GEOMETRY OF THE FISHER–RAO METRIC ON THE SPACE OF SMOOTH DENSITIES ON A COMPACT MANIFOLD

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Abstract. It is known that on a closed manifold of dimension greater than one, every smooth weak Riemannian metric on the space of smooth positive densities that is invariant under the action of the diffeomorphism group, is of the form

$$G_\mu(\alpha,\beta) = C_1(\mu(M)) \int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\,\mu + C_2(\mu(M)) \int_M \alpha \cdot \int_M \beta$$

for some smooth functions C_1, C_2 of the total volume $\mu(M)$. Here we determine the geodesics and the curvature of this metric and study geodesic and metric completeness.

1. Introduction. The Fisher–Rao metric on the space $\mathrm{Prob}(M)$ of probability densities is invariant under the action of the diffeomorphism group $\mathrm{Diff}(M)$. Restricted to finite-dimensional submanifolds of $\mathrm{Prob}(M)$, so-called statistical manifolds, it is called Fisher's information metric [1]. A uniqueness result was established [10, p. 156] for Fisher's information metric on finite sample spaces and [2] extended it to infinite sample spaces. The Fisher–Rao metric on the infinite-dimensional manifold of all positive probability densities was studied in [5], including the computation of its curvature. In [3] it was proved that any $\mathrm{Diff}(M)$-invariant Riemannian metric on the space $\mathrm{Dens}_+(M)$ of smooth positive densities on a compact manifold M without boundary is of the form

$$G_\mu(\alpha,\beta) = C_1(\mu(M)) \int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\,\mu + C_2(\mu(M)) \int_M \alpha \cdot \int_M \beta \tag{1}$$

for some smooth functions C_1, C_2 of the total volume $\mu(M)$. This implies that the Fisher–Rao metric on $\mathrm{Prob}(M)$ is, up to a multiplicative constant, the unique $\mathrm{Diff}(M)$-invariant metric.

2. The setting. Let M^m be a smooth compact manifold. It may have boundary or it may even be a manifold with corners; i.e., modelled on open subsets of quadrants in $\mathbb{R}^m$. For a detailed description of the line bundle of smooth densities we refer to [3] or [9, 10.2]. We let $\mathrm{Dens}_+(M)$ denote the space of smooth positive densities on M, i.e., $\mathrm{Dens}_+(M) = \{\mu \in \Gamma(\mathrm{Vol}(M)) : \mu(x) > 0\ \forall x \in M\}$. Let $\mathrm{Prob}(M)$ be the subspace of positive densities with integral 1 on M. Both spaces are smooth Fréchet manifolds; in particular they are open subsets of the affine spaces of all densities and densities of integral 1 respectively. For $\mu \in \mathrm{Dens}_+(M)$ we have

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$T_\mu \operatorname{Dens}_+(M) = \Gamma(\operatorname{Vol}(M))$ and for $\mu \in \operatorname{Prob}(M)$ we have

$$T_\mu \operatorname{Prob}(M) = \{\alpha \in \Gamma(\operatorname{Vol}(M)) : \int_M \alpha = 0\}.$$

The Fisher–Rao metric, given by $G^{\mathrm{FR}}_\mu(\alpha,\beta) = \int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\mu$ is a Riemannian metric on $\operatorname{Prob}(M)$; it is invariant under the natural action of the group $\operatorname{Diff}(M)$ of all diffeomorphisms of M. If M is compact without boundary of dimension ≥ 2, the Fisher-Rao metric is the unique $\operatorname{Diff}(M)$-invariant metric up to a multiplicative constant. This follows, since any $\operatorname{Diff}(M)$-invariant Riemannian metric on $\operatorname{Dens}_+(M)$ is of the form (1) as proved in [3].

3. Overview. We will study four different representations of the metric G in (1).

$$\operatorname{Dens}_+(M) \xrightarrow{R} C^\infty(M,\mathbb{R}_{>0}) \xrightarrow{\Phi} \mathbb{R}_{>0} \times S \cap C^\infty_{>0} \xrightarrow{W\times \mathrm{Id}} (W_-,W_+) \times S \cap C^\infty_{>0}\,.$$

The first representation is G itself on the space $\operatorname{Dens}_+(M)$. Next we fix a density $\mu_0 \in \operatorname{Prob}(M)$ and consider the mapping

$$R : \operatorname{Dens}_+(M) \to C^\infty(M,\mathbb{R}_{>0})\,, \qquad R(\mu) = f = \sqrt{\frac{\mu}{\mu_0}}\,.$$

This map is a diffeomorphism with inverse $R^{-1}(f) = f^2\mu_0$, and we will denote the induced metric by $\tilde G = \left(R^{-1}\right)^* G$; it is given by the formula

$$\tilde G_f(h,k) = 4C_1(\|f\|^2)\langle h,k\rangle + 4C_2(\|f\|^2)\langle f,h\rangle\langle f,k\rangle\,,$$

with $\|f\|^2 = \int_M f^2\mu_0$ denoting the $L^2(\mu_0)$-norm, and this formula makes sense for $f \in C^\infty(M,\mathbb{R}) \setminus \{0\}$. See Sect. **5** for calculations.

Next we take the pre-Hilbert space $(C^\infty(M,\mathbb{R}), \langle\ ,\ \rangle_{L^2(\mu_0)})$ and pass to polar coordinates. Let $S = \{\varphi \in L^2(M,\mathbb{R}) : \int_M \varphi^2\mu_0 = 1\}$ denote the L^2-sphere. Then

$$\Phi : C^\infty(M,\mathbb{R}_{>0}) \to \mathbb{R}_{>0} \times (S \cap C^\infty_{>0})\,, \qquad \Phi(f) = (r,\varphi) = \left(\|f\|, \frac{f}{\|f\|}\right)\,,$$

is a diffeomorphism, where $C^\infty_{>0} = C^\infty(M,\mathbb{R}_{>0})$; its inverse is $\Phi^{-1}(r,\varphi) = r.\varphi$. We set $\bar G = \left(\Phi^{-1}\right)^* \tilde G$; the metric $\bar G$ has the expression

$$\bar G_{r,\varphi} = g_1(r)\langle d\varphi, d\varphi\rangle + g_2(r)dr^2\,,$$

with $g_1(r) = 4C_1(r^2)r^2$ and $g_2(r) = 4\left(C_1(r^2) + C_2(r^2)r^2\right)$. Finally we change the coordinate r diffeomorphically to

$$s = W(r) = 2\int_1^r \sqrt{g_2(\rho)}\,d\rho\,.$$

Then, defining $a(s) = 4C_1(r(s)^2)r(s)^2$, we have

$$\bar G_{s,\varphi} = a(s)\langle d\varphi, d\varphi\rangle + ds^2\,.$$

We will use $\bar G$ to denote the metric in both (r,φ) and (s,φ) coordinates. Let $W_- = \lim_{r\to 0+} W(r)$ and $W_+ = \lim_{r\to\infty} W(r)$. Then $W : \mathbb{R}_{>0} \to (W_-,W_+)$ is a diffeomorphism. This completes the first row in Fig. 1. The geodesic equation of G in the various representations will be derived in Sect. **5**.

