\hat{Q}_\delta^i(k)x^*(k). \tag{35}$$

Then, the term ΔJ_{K1}^i can be calculated as follows:

$$\begin{aligned} \Delta J_{K1}^i \;&=\; \hat{J}_K^i(u^{i*}, u^{-i*}) - J_K^i(u^{i*}, u^{-i*}) \\ &=\; \sum\nolimits_{k=0}^K x^T(k)\hat{Q}_\delta^i x(k) - \sum\nolimits_{k=0}^K x^{*T}(k)\hat{Q}_\delta^i x^*(k) \\ &=\; \sum\nolimits_{k=0}^K (x-x^*)^T\hat{Q}_\delta^i(x-x^*+2x^*) \\ &\leq\; \sum\nolimits_{k=0}^K \{\|x-x^*\|_{\hat{Q}_\delta^i}^2 + 2\|x-x^*\|\left\|\hat{Q}_\delta^i x^*\right\|\} \end{aligned}$$

$$\leq \lambda_{\max}\{\hat{Q}^i_\delta\}\left\{\sum_{k=0}^{K}\{\|x-x^*\|^2\}+\sum_{k=0}^{K}\{2\|x-x^*\|\|x^*\|\}\right\}$$
$$\leq \lambda_{\max}\{\hat{Q}^i_\delta\}\left\{\sum_{k=0}^{K}\{\|z\|^2\}+2\sqrt{\sum_{k=0}^{K}\|z\|^2}\sqrt{\sum_{k=0}^{K}\|x^*\|^2}\right\}. \quad (36)$$

It should be noticed that the Cauchy inequality is used when deriving the last inequality. According to the definition of $z(k)$

$$\delta z(k)=\tilde{A}z(k)+\bar{B}Ve_w(k)+H_1d_u(k), \quad (37)$$

with $\tilde{A}=A-\sum_{i=1}^{S}B^iP^i(k)$, calculating the difference of $V_1(k)$ along (37) in the delta domain yields

$$\begin{aligned}\delta V_1(k) = & \frac{1}{T_s}\{z^T(k)(T_s\tilde{A}+I)^TS_1(k+1)(T_s\tilde{A}+I)z(k)\\ &+T_s^2e_w^T(k)V^T\bar{B}^TS_1(k+1)\bar{B}Ve_w(k)+T_s^2d_u^T(k)H_1^TS_1(k+1)H_1d_u(k)\\ &+T_sz^T(k)(T_s\tilde{A}+I)^TS_1(k+1)\bar{B}Ve_w(k)+T_se_w^T(k)V^T\bar{B}^TS_1(k+1)(T_s\tilde{A}+I)z(k)\\ &+T_s\, z^T(k)(T_s\tilde{A}+I)^TS_1(k+1)H_1d_u(k)+T_s\, d_u^T(k)H_1^TS_1(k+1)(T_s\tilde{A}+I)z(k)\\ &+T_s^2e_w^T(k)V^T\bar{B}^TS_1(k+1)H_1d_u(k)+T_s^2d_u^T(k)H_1^TS_1(k+1)\bar{B}Ve_w(k)\\ &+z^T(k)L_1z(k)-z^T(k)S_1(k)z(k)\ \}-\frac{1}{T_s}z^T(k)L_1z(k). \quad (38)\end{aligned}$$

It follows from Lemma 2 that

$$\begin{aligned}\delta V_1(k) \leq & \frac{1}{T_s}z^T(k)\{(T_s\tilde{A}+I)^TS_1(k+1)(T_s\tilde{A}+I)-S_1(k)+L_1+(T_s\tilde{A}+I)^T\Lambda_1(T_s\tilde{A}+I)\\ &+(T_s\tilde{A}+I)^T\Lambda_2(T_s\tilde{A}+I)\}z(k)+T_s\{e_w^T(k)V^T\bar{B}^TS_1(k+1)\bar{B}Ve_w(k)\\ &+d_u^T(k)H_1^TS_1(k+1)H_1d_u(k)+e_w^T(k)V^T\bar{B}^TS_1(k+1)\Lambda_1^{-1}S_1(k+1)\bar{B}Ve_w(k)\\ &+d_u^T(k)H_1^TS_1(k+1)\Lambda_2^{-1}S_1(k+1)H_1d_u(k)+e_w^T(k)V^T\bar{B}^T\Lambda_3\bar{B}Ve_w(k)\\ &+d_u^T(k)H_1^TS_1(k+1)\Lambda_3^{-1}S_1(k+1)H_1d_u(k)\}-\frac{1}{T_s}z^T(k)L_1z(k). \quad (39)\end{aligned}$$

Select $S_1(k)$ such that

$$\begin{aligned}&(T_s\tilde{A}+I)^TS_1(k+1)(T_s\tilde{A}+I)-S_1(k)+L_1\\ &+(T_s\tilde{A}+I)^T\Lambda_1(T_s\tilde{A}+I)+(T_s\tilde{A}+I)^T\Lambda_2(T_s\tilde{A}+I)=0. \quad (40)\end{aligned}$$

