Boundary-Domain Integral Equation Systems for the Stokes System with Variable Viscosity and Diffusion Equation in Inhomogeneous Media

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A thesis submitted for the degree of

Doctor of Philosophy

November 2016

Esta tesis está dedicada a mi madre, María Elena Portillo Valdés, por su constante apoyo y cariño incondicional.

This thesis is solely dedicated to my mother, María Elena Portillo Valdés, due to her constant love and support.

Acknowledgements

First of all, I would like to thank my supervisor, Sergey Mikhailov, for giving me the opportunity to have an interview to explain and show my mathematical background; for being enormously patient; and for bringing up so many opportunities to publish and present my work.

Secondly, I would like ackowledge the funding received from the Department of Mathematical Sciences from Brunel University of London for my PhD and conferences as well as for providing me with the great opportunity to teach in Higher Education. In particular, I would like to thank very much to Steven Noble and Susan Browne for being very welcoming and supportive throughout all my teaching activities and for listening to all my teaching ideas.

As well, I would like to mention the support received from my dear PhD colleagues, especially, Chris Knapp, Hammed Hasseli and Layal Hakim.

In addition, I would like to show my gratitude to my secondary school teachers, in particular, to Claudio Bravo, María Luisa Sanchez for giving me plenty of enthusiasm in Mathematics; and to Angel Martinez Martinez, the counsellor of my secondary school, who offered infinite support through all the difficult moments.

Furthermore, I would like to thank my university professor, Prof. Sanz Serna for all his support given for the PhD applications.

To conclude, I would like to enormously acknowledge the support received from my Ornitorringos and my Terrelisticos, as from any other of my closed friends.

With a particular mention to Laura de la Fuente, Ignacio Portillo, Daniel de Una, Amaia Lesta, Julia Mosquera, Hector Izquierdo, Coral Pizarro, Beatriz Esgueva, Natalia Abejon, Marisol Herranz, Lorena Herranz and Raquel Gonzalez.

Last but not least, I highly appreciate the support received from my sister, Edurne, my brother, Enrique, my uncle and my father.

Abstract

The importance of the Stokes system stems from the fact that the Stokes system is the stationary linearised form of the Navier Stokes system [Te01, Chapter 1]. This linearisation is allowed when neglecting the inertial terms at a low Reinolds numbers $Re << 1$. The Stokes system essentially models the behaviour of a non-turbulent viscous fluid. The mixed interior boundary value problem related to the compressible Stokes system is reduced to two different BDIES which are equivalent to the original boundary value problem. These boundary-domain integral equation systems (BDIES) can be expressed in terms of surface and volume parametrix-based potential type operators whose properties are also analysed in appropriate Sobolev spaces. The invertibility and Fredholm properties related to the matrix operators that define the BDIES are also presented.

Furthermore, we also consider the mixed compressible Stokes system with variable viscosity in unbounded domains. An analysis of the similarities and differences with regards to the bounded domain case is presented. Furthermore, we outline the mapping properties of the surface and volume parametrix-based potentials in weighted Sobolev spaces. Equivalence and invertibility results still hold under certain decay conditions on the variable coefficient

The last part of the thesis refers to the mixed boundary value problem for the stationary heat transfer partial differential equation with variable coefficient. This BVP is reduced to a system of direct segregated parametrix-based Boundary-Domain Integral Equations (BDIEs). We use a parametrix different from the one employed by Chkadua, Mikhailov and Natroshvili in the paper [CMN09]. Mapping properties of the potential type integral operators appearing in these equations are presented in appropriate Sobolev spaces. We prove the equivalence between the original BVP and the corresponding BDIE system. The invertibility and Fredholm properties of the boundary-domain integral operators are also analysed in both bounded and unbounded domains.

Biography

I was born in Burgos, Spain. I studied in the Secondary School, Juan Martín "El Empecinado". Throughout my life, I have always had very clear I wanted to teach.

Since I was 14 years old I was very passionate about Mathematics. I studied a 5 years integrated Master of Mathematics degree at the University of Valladolid, which in several ocassions, demotivated me due to the level of difficulty of the exams. In addition, during the forth year, I studied at the University of Versailles-Saint Quentin, where I wrote my Master thesis in "Maxwell Equations with applications to microwave modelling".

After I graduated, I came to the UK and worked as a waiter, cover teacher, private tutor, form tutor and teaching assistant. One year afterwards, I was awarded a studentship to study the PhD in "Boundary Domain Integral Equations for Stokes System" under the supervision of Prof. Sergey Mikhailov and Dr. Michael Warby. My research interests are Integral Equations, Operator Theory, Quantum Mechanics, Partial Differential Equations and Numerical Analysis of PDEs and Integral Equations, Curriculum in Mathematics and Enabling Learning in Mathematics.

My interests are not only centered in Mathematics, but also I like playing piano, painting with oils, travelling, politics and nature.

During my PhD, I took part in 9 conferences in which I presented my work. Furthermore, I have two publications.

PUBLICATIONS

- S.E. Mikhailov and C.F.Portillo, *BDIE System to the Mixed BVP for the Stokes Equations with Variable Viscosity.* In: Integral Methods in Science and Engineering: Theoretical and Computational Advances. C. Constanda and A. Kirsh, eds., Springer (Birkhuser): Boston, ISBN 978-3-319-16727-5, 2015, DOI: 10.1007/978-3-319-16727-5_33.
- S.E. Mikhailov and C.F.Portillo, *A New Family of Boundary-Domain Integral Equations for a Mixed Elliptic BVP with Variable Coefficient.* Proceedings of the 10th UK Conference on Boundary Integral Methods, 13-14 July 2015 (Edited by P. Harris), ISBN 978-1-910172-05-6, 2015, 76-84.

TALKS

- *A New Family of Boundary-Domain Integral Equation Systems for a Mixed Elliptic Boundary Value Problem with Variable Coefficient in Unbounded Domains* 14th International Conference on Integral Methods in Science and Engineering (IMSE 2016), University of Padua, July 2016.
- *Boundary Domain Integral Equations for the Mixed Compressible Stokes System with Variable Viscosity in Bounded Domains,* MAFELAP 2016, The Mathematics of Finite Elements and its Applications, Brunel University London, June 2016.
- *A New Family of Boundary Domain Integral Equations for a Mixed Elliptic BVP with Variable Coefficient on a Bounded and Unbounded Domains BAMC*, British Colloquium of Applied Mathematics, University of Oxford, April 2016.
- *A New Family of Boundary-Domain Integral Equation Systems for a Mixed Elliptic Boundary Value Problem with Variable Coefficient, Exterior Problems,* 10th UK

Conference in Boundary Integral Methods *UKBIM10*, University of Brighton, July 2015.

- *A New Family of Boundary-Domain Integral Equation Systems for a Mixed Elliptic Boundary Value Problem with Variable Coefficient in Bounded Domains,* Brunel Student Symposium, Brunel University London, April 2015.

- *A New Family of Boundary-Domain Integral Equation Systems for a Mixed Elliptic Boundary Value Problem with Variable Coefficient,* Joint British Mathematical Colloquium - British Applied Mathematics Colloquium, University of Cambridge, April 2015.

- *Boundary Domain Integral Equations for the Mixed Interior Boundary Value Problem for the Stokes System with Variable Coefficient,* London Mathematical Society, London, December 2014.

- *Boundary Domain Integral Equations for the Mixed Boundary value Problem for the Stokes System with Variable Coefficient,* 13th International Conference on Integral Methods for Science and Engineering *IMSE*, Karlsruhe (Germany), July 2014.

- Green Identities for the Stokes System with Variable Coefficient. Brunel Student Symposium, Brunel University London, March 2014.

Contents

Chapter 1

Introduction

The Stokes system of partial differential equations is derived from the linearised steady-state Navier Stokes system. This line highlights the importance of the Stokes system as the main step to understand the popular Navier Stokes system whose study is highly encouraged and rewarded by the Clay Institute which offers a million dollars for the sophisticated proofs of existence, uniqueness and regularity of the solutions.

Needless to say, that if such amount of money is involved is because of the numerous applications in Science and Engineering such as Oceanography, Climatology or Magnetofluidynamics.

The Stokes system models the motion of a laminar viscous fluid, that is, a fluid whose motion does not depend on the time. A graphical picture of this scenario, would be a calm river.