Since $\bar G$ induces the canonical metric on (W_-,W_+), a necessary condition for $\bar G$ to be complete is $(W_-,W_+) = (-\infty,+\infty)$. Rewritten in terms of the functions C_1 and C_2 this becomes

$$W_+ = \infty \Leftrightarrow \left(\int_1^\infty r^{-1/2}\sqrt{C_1(r)}\,dr = \infty \text{ or } \int_1^\infty \sqrt{C_2(r)}\,dr = \infty\right)\,,$$

$$\begin{tikzcd}
\mathrm{Dens}_+(M) \arrow[r, "R"] \arrow[d] & C^\infty(M,\mathbb{R}_{>0}) \arrow[r, "\Phi"] \arrow[d] & \mathbb{R}_{>0}\times S\cap C^\infty_{>0} \arrow[r, "W\times\mathrm{Id}"] \arrow[d] & (W_-,W_+)\times S\cap C^\infty_{>0} \arrow[d] \\
\Gamma_{C^1}(\mathrm{Vol}(M))\backslash\{0\} \arrow[d] & C^\infty(M,\mathbb{R})\backslash\{0\} \arrow[l, "R^{-1}"'] \arrow[r, "\Phi"] \arrow[d] & \mathbb{R}_{>0}\times S\cap C^\infty \arrow[r, "W\times\mathrm{Id}"] \arrow[d] & \mathbb{R}\times S\cap C^\infty \arrow[d] \\
\Gamma_{L^1}(\mathrm{Vol}(M))\backslash\{0\} \arrow[r, "R"] & L^2(M,\mathbb{R})\backslash\{0\} \arrow[r, "\Phi"] & \mathbb{R}_{>0}\times S \arrow[r, "W\times\mathrm{Id}"] & \mathbb{R}\times S
\end{tikzcd}$$

FIGURE 1. Representations of $\mathrm{Dens}_+(M)$ and its completions. In the second and third rows we assume that $(W_-,W_+)=(-\infty,+\infty)$ and we note that R is a diffeomorphism only in the first row.

and similarly for $W_-=-\infty$, with the limits of integration being 0 and 1.

We now assume that $(W_-,W_+)=(-\infty,+\infty)$. The metrics $\bar{G}$ and $\tilde{G}$ can be extended to the spaces $\mathbb{R}\times S\cap C^\infty$ and $C^\infty(M,\mathbb{R})\setminus\{0\}$ and the last two maps in the diagram

$$\Gamma_{C^1}(\mathrm{Vol}(M))\backslash\{0\} \xleftarrow{R^{-1}} C^\infty(M,\mathbb{R})\backslash\{0\} \xrightarrow{\Phi} \mathbb{R}_{>0}\times S\cap C^\infty \xrightarrow{W\times\mathrm{Id}} \mathbb{R}\times S\cap C^\infty$$

are bijections. The extension of R^{-1} is given by $R^{-1}(f)=f|f|\mu_0$; it does not map into smooth densities any more, but only into C^1-sections of the volume bundle; moreover R^{-1} is not surjective. The last two maps, Φ and $W\times\mathrm{Id}$, are diffeomorphisms. The following will be shown in Sect. **7**: $(W_-,W_+)=(-\infty,+\infty)$ implies that $(\mathbb{R}\times S\cap C^\infty,\bar{G})$ is geodesically complete and hence so are $(\mathbb{R}_{>0}\times S\cap C^\infty,\bar{G})$ and $(C^\infty(M,\mathbb{R})\setminus\{0\},\tilde{G})$.

Finally we consider the metric completions, still assuming that $(W_-,W_+)=(-\infty,+\infty)$. For $\bar{G}$ this is $\mathbb{R}\times S$ or $\mathbb{R}_{>0}\times S$ in (s,φ) or (r,φ)-coordinates, respectively, as shown in Sect. **7**. The metrics and maps can be extended to

$$\Gamma_{L^1}(\mathrm{Vol}(M))\setminus\{0\} \xrightarrow{R} L^2(M,\mathbb{R})\setminus\{0\} \xrightarrow{\Phi} \mathbb{R}_{>0}\times S \xrightarrow{W\times\mathrm{Id}} \mathbb{R}\times S\ .$$

Here Γ_{L^1} denotes the space of L^1-sections. The extension of R is given by $R(\mu)=\mathrm{sgn}(\mu)\sqrt{|\mu|/\mu_0}$ and its inverse is $R^{-1}(f)=f|f|\mu_0$ as before. The last two maps are diffeomorphisms and hence $(L^2(M,\mathbb{R})\setminus\{0\},\tilde{G})$ is metrically complete. The extension of R is bijective, but not a diffeomorphism. It is continuous, but not C^1, and its inverse is C^1, but not C^2; furthermore $DR^{-1}(f)$ is not surjective if $f=0$ on a set of positive measure. However we can use R to pull back the geodesic distance function from $L^2(M,\mathbb{R})\setminus\{0\}$ to $\Gamma_{L^1}(\mathrm{Vol}(M))\setminus\{0\}$ to obtain a complete metric on the latter space, that is compatible with the standard topology.

4. **Remark on the inverse R^{-1}.** There is more than one choice for the extension of $R^{-1}(f)=f^2\mu_0$ from $C^\infty(M,\mathbb{R}_{>0})$ to $C^\infty(M,\mathbb{R})$. The choice $R^{-1}(f)=f|f|\mu_0$ remains injective and can be further extended to a bijection on the metric completion $L^2(M,\mathbb{R})\setminus\{0\}$. We can consider the equally natural extension given by

$$Q: C^\infty(M,\mathbb{R})\to\Gamma_{\geq 0}(\mathrm{Vol}(M))\,,\quad f\mapsto f^2\mu_0\,,$$

into the space of smooth, nonnegative sections. The map Q is not surjective; see [7] for a discussion of smooth non-negative functions admitting smooth square roots.

The image of Q looks somewhat like the orbit space of a discrete reflection group; An example of a codimension 1 wall of the image could be $\{f^2\mu_0 : f \in C^\infty(M,\mathbb{R}), f(x) = 0\}$ for one fixed point $x \in M$. Since this is dense in the L^2-completion of $T_f C^\infty(M,\mathbb{R})$ with respect to $\tilde{G}_f$, we do not have a reflection at this wall. Fixing $\varphi_0 \in S \cap C^\infty$ and considering $\{(r,\varphi) \in \mathbb{R}_{>0} \times S \cap C^\infty : \langle \varphi_0, \varphi\rangle = 0\}$ we can write the orthogonal reflection $(r, t_1\varphi_0 + t_2\varphi) \mapsto (r, -t_1\varphi_0 + t_2\varphi)$. Geodesics in $(C^\infty(M,\mathbb{R}), \tilde{G})$ are mapped by R^{-1} to curves that are geodesics in the interior $\Gamma_{>0}(\mathrm{Vol}(M))$, and that are reflected following Snell's law at any hyperplanes in the boundary for which the angle makes sense.