Then

$$\begin{aligned}\delta V_1(k) \leq\ & T_s\{e_w^T(k)V^T\bar{B}^T\tilde{S}_1\bar{B}Ve_w(k)+d_u^T(k)H_1^TS_1H_1d_u(k)\\ &+e_w(k)^TV^T\bar{B}^T\tilde{S}_1\Lambda_1^{-1}\tilde{S}_1\bar{B}Ve_w(k)+d_u^T(k)H_1^T\tilde{S}_1\Lambda_2^{-1}\tilde{S}_1H_1d_u(k)\\ &+e_w^T(k)V^T\bar{B}^T\Lambda_3\bar{B}Ve_w(k)+d_u^T(k)H_1^T\tilde{S}_1\Lambda_3^{-1}\tilde{S}_1H_1d_u(k)\}-\frac{1}{T_s}z^T(k)L_1z(k)\\ \leq\ & \nu_1(k)-\frac{1}{T_s}z^T(k)L_1z(k), \quad (41)\end{aligned}$$

where

$$\begin{aligned}\nu_1(k) &= \lambda_1\|e_w(k)\|^2+\lambda_2\|d_u(k)\|^2,\\ \lambda_1 &= \lambda_{\max}\{T_sV^T\bar{B}^T(\tilde{S}_1+\tilde{S}_1\Lambda_1^{-1}\tilde{S}_1+\Lambda_3)\bar{B}V\},\\ \lambda_2 &= \lambda_{\max}\{T_sH_1^T(\tilde{S}_1+\tilde{S}_1\Lambda_2^{-1}\tilde{S}_1+\tilde{S}_1\Lambda_3^{-1}\tilde{S}_1)H_1\}.\end{aligned}$$

Furthermore, we have

$$\sum_{k=0}^{K}\delta V_1(k) \leq \sum_{k=0}^{K}\nu_1(k)-\frac{1}{T_s}\sum_{k=0}^{\infty}z^T(k)L_1z(k), \quad (42)$$

which leads to

$$\begin{aligned}\frac{1}{T_s}\lambda_{\min}\{L_1\}\sum\nolimits_{k=0}^{K}\|z(k)\|^2 &\le \frac{1}{T_s}\sum\nolimits_{k=0}^{K} z^T(k)L_1z(k)\\ &\le \sum\nolimits_{k=0}^{K}\nu_1(k)-1/T_sV_1(K)+1/T_sV_1(0)\\ &\le \sum\nolimits_{k=0}^{K}\nu_1(k)+1/T_sV_1(0). \end{aligned}\tag{43}$$

Then, we can conclude that $\sum_{k=0}^{K}\|z(k)\|^2 \le T_s\lambda_{\min}\{L_1\}^{-1}K\{\lambda_1\beta_2+\lambda_2\beta_1\}+\lambda_{\min}\{L_1\}^{-1}V_1(0)$. The upper bound for ΔJ_{K1}^i can then be obtained

$$\begin{aligned}\Delta J_{K1}^i \le\ & \lambda_{\max}\{\hat{Q}_\delta^i\}\{T_s\lambda_{\min}\{L_1\}^{-1}K\varsigma_1+\lambda_{\min}\{L_1\}^{-1}V_1(0)\\ &+2\sqrt{\beta_3\{T_s\lambda_{\min}\{L_1\}^{-1}K\varsigma_1+\lambda_{\min}\{L_1\}^{-1}V_1(0)\}}\}.\end{aligned}\tag{44}$$

Secondly, we are ready to estimate the term ΔJ_{K2}^i. The cost functions without and with noises are shown as follows:

$$\begin{cases} J_K^i(u^i,u^{-i*})=\sum\nolimits_{k=0}^{K}\Big\{\tilde{x}^T(k)\tilde{Q}_\delta^i\tilde{x}(k)+u^{iT}(k)R_\delta^{ii}u^i(k)\Big\},\\ \hat{J}_K^i(u^i,u^{-i*})=\sum\nolimits_{k=0}^{K}\Big\{\check{x}^T(k)\tilde{Q}_\delta^i\check{x}(k)+u^{iT}(k)R_\delta^{ii}u^i(k)\Big\},\end{cases}\tag{45}$$

where $\tilde{Q}_\delta^i=Q_\delta^i+\sum_{j\neq i}^{S}P^{jT}R_\delta^{ij}P^j$. The term ΔJ_{K2}^i can be rewritten as

$$\begin{aligned}\Delta J_{K2}^i=&\sum\nolimits_{k=0}^{K}\Big\{\{\tilde{x}(k)-\check{x}(k)\}^T\tilde{Q}_\delta^i\{\tilde{x}(k)-\check{x}(k)+2\check{x}(k)\}\Big\}\\ &+\sum\nolimits_{k=0}^{K}\Big\{\{u^i(\tilde{x})-u^i(\check{x})\}^TR_\delta^{ii}\{u^i(\tilde{x})-u^i(\check{x})+2u^i(\check{x})\}\Big\}\\ &\le\sum\nolimits_{k=0}^{K}\Big\{\|\tilde{x}(k)-\check{x}(k)\|_{\tilde{Q}_\delta^i}^2+2\|\tilde{x}(k)-\check{x}(k)\|\left\|\tilde{Q}_\delta^i\check{x}(k)\right\|\Big\}\\ &+\sum\nolimits_{k=0}^{K}\Big\{\left\|u^i(\tilde{x})-u^i(\check{x})\right\|_{R_\delta^{ii}}^2+2\left\|u^i(\tilde{x})-u^i(\check{x})\right\|\left\|R_\delta^{ii}u^i(\check{x})\right\|\Big\}\\ &\le\lambda_{\max}\{\tilde{Q}_\delta^i\}\left\{\sum\nolimits_{k=0}^{K}\|\tilde{z}(k)\|^2+2\sqrt{\sum\nolimits_{k=0}^{K}\|\tilde{z}(k)\|^2}\sqrt{\sum\nolimits_{k=0}^{K}\|\check{x}\|^2}\right\}\\ &+\lambda_{\max}\{R_\delta^{ii}\}\left\{\sum\nolimits_{k=0}^{K}\left\|u^i(\tilde{x})-u^i(\check{x})\right\|^2+2\sqrt{\sum\nolimits_{k=0}^{K}\|u^i(\tilde{x})-u^i(\check{x})\|^2}\sqrt{\sum\nolimits_{k=0}^{K}\|u^i(\check{x})\|^2}\right\}\\ &\le\lambda_{\max}\{\tilde{Q}_\delta^i\}\left\{\sum\nolimits_{k=0}^{K}\|\tilde{z}(k)\|^2+2\sqrt{\sum\nolimits_{k=0}^{K}\|\tilde{z}(k)\|^2}\sqrt{\sum\nolimits_{k=0}^{K}\|\check{x}\|^2}\right\}+4\lambda_{\max}\{R_\delta^{ii}\}K\bar{\delta}^{i2}.\end{aligned}\tag{46}$$