The case of variable viscosity, as in general for any variable coefficient, refers to non homogeneous media, in this case, the viscosity of the fluid depends on the point within the fluid. A possible scenario to illustrate this situation could be a river of lava. The higher the temperature of the lava, the lower the viscosity. Therefore the fluid will tend to move slower as the viscosity increases.

The Stokes system also models how the fluid behaves when it encountes an obstacle. Returning to the river of lava example, it could happen that the lava comes accross with a house or a rock. Thanks to the Stokes system with variable viscosity we could predict the possible directions the lava could take around the around the building and maybe predict how much time we have to save the building before it is consumed by the heat. Mathematically, this is the most general approach for the Stokes system, when the domain is not simply connected and it can be easily derived from the results of this thesis.

1.1 Arrangement of the thesis

Chapter 1 Literature review

In this chapter, we will go through some of the most influential authors on the study of the incompressible and compressible Stokes system for the constant viscosity case, boundary integral equations and boundary-domain integral equations. Results on the fundamental solution, theory of hydrodynamic potentials, Green identities, existence and uniqueness of Dirichlet, Neumann-traction and mixed boundary value problems are presented.

Chapter 2 BDIES for the compressible Stokes system in bounded domains

In this chapter, we introduce an appropriate parametrix for the compressible Stokes system in order to deduce two equivalent boundary domain integral equation systems (BDIES) to the mixed compressible Stokes problem. We study in detail the relationships of the new parametrix-based volume and surface potentials to obtain mapping properties. Theorems of equivalence, Fredholm and invertibility properties are proved at the end of the chapter.

Chapter 3 BDIES for the compressible Stokes system in exterior domains

In this chapter, we follow the same route as in Chapter two to obtain boundary domain integral equation systems, however, this time in unbounded domains. We prove mapping properties in weighted Sobolev spaces under certain decay conditions on the variable coefficient. Theorems of equivalency, Fredholm properties and invertibility are proved at the end of the chapter.

Chapter 4 A new family of BDIES for a scalar mixed elliptic interior BVP

In this chapter, we consider a scalar partial differential equation $A(x, \partial_x; a(x))u = f$, where $a(x)$ is the variable coefficient. For this scalar equation, a parametrix of the form $P^y(x, y; a(y))$ for the operator $A(x, \partial_x; a(x))$ has already been studied in [CMN09]. Here, we introduce parametrices of the form $P^x(x, y; a(x))$ for the same operator $A(x, \partial_x; a(x))$. This parametrix leads to a new family of boundary domain integral equations. A system of BDIES is derived. Results on equivalence of the BDIES and the mixed BVP are shown on Sobolev spaces. Mapping properties of the surface and volume potentials based on this new parametrix are proven.

Chapter 5 A new family of BDIES for a scalar mixed elliptic exterior BVP

Following the introduction of the previous chapter, we tackle the same mixed boundary value problem in a unbounded domain. We derive an analogous system of BDIEs, prove equivalence and invertibility. We analyse the obstacles to overcome for unbounded domains to prove similar results as in chapter 4 for bounded domains.

Chapter 6 Conclusions and further work

In this chapter, we present a summary of the conclusions drwan from the results as well as open problems to be studied in the future.

1.2 Literature Review

Although the first construction of hydrodynamical potentials is owed to Lichtenstein and Odqvist, see [Li27] and [Od30]. However, the first author gathering an exhaustive description of the potential theory applied to the Stokes system is given in [La69]. The importance of the hydrodynamic potential theory stems from the fact that it only differs from the harmonic potential theory in the kernels of the potentials. Therefore, as the potential theory has been extensively studied during the XIX and XX century, similar results can be obtained for the case of the Stokes system.

The derivation of the fundamental solution using the Fourier transform and the Helmholtz decomposition is given in [La69, p.50-p.51]. This has a double great advantage. On one hand, an explicit fundamental solution allows to use fast and robust numerical methods in order to approximate the solution such as the boundary element method (BEM) [Ste07, Chapter 10]. On the other hand, the Helmholtz decomposition, see e.g. [Bo04, Appendix

2.5], allows to understand in depth the properties of the solutions of the Navier Stokes equations (cf. [So01]).

An integral representation formulae for the velocity and pressure, for an incompressible fluid with constant viscosity is also presented in [La69]. The third Green identites are then used to derive integral equations for the Dirichlet problem and Neumann-traction problem for the Stokes system. The main results are shown in [La69, Section 3.3], where there is a further investigation of the solvability and uniqueness of the solution for both aforementioned problems. Nevertheless, there is not much detail about the spaces where this unique solvability is discussed. Thus, in the following sections a functional approach is used to study the existence in the classical spaces of continuous functions and in some weaker classes of Sobolev spaces.

In broad words, Ladyzhenskaya develops an extensive study of the Stokes system mainly using a functional approach rather than from the point of view of boundary integral equations or the Fredholm alternative. To understand in depth both approaches, it is essential to study first the mapping properties of the surface and volume (newtonian) hydrodynamic potentials.

M. Costabel presents in [Co88] the elementary results of continuity and positivity of the boundary potentials and newtonian potentials in the general case of a second order elliptic operator. Furthermore, he shows some elementary results of uniqueness using the variational approach in Sobolev and Lebesgue spaces over Lipschitz domains, via Lax-Milgram Lemma.

W. Wenland and G. Hsiao, in [HsWe08] gather most of the boundary integral operators mapping properties for various partial differential equations, in particular for the incompressible Stokes system. A table with the compatibility conditions for the interior and exterior incompressible Stokes, with Dirichlet and Neumann boundary conditions can be found in [HsWe08, Table 2.3.3]. Variational formulations for the Stokes system are also deduced for the Dirichlet and Neumann, interior and exterior boundary value problems. In addition, in this book, results on Fredholm theorems and Fredholm properties of the potentials are presented.

Furthermore, I would like to highlight Theorem 2.3.2. from [HsWe08]. This theorem, with versions in [KoPo04] and [ReSt03], characterises the eigenspaces of the direct value of the single layer potential and hypersingular operator for the constant coefficient case.

Existence, non uniqueness and uniqueness for the compressible Stokes with constant rate of expansion, this means the divergence of the velocity field remains constant, are discussed in [Ko07] using classical spaces of continuous functions.

The great advantage of applying the BEM in the homogeneous constant coefficient case is the fact that we can reduce a boundary value problem for a partial differential equation (PDE) defined in a three dimensional domain to a integral equation over the boundary of the domain. Computationally, the complexity considerably decreases since we reduce the dimensionality of the problem. Consequently, some algorithms involving boundary elements are able to approximate the solution of such boundary value problems - homogeneous with constant coefficient - much more rapidly than, for example, with the finite element method (FEM).

Following the same approach as in [McL00, Chapters 6 & 7], it is possible to input the fundamental solution and the right hand side of the PDE *with constant coefficient*, into the second Green identity to obtain a integral representation formula, third Green identity, for the solution, its trace and its conormal derivative (or traction in the case of the Stokes system). The solution of the boundary value problem will satisfy these third Green identities in the domain. Then, some extensions to the boundary data are introduced in order to completely segregate the trace and conormal derivative from the solution function, [McL00, Theorem 7.9]. Using this approach, one can derive integral equations for the Dirichlet and Neumann problem, or systems for the case of the mixed problem.

The subsequent essential steps are: proving the equivalence between the original boundary value problem and the boundary integral equation system (BIES) and showing the invertibility of the operators that define the boundary integral equation (BIE).

Furthermore, since we work with Sobolev spaces in bounded domains, we can apply the Rellich compactness theorem to prove compact properties of integral operators related with embeddings of Sobolev spaces. The importance of the compactness property stems from the fact that it can be very useful at the time of applying Fredholm alternative theorems, (cf., [McL00]) to prove uniqueness of a BIE.

In general, it is essential to have an explicit fundamental solution in order to use BEM for numerical approximations. Examples of numerical approximation of boundary domain integral equations BDIEs) can be found in [GMR13, MiMo11, Mi06].

For elliptic equations and systems, even though the fundamental solution may exist, see [Ru06, Theorem 8.4 and Theorem 8.5], it is not always known explicitly. This is the most common scenario when the PDE has variable coefficients.