5. Geodesics of the Fisher-Rao metric on $\mathrm{Dens}_+(M)$**.** In [5] it was shown that $\mathrm{Prob}(M)$ has constant sectional curvature for the Fisher-Rao metric. For fixed $\mu_0 \in \mathrm{Prob}(M)$ we consider the mapping

$$R : \mathrm{Dens}_+(M) \to C^\infty(M,\mathbb{R}_{>0}), \qquad R(\mu) = \sqrt{\frac{\mu}{\mu_0}}\,.$$

The inverse $R^{-1} : C^\infty(M,\mathbb{R}_{>0}) \to \mathrm{Dens}_+(M)$ is given by $R^{-1}(f) = f^2\mu_0$; its tangent mapping is $T_f R^{-1}.h = 2fh\mu_0$.

Remark. In [6] it was shown that for $C_1 \equiv 1$ and $C_2 \equiv 0$ the rescaled map $R(\nu) = 2\sqrt{\frac{\mu}{\mu_0}}$ is an isometric diffeomorphism from $\mathrm{Prob}(M)$ onto the open subset $C^\infty(M,\mathbb{R}_{>0}) \cap \{f : \int f^2\mu_0 = 4\}$ of the L^2-sphere of radius 2 in the pre-Hilbert space $(C^\infty(M,\mathbb{R}), \langle\ ,\ \rangle_{L^2(\mu_0)})$. For a general function C_1 the same holds for $R(\mu) = \lambda\sqrt{\frac{\mu}{\mu_0}}$ and the L^2-sphere of radius λ, where $\lambda > 0$ is a solution of the equation $\lambda^2 = 4C_1(\lambda^{-2})$.

The Fisher–Rao metric induces the following metric on the open convex cone $C^\infty(M,\mathbb{R}_{>0}) \subset C^\infty(M,\mathbb{R})$:

$$
\begin{aligned}
\text{(a)}\quad \big((R^{-1})^*G\big)_f(h,k) &= G_{R^{-1}(f)}(T_fR^{-1}.h, T_fR^{-1}.k) = G_{f^2\mu_0}(2fh\mu_0, 2fk\mu_0)\\
&= C_1(\|f\|^2_{L^2(\mu_0)})\int \frac{2fh\mu_0}{f^2\mu_0}\frac{2fk\mu_0}{f^2\mu_0} f^2\mu_0 + C_2(\|f\|^2_{L^2(\mu_0)})\int 2fh\mu_0\cdot\int 2fk\mu_0\\
&= 4C_1(\|f\|^2)\int hk\mu_0 + 4C_2(\|f\|^2)\int fh\mu_0\cdot\int fk\mu_0\\
&= 4C_1(\|f\|^2)\langle h,k\rangle + 4C_2(\|f\|^2)\langle f,h\rangle\langle f,k\rangle\\
&= 4C_1(\|f\|^2)\Big\langle h - \frac{\langle f,h\rangle}{\|f\|^2}f, k - \frac{\langle f,k\rangle}{\|f\|^2}f\Big\rangle +\\
&\qquad + 4\big(C_2(\|f\|^2).\|f\|^2 + C_1(\|f\|^2)\big)\Big\langle\frac{f}{\|f\|},h\Big\rangle\Big\langle\frac{f}{\|f\|},k\Big\rangle,
\end{aligned}
$$

where in the last expression we split h and k into the parts perpendicular to f and multiples of f.

We now switch to *polar coordinates on the pre-Hilbert space*: Let $S = \{\varphi \in L^2(M,\mathbb{R}) : \int\varphi^2\mu_0 = 1\}$ denote the sphere, and let $S \cap C^\infty_{>0}$ be the intersection with the positive cone. Then $C^\infty(M,\mathbb{R}) \setminus \{0\} \cong \mathbb{R}_{>0} \times S \cap C^\infty$ via

$$\Phi : C^\infty(M,\mathbb{R}) \setminus \{0\} \to \mathbb{R}_{>0} \times S\,, \qquad \Phi(f) = (r,\varphi) = \left(\|f\|, \frac{f}{\|f\|}\right).$$

Note that $\Phi(C^\infty(M,\mathbb{R}_{>0})) = \mathbb{R}_{>0} \times S \cap C^\infty_{>0}$. We have $f = \Phi^{-1}(r,\varphi) = r.\varphi$ thus $df = r\,d\varphi + \varphi\,dr$, where $r\,d\varphi(h) = h - \langle\varphi,h\rangle\varphi$ is the orthogonal projection onto the tangent space of S at φ and $dr(h) = \langle \varphi, h\rangle$. The Euclidean (pre-Hilbert) metric in polar coordinates is given by

$$\begin{aligned}\langle df, df\rangle &= \langle \varphi.dr + r.d\varphi, \varphi.dr + r.d\varphi\rangle = \langle\varphi,\varphi\rangle dr^2 + 2r.\langle\varphi, d\varphi\rangle.dr + r^2\langle d\varphi, d\varphi\rangle \\ &= dr^2 + r^2\langle d\varphi, d\varphi\rangle\,.\end{aligned}$$

The pullback metric is then

$$\text{(b)}\qquad \begin{aligned}\bar G = \Big((\Phi^{-1})^*\tilde G\Big) &= 4C_1(r^2)r^2\langle d\varphi, d\varphi\rangle + 4\left(C_2(r^2)r^2 + C_1(r^2)\right)dr^2 \\ &= g_1(r)\langle d\varphi, d\varphi\rangle + g_2(r)dr^2 \\ &= a(s)\langle d\varphi, d\varphi\rangle + ds^2\,,\end{aligned}$$

where we introduced the functions

$$g_1(r) = 4C_1(r^2)r^2 \quad\text{and}\quad g_2(r) = 4\left(C_2(r^2)r^2 + C_1(r^2)\right)\,,$$

and where in the last expression we changed the coordinate r diffeomorphically to

$$s(r) = 2\int_1^r \sqrt{C_2(\rho^2)\rho^2 + C_1(\rho^2)}\,d\rho \qquad\text{and let } a(s) = 4C_1(r(s)^2)r(s)^2.$$

The resulting metric is a radius dependent scaling of the metric on the sphere times a different radius dependent scaling of the metric on $\mathbb{R}_{>0}$. Note that the metric (b) (as well as the metric in the last expression of (a)) is actually well-defined on $C^\infty(M,\mathbb{R})\setminus\{0\} \cong \mathbb{R}_{>0}\times S\cap C^\infty$; this leads to a (partial) geodesic completion of $(\mathrm{Dens}_+(M), G)$.