Note that the last inequality holds from the Clarkson's inequalities in [40] Similarly, according to the definition of $\tilde{z}(k)$

$$\delta\tilde{z}(k)=\tilde{A}^i\tilde{z}(k)+B^iu^i(\tilde{x})-B^iu^i(\check{x})+\bar{B}Ve_w(k)+H_1d_u(k)\tag{47}$$

with $\tilde{A}^i=A-\sum_{j\neq i}^{S}B^jP^j$, calculating the difference of $V_2^i(k)$ along (47) in the delta domain yields

$$\begin{aligned}V_2^i(k)\ =\ &\frac{1}{T_s}\{V_2^i(k+1)-V_2^i(k)\}\\ =\ &\frac{1}{T_s}\{(T_s\tilde{A}^i+I)\tilde{z}(k)+T_sB^iu^i(\tilde{x})-T_sB^iu^i(\check{x})+T_s\bar{B}Ve_w(k)+T_sH_1d_u(k)\}^T\\ &\times S_2^i(k+1)\{(T_s\tilde{A}^i+I)\tilde{z}(k)+T_sB^iu^i(\tilde{x})-T_sB^iu^i(\check{x})+T_s\bar{B}Ve_w(k)\\ &+T_sH_1d_u(k)\}-\frac{1}{T_s}\tilde{z}^T(k)S_2^i(k)\tilde{z}(k)\end{aligned}\tag{48}$$

Again, it follows from Lemma 2 that

$$\begin{aligned} V_2^i(k) &\le \frac{1}{T_s}\tilde{z}^T(k)\{(T_s\tilde{A}^i+I)^T S_2^i(k+1)(T_s\tilde{A}^i+I)) \\ &\quad +(T_s\tilde{A}^i+I)^T(\tilde{\Lambda}_1+\tilde{\Lambda}_2+\tilde{\Lambda}_3)(T_s\tilde{A}^i+I+L_2^i-S_2^i(k)\}\tilde{z}(k) \\ &\quad +\left\|u^i(\tilde{x})-u^i(\check{x})\right\|^2_{\kappa_{1,i}/4}+\lambda_{3,i}\left\|d_u(k)\right\|^2+\lambda_{4,i}\left\|e_w(k)\right\|^2-\frac{1}{T_s}\tilde{z}^T(k)L_2^i\tilde{z}(k) \\ &\le \frac{1}{T_s}\tilde{z}^T(k)\{(T_s\tilde{A}^i+I)^T S_2^i(k+1)(T_s\tilde{A}^i+I)+(T_s\tilde{A}^i+I)^T(\tilde{\Lambda}_1+\tilde{\Lambda}_2+\tilde{\Lambda}_3)(T_s\tilde{A}^i+I) \\ &\quad +L_2^i-S_2^i(k)\}\tilde{z}(k)+\lambda_{3,i}\left\|d_u(k)\right\|^2+\lambda_{4,i}\left\|e_w(k)\right\|^2+\kappa_{1,i}\bar{\delta}^{i2}-\frac{1}{T_s}\tilde{z}^T(k)L_2^i\tilde{z}(k), \end{aligned} \tag{49}$$

where

$$\begin{cases} \tilde{S}_2^i := \{\tilde{S}_2^i - S_2^i(k) > 0, \forall k \in [0,K]\}, \\ \kappa_{1,i} = 4T_s B^{iT}(\tilde{S}_2^i+\tilde{S}_2^i\tilde{\Lambda}_1^{-1}\tilde{S}_2^i+\tilde{\Lambda}_4+\tilde{\Lambda}_5)B^i, \\ \lambda_{3,i} = \lambda_{\max}\left\{T_s V^T\bar{B}^T(\tilde{S}_2^i+\tilde{S}_2^i\tilde{\Lambda}_2^{-1}\tilde{S}_2^i+\tilde{S}_2^i\tilde{\Lambda}_4^{-1}S_2^i+\tilde{\Lambda}_6)\bar{B}V\right\}, \\ \lambda_{4,i} = \lambda_{\max}\left\{T_s H_1^T(\tilde{S}_2^i+\tilde{S}_2^i\tilde{\Lambda}_3^{-1}\tilde{S}_2^i+\tilde{S}_2^i\tilde{\Lambda}_5^{-1}\tilde{S}_2^i+\tilde{S}_2^i\tilde{\Lambda}_6^{-1}\tilde{S}_2^i)H_1\right\}. \end{cases}$$

It is worth mentioning that we use the Clarkson's inequalities again when deriving (49). Next, select $S_2^i(k)$ such that

$$(T_s\tilde{A}^i+I)^T S_2^i(k+1)(T_s\tilde{A}^i+I)+(T_s\tilde{A}^i+I)^T(\tilde{\Lambda}_1+\tilde{\Lambda}_2+\tilde{\Lambda}_3)(T_s\tilde{A}^i+I)+L_2^i-S_2^i(k)=0. \tag{50}$$