Although fundamental solutions might not be available for the variable coefficient case; if the corresponding PDE with constant coefficient has a fundamental solution explicitly known, it might be possible to construct a parametrix or Levi function (cf. [CMN09, Mi02, MiPo15-I]). This parametrix plays the role of an approximation to the fundamental solution. It can substituted into the second Green identity to obtain integral representation formulas and from there, deduce an integral equation. However, in contrast with the constant coefficient case, the integral equations derived will be not only defined on the boundary but also within the domain leading to BDIEs.

Boundary value problems (BVPs) with variable coefficients normally arise in the context of non-homogenenous media such as a material with heterogeneous electrical conductivity or a fluid with different temperatures.

BDIEs and parametrices are well studied nowadays for scalar equations for elliptic boundary value problems, e.g., [CMN09, MiPo15-II, CMN13] and references therein. Nevertheless, little is known about other types of BVPs. For instance: the Stokes system is elliptic in the sense of Douglis - Nirenberg but not in the sense of Petrovski and therefore the analysis of the Stokes system with variable coefficient remains open, see [KoPo04, HsWe08].

Chapter 2

BDIES for the compressible Stokes system in bounded domains

2.1 Introduction

Boundary integral equations and the hydrodynamic potential theory for the Stokes system with constant viscosity have been extensively studied by numerous authors, e.g., [La69, LiMa73, HsWe08, ReSt03, Ste07, KoWe06, WeZh91].

Although the compressible Stokes System with variable viscosity has been extensively studied, it has not yet been reduced to BDIES following a similar approach as in [CMN09]. In contrast to [CMN09], the BVP approached in this chapter consists of a system of four equations with four unknowns: the three component velocity field and the scalar pressure field.

In the case of constant viscosity, fundamental solutions for both, velocity and pressure, are available. Notwithstanding, these fundamental solutions are not available in the variable coefficient case for which a *parametrix* (Levi function), (see e.g., [CMN09, Mi02, MiPo15-I, MiPo15-II]) is needed in order to derive the (BDIES).

However, a parametrix for a certain PDE is not unique and neither is it in the case of a PDE system. Therefore, the choice of an appropriate parametrix is not a trivial decision at all. In [MiPo15-I], we develop BDIES for the mixed imcompressible Stokes problem defined over a bounded domain. Equivalence between the BVP-BDIES is shown, however, invertibility results are not proved.

In this chapter, we derive two BDIES equivalent to the original mixed compressible Stokes system defined on a bounded domain. Furthermore, mapping properties of the hydrodynamic surface and volume potentials are shown. The main results are the equivalence theorems and the invertibility theorems of the operators defined by the BDIES.

2.2 Preliminaries

Let $\Omega = \Omega^+$ be a *bounded* and simply connected domain and let $\Omega^- := \mathbb{R}^3 \smallsetminus \overline{\Omega}^+$. We will assume that the boundary $S := \partial\Omega$ is simply connected, closed and infinitely differentiable, $S \in \mathcal{C}^\infty$. Furthermore, $S := \overline{S}_N \cup \overline{S}_D$ where both S_N and S_D are non-empty, connected disjoint manifolds of S. The border of these two submanifolds is also infinitely differentiable, $\partial S_N = \partial S_D \in \mathcal{C}^\infty$.

Let $\boldsymbol{v}$ be the velocity vector field; p the pressure scalar field and $\mu \in \mathcal{C}^\infty(\Omega)$ be the variable kinematic viscosity of the fluid such that $\mu(\boldsymbol{x}) > c > 0$.

The Stokes operator is defined as

$$\begin{aligned}\mathcal{A}_j(p, \boldsymbol{v})(\boldsymbol{x}) :&= \frac{\partial}{\partial x_i}\sigma_{ji}(p, \boldsymbol{v})(\boldsymbol{x}) \\ &= \frac{\partial}{\partial x_i}\left(\mu(\boldsymbol{x})\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\delta_i^j \text{div}\boldsymbol{v}\right)\right) - \frac{\partial p}{\partial x_j},\ j, i \in \{1, 2, 3\},\end{aligned} \tag{2.1}$$

where δ_i^j is Kronecker symbol. Here and henceforth we assume the Einstein summation in repeated indices from 1 to 3. We also denote the Stokes operator as $\boldsymbol{\mathcal{A}} = \{\mathcal{A}_j\}_{j=1}^3$. Ocassionally, we may use the following notation for derivative operators: $\partial_j = \partial_{x_j} := \dfrac{\partial}{\partial x_j}$ with $j = 1, 2, 3$; $\nabla := (\partial_1, \partial_2, \partial_3)$.

For a compressible fluid div$\boldsymbol{v} = g$, which gives the following stress tensor operator and the Stokes operator, respectively, to

$$\begin{aligned}\sigma_{ji}(p, \boldsymbol{v})(\boldsymbol{x}) &= -\delta_i^j p(\boldsymbol{x}) + \mu(\boldsymbol{x})\left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i} - \frac{2}{3}\delta_i^j g\right), \\ \mathcal{A}_j(p, \boldsymbol{v})(\boldsymbol{x}) &= \frac{\partial}{\partial x_i}\left(\mu(\boldsymbol{x})\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\delta_i^j g\right)\right) - \frac{\partial p}{\partial x_j},\ j, i \in \{1, 2, 3\}.\end{aligned}$$

In what follows $H^s(\Omega)$, $H^s(S)$ are the Bessel potential spaces, where $s \in \mathbb{R}$ is an arbitrary real number (see, e.g., [LiMa73], [McL00]). We recall that H^s coincide with the Sobolev–Slobodetski spaces W_2^s for any non-negative s. Let $H_K^s := \{g \in H^1(\mathbb{R}^3) : \operatorname{supp}(g) \subseteq K\}$ where K is a compact subset of $\mathbb{R}^3$. In what follows we use the bold notation: $\boldsymbol{H}^s(\Omega) = [H^s(\Omega)]^3$ for 3-dimensional vector spaces. We denote by $\widetilde{\boldsymbol{H}}^s(\Omega)$ the subspace of $\boldsymbol{H}^s(\mathbb{R}^3)$, $\widetilde{\boldsymbol{H}}^s(\Omega) := \{\boldsymbol{g} : \ \boldsymbol{g} \in \boldsymbol{H}^s(\mathbb{R}^3), \ \operatorname{supp} \boldsymbol{g} \subset \overline{\Omega}\}$; similarly, $\widetilde{\boldsymbol{H}}^s(S_1) = \{\boldsymbol{g} \in \boldsymbol{H}^s(S), \ \operatorname{supp} \boldsymbol{g} \subset \overline{S}_1\}$ is the Sobolev space of functions having support in $S_1 \subset S$.

We will also make use of the following space, (cf. e.g. [Co88] [CMN09])

$$\boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) := \{(p, \boldsymbol{v}) \in L_2(\Omega) \times \boldsymbol{H}^1(\Omega) : \boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) \in \boldsymbol{L}_2(\Omega)\},$$

endowed with the norm

$$\|(p, \boldsymbol{v})\|_{\boldsymbol{H}^{1,0}(\Omega; L)} := \left(\|p\|^2_{L_2(\Omega)} + \|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{\mathcal{A}}(p, \boldsymbol{v})\|^2_{\boldsymbol{L}_2(\Omega)} \right)^{1/2}.$$

The operator $\boldsymbol{\mathcal{A}}$ acting on $(p, \boldsymbol{v})$ is well defined in the weak sense provided $\mu(\boldsymbol{x}) \in L^\infty(\Omega)$ as

$$\langle \boldsymbol{\mathcal{A}}(p, \boldsymbol{v}), \boldsymbol{u} \rangle_\Omega := -\mathcal{E}((p, \boldsymbol{v}), \boldsymbol{u}), \qquad \forall \boldsymbol{u} \in \widetilde{\boldsymbol{H}}^1(\Omega),$$

where the form $\mathcal{E} : \left[L^2(\Omega) \times \boldsymbol{H}^1(\Omega)\right] \times \widetilde{\boldsymbol{H}}^1(\Omega) \longrightarrow \mathbb{R}$ is defined as

$$\mathcal{E}\left((p, \boldsymbol{v}), \boldsymbol{u}\right) := \int_\Omega E\left((p, \boldsymbol{v}), \boldsymbol{u}\right)(\boldsymbol{x})\, dx, \tag{2.2}$$

and the function $E\left((p, \boldsymbol{v}), \boldsymbol{u}\right)$ is defined as

$$\begin{aligned} E\left((p, \boldsymbol{v}), \boldsymbol{u}\right)(\boldsymbol{x}) : = \ & \frac{1}{2}\mu(\boldsymbol{x}) \left(\frac{\partial u_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial u_j(\boldsymbol{x})}{\partial x_i} \right) \left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i} \right) \\ & - \frac{2}{3}\mu(\boldsymbol{x}) \mathrm{div} div \boldsymbol{v}(\boldsymbol{x})\, \mathrm{div} \boldsymbol{u}(\boldsymbol{x}) - p(\boldsymbol{x}) \mathrm{div} \boldsymbol{u}(\boldsymbol{x}). \end{aligned} \tag{2.3}$$