Geodesics for the metric (b) follow great circles on the sphere with some time dependent stretching, since reflection at any hyperplane containing this great circle is an isometry.

Let us derive the geodesic equation. Let $[0,1]\times(-\varepsilon,\varepsilon)\ni(t,s)\mapsto(r(t,s),\varphi(t,s))$ be a smooth variation with fixed ends of a curve $(r(t,0),\varphi(t,0))$. The energy of the curve and its derivative with respect to the variation parameter s are as follows, where ∇^S is the covariant derivative on the sphere S.

$$\begin{aligned}E(r,\varphi) &= \int_0^1\left(\frac12 g_1(r)\langle\varphi_t,\varphi_t\rangle + \frac12 g_2(r).r_t^2\right)dt \\ \partial_s E(r,\varphi) &= \int_0^1\Big(\frac12 g_1'(r).r_s\langle\varphi_t,\varphi_t\rangle + g_1(r)\langle\nabla^S_{\partial_s}\varphi_t,\varphi_t\rangle + \\ &\qquad + \frac12 g_2'(r).r_s.r_t^2 + g_2(r).r_t.r_{ts}\Big)dt \\ &= \int_0^1\Big(\frac12 g_1'(r).r_s\langle\varphi_t,\varphi_t\rangle - g_1'(r).r_t\langle\varphi_s,\varphi_t\rangle - g_1(r)\langle\varphi_s,\nabla^S_{\partial_t}\varphi_t\rangle + \\ &\qquad + \frac12 g_2'(r).r_s.r_t^2 - g_2'(r).r_t^2.r_s - g_2(r).r_{tt}.r_s\Big)dt \\ &= \int_0^1\Big(\frac12 g_1'(r)\langle\varphi_t,\varphi_t\rangle - \frac12 g_2'(r).r_t^2 - g_2(r).r_{tt}\Big)r_s \\ &\qquad - \Big(g_1'(r).r_t\langle\varphi_s,\varphi_t\rangle + g_1(r)\langle\varphi_s,\nabla^S_{\partial_t}\varphi_t\rangle\Big)dt\,.\end{aligned}$$

Thus the geodesic equation is

$$\text{(c)}\qquad \boxed{\begin{aligned}\nabla^S_{\partial_t}\varphi_t &= -\partial_t\left(\log g_1(r)\right)\varphi_t \\ r_{tt} &= \frac12\frac{g_1'(r)}{g_2(r)}\langle\varphi_t,\varphi_t\rangle - \frac12\partial_t\left(\log g_2(r)\right)r_t\end{aligned}}$$

Some simplifications:

$$\begin{aligned}
\partial_t \langle \varphi_t, \varphi_t \rangle &= 2\langle \nabla_{\partial_t} \varphi_t, \varphi_t \rangle = -2\, \partial_t \left(\log g_1(r) \right) \langle \varphi_t, \varphi_t \rangle \\
\partial_t \left(\log \langle \varphi_t, \varphi_t \rangle \right) &= -2\, \partial_t \left(\log g_1(r) \right) \\
\log(\|\varphi_t\|^2) &= -2 \log g_1(r) + 2 \log C_0 \qquad \text{with} \qquad C_0 = g_1(r)\, \|\varphi_t\| \,,
\end{aligned}$$

which describes the speed of $\varphi(t)$ along the great circle in terms of $r(t)$; the geodesic equation (c) simplifies to

$$\text{(d)} \qquad \boxed{\begin{aligned}
&\nabla^S_{\partial_t} \varphi_t = -\partial_t \left(\log g_1(r) \right) \varphi_t \\
&\qquad r_{tt} = \frac{C_0^2}{2} \frac{g_1'(r)}{g_1(r)^2 g_2(r)} - \frac{1}{2} \partial_t \left(\log g_2(r) \right) r_t
\end{aligned}}$$

with $g_1(r) = 4C_1(r^2) r^2$ and $g_2(r) = 4\left(C_2(r^2) r^2 + C_1(r^2) \right)$.

We can solve the equation for φ explicitely. Given initial conditions φ_0, ψ_0, the geodesic $\tilde\varphi(t)$ on the sphere with radius 1 satisfying $\tilde\varphi(0) = \varphi_0$, $\tilde\varphi_t(0) = \psi_0$ is

$$\tilde\varphi(t) = \cos(\|\psi_0\| t) \varphi_0 + \sin(\|\psi_0\| t) \frac{\psi_0}{\|\psi_0\|} \,.$$

We are looking for a reparametrization $\varphi(t) = \tilde\varphi(\alpha(t))$. Inserting this into the geodesic equation we obtain

$$\begin{aligned}
\partial_t^2 \left(\tilde\varphi(\alpha) \right) - \left\langle \partial_t^2 \left(\tilde\varphi(\alpha) \right), \frac{\tilde\varphi(\alpha)}{\|\tilde\varphi(\alpha)\|} \right\rangle \tilde\varphi(\alpha) &= -\partial_t \left(\log g_1(r) \right) \partial_t \left(\tilde\varphi(\alpha) \right) \\
\left(\nabla^S_{\partial_t} \tilde\varphi_t \right)(\alpha) \alpha_t^2 + \tilde\varphi_t(\alpha) \alpha_{tt} - \left\langle \tilde\varphi_t(\alpha) \alpha_{tt}, \frac{\tilde\varphi(\alpha)}{\|\tilde\varphi(\alpha)\|} \right\rangle \tilde\varphi(\alpha) &= -\partial_t \left(\log g_1(r) \right) \tilde\varphi_t(\alpha) \alpha_t \\
\alpha_{tt} &= \partial_t \left(\log g_1(r) \right) \alpha_t \,.
\end{aligned}$$

With intial conditions $\alpha(0) = 0$ and $\alpha_t(0) = 1$ this equation has the solution

$$\alpha(t) = g_1(r_0) \int_0^t \frac{1}{g_1(r(\tau))} \, d\tau \,,$$

where $r_0 = r(0)$ is the initial condition for the r-component of the geodesic.