Then,

$$\sum_{k=0}^{K}\delta V_2^i(k) \le \sum_{k=0}^{K}\nu_2(k)-\frac{1}{T_s}\sum_{k=0}^{K}\tilde{z}^T(k)L_2^i\tilde{z}(k), \tag{51}$$

where $\nu_2 = \lambda_{3,i}\|d_u(k)\|^2+\lambda_{4,i}\|e_w(k)\|^2++\kappa_{1,i}\bar{\delta}^{i2}$. Note that (51) can be rewritten as

$$\begin{aligned} \frac{1}{T_s}\lambda_{\min}\{L_2^i\}\sum_{k=0}^{K}\|\tilde{z}(k)\|^2 &\le \frac{1}{T_s}\sum_{k=0}^{K}\tilde{z}^T(k)L_2^i\tilde{z}(k) \\ &\le \sum_{k=0}^{K}\nu_2(k)-1/T_sV_2^i(K)+1/T_sV_2^i(0) \\ &\le \sum_{k=0}^{K}\nu_2(k)+1/T_sV_2^i(0). \end{aligned} \tag{52}$$

Hence, it is concluded that $\sum_{k=0}^{K}\|\tilde{z}(k)\|^2 \le T_s\lambda_{\min}\{L_2^i\}^{-1}K\{\lambda_{3,i}\beta_2+\lambda_{4,i}\beta_1+\kappa_{1,i}\bar{\delta}^{i2}\}+\lambda_{\min}\{L_2^i\}^{-1}V_2^i(0)$. Thus, the upper bound for ΔJ_{K2}^i can be obtained as

$$\begin{aligned} \Delta J_{K2}^i &\le \lambda_{\max}\{\tilde{Q}_\delta^i\}\{T_s\lambda_{\min}\{L_2^i\}^{-1}K\varsigma_{2,i}+\lambda_{\min}\{L_2^i\}^{-1}V_2^i(0) \\ &\quad +2\sqrt{\beta_4\{T_s\lambda_{\min}\{L_2^i\}^{-1}K\varsigma_{2,i}+\lambda_{\min}\{L_2^i\}^{-1}V_2^i(0)\}}\}+4\lambda_{\max}\{R_\delta^{ii}\}K\bar{\delta}^{i2}. \end{aligned} \tag{53}$$

Therefore, the proof of this theorem is complete. ■

Remark 6: According to [37], the set of riccati recursions (29) will be convergent as the time goes to infinity if $P^i(k),\ i\in\mathbf{S}$ remains constant over time and all eignevalues of $\tilde{A}$ and $\tilde{A}^i$ lie within the stability boundary in the delta-domain.

Remark 7: From Theorem 3, the following argument can easily be verified, i.e., if $e_w(k)=0$, $d_u(k)=0$ and $z(0)=\tilde{z}(0)=0$, the so-called ϵ-NE will reduce to the pure NE with $\epsilon_K^i\equiv 0$.

Remark 8: Note that (29) yields the typical delta domain Lyapunov equation which is actually a unified form of the discrete and continuous Lyapunov equations. For example, if we choose $L_1=T_s\hat{L}_1$, $\Lambda_1=T_s\hat{\Lambda}_1$ and $\Lambda_2=T_s\hat{\Lambda}_2$, the continuous domain Lyapunov equation in [38] can be obtained with $T_s\to 0$

$$\dot{S}_1(t)+\tilde{A}_t^T S_1(t)+S_1(t)\tilde{A}_t+\left\{\hat{L}_1+\hat{\Lambda}_1+\hat{\Lambda}_2\right\}=0. \tag{54}$$

On the other hand, by using $\tilde{A} = (\tilde{A}_z - I)/T_s$ and setting $T_s = 1$, one has the discrete Lyapunov equation as in [38]

$$S_1(k) = \tilde{A}_z^T S_1(k+1)\tilde{A}_z + \left\{ L_1 + \tilde{A}_z^T \Lambda_1 \tilde{A}_z + \tilde{A}_z^T \Lambda_2 \tilde{A}_z \right\}. \tag{55}$$

Remark 9: It should be noticed that the parameters which are used to calculate the epsilon level for ϵ-NE can all be obtained in prior or be deduced. For example, the parameter β_4 can be rewritten as $\beta_4 = \beta_3 + \Delta\beta$, and $\Delta\beta$ can be obtained by using similar techniques with deriving ΔJ^i_{K2}.

The proposed methods can be outlined as the following two algorithms.

Algorithm 1 The calculation of the composite control strategies

1: Set $k = K$, then $Z^i(K) = Q^K_\delta$ is available
2: Calculate the matrices $\Theta(k)$. If $\Theta(k)$ is invertible and $R^{ii}_\delta + T_s B^{iT} Z^i(k+1) B^i > 0$, then we obtain the feedback gains $P^i(k)$ by using (19).
3: Solve the backward RDEs of (18) to get $Z^i(k)$.
4: If $k \neq 0$, set $k = k - 1$ and go back to Step 2, else stop the algorithm
5: Solve the LMI (14) to get the gain of the disturbance observer L.
6: Substitute L into (5) and get the estimation for the matched disturbance $\hat{d}_m(k)$
7: Calculate the composite control strategy by using (10).

It should be noticed that, once the composite control strategies have been derived, the scalar β_2 and β_3 could be obtained. Then, the calculation of the ϵ level is summarized in the following algorithm.