For sufficiently smooth functions $(p, \boldsymbol{v}) \in \boldsymbol{H}^{s-1}(\Omega^\pm) \times H^s(\Omega^\pm)$ with $s > 3/2$, we can define the classical traction operators on the boundary S as

$$T_i^\pm(p, \boldsymbol{v})(\boldsymbol{x}) := \gamma^\pm \sigma_{ij}(p, \boldsymbol{v})(\boldsymbol{x})\, n_j(\boldsymbol{x}), \tag{2.4}$$

where $n_j(\boldsymbol{x})$ denote components of the unit outward normal vector $\boldsymbol{n}(\boldsymbol{x})$ to the boundary S of the domain Ω and $\gamma^{\pm}(\,\cdot\,)$ denote the trace operators from inside and outside Ω.

Traction operators (2.4) can be continuously extended to the *canonical* traction operators $\boldsymbol{T}^{\pm} : \boldsymbol{H}^{1,0}(\Omega^{\pm}, \boldsymbol{\mathcal{A}}) \to \boldsymbol{H}^{-1/2}(S)$ defined in the weak form similar to [Co88, Mi11, CMN09] as

$$\langle \boldsymbol{T}^{\pm}(p,\boldsymbol{v}), \boldsymbol{w}\rangle_S := \pm \int_{\Omega^{\pm}} \left[\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})\boldsymbol{\gamma}^{-1}\boldsymbol{w} + E\left((p,\boldsymbol{v}), \boldsymbol{\gamma}^{-1}\boldsymbol{w}\right)\right]\,dx,$$
$$\forall\,(p,\boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega^{\pm}, \boldsymbol{\mathcal{A}}),\ \forall\,\boldsymbol{w} \in \boldsymbol{H}^{1/2}(S).$$

Here the operator $\boldsymbol{\gamma}^{-1} : \boldsymbol{H}^{1/2}(S) \to \boldsymbol{H}^{1}(\mathbb{R}^3)$ denotes a continuous right inverse of the trace operator $\boldsymbol{\gamma} : \boldsymbol{H}^{1}(\mathbb{R}^3) \to \boldsymbol{H}^{1/2}(S)$.

Furthermore, if $(p,\boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$ and $\boldsymbol{u} \in \boldsymbol{H}^{1}(\Omega)$, the following first Green identity holds, cf. [Co88, Mi11, CMN09, MiPo15-I],

$$\langle \boldsymbol{T}^{+}(p,\boldsymbol{v}), \boldsymbol{\gamma}^{+}\boldsymbol{u}\rangle_S = \int_{\Omega}[\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})\boldsymbol{u} + E\left((p,\boldsymbol{v}),\boldsymbol{u}\right)(\boldsymbol{x})]dx. \tag{2.5}$$

Applying the identity (2.5) to the pairs $(p,\boldsymbol{v}),(q,\boldsymbol{u}) \in \boldsymbol{H}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$ with exchanged roles and subtracting the one from the other, we arrive at the second Green identity, cf. [McL00, Mi11],

$$\int_{\Omega}\left[\mathcal{A}_j(p,\boldsymbol{v})u_j - \mathcal{A}_j(q,\boldsymbol{u})v_j + q\,\mathrm{div}\,\boldsymbol{v} - p\,\mathrm{div}\,\boldsymbol{u}\right]\,dx =$$
$$\langle \boldsymbol{T}^{+}(p,\boldsymbol{v}), \boldsymbol{\gamma}^{+}\boldsymbol{u}\rangle_S - \langle \boldsymbol{T}^{+}(q,\boldsymbol{u}), \boldsymbol{\gamma}^{+}\boldsymbol{v}\rangle_S. \tag{2.6}$$

Now we are ready to define the mixed BVP for which we aim to derive equivalent BDIES and investigate the existence and uniqueness of their solutions.

For $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$, $g \in L^2(\Omega)$, $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(S_D)$ *and* $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(S_N)$, *find* $(p,\boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$ *such that:*

$$\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}),\ \ \boldsymbol{x} \in \Omega, \tag{2.7a}$$

$$\mathrm{div}(\boldsymbol{v})(\boldsymbol{x}) = g(\boldsymbol{x}),\ \ \boldsymbol{x} \in \Omega, \tag{2.7b}$$

$$r_{S_D}\boldsymbol{\gamma}^{+}\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\varphi}_0(\boldsymbol{x}),\ \boldsymbol{x} \in S_D, \tag{2.7c}$$

$$r_{S_N}\boldsymbol{T}^{+}(p,\boldsymbol{v})(\boldsymbol{x}) = \boldsymbol{\psi}_0(\boldsymbol{x}),\ \boldsymbol{x} \in S_N. \tag{2.7d}$$

Applying the first Green identity it is easy to prove the following uniqueness result.

Theorem 2.1. *Mixed BVP* (2.7) *has at most one solution in the space* $\boldsymbol{H}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$.

Proof. Let us suppose that there are two possible solutions: $(p_1, \boldsymbol{v}_1)$ and $(p_2, \boldsymbol{v}_2)$ belonging to the space $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$, that satisfy the BVP (2.7). Then, the pair $(p, \boldsymbol{v}) := (p_2, \boldsymbol{v}_2) - (p_1, \boldsymbol{v}_1)$ also belongs to the space $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$ and satisfies the following homogeneous mixed BVP

$$\boldsymbol{\mathcal{A}}(p, \boldsymbol{v})(\boldsymbol{x}) = \boldsymbol{0}, \ \ \boldsymbol{x} \in \Omega, \tag{2.8a}$$

$$\operatorname{div}(\boldsymbol{v})(\boldsymbol{x}) = 0, \ \ \boldsymbol{x} \in \Omega, \tag{2.8b}$$

$$r_{S_D} \gamma^+ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{0}, \ \ \boldsymbol{x} \in S_D, \tag{2.8c}$$

$$r_{S_N} \boldsymbol{T}^+(p, \boldsymbol{v})(\boldsymbol{x}) = \boldsymbol{0}, \ \ \boldsymbol{x} \in S_N. \tag{2.8d}$$

The first Green identity (2.5) holds for any $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and for any pair $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$. Hence, we can choose $\boldsymbol{u} \in \boldsymbol{H}^1_{0,div}(\Omega; S_D) \subset \boldsymbol{H}^1(\Omega)$, where the space $\boldsymbol{H}^1_{0,div}(\Omega; S_D)$ is defined as

$$\boldsymbol{H}^1_{0,div}(\Omega; S_D) := \{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{\gamma}^+_{S_D} \boldsymbol{u} = \boldsymbol{0}, \ \ \operatorname{div} \boldsymbol{u} = 0 \ \ in \ \Omega\}.$$

Due to (2.8a), $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$. Consequently, the first Green identity can be applied to $\boldsymbol{u} \in \boldsymbol{H}^1_{0,div}(\Omega; S_D)$ and $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$,

$$\int_\Omega \frac{1}{2}\mu(\boldsymbol{x}) \left(\frac{\partial u_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial u_j(\boldsymbol{x})}{\partial x_i}\right) \left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i}\right) dx = 0. \tag{2.9}$$

In particular, one could choose $\boldsymbol{u} := \boldsymbol{v}$ since $\boldsymbol{v} \in \boldsymbol{H}^1_{0,div}(\Omega; S_D)$. Then, the first Green identity now reads:

$$\int_\Omega \frac{1}{2}\mu(\boldsymbol{x}) \left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i}\right)^2 dx \ = \ 0.$$

As $\mu(\boldsymbol{x}) > 0$, the only possibility is that $\boldsymbol{v}(\boldsymbol{x}) \ = \ \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$, i.e., $\boldsymbol{v}$ is a rigid movement, [McL00, Lemma 10.5]. Nevertheless, taking into account the Dirichlet condition (2.8c), we deduce that $\boldsymbol{v} \equiv \boldsymbol{0}$. Hence, $\boldsymbol{v}_1 = \boldsymbol{v}_2$.