If the metric is written in the form $\bar G = ds^2 + a(s) \langle d\varphi, d\varphi \rangle$, equation (d) becomes

$$s_{tt} = \frac{C_0^2}{2} \frac{a'(s)}{a(s)^2} \,, \qquad \text{for } C_0 = a(s) \|\varphi_t\| \,,$$

where $\varphi(t)$ is given explicitly as above. This can be integrated into the form

$$\text{(e)} \qquad \boxed{s_t^2 = -\frac{C_0^2}{a(s)} + C_1 \,, \quad C_1 \text{ a constant.}}$$

6. Relation to hypersurfaces of revolution. We consider the metric $\bar G$ on $(W_-, W_+) \times S \cap C^\infty$ where $\bar G_{s,\varphi} = a(s) \langle d\varphi, d\varphi \rangle + ds^2$ and $a(s) = 4C_1(r(s)^2) r(s)^2$. Then the map Ψ is an isometric embedding (remember $\langle \varphi, d\varphi \rangle = 0$ on $S \cap C^\infty$),

$$\begin{aligned}
&\Psi : ((W_-, W_+) \times S \cap C^\infty, \bar G) \to \left(\mathbb{R} \times C^\infty(M, \mathbb{R}), du^2 + \langle df, df \rangle \right) , \\
&\Psi(s, \varphi) = \Big(\int_0^s \sqrt{1 - \frac{a'(\sigma)^2}{4a(\sigma)}} \, d\sigma \,,\ \sqrt{a(s)} \varphi \Big) ,
\end{aligned}$$

In fact it is defined and smooth only on the open subset

$$\left\{ (s, \varphi) \in (W_-, W_+) \times S \cap C^\infty : a'(s)^2 < 4a(s) \right\} \,.$$

We will see in Sect. **8** that the condition $a'(s)^2 < 4a(s)$ is equivalent to a sign condition on the sectional curvature; to be precise

$$a'(s)^2 < 4a(s) \Leftrightarrow \mathrm{Sec}_{(s,\varphi)}(\mathrm{span}(X,Y)) > 0\,,$$

where $X, Y \in T_\varphi S$ is an $\bar{G}$-orthonormal pair of tangent vectors. Fix some $\varphi_0 \in S \cap C^\infty$ and consider the generating curve

$$\gamma(s) = \Big(\int_0^s \sqrt{1 - \frac{a'(\sigma)^2}{4a(\sigma)}}\, d\sigma\ ,\ \sqrt{a(s)}\varphi_0 \Big) \in \mathbb{R} \times C^\infty(M,\mathbb{R})\,;$$

then $\gamma(s)$ is already arc-length parametrized!

Any arc-length parameterized curve $I \ni s \mapsto (c_1(s), c_2(s))$ in $\mathbb{R}^2$ generates a hypersurface of revolution

$$\{(c_1(s), c_2(s)\varphi) : s \in I, \varphi \in S \cap C^\infty\} \subset \mathbb{R} \times C^\infty(M,\mathbb{R})\,,$$

and the induced metric in the (s,φ)-parameterization is $c_2(s)^2\langle d\varphi, d\varphi\rangle + ds^2$.

This suggests that the moduli space of hypersurfaces of revolution is naturally embedded in the moduli space of all metrics of the form $a(s)\langle d\varphi, d\varphi\rangle + ds^2$. Let us make this more precise in an example: In the case of $S = S^1$ and the tractrix (c_1, c_2), the surface of revolution is the pseudosphere (curvature -1) whose universal cover is only part of the hyperbolic plane. But in polar coordinates we get a space whose universal cover is the whole hyperbolic plane. In detail: the arc-length parametrization of the tractrix and the induced metric are

$$\begin{aligned}
&c_1(s) = \int_0^s \sqrt{1 - e^{-2\sigma}}\, d\sigma = \mathrm{Arcosh}\left(e^s\right) - \sqrt{1 - e^{-2s}}, \quad c_2(s) = e^{-s}, \quad s > 0\\
&a(s)\, d\varphi^2 + ds^2 = e^{-2s} d\varphi^2 + ds^2, \qquad s \in \mathbb{R}\,.
\end{aligned}$$

7. Completeness. In this section we assume that $(W_-, W_+) = (-\infty, +\infty)$, which is a necessary and sufficient condition for completeness. First we have the following estimate for the geodesic distance dist of the metric $\bar{G}$, which is valid on bounded metric balls. Let dist_S denote the geodesic distance on S with respect to the standard metric.

Lemma. *Let* $(W_-, W_+) = (-\infty, +\infty)$*,* $(s_0, \varphi_0) \in \mathbb{R} \times S$ *and* $R > 0$*. Then there exists* $C > 0$*, such that*

$$\begin{aligned}
C^{-1}\left(\mathrm{dist}_S(\varphi_1,\varphi_2) + |s_1 - s_2|\right) &\le \mathrm{dist}\left((s_1,\varphi_1),(s_2,\varphi_2)\right) \le\\
&\le C\left(\mathrm{dist}_S(\varphi_1,\varphi_2) + |s_1 - s_2|\right)\,,
\end{aligned}$$

holds for all (s_i, φ_i) *with* $\mathrm{dist}\left((s_0,\varphi_0),(s_i,\varphi_i)\right) < R$*,* $i = 1, 2$*.*

Proof. First we observe that

$$|s_1 - s_2| \le \int_0^1 |s_t(t)|\, dt \le \int_0^1 \sqrt{a(s)\|\varphi_t\|^2 + s_t^2}\, dt = \mathrm{Len}(s,\varphi)\,,$$

and hence by taking the infimum over all paths,

$$|s_1 - s_2| \le \mathrm{dist}\left((s_1,\varphi_1),(s_2,\varphi_2)\right) < 2R\,.$$

Thus s is bounded on bounded geodesic balls.

Now let (s_i, φ_i) be chosen according to the assumptions and let $(s(t), \varphi(t))$ be a path connecting (s_1, φ_1) and (s_2, φ_2) with $\mathrm{Len}(s,\varphi) < 2\,\mathrm{dist}\left((s_1,\varphi_1),(s_2,\varphi_2)\right)$. Then for $t \in [0,1]$,

$$\mathrm{dist}\left((s_0,\varphi_0),(s(t),\varphi(t))\right) \le \mathrm{dist}\left((s_0,\varphi_0),(s_1,\varphi_1)\right) + 2\,\mathrm{dist}\left((s_1,\varphi_1),(s_2,\varphi_2)\right) \le 5R\,.$$

In particular the path remains in a bounded geodesic ball.

Thus there exists a constant $c > 0$, such that $c^{-1} \le a(s) \le c$ holds along $(s(t), \varphi(t))$. From there we obtain

$$C^{-1} \int_0^1 \|\varphi_t\|^2 + s_t^2 \, dt \le \int_0^1 a(s) \|\varphi_t\|^2 + s_t^2 \, dt \le C \int_0^1 \|\varphi_t\|^2 + s_t^2 \, dt \,,$$

for $C = \max(c, 1)$ and by taking the infimum over paths connecting (s_1, φ_1) and (s_2, φ_2) the desired result follows. □

Proposition. *If $(W_-, W_+) = (-\infty, +\infty)$, the space $(\mathbb{R} \times S, \bar{G})$ is metrically and geodesically complete. The subspace $(\mathbb{R} \times S \cap C^\infty, \bar{G})$ is geodesically complete.*

Proof. Given a Cauchy sequence $(s_n, \varphi_n)_{n\in\mathbb{N}}$ in $\mathbb{R} \times S$ with respect to the geodesic distance, the lemma shows that $(s_n)_{n\in\mathbb{N}}$ and $(\varphi_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathbb{R}$ and S respectively. Hence they have limits s and φ and by the lemma the sequence $(s_n, \varphi_n)_{n\in\mathbb{N}}$ converges to (s, φ) in the geodesic distance as well. It is shown in [8, Prop. 6.5] that a metrically complete, strong Riemannian manifold is geodesically complete.