Algorithm 2 The calculation of the ϵ level

1: Select positive definite matrices $\{\Lambda_j\}_{j=1}^3$, L_1, $\tilde{L}^i_2$, $\{\tilde{\Lambda}^i_j\}_{j=1}^6$ $(i \in \mathbf{S})$
2: Set $k = K$, then $S_1(K) = 0$ and $S^i_2(K) = 0$ are available
3: Solve the backward recursions (29) to get $S_1(k)$ and $S^i_2(k)$
4: If $S_1(k) > 0$, $S^i_2(k) > 0$, then we can move to the next procedure, else jump to Step 1
5: If $k \neq 0$, set $k = k - 1$, else stop the algorithm
6: Calculate $\tilde{S}_1$, $\tilde{S}^i_2$, λ_1, λ_2, $\lambda_{3,i}$, $\lambda_{4,i}$, $\kappa_{1,i}$, ς_1, $\varsigma_{2,i}$, $\hat{Q}^i_\delta$ and $\tilde{Q}^i_\delta$ according to (28). The epsilon level can be obtained by (31).

Up to now, the estimation for the upper bound of the ϵ-NE has been provided. In the following corollary, we will present an upper bound estimation for system (Σ_e) with an average cost.

Corollary 1: Consider the system (Σ_e) with the average cost defined by $J^i_{av,K}(u^i, u^{-i}) := \frac{1}{K} J^i_K(u^i, u^{-i})$. If the Nash strategies u^{i*} for any K yields an ϵ-NE of the delta domain LQ game, we have

$$J^i_{av,K}(u^{i*}, u^{-i*}) \leq J^i_{av,K}(u^i, u^{-i*}) + \epsilon^i_{av,K}, \tag{56}$$

with

$$\epsilon^i_{av,K} = \lambda_{\max}\{\hat{Q}^i_\delta\} T_s \lambda_{\min}\{L_1\}^{-1}\varsigma_1 + \lambda_{\max}\{\tilde{Q}^i_\delta\}\{T_s \lambda_{\min}\{L^i_2\}^{-1}\varsigma_{2,i} + 4\lambda_{\max}\{R^{ii}_\delta\}\bar{\delta}^{i2} + O(\frac{1}{\sqrt{K}}). \tag{57}$$

Proof: The proof follows directly from Theorem 3 and is therefore omitted here for brevity. ∎

Remark 10: Compared with the results in Theorem 2, it should be noticed that the ϵ-level in Corollary 1 is not dependent on the initial value $x(0)$ as $K \to \infty$. The reason is that costs incurred in the early stages do not matter since their contribution to the average cost per stage is reduced to zero as $K \to \infty$ for any fixed K_k, i.e.,

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K_k-1} \left\{ x^T(k) Q^i_\delta x(k) + \sum_{i=1}^{S} u^{jT}(k) R^{ij}_\delta u^j(k) \right\} = 0.$$

Remark 11: In most of the existing literature, it is implicitly assumed that the underlying dynamics of the LQ game evolves in the ideal environment without disturbances. Recently, it has been pointed out in [27] and [28] that such an assumption is impractical because that disturbances could perturb the outcome of the game. Under the disturbances, the classic pure NE cannot be used to describe the outcome of the game anymore. Therefore, the most distinguishing feature of our paper over the existing literature is that the impact from disturbance on the NE has been explicitly expressed in the delta domain. By using the analytical methods in our paper, the ϵ-NE could be found which is more practical since, in practice, all the control system are operated subjected to disturbances.

V. A Numerical Simulation

In this section, we aim to demonstrate the validity and applicability of the proposed method. For this purpose, we discuss the disturbance-observer-based composite control problem for a two-area interconnected power system. Our objective is to control the load frequency control system such that the outputs are kept at the desired setting, while maintaining robustness against load disturbances. As in [39], the basic parameters of the power system are shown in Table I.

TABLE I
Parameters of Two-Area Interconnected Power System

Area	T_{P_i}	K_{P_i}	T_{T_i}	T_{G_i}	R_i	K_{E_i}	K_{B_i}	$K_{S_{ij}}$
1	20	120	0.3	0.08	2.4	10	0.41	0.55
2	25	112.5	0.33	0.072	2.7	9	0.37	0.65

Consider the following two-area interconnected power system:

$$\dot{x}(t) = Ax_i(t) + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \left\{ \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \Delta\tilde{P}_d(t) \right\} + F\Delta P_d(t),$$

where

$$\begin{aligned}
x(t) &= \begin{bmatrix} x_1^T(t) & x_2^T(t) \end{bmatrix}^T, \quad A = \begin{bmatrix} A_1 & E_{12} \\ E_{21} & A_2 \end{bmatrix}, \\
A_i &= \begin{bmatrix} \frac{1}{T_{P_i}} & \frac{K_{P_i}}{T_{P_i}} & 0 & 0 & -\frac{K_{P_i}}{2\pi T_{P_i}} \sum_{j\neq i} K_{sij} \\ 0 & -\frac{1}{T_{T_i}} & \frac{1}{T_{T_i}} & 0 & 0 \\ -\frac{1}{R_i T_{G_i}} & 0 & -\frac{1}{T_{G_i}} & -\frac{1}{T_{G_i}} & 0 \\ K_{E_i}K_{B_i} & 0 & 0 & 0 & \frac{K_{E_i}}{2\pi} \sum_{j\neq i} K_{sij} \\ 2\pi & 0 & 0 & 0 & 0 \end{bmatrix}, \\
B_i &= \begin{bmatrix} 0 & 0 & \frac{1}{T_{G_i}} & 0 & 0 \end{bmatrix}^T, \\
E_{ij} &= \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{K_{P_i}}{2\pi T_{P_i}} \sum_{j\neq i} K_{sij} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{K_{E_i}}{2\pi} \sum_{j\neq i} K_{sij} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad x_i(t) = \begin{bmatrix} \Delta f_i(t) \\ \Delta P_{g_i}(t) \\ \Delta X_{g_i}(t) \\ \Delta E_i(t) \\ \Delta \delta_i(t) \end{bmatrix}, \ i, j = 1, 2
\end{aligned}$$