Considering now $\boldsymbol{v} \equiv \boldsymbol{0}$ and keeping in mind the Neumann-traction condition (2.8d), it is easy to conclude that $p_1 = p_2$. □

2.3 Parametrix and Remainder

When $\mu(\boldsymbol{x}) = 1$, the operator $\boldsymbol{\mathcal{A}}$ becomes the constant-coefficient Stokes operator $\mathring{\boldsymbol{\mathcal{A}}}$, for which we know an explicit fundamental solution defined by the pair of fields $(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)$, where $\mathring{u}_j^k$ represent components of the incompressible velocity fundamental solution and $\mathring{q}^k$ represent the components of the pressure fundamental solution (see e.g. [La69], [KoWe06], [HsWe08]).

$$\mathring{q}^k(\boldsymbol{x},\boldsymbol{y}) = \frac{(x_k - y_k)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3},$$
$$\mathring{u}_j^k(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{8\pi}\left\{\frac{\delta_j^k}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{(x_j-y_j)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right\}, \quad j,k \in \{1,2,3\}.$$

Therefore, $(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)$ satisfy

$$\mathring{\mathcal{A}}_j(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)(\boldsymbol{x}) = \sum_{i=1}^{3}\frac{\partial^2 \mathring{u}_j^k}{\partial x_i^2} - \frac{\partial \mathring{q}^k}{\partial x_j} = \delta_j^k \delta(\boldsymbol{x}-\boldsymbol{y}).$$

Let us denote $\mathring{\sigma}_{ij}(p,\boldsymbol{v}) := \sigma_{ij}(p,\boldsymbol{v})|_{\mu=1}$. Then, in the particular case $\mu = 1$, the stress tensor $\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)(\boldsymbol{x}-\boldsymbol{y})$ reads as

$$\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)(\boldsymbol{x}-\boldsymbol{y}) = \frac{3}{4\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^5},$$

and the boundary traction becomes

$$\begin{aligned}\mathring{T}_i(\boldsymbol{x};\mathring{q}^k, \mathring{\boldsymbol{u}}^k)(\boldsymbol{x},\boldsymbol{y}) :&= \mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)(\boldsymbol{x}-\boldsymbol{y})\, n_j(\boldsymbol{x}) \\ &= \frac{3}{4\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^5}\, n_j(\boldsymbol{x}).\end{aligned}$$

Let us define a pair of functions $(q^k, \boldsymbol{u}^k)_{k=1,2,3}$ as

$$q^k(\boldsymbol{x},\boldsymbol{y}) = \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{q}^k(\boldsymbol{x},\boldsymbol{y}) = \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\frac{x_k-y_k}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}, \quad j,k \in \{1,2,3\}. \tag{2.10}$$

$$u_j^k(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\mathring{u}_j^k(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{8\pi\mu(\boldsymbol{y})}\left\{\frac{\delta_j^k}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{(x_j-y_j)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right\}, \tag{2.11}$$

Then,

$$\sigma_{ij}(\boldsymbol{x};q^k,\boldsymbol{u}^k)(\boldsymbol{x},\boldsymbol{y}) = \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)(\boldsymbol{x}-\boldsymbol{y}),$$
$$T_i(\boldsymbol{x};q^k,\boldsymbol{u}^k)(\boldsymbol{x},\boldsymbol{y}) := \sigma_{ij}(\boldsymbol{x};q^k,\boldsymbol{u}^k)(\boldsymbol{x},\boldsymbol{y})\, n_j(\boldsymbol{x}) = \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{T}_i(\boldsymbol{x};\mathring{q}^k, \mathring{\boldsymbol{u}}^k)(\boldsymbol{x},\boldsymbol{y}).$$

Substituting (2.11)-(2.10) in the Stokes system with variable coefficient (2.1) gives

$$\mathcal{A}_j(\boldsymbol{x}; q^k, \boldsymbol{u}^k)(\boldsymbol{x}, \boldsymbol{y}) = \delta_j^k \delta(\boldsymbol{x} - \boldsymbol{y}) + R_{kj}(\boldsymbol{x}, \boldsymbol{y}), \tag{2.12}$$

where

$$R_{kj}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})} \frac{\partial \mu(\boldsymbol{x})}{\partial x_i} \mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)(\boldsymbol{x} - \boldsymbol{y}) = \mathcal{O}(|\boldsymbol{x} - \boldsymbol{y}|)^{-2})$$

is a weakly singular remainder. This implies that $(q^k, \boldsymbol{u}^k)$ is a parametrix of the operator $\boldsymbol{\mathcal{A}}$.

2.4 Hydrodynamic parametrix-based potentials

2.4.1 Volume and surface potentials

Let us define the parametrix-based Newton-type and remainder vector potentials

$$\mathcal{U}_k \boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{U}_{kj} \rho_j(\boldsymbol{y}) := \int_\Omega u_j^k(\boldsymbol{x}, \boldsymbol{y}) \rho_j(\boldsymbol{x}) dx,$$
$$\mathcal{R}_k \boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{R}_{kj} \rho_j(\boldsymbol{y}) := \int_\Omega R_{kj}(\boldsymbol{x}, \boldsymbol{y}) \rho_j(\boldsymbol{x}) dx, \quad \boldsymbol{y} \in \mathbb{R}^3,$$

for the velocity, and the scalar Newton-type pressure and remainder potentials

$$\mathcal{Q} \boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{Q}_j \rho_j(\boldsymbol{y}) := \int_\Omega q^j(\boldsymbol{x}, \boldsymbol{y}) \rho_j(\boldsymbol{x}) dx, \tag{2.13}$$
$$\boldsymbol{\mathcal{Q}} \rho(\boldsymbol{y}) = \mathcal{Q}_j \rho(\boldsymbol{y}) := \int_\Omega q^j(\boldsymbol{x}, \boldsymbol{y}) \rho(\boldsymbol{x}) dx, \tag{2.14}$$
$$\mathcal{R}^\bullet \boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{R}_j^\bullet \rho_j(\boldsymbol{y}) := 2\, v.p. \int_\Omega \frac{\partial \mathring{q}^j(\boldsymbol{x}, \boldsymbol{y})}{\partial x_i} \frac{\partial \mu(\boldsymbol{x})}{\partial x_i} \rho_j(\boldsymbol{x}) dx - \frac{4}{3} \rho_j \frac{\partial \mu}{\partial y_j}, \ \boldsymbol{y} \in \mathbb{R}^3, \tag{2.15}$$

for the pressure. The integral in (2.15) is understood as a 3D strongly singular integral in the Cauchy sense.

For the velocity, let us also define the parametrix-based single layer potential, double layer potential and their respective direct values on the boundary, as follows:

$$V_k \boldsymbol{\rho}(\boldsymbol{y}) = V_{kj} \rho_j(\boldsymbol{y}) := -\int_S u_j^k(\boldsymbol{x}, \boldsymbol{y}) \rho_j(\boldsymbol{x})\, dS(\boldsymbol{x}), \ \boldsymbol{y} \notin S,$$
$$W_k \boldsymbol{\rho}(\boldsymbol{y}) = W_{kj} \rho_j(\boldsymbol{y}) := -\int_S T_j(\boldsymbol{x}; q^k, \boldsymbol{u}^k)(\boldsymbol{x}, \boldsymbol{y}) \rho_j(\boldsymbol{x})\, dS(\boldsymbol{x}), \ \boldsymbol{y} \notin S,$$
$$\mathcal{V}_k \boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{V}_{kj} \rho_j(\boldsymbol{y}) := -\int_S u_j^k(\boldsymbol{x}, \boldsymbol{y}) \rho_j(\boldsymbol{x})\, dS(\boldsymbol{x}), \ \boldsymbol{y} \in S,$$
$$\mathcal{W}_k \boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{W}_{kj} \rho_j(\boldsymbol{y}) := -\int_S T_j(\boldsymbol{x}; q^k, \boldsymbol{u}^k)(\boldsymbol{x}, \boldsymbol{y}) \rho_j(\boldsymbol{x})\, dS(\boldsymbol{x}), \ \boldsymbol{y} \in S.$$