Since the φ-part of a geodesic in $\mathbb{R} \times S$ is a reparametrization of a great circle, if the initial conditions lie in $\mathbb{R} \times S \cap C^\infty$, so will the whole geodesic. Hence $\mathbb{R} \times S \cap C^\infty$ is geodesically complete. □

The map $\Phi \circ W \times \mathrm{Id} : L^2(M, \mathbb{R}) \setminus \{0\} \to \mathbb{R} \times S$ is a diffeomorphism and an isometry with respect to the metrics $\tilde{G}$ and $\bar{G}$.

Corollary. *If $(W_-, W_+) = (-\infty, +\infty)$, the space $(L^2(M, \mathbb{R}) \setminus \{0\}, \tilde{G})$ is metrically and geodesically complete. The subset $(C^\infty(M, \mathbb{R}) \setminus \{0\}, \tilde{G})$ is geodesically complete.*

It remains to consider the existence of minimal geodesics.

Theorem. *If $(W_-, W_+) = (-\infty, +\infty)$, then any two points (s_0, φ_0) and (s_1, φ_1) in $\mathbb{R} \times S$ can be joined by a minimal geodesic. If φ_0 and φ_1 lie in $S \cap C^\infty$, then the minimal geodesic also lies in $\mathbb{R} \times S \cap C^\infty$.*

Proof. If φ_0 and φ_1 are linearly independent, we consider the 2-space $V = V(\varphi_0, \varphi_1)$ spanned by φ_0 and φ_1 in L^2. Then $\mathbb{R} \times V \cap S$ is totally geodesic since it is the fixed point set of the isometry $(s, \varphi) \mapsto (s, \mathfrak{s}_V(\varphi))$ where $\mathfrak{s}_V$ is the orthogonal reflection at V. Thus there is exists a minimizing geodesic between (s_0, φ_0) and (s_1, φ_1) in the complete 3-dimensional Riemannian submanifold $\mathbb{R} \times V \cap S$. This geodesic is also length-minimizing in the strong Hilbert manifold $\mathbb{R} \times S$ by the following argument:

Given any smooth curve $c = (s, \varphi) : [0, 1] \to \mathbb{R} \times S$ between these two points, there is a subdivision $0 = t_0 < t_1 < \cdots < t_N = 1$ such that the piecewise geodesic c_1 which first runs along a geodesic from $c(t_0)$ to $c(t_1)$, then to $c(t_2)$, …, and finally to $c(t_N)$, has length $\mathrm{Len}(c_1) \le \mathrm{Len}(c)$. This piecewise geodesic now lies in the totally geodesic $(N+2)$-dimensional submanifold $\mathbb{R} \times V(\varphi(t_0), \ldots, \varphi(t_N)) \cap S$. Thus there exists a geodesic c_2 between the two points (s_0, φ_0) and (s_1, φ_1) which is length-minimizing in this $(N+2)$-dimensional submanifold. Therefore $\mathrm{Len}(c_2) \le \mathrm{Len}(c_1) \le \mathrm{Len}(c)$. Moreover, $c_2 = (s \circ c_2, \varphi \circ c_2)$ lies in $\mathbb{R} \times V(\varphi_0, (\varphi \circ c_2)'(0)) \cap S$ which also contains φ_1, thus c_2 lies in $\mathbb{R} \times V(\varphi_0, \varphi_1) \cap S$.

If $\varphi_0 = \varphi_1$, then $\mathbb{R} \times \{\varphi_0\}$ is a minimal geodesic. If $\varphi_0 = -\varphi_0$ we choose a great circle between them which lies in a 2-space V and proceed as above. When

$\varphi_0, \varphi_1 \in C^\infty$, then the 3-dimensional submanifold $\mathbb{R} \times V \cap S$ lies in $\mathbb{R} \times S \cap C^\infty$ and hence so does the minimal geodesic. $\square$

8. Covariant derivative and curvature. In this section we will write $I = (W_-, W_+)$. In order to calculate the covariant derivative we consider the infinite-dimensional manifold $I \times S$ with the metric $\bar{G} = ds^2 + a(s)\langle d\varphi, d\varphi\rangle$ smooth vector fields $f(s,\varphi)\partial_s + X(s,\varphi)$ where $X(s,\ ) \in \mathfrak{X}(S)$ is a smooth vector field on the Hilbert sphere S. We denote by ∇^S the covariant derivative on S and get

$$\begin{aligned}
&\partial_s \bar{G}\big(g\partial_s + Y, h\partial_s + Z\big) = \partial_s\big(gh + a\langle Y, Z\rangle\big) = \\
&\quad = g_s h + g h_s + a_s\langle Y, Z\rangle + a\langle Y_s, Z\rangle + a\langle Y, Z_s\rangle \\
&\quad = \bar{G}\big(g_s\partial_s + \frac{a_s}{2a}Y + Y_s, h\partial_s + Z\big) + \bar{G}\big(g\partial_s + Y, h_s\partial_s + \frac{a_s}{2a}Z + Z_s\big) \\
&X\bar{G}\big(g\partial_s + Y, h\partial_s + Z\big) = X\big(gh + a\langle Y, Z\rangle\big) \\
&\quad = dg(X).h + g.dh(X) + a\langle \nabla^S_X Y, Z\rangle + a\langle Y, \nabla^S_X Z, \rangle \\
&\quad = \bar{G}\big(dg(X)\partial_s + \nabla^S_X Y, h\partial_s + Z\big) + \bar{G}\big(g\partial_s + Y, dh(X)\partial_s + \nabla^S_X Z\big)\,.
\end{aligned}$$

Thus the following covariant derivative on $I \times S$, which is not the Levi-Civita covariant derivative,

$$\bar{\nabla}_{f\partial_s + X}(g\partial_s + Y) = f.g_s\partial_s + f\frac{a_s}{2a}Y + fY_s + dg(X)\partial_s + \nabla^S_X Y\,,$$

respects the metric $ds^2 + a\langle d\varphi, d\varphi\rangle$. But it has torsion which is given by

$$\begin{aligned}
&\operatorname{Tor}(f\partial_s + X, g\partial_s + Y) = \\
&\quad = \bar{\nabla}_{f\partial_s + X}(g\partial_s + Y) - \bar{\nabla}_{g\partial_s + Y}(f\partial_s + X) - [f\partial_s + X, g\partial_s + Y] = \\
&\quad = \frac{a_s}{2a}(fY - gX)\,.
\end{aligned}$$