$\Delta P_d(t)$ is the vector of load disturbance, $\Delta\tilde{P}_d(t)$ is the disturbance in the control channel, $\Delta f_i(t)$, $\Delta P_{g_i}(t)$, $\Delta X_{g_i}(t)$, $\Delta E_i(t)$ and $\Delta\delta_i(t)$ are the changes of frequency, power output, governor valve position, integral control and rotor angle deviation, respectively. T_{G_i}, T_{t_i} and T_{p_i} are the time constants of governor, turbine and power system, respectively. K_{p_i}, R_i, K_{E_i} and K_{B_i} are the power system gain, speed regulation coefficient, integral control gain and frequency bias factor, respectively. K_{sij} is the interconnection gain between area i and j $(i \neq j)$.

First, let us design the composite controller with $K = 100$, $\gamma = 2$ and $T_s = 0.05s$. That is, we aim to deal with the problems O1) and O2). The parameters of the cost function are chosen as $Q_\delta^{100} = Q_\delta^1 = Q_\delta^2 = 1$ and $R_\delta^{11} = R_\delta^{12} = R_\delta^{21} = R_\delta^{22} = 0.1$. We assume that the unmatched disturbance $\Delta P_{d_i}(t) = 0$ and the matched disturbance $\Delta\tilde{P}_{d_i}(t)$ is given by

$$\begin{cases} \delta w(k) = \begin{bmatrix} 0.8776 & 0.4794 \\ -0.4794 & 0.8776 \end{bmatrix} w(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \dfrac{1}{k}, \\ \Delta\tilde{P}_d(t) = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} w(k),\ w(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T. \end{cases} \tag{58}$$

Solving the LMI condition (14) in Theorem 1 by using the LMI Toolbox, one has

$$L = \begin{bmatrix} -0.1076 & -0.2382 & -0.0216 & -41.7887 & -27.3024 & 0.1435 & -0.0100 & -0.0008 & -1.3946 & -1.5204 \\ 0.1125 & 0.0010 & 0.0001 & 0.5698 & 0.1142 & -0.1500 & -0.2394 & -0.0200 & -55.6887 & -36.3582 \end{bmatrix}.$$

The initial conditions are set as $e_w(0) = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix}^T$ and $x(0) = \begin{bmatrix} 0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}^T$. By using (19) in Theorem 2, we depict the state responses in Fig. 2a and Fig. 2b, respectively. It follows from the simulation results that the disturbance-observer-based composite control method can significantly improve the control results, which further confirms the advantage of the proposed control scheme.

Next, let us estimate an upper bound of epsilon for ϵ-NE. Set $\Delta\tilde{P}_d(t) = 0$ and $\Delta P_d(t) = [1/k, \dots, 1/k]^T$. The other parameters are chosen as $F = 0.5$, $Q_\delta^{100} = Q_\delta^1 = Q_\delta^2 = 1$, $R_\delta^{11} = R_\delta^{12} = R_\delta^{21} = R_\delta^{22} = 40$, $z(0) = \tilde{z}(0) = 0$, and $L_1 = L_2^1 = L_2^2 = \Lambda_1 = \Lambda_2 = \Lambda_3 = \Lambda_4 = \Lambda_5 = \Lambda_6 = I_{10\times 10}$. The specific values on the right-hand side of (31) are $\beta_1 = 10$, $\beta_3 = 49.1925$, $\beta_4 = 28839$, $\bar{\delta}^1 = \bar{\delta}^2 = 0.005$, $\varsigma_{2,1} = 531.9233$, $\varsigma_{2,2} = 532.9554$, $\varsigma_1 = 199.8622$,
$\lambda_{\max}\{\hat{Q}_\delta^1\} = \lambda_{\max}\{\hat{Q}_\delta^2\} = \lambda_{\max}\{\tilde{Q}_\delta^1\} = 6.2050$, $\lambda_{\max}\{\tilde{Q}_\delta^2\} = 5.2186$. It follows from (31) in Theorem 3 that $\epsilon_{100}^1 = 698.7544$ and $\epsilon_{100}^2 = 694.1752$. Since the cost functions are $\hat{J}_{100}^{1*} = 2.2571 \times 10^4$ and $\hat{J}_{100}^{2*} = 2.3054 \times 10^4$, it is not difficult to see that the obtained upper limits of a possible deviation of an ϵ-equilibrium from the cost functions are 3.10% and 3.01%, respectively.

VI. Conclusion

In this paper, a novel composite control scheme has been given for a class of delta domain LQ games with disturbances. In the presence of the disturbances, a disturbance-observer-based composite control method has been proposed, where the disturbance has been counteracted and the individual cost function of each player has been minimized. The ϵ-NE has been employed to characterize the dynamic coupling of the disturbance observer and LQ games, and an upper bound for the epsilon level has been obtained. Finally, a simulation has been provided to demonstrate the feasibility of the proposed control scheme.