For pressure in the variable coefficient Stokes system, we will need the following single-layer and double layer potentials:

$$\mathcal{P}\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{P}_j\rho_j(\boldsymbol{y}) := -\int_S \mathring{q}^j(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})dS(\boldsymbol{x}),$$
$$\Pi\boldsymbol{\rho}(\boldsymbol{y}) = \Pi_j\rho_j(\boldsymbol{y}) := -2\int_S \frac{\partial\mathring{q}^j(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{x})}\mu(\boldsymbol{x})\rho_j(\boldsymbol{x})dS(\boldsymbol{x}),\ \boldsymbol{y}\notin S.$$

Let us also denote

$$\mathcal{W}'_k\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{W}'_{kj}\rho_j(\boldsymbol{y}) := -\int_S T_j(\boldsymbol{y};q^k,\boldsymbol{u}^k)(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,dS(\boldsymbol{x}),\ \ \boldsymbol{y}\in S,$$
$$\mathcal{L}^{\pm}_k\boldsymbol{\rho}(\boldsymbol{y}) := T^{\pm}_k(\Pi\boldsymbol{\rho},\boldsymbol{W}\boldsymbol{\rho})(\boldsymbol{y}),\quad \boldsymbol{y}\in S,$$

where $T^{\pm}_k$ are the traction operators for the *compressible* fluid.

2.4.2 Mapping properties

The parametrix-based integral operators, depending on the variable coefficient $\mu(\boldsymbol{y})$, can be expressed in terms of the corresponding integral operators for the constant coefficient case, $\mu = 1$,

$$\mathcal{U}_k\boldsymbol{\rho}(\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\mathring{\mathcal{U}}_k\boldsymbol{\rho}(\boldsymbol{y}), \tag{2.16}$$
$$\mathcal{R}_k\boldsymbol{\rho}(\boldsymbol{y}) = \frac{-1}{\mu(\boldsymbol{y})}\Big[\frac{\partial}{\partial y_j}\mathring{\mathcal{U}}_{ki}(\rho_j\partial_i\mu)(\boldsymbol{y}) + \frac{\partial}{\partial y_i}\mathring{\mathcal{U}}_{kj}(\rho_j\partial_i\mu)(\boldsymbol{y}) - \mathring{\mathcal{Q}}_k(\rho_j\partial_j\mu)(\boldsymbol{y})\Big], \tag{2.17}$$
$$\mathcal{Q}_j\rho(\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\mathring{\mathcal{Q}}_j(\mu\rho)(\boldsymbol{y}), \tag{2.18}$$
$$\mathcal{R}^{\bullet}_j\rho_j(\boldsymbol{y}) = -2\frac{\partial}{\partial y_i}\mathring{\mathcal{Q}}_j(\rho_j\partial_i\mu)(\boldsymbol{y}) - 2\rho_j(\boldsymbol{y})\frac{\partial\mu}{\partial y_j}(\boldsymbol{y}), \tag{2.19}$$
$$V_k\boldsymbol{\rho}(\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\mathring{V}_k\boldsymbol{\rho}(\boldsymbol{y}),\qquad W_k\boldsymbol{\rho}(\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\mathring{W}_k(\mu\boldsymbol{\rho})(\boldsymbol{y}), \tag{2.20}$$
$$\mathcal{V}_k\boldsymbol{\rho}(\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\mathring{\mathcal{V}}_k\boldsymbol{\rho}(\boldsymbol{y}),\qquad \mathcal{W}_k\boldsymbol{\rho}(\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\mathring{\mathcal{W}}_k(\mu\boldsymbol{\rho})(\boldsymbol{y}), \tag{2.21}$$
$$\mathcal{P}_j\rho_j(\boldsymbol{y}) = \mathring{\mathcal{P}}_j\rho_j(\boldsymbol{y}),\qquad \Pi_j\rho_j(\boldsymbol{y}) = \mathring{\Pi}_j(\mu\rho_j)(\boldsymbol{y}), \tag{2.22}$$
$$\mathcal{W}'_k\boldsymbol{\rho} = \mathring{\mathcal{W}}'_k\boldsymbol{\rho} - \left(\frac{\partial_i\mu}{\mu}\mathring{\mathcal{V}}_k\boldsymbol{\rho} + \frac{\partial_k\mu}{\mu}\mathring{\mathcal{V}}_i\boldsymbol{\rho} - \frac{2}{3}\delta^k_i\frac{\partial_j\mu}{\mu}\mathring{\mathcal{V}}_j\boldsymbol{\rho}\right)n_i, \tag{2.23}$$
$$\widehat{\mathcal{L}}_k(\boldsymbol{\tau}) := \mathring{\mathcal{L}}_k(\mu\boldsymbol{\tau}). \tag{2.24}$$

Note that the velocity potentials defined above are *not incompressible for the variable coefficient* $\mu(\boldsymbol{y})$. The following assertions of this section are well-known for the constant coefficient case, see e.g. [KoWe06, HsWe08]. Then, by relations (2.16)-(2.23) we obtain their counterparts for the variable-coefficient case.

Theorem 2.2. *The following operators are continuous:*

$$\mathcal{U}_{ik} : \widetilde{\boldsymbol{H}}^s(\Omega) \to \boldsymbol{H}^{s+2}(\Omega), \quad s \in \mathbb{R}, \tag{2.25}$$

$$\mathcal{U}_{ik} : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s+2}(\Omega), \quad s > -1/2, \tag{2.26}$$

$$\mathcal{R}_{ik} : \widetilde{\boldsymbol{H}}^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{2.27}$$

$$\mathcal{R}_{ik} : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), \quad s > -1/2, \tag{2.28}$$

$$\mathcal{Q}_k : \widetilde{\boldsymbol{H}}^s(\Omega) \to H^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{2.29}$$

$$\mathcal{Q}_k : \boldsymbol{H}^s(\Omega) \to H^{s+1}(\Omega), \quad s > -1/2, \tag{2.30}$$

$$\mathcal{R}^\bullet_k : \widetilde{\boldsymbol{H}}^s(\Omega) \to H^s(\Omega), \quad s \in \mathbb{R}. \tag{2.31}$$

$$\mathcal{R}^\bullet_k : \boldsymbol{H}^s(\Omega) \to H^s(\Omega), \quad s > -1/2. \tag{2.32}$$

Proof. Since the surface S is infinitely differentiable, the operators $\boldsymbol{\mathcal{U}}$ and $\mathcal{Q}$ are respectively pseudodifferential operators of order -2 and -1, see [HsWe08, Lemma 5.6.6. and Section 9.1.3]. Then, the continuity of the operators $\boldsymbol{\mathcal{U}}$ and $\mathcal{Q}$ from the 'tilde spaces' immediately follows by virtue of the mapping properties of the pseudodifferential operators (see, e.g. [Es81, McPr86]). Alternatively, these mapping properties are well studied for the constant coefficient case, i.e. operators $\mathring{\boldsymbol{\mathcal{U}}}$ and $\mathring{\mathcal{Q}}$, see [HsWe08, Lemma 5.6.6]. Consequently, the respective mapping properties for the remainder operators (2.27) and (2.31) immediately follow by considering the relation (2.17).

For the remaining part of the proof, we shall assume that $s \in (-1/2, 1/2)$. In this case, $H^s(\Omega) = \widetilde{H}^s(\Omega)$. Hence, the continuity of the operator (2.26) immediately follows from the continuity of (2.25).

Let us consider now that $s \in \left(1/2, \frac{3}{2}\right)$. Then, let $\boldsymbol{g} = (g_1, g_2, g_3)$, $\boldsymbol{g} \in \boldsymbol{H}^s(\Omega)$. It is well known that $\partial_j g_i \in H^{s-1}(\Omega)$ and that $\gamma^+\boldsymbol{g} \in \boldsymbol{H}^{s-1/2}(S)$ due to the continuity of the ∂_j operator and the trace theorem. Consequently, it is possible to use the representation obtained by integrating by parts, (see [CMN09, Theorem 3.8])

$$\partial_j \mathring{\mathcal{U}}_{ik} g_k = \mathring{\mathcal{U}}_{ik}(\partial_j g_k) + \mathring{V}_{ik}(\gamma^+ g_k n_j), \qquad i,j,k \in \{1,2,3\} \tag{2.33}$$

where n_j denotes the components of the normal vector to the surface S directed outwards the domain.