To remove the torsion we consider the endomorphisms

$$\begin{gathered}
\operatorname{Tor}_{f\partial_s + X}, \operatorname{Tor}^\top_{f\partial_s + X} : T(I \times S) \to T(I \times S)\,, \\
\operatorname{Tor}_{f\partial_s + X}(g\partial_s + Y) = \operatorname{Tor}(f\partial_s + X, g\partial_s + Y)\,, \\
\bar{G}\big(\operatorname{Tor}^\top_{f\partial_s + X}(g\partial_s + Y), h\partial_s + Z) = \bar{G}\big(g\partial_s + Y, \operatorname{Tor}_{f\partial_s + X}(h\partial_s + Z)\big)
\end{gathered}$$

The endomorphism

$$\begin{aligned}
&A_{f\partial_s + X}(g\partial_s + Y) := \\
&\quad = \tfrac{1}{2}\big(\operatorname{Tor}(f\partial_s + X, g\partial_s + Y) - \operatorname{Tor}^\top_{f\partial_s + X}(g\partial_s + Y) - \operatorname{Tor}^\top_{g\partial_s + Y}(f\partial_s + X)\big)
\end{aligned}$$

is then $\bar{G}$-skew, so that

$$\nabla_{f\partial_s + X}(g\partial_s + Y) = \bar{\nabla}_{f\partial_s + X}(g\partial_s + Y) - A_{f\partial_s + X}(g\partial_s + Y)$$

still respects $\bar{G}$ and is now torsion free. In detail we get

$$\begin{aligned}
\operatorname{Tor}^\top_{f\partial_s + X}(g\partial_s + Y) &= -\frac{a_s}{2}\langle X, Y\rangle\partial_s + \frac{a_s}{2a}fY \\
A_{f\partial_s + X}(g\partial_s + Y) &= \frac{a_s}{2}\langle X, Y\rangle\partial_s - \frac{a_s}{2a}gX\,,
\end{aligned}$$

so that ∇ is the Levi-Civita connection of $\bar{G}$:

$$\boxed{\begin{aligned}
\nabla_{f\partial_s + X}(g\partial_s + Y) = \big(&f.g_s + dg(X) - \frac{a_s}{2}\langle X, Y\rangle\big)\partial_s \\
&+ \frac{a_s}{2a}(fY + gX) + fY_s + \nabla^S_X Y\,.
\end{aligned}}$$

For the curvature computation we assume from now on that all vector fields of the form $f\partial_s + X$ have f constant and $X = X(\varphi)$ so that in this case

$$\begin{aligned}
&\nabla_{f\partial_s+X}(g\partial_s + Y) = -\frac{a_s}{2}\langle X,Y\rangle\partial_s + \frac{a_s}{2a}(fY+gX) + \nabla^S_X Y\,,\\
&[f\partial_s + X, g\partial_s + Y] = [X,Y]^S\,,
\end{aligned}$$

in order to obtain

$$\begin{aligned}
&\nabla_{f\partial_s+X}\nabla_{g\partial_s+Y}(h\partial_s + Z) = \nabla_{f\partial_s+X}\big(-\frac{a_s}{2}\langle Y,Z\rangle\partial_s + \frac{a_s}{2a}(gZ+hY) + \nabla^S_Y Z\big)\\
&= \Big(-f\frac{a_{ss}}{2}\langle Y,Z\rangle - \frac{a_s}{2}\langle\nabla^S_X Y,Z\rangle - \frac{a_s}{2}\langle Y,\nabla^S_X Z\rangle\\
&\qquad\quad - \frac{a_s^2}{4a}g\langle X,Z\rangle - \frac{a_s^2}{4a}h\langle X,Y\rangle - \frac{a_s}{2}\langle X,\nabla^S_Y Z\rangle\Big)\partial_s\\
&\quad + \frac{a_s^2}{4a^2}fgZ + \frac{a_s^2}{4a^2}fhY + \frac{a_s}{2a}f\nabla^S_Y Z - \frac{a_s^2}{4a}\langle Y,Z\rangle X\\
&\quad + \big(\frac{a_s}{2a}\big)_s fgZ + \big(\frac{a_s}{2a}\big)_s fhY + \frac{a_s}{2a}g\nabla^S_X Z + \frac{a_s}{2a}h\nabla^S_X Y + \nabla^S_X\nabla^S_Y Z\\
&-\nabla_{[f\partial_s+X,g\partial_s+Y]}(h\partial_s+Z) = -\nabla_{[X,Y]^S}(h\partial_s+Z)\\
&= +\frac{a_s}{2}\langle[X,Y]^S,Z\rangle\partial_s - \frac{a_s}{2a}h[X,Y]^S - \nabla^S_{[X,Y]^S}Z
\end{aligned}$$

Summing up we obtain the curvature (for general vector fields, since curvature is of tensorial character)

$$\begin{aligned}
&\mathcal{R}(f\partial_s+X,g\partial_s+Y)(h\partial_s+Z) =\\
&= \big(\frac{a_{ss}}{2} - \frac{a_s^2}{4a}\big)\langle gX - fY, Z\rangle\partial_s + \mathcal{R}^S(X,Y)Z\\
&\quad - \big(\big(\frac{a_s}{2a}\big)_s + \frac{a_s^2}{4a^2}\big)h(gX-fY) + \frac{a_s^2}{4a}\big(\langle X,Z\rangle Y - \langle Y,Z\rangle X\big)\,.
\end{aligned}$$

and the numerator for sectional curvature

$$\begin{aligned}
&\bar G\big(\mathcal{R}(f\partial_s+X,g\partial_s+Y)(g\partial_s+Y), f\partial_s+X\big) = a\langle\mathcal{R}^S(X,Y)Y,X\rangle\\
&\quad - \big(\frac{a_{ss}}{2} - \frac{a_s^2}{4a}\big)\big\langle gX-fY, gX-fY\big\rangle + \frac{a_s^2}{4}\big(\langle X,Y\rangle^2 - \langle Y,Y\rangle\langle X,X\rangle\big)\,.
\end{aligned}$$

Let us take $X, Y \in T_\varphi S$ with $\langle X,Y\rangle = 0$ and $\langle X,X\rangle = \langle Y,Y\rangle = 1/a(s)$, then

$$\mathrm{Sec}_{(s,\varphi)}(\mathrm{span}(X,Y)) = \frac{1}{a} - \frac{a_s^2}{4a^2}\,,\qquad \mathrm{Sec}_{(s,\varphi)}(\mathrm{span}(\partial_s,Y)) = -\frac{a_{ss}}{2a} + \frac{a_s^2}{4a^2}\,,$$

are all the possible sectional curvatures. Compare this with the formulae for the principal curvatures of a hypersurface of revolution in [4].