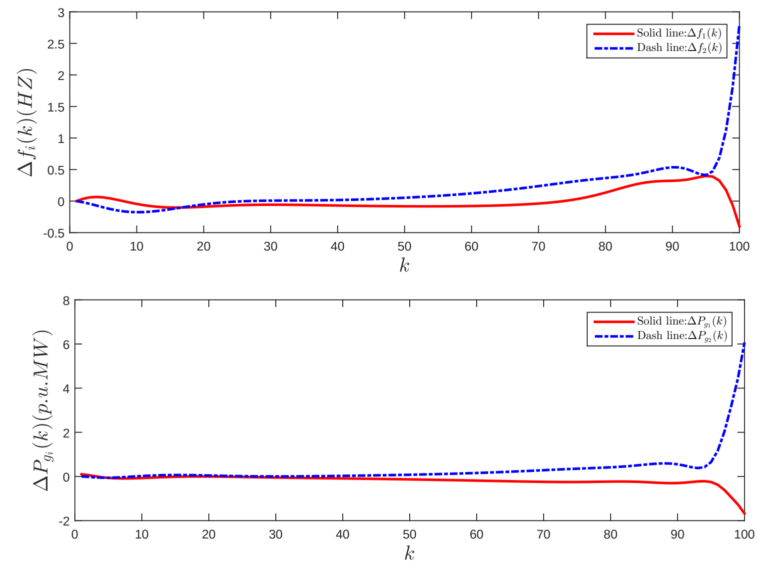
Acknowledgement

This work was supported in part by the National Natural Science Foundation of China under Grants 61273156.

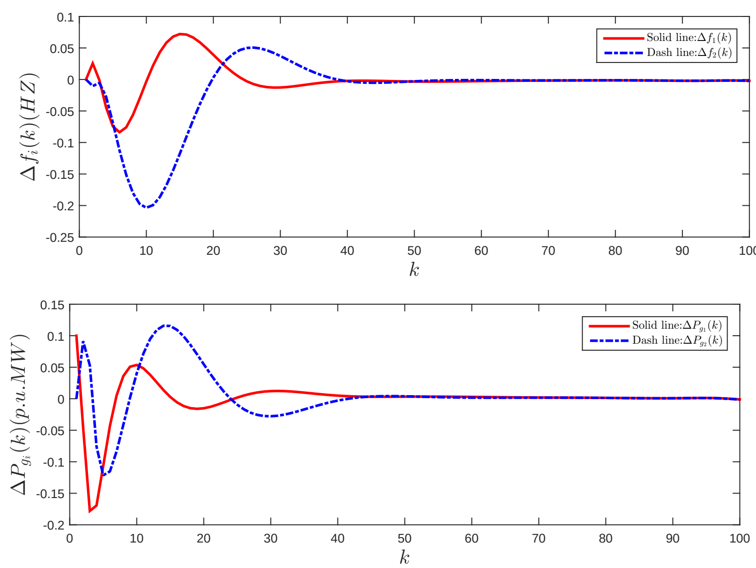
References

[1] S. Amin, G. A. Schwartz, and S. S. Sastry, Security of interdependent and identical networked control systems, *Automatica*, vol. 49, pp. 186–192, 2013.

[2] W. Zhang, H. Zhang, and B. Chen, Stochastic H_2/H_∞ control with (x, u, v)-dependent noise: finite horizon case, *Automatica*, vol. 42, pp. 1–6, 2006.



(a) The state responses without disturbance observer



(b) The state responses with disturbance observer

Fig. 2. The comparison of the state vectors

[3] M. H. Manshaei, Q. Zhu, T. Alpcan, T. Basar, J. P. Hubaux, Game theory meets network security and privacy, *ACM Computing Surveys*, vol. 45, pp. 533–545, 2013.

[4] M. Jun and T. Basar, Control over TCP-like lossy networks: A dynamic game approach, *Proceedings of the 2013 American Control Conference*, Washington, U.S.A., pp. 1578–1583, 2013.

[5] Y. Yuan, F. Sun, and H. Liu, Resilient control of cyber-physical systems against intelligent attacker: a hierarchal stackelberg game approach, *International Journal of Systems Science*, pp. 1–11, 2015.

[6] Y. Yuan, H. Yuan, L. Guo, H. Yang, and S. Sun, Resilient control of networked control system under DoS attacks: A unified game approach, *IEEE Transactions on Industrial Informatics*, doi: 10.1109/TII.2016.2542208

[7] Z. Yan, G. Zhang, and J. Wang, Infinite horizon H-two/H-infinity control for descriptor systems: Nash game approach, *Journal of control theory and applications*, vol. 10, pp. 159-165, 2012.

[8] B. Chen and W. Zhang, Stochastic $H2/H_\infty$ control with state-dependent noise, *IEEE Transactions on Automatic Control*, vol.49, pp. 45-57, 2004.

[9] H. Yang, Y. Xia, P. Shi, and L. Zhao, *Analysis and Synthesis of Delta Operator Systems*. Springer, New York, 2012.