Keeping in mind the mapping properties V_{ik} and $\mathcal{U}_{ik}$, provided by Theorems 2.2 and 2.5, we can deduce that $\partial_j \mathring{\mathcal{U}}_{ik} g_k \in H^{s+1}(\Omega)$ is continuous for $j \in \{1,2,3\}$. Consequently, the continuity of the operator (2.26) immediately follows from relations (2.16) and (2.20), for $s \in (1/2, 3/2)$.

Furthermore, one can prove by induction on $k \in \mathbb{N}$, using the representation provided by the identity (2.33) and the fact that the operator (2.26) is continuous for $s \in (-1/2, 1/2)$, that the operator (2.26) is also continuous for $s \in (k-1/2, k+1/2)$. The continuity of the operator (2.26) for the cases $s = k + 1/2$ is proven by applying the theory of interpolation of Bessel potential spaces (see, e.g. [Tr78, Chapter 4]).

The continuity of the operator (2.30) can be proven following a similar argument.

Consequently, the respective mapping properties for the remainder operators (2.28) and (2.32) immediately follow from the continuity of the operators (2.30), (2.26) and the relation (2.17). □

The following corollary reflects the mapping property of the vector operator $\mathring{\boldsymbol{\mathcal{Q}}}$ which transforms a scalar function into a vector as opposed as the scalar operator $\mathring{\mathcal{Q}}$, which transforms a vector function into a scalar function, whose mapping properties are already well known, see e.g. [HsWe08, Lemma 5.6.6.] for the constant coefficient case and presented in the previous theorem for the variable coefficient case.

Corollary 2.3. *The following operators are continuous*

$$\mathring{\mathcal{Q}}_k : \widetilde{H}^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{2.34}$$

$$\mathring{\mathcal{Q}}_k : H^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), \quad s > -1/2. \tag{2.35}$$

Proof. Let us consider $\phi \in \widetilde{H}^s(\Omega)$. We denote by

$$E_\Delta(x,y) = \frac{-1}{4\pi|x-y|},$$

the fundamental solution of the three-dimensional Laplace equation. Note the following property

$$\mathring{q}^k = \frac{\partial E_\Delta}{\partial x_k} = -\frac{\partial E_\Delta}{\partial y_k}. \tag{2.36}$$

The newtonian volume potential for the Laplace equation is defined as

$$P_\Delta \phi(y) = \int_\Omega E_\Delta(x,y)\,\phi(x)\,dx, \tag{2.37}$$

and solves the Poisson equation $\Delta\omega = \phi$ in Ω. It is well known that $\mathcal{P}_\Delta$ has the following mapping properties, see [CMN09, Theorem 3.8]:

$$\mathcal{P}_\Delta : \widetilde{H}^s(\Omega) \longrightarrow H^{s+2}(\Omega), \;\; s \in \mathbb{R}, \tag{2.38}$$

$$\mathcal{P}_\Delta : H^s(\Omega) \longrightarrow H^{s+2}(\Omega), \;\; s > \frac{-1}{2}. \tag{2.39}$$

Let us take into account the relation (2.36) to deduce,

$$\begin{aligned}\mathring{\mathcal{Q}}_k\phi &= \int_\Omega \mathring{q}^k(x,y)\phi(x)\,dx = \int_\Omega \frac{\partial E_\Delta}{\partial x_k}(x,y)\phi(x)\,dx \\ &= -\frac{\partial}{\partial y_k}\int_\Omega E_\Delta(x,y)\phi(x)\,dx = -\frac{\partial \mathcal{P}_\Delta\phi}{\partial y_k}.\end{aligned}$$

By virtue of the mapping properties (2.38) and (2.39), $\mathcal{P}_\Delta\phi \in H^{s+2}(\Omega)$ and hence

$$-\frac{\partial(\mathcal{P}_\Delta\phi)}{\partial y_k} \in H^{s+1}(\Omega),$$

from where it follows the result. □

Theorem 2.4. *The following operators, with $s > 1/2$,*

$$\mathcal{R}_{ik} : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^s(\Omega), \qquad \mathcal{R}^\bullet_k : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s-1}(\Omega),$$

$$\gamma^+\mathcal{R}_{ik} : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s-1/2}(S), \qquad T^\pm_{ik}(\mathcal{R}^\bullet, \mathcal{R}) : \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \to \boldsymbol{H}^{-1/2}(S)$$

are compact.

Proof. The proof of the compactness for the operators $\mathcal{R}_{ik}, \gamma^+\mathcal{R}_{ik}$ and $\mathcal{R}^\bullet_k$ immediately follows from Theorem 2.2 and the trace theorem along with the Rellich compact embedding theorem. To prove the compactness of the operator $T^\pm_{ik}(\mathcal{R}^\bullet, \mathcal{R})$ we consider a function $\boldsymbol{g} \in \boldsymbol{H}^1(\Omega)$. Then, $(\mathcal{R}^\bullet\boldsymbol{g}, \mathcal{R}\boldsymbol{g}) \in H^1(\Omega) \times \boldsymbol{H}^2(\Omega)$ and hence, $(\mathcal{R}^\bullet\boldsymbol{g}, \mathcal{R}\boldsymbol{g}) \in \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$.

The operator $\boldsymbol{T}^\pm$ is the composite of a differential operator, of order 1 with respect to the first variable and of order 2 with respect to the second variable, and the trace operator $\boldsymbol{\gamma}^\pm$ which reduces the regularity by 1/2 according to the Trace Theorem. Therefore, $T^\pm_{ik}(\mathcal{R}^\bullet\boldsymbol{g}, \mathcal{R}\boldsymbol{g}) \in \boldsymbol{H}^{1/2}(S)$. Then, the compactness follows from the Rellich compact embedding $\boldsymbol{H}^{1/2}(S) \subset \boldsymbol{H}^{-1/2}(S)$. □

The theorems in the remainder of this section are well known for the constant coefficient case, see e.g. [KoWe06, HsWe08]. Then by relations (2.16)-(2.23) we obtain their counterparts for the variable-coefficient case.

Theorem 2.5. *Let $s \in \mathbb{R}$. Let S_1 and S_2 be two non empty manifolds on S with smooth boundary ∂S_1 and ∂S_2, respectively. Then, the following operators are continuous:*

$$V_{ik} : \boldsymbol{H}^s(S) \to \boldsymbol{H}^{s+\frac{3}{2}}(\Omega), \qquad W_{ik} : \boldsymbol{H}^s(S) \to \boldsymbol{H}^{s+1/2}(\Omega),$$

$$\mathcal{V}_{ik} : \boldsymbol{H}^s(S) \to \boldsymbol{H}^{s+1}(S), \qquad \mathcal{W}_{ik} : \boldsymbol{H}^s(S) \to \boldsymbol{H}^{s+1}(S),$$

$$r_{S_2}\mathcal{V}_{ik} : \widetilde{\boldsymbol{H}}^s(S_1) \to \boldsymbol{H}^{s+1}(S_2), \qquad r_{S_2}\mathcal{W}_{ik} : \widetilde{\boldsymbol{H}}^s(S_1) \to \boldsymbol{H}^{s+1}(S_2),$$

$$\mathcal{L}^\pm_{ik} : \boldsymbol{H}^s(S) \to \boldsymbol{H}^{s-1}(S), \qquad \mathcal{W}'_{ik} : \boldsymbol{H}^s(S) \to \boldsymbol{H}^{s+1}(S).$$

Proof. The theorem follows from the relations (2.16)-(2.23) and the continuity of the counterpart operators for the constant coefficient case, see e.g. [KoWe06, HsWe08]. □

Theorem 2.6. *Let $s \in \mathbb{R}$, let S_1 and S_2 be two non-empty manifolds with smooth boundaries, ∂S_1 and ∂S_2, respectively. Then, the following operators are compact:*

$$r_{S_2}\mathcal{V}_{ik} : \widetilde{\boldsymbol{H}}^s(S_1) \longrightarrow \boldsymbol{H}^s(S_2),$$

$$r_{S_2}\mathcal{W}_{ik} : \widetilde{\boldsymbol{H}}^s(S_1) \longrightarrow \boldsymbol{H}^s(S_2),$$

$$r_{S_2}\mathcal{W}'_{ik} : \widetilde{\boldsymbol{H}}^s(S_1) \longrightarrow \boldsymbol{H}^s(S_2).$$