9. Example. The simplest case is the choice $C_1(\lambda) = \frac{1}{\lambda}$ and $C_2(\lambda) = 0$. The Riemannian metric is

$$G_\mu(\alpha,\beta) = \frac{1}{\mu(M)}\int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\mu\,.$$

Then $g_1(r) = 4$ and $g_2(r) = \frac{4}{r^2}$. This metric is geodesically complete on $C^\infty(M,\mathbb{R})\setminus\{0\}$. The geodesic equation (d) simplifies to

$$r_{tt} = \frac{r_t^2}{r}\,.$$

This ODE can be solved explicitely and the solution is given by

$$r(t) = r(0)\exp\left(\frac{r_t(0)}{r(0)}t\right)\,.$$

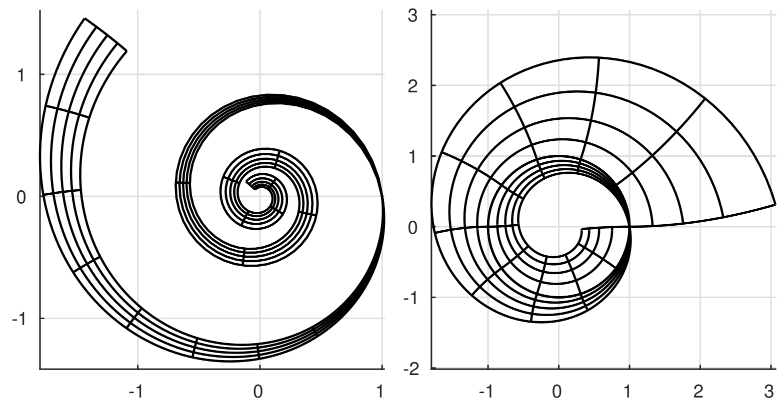

FIGURE 2. Fixing $\varphi(0)$, $\varphi_t(0)$ with $\|\varphi_t(0)\| = 1$, the figure shows geodesics $r(t).\varphi(t)$ starting at $r(0) = 1$ for various choices of $r_t(0)$; the geodesics are shown in the orthonormal basis $\{\varphi(0), \varphi_t(0)\}$. A periodic geodesic can be seen on the right.

The reparamterization map is $\alpha(t) = t$ and thus the geodesic

$$\varphi(t) = \cos\left(\|\varphi_t(0)\|t\right) + \sin\left(\|\varphi_t(0)\|t\right) \frac{\varphi_t(0)}{\|\varphi_t(0)\|} \,,$$

describes a great circle on the sphere with the standard parametrization. Note that geodesics with $r_t(0) = 0$ are closed with period $2\pi/\|\varphi_t(0)\|$. The spiraling behaviour of the geodesics can be seen in Fig. 2.

10. **Example.** By setting $C_1(\lambda) = 1$ and $C_2(\lambda) = 0$ we obtain the Fisher–Rao metric on the space of all densities. The Riemannian metric is

$$G_\mu(\alpha, \beta) = \int_M \frac{\alpha}{\mu} \frac{\beta}{\mu} \mu \,.$$

In this case $g_1(r) = 4r^2$ and $g_2(r) = 4$. The metric is incomplete towards 0 on $C^\infty(M, \mathbb{R}) \setminus \{0\}$. The pullback metric (b) is

$$\tilde{G} = 4r^2 \langle d\varphi, d\varphi \rangle + 4dr^2 \,,$$

and hence geodesics are straight lines in $C^\infty(M, \mathbb{R}) \setminus \{0\}$. In terms of the variables (r, φ), the geodesic equation (d) for r is

$$r_{tt} = \frac{C_0^2}{8} \frac{1}{r^3} \,,$$

with $C_0 = 4r(0)^2 \|\varphi_t(0)\|$.

11. **Example.** Setting $C_1(\lambda) = 1$ and $C_2(\lambda) = 1$ we obtain the extended metric

$$G_\mu(\alpha, \beta) = \int_M \frac{\alpha}{\mu} \frac{\beta}{\mu} \mu + \int_M \alpha \int_M \beta \,.$$

In this case $g_1(r) = 4r^2$ and $g_2(r) = 4r^2 + 4$. The geodesic equation (d) is

$$r_{tt} = \frac{C_0^2 - 16r^4 r_t^2}{r^3 \left(1 + r^2\right)} \,.$$

The metric on $C^\infty(M, \mathbb{R}) \setminus \{0\}$ is incomplete towards 0. Geodesics for the metric can be seen in Fig. 3. Note that only the geodesic going straight into the origin seems to be incomplete.

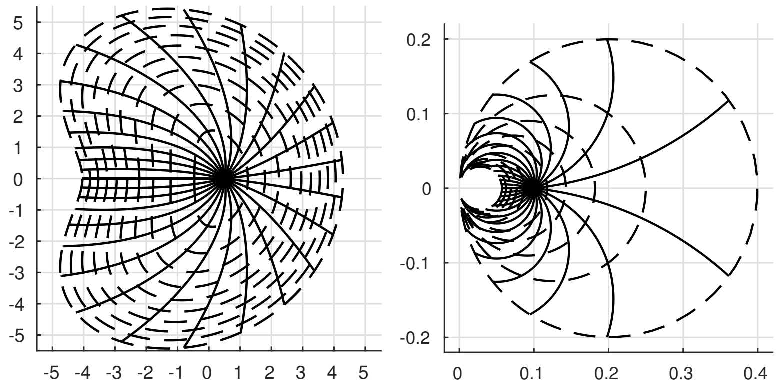

FIGURE 3. Fixing $\varphi(0)$, $\varphi_t(0)$ with $\|\varphi_t(0)\| = 1$, the figure shows geodesics $r(t).\varphi(t)$ for various choices of $r_t(0)$; on the left the extended Fisher–Rao metric with $C_1 = C_2 = 1$ with geodesics starting from $r(0) = 1$; on the right the metric with $C_1 = \frac{1}{r^2}$ with geodesics starting from $r(0) = 0.1$.

12. Example. Setting $C_1(\lambda) = \frac{1}{\lambda^2}$ and $C_2(\lambda) = 0$ we obtain the metric

$$G_\mu(\alpha, \beta) = \frac{1}{\mu(M)^2} \int_M \frac{\alpha}{\mu} \frac{\beta}{\mu} \mu \,,$$

which is complete towards 0, but incomplete towards infinity on $C^\infty(M, \mathbb{R}) \setminus \{0\}$. We have $g_1(r) = 4/r^2$ and $g_2(r) = 4/r^2$. The geodesic equation (d) is

$$r_{tt} = \frac{16 r_t^2 - C_0^2 r^4}{16 r} \,.$$

Examples of geodesics can be seen in Fig. 3. Note that the geodesic ball extends more towards infinity than towards the origin.

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