[10] Y. Imura and D. S. Naidu, Unified approach for open-loop optimal control, *Optimal Control Applications and Methods*, vol. 28, pp. 59–75, 2007.
[11] R. H. Middleton and G. C. Goodwin, Improved finite word length characteristics in digital control using delta operators, *IEEE Transactions on Automatic Control*, vol. 31, pp. 1015–1021, 1986.
[12] Y. Xia, M. Fu, and G. Liu, Robust sliding-mode control for uncertain time-delay systems based on delta operator, *IEEE Transaction on Industrial Electronics*, vol. 56, pp. 3646–3655, 2009.
[13] J. Qiu, Y. Xia, H. Yang, and J. Zhang, Robust stabilisation for a class of discrete-time systems with time-varying delays via delta operators, *IET Control Theory & Applications*, vol. 2, pp. 87–93, 2008.
[14] H. Liu, L. Guo, and Y. Zhang, An anti-disturbance PD control scheme for attitude control and stabilization of flexible spacecrafts, *Nonlinear Dynamics*, vol. 67, pp. 2081–2088, 2012.
[15] X. Wei, N. Chen, C. Deng, X. Liu, and M. Tang, Composite stratified anti-disturbance control for a class of MIMO discrete-time systems with nonlinearity, *International Journal of Robust and Nonlinear Control*, vol. 22, pp. 453–472, 2012.
[16] H. Ogawa, M. Ono, Y. Masukake, and Y. Ishida, Two Stage Control Method Based on an Optimal Control System, *Proceedings of 2009 International Conference on Computer and Automation Engineering*, Bangkok, Thailand, pp. 12–15, 2009.
[17] W. Chen, Nonlinear disturbance observer-enhanced dynamic inversion control of missiles, *Journal of Guidance, Control, and Dynamics*, vol. 26, pp. 161–166, 2003.
[18] W. Chen, D. Ballance, P. Gawthrop, and J. O'Reilly, A nonlinear disturbance observer for robotic manipulators, *IEEE Transactions on Industrial Electronics*, vol. 47, pp. 932–938, 2000.
[19] H. Shen, Y. Zhu, L. Zhang, and J. H. Park, Extended dissipative state estimation for Markov jump neural networks with unreliable links, *IEEE Transactions on Neural Networks and Learning Systems*, doi: 10.1109/TNNLS.2015.2511196.
[20] H. Shen, Z. Wu, and J. H. Park, Reliable mixed passive and H_∞ filtering for semi-Markov jump systems with randomly occurring uncertainties and sensor failures, *International Journal of Robust and Nonlinear Control*, vol. 25, pp. 3231-3251, 2015
[21] C. E. Hall and Y. B. Shtessel, Sliding mode disturbance observer-based control for a reusable launch vehicle, *Journal of Guidance, Control, and Dynamics*, vol. 29, pp. 1315–1328, 2006.
[22] X. Chen, J. Yang, S. Li, and Q. Li, Disturbance observer based multi-variable control of ball mill grinding circuits, *Journal of Process Control*, vol. 19, pp. 1205–1213, 2009.
[23] S. Li and Z. Liu, Adaptive speed control for permanent-magnet synchronous motor system with variations of load inertia, *IEEE Transactions on Industrial Electronics*, vol. 56, pp. 3050–3059, 2009.
[24] S. Li, J. Yang, W. Chen, and X. Chen, *Disturbance observer-based control: methods and applications*, CRC press, Florida, 2014.
[25] S. Coogan, L. J. Ratliff, D. Calderone, and S. S. Sastry, Energy management via pricing in LQ dynamic games, *Proceedings of 2013 American Control Conference*, Washington DC, US, pp. 443–448, 2013.
[26] X. Tan and B. C. Jose, Adaptive noncooperative-person games with unknown general quadratic objectives, *IEEE Transactions on Control Systems Technology*, vol. 18, pp. 1033–1043, 2010.
[27] J. Manuel and A. Poznyak, ε equilibrium in LQ differential games with bounded uncertain disturbances: robustness of standard strategies and new strategies with adaptation, *International Journal of Control*, vol. 79, pp. 786–797, 2006.
[28] M. Jimenez and A. Poznyak, Quasi-equilibrium in LQ differential games with bounded uncertain disturbances: robust and adaptive strategies with pre-identification via sliding mode technique, *International journal of Systems Science*, vol. 38, pp. 585–599, 2007.
[29] M. Abouheaf, F. Lewis, K. Vamvoudakis, S. Haesaert, and R. Babuska, Multi-agent discrete-time graphical games and reinforcement learning solutions, *Automatica*, vol. 50, pp. 3038-3053, 2014.
[30] Z. Wang, X. Wang, X. Liu, and M. Huang, Optimal state feedback control for wireless networked control systems with decentralised controllers, *IET Control Theory & Applications*, vol. 9, pp. 852-862, 2014.
[31] X. Wei, N. Chen, and W. Li, Composite adaptive disturbance observer-based control for a class of nonlinear systems with multisource disturbance, *International Journal of Adaptive Control and Signal Processing*, vol. 27, pp. 199–208, 2013.
[32] S. Cao and L. Guo, Multi-objective robust initial alignment algorithm for Inertial Navigation System with multiple disturbances, *Aerospace Science and Technology*, vol. 21, pp. 1–6, 2012.
[33] B. Barmish and G. Leitmann, On ultimate boundedness control of uncertain systems in the absence of matching assumptions, *IEEE Transactions on Automatic Control*, vol. 27, pp. 153-158, 1982.
[34] H. Yang, A. Zolotas, W. H. Michail, and S. Li, Robust control of nonlinear MAGLEV suspension system with mismatched uncertainties via DOBC approach, *ISA transactions*, vol. 50, pp. 389-396, 2011.
[35] R. Errouissi, M. Ouhrouche, W. H. Chen, and A. M. Trzynadlowsk, Robust nonlinear predictive controller for permanent-magnet synchronous motors with an optimized cost function, vol. 59, pp. 2849-2858, 2012.
[36] T. Basar and G. J. Olsder, *Dynamic noncooperative game theory*, Academic press, London, 1995.
[37] R. H. Middleton, and G. C. Goodwin, *Digital Control and Estimation: A Unified Approach*, Prentice Hall, New Jersey, 1990.
[38] H. Sandberg, *Linear Time-Varying Systems: Modeling and Reduction*. PhD thesis, Department of Automatic Control, Lund Institute of Technoloy, Lund (Sweden).

[39] Y. Mi, Y. Fu, C. Wang, P. Wang, Decentralized sliding mode load frequency control for multi-area power systems, *IEEE Transactions on Power Systems*, vol. 28, pp. 4301–4309, 2013.

[40] R. P. Boas, Some uniformly convex spaces, *Bulletion of American Mathematical Society*, vol. 46, pp. 304–311, 1940.