Proof. The proof follows by applying the Rellich compactness embedding to the mapping properties of the operators $\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{W}}$ and $\boldsymbol{\mathcal{W}'}$ given by Theorem 2.5. $\square$

Theorem 2.7. *The following operators are continuous*

$$(\mathcal{P}, \boldsymbol{V}) : \boldsymbol{H}^{-1/2}(S) \longrightarrow \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}), \tag{2.40}$$

$$(\Pi, \boldsymbol{W}) : \boldsymbol{H}^{1/2}(S) \longrightarrow \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}), \tag{2.41}$$

$$(\mathcal{Q}, \boldsymbol{\mathcal{U}}) : \boldsymbol{L}^{2}(\Omega) \longrightarrow \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}), \tag{2.42}$$

$$(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}}) : \boldsymbol{H}^{1}(\Omega) \longrightarrow \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}), \tag{2.43}$$

$$(\frac{4\mu}{3} I, \boldsymbol{Q}) : L^{2}(\Omega) \longrightarrow \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}). \tag{2.44}$$

Proof. To prove that an arbitrary pair $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$, we need to see that $(p, \boldsymbol{v}) \in L^2(\Omega) \times \boldsymbol{H}^1(\Omega)$ and $\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) \in \boldsymbol{L}^2(\Omega)$.

By expanding the operator $\mathcal{A}_j(\boldsymbol{y}; p, \boldsymbol{v})$

$$\mathcal{A}_j(\boldsymbol{y}; p, \boldsymbol{v}) = \mathring{\mathcal{A}}_j(\boldsymbol{y}; p, \mu\boldsymbol{v}) - \frac{\partial}{\partial y_i}\left[v_j \frac{\partial \mu}{\partial y_i} + v_i \frac{\partial \mu}{\partial y_j} - \frac{2}{3}\delta_i^j v_l \frac{\partial \mu}{\partial y_l}\right], \tag{2.45}$$

we can see that if $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, then the second them in (2.45) belongs to $\boldsymbol{L}^2(\Omega)$. Therefore, we only need to check that $\mathring{\mathcal{A}}_j(\boldsymbol{y}; p, \mu\boldsymbol{v}) \in L^2(\Omega)$.

We will use this argument in what follows. First, let us prove the corresponding mapping property for the pair the pair (2.40). Let $\boldsymbol{\Psi} \in \boldsymbol{H}^{-1/2}(S)$. Then, $(\mathcal{P}\boldsymbol{\Psi}, \boldsymbol{\mathcal{V}}\boldsymbol{\Psi}) \in L^2(\Omega) \times \boldsymbol{H}^1(\Omega)$ by virtue of Theorems 2.5 and 2.11. Now, $\mathring{\mathcal{A}}_j(\mathcal{P}\boldsymbol{\Psi}, \mu\boldsymbol{V}\boldsymbol{\Psi}) = \mathring{\mathcal{A}}_j(\mathring{\mathcal{P}}\boldsymbol{\Psi}, \mathring{\boldsymbol{V}}\boldsymbol{\Psi})$ by applying relations (2.20) and (2.22). As $(\mathring{\mathcal{P}}, \mathring{\boldsymbol{V}})$ is the single layer potential for the Stokes operator with constant viscosity $\mu = 1$, we obtain $\mathring{\mathcal{A}}_j(\mathring{\mathcal{P}}\boldsymbol{\Psi}, \mathring{\boldsymbol{V}}\boldsymbol{\Psi}) = 0$, what completes the proof for the pair (2.40).

Let us now prove it for the operator (2.41). Let $\boldsymbol{\Phi} \in \boldsymbol{H}^{1/2}(S)$. By virtue of Theorems 2.5 and 2.11, $(\Pi\boldsymbol{\Phi}, \boldsymbol{W}\boldsymbol{\Phi}) \in L^2(\Omega) \times \boldsymbol{H}^1(\Omega)$. Moreover, by applying relations (2.20) and (2.22) we deduce $\mathring{\mathcal{A}}_j(\Pi\boldsymbol{\Phi}, \mu\boldsymbol{W}\boldsymbol{\Phi}) = \mathring{\mathcal{A}}_j(\mathring{\Pi}(\mu\boldsymbol{\Phi}), \mathring{\boldsymbol{W}}(\mu\boldsymbol{\Phi})) = 0$, since $(\mathring{\Pi}, \mathring{\boldsymbol{W}})$ is the double layer potential for the Stokes operator with constant viscosity $\mu = 1$, which completes the proof for the operator (2.41).

For the operator (2.42), we follow again a similar argumet. Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, taking into account the mapping properties of the volume potentials, see Theorem 2.2 and relation (2.16), we deduce that $\mathring{\mathcal{A}}_j(\mathcal{Q}\boldsymbol{f}, \mu\boldsymbol{\mathcal{U}}\boldsymbol{f}) = \mathring{\mathcal{A}}_j(\mathring{\mathcal{Q}}\boldsymbol{f}, \mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{f}) = \boldsymbol{f}$ since $(\mathring{\mathcal{Q}}, \mathring{\boldsymbol{\mathcal{U}}})$, what completes the proof for the operator (2.42).

In the case of the operator (2.43), the situation is easier due to the extra regularity. Let $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, then $(\mathcal{R}^\bullet\boldsymbol{v}, \boldsymbol{\mathcal{R}}\boldsymbol{v}) \in H^1(\Omega) \times \boldsymbol{H}^2(\Omega)$ by virtue of Theorem 2.2. Hence, $\boldsymbol{\mathcal{A}}(\mathcal{R}^\bullet\boldsymbol{v}, \boldsymbol{\mathcal{R}}\boldsymbol{v}) \in \boldsymbol{L}^2(\Omega)$.

Let us prove the corresponding property for the operator (2.44). Let $g \in L^2(\Omega)$, then by virtue of Corollary 2.3, the pair $(\frac{4\mu}{3}g, \boldsymbol{\mathcal{Q}}g) \in L^2(\Omega) \times \boldsymbol{H}^1(\Omega)$. Now, applying the relation (2.18), we obtain

$$\begin{aligned}
\mathring{\mathcal{A}}_j(\frac{4}{3}g\mu, \mathring{\boldsymbol{\mathcal{Q}}}(\mu g)) &= \frac{\partial}{\partial y_i}\left(\frac{\partial \mathring{\mathcal{Q}}_j(\mu g)}{\partial y_i} + \frac{\partial \mathring{\mathcal{Q}}_i(\mu g)}{\partial y_j} - \frac{2}{3}\delta_i^j div \boldsymbol{\mathcal{Q}}(\mu g)\right) - \frac{4}{3}\frac{\partial(\mu g)}{\partial y_j} \\
&= \frac{\partial}{\partial y_i}\left(2\frac{\partial \mathring{\mathcal{Q}}_i(\mu g)}{\partial y_j} - \frac{2}{3}\delta_i^j(\mu g)\right) - \frac{4}{3}\frac{\partial(\mu g)}{\partial y_j} \\
&= 2\frac{\partial}{\partial y_j}\left(\frac{\partial \mathring{\mathcal{Q}}_i(\mu g)}{\partial y_i}\right) - 2\frac{\partial(\mu g)}{\partial y_j} = 0,
\end{aligned} \tag{2.46}$$

since, see [Bo04, Appendix A1],

$$\frac{\partial \mathring{\mathcal{Q}}_i(\mu g)}{\partial y_i} = \mu g,$$

which completes the proof of the theorem. □

Theorem 2.8. *If* $\boldsymbol{\tau} \in \boldsymbol{H}^{1/2}(S)$, $\boldsymbol{\rho} \in \boldsymbol{H}^{-1/2}(S)$, *then the following jump relations hold:*

$$\gamma^{\pm}V_k\boldsymbol{\rho} = \mathcal{V}_k\boldsymbol{\rho}, \qquad \gamma^{\pm}W_k\boldsymbol{\tau} = \mp\frac{1}{2}\tau_k + \mathcal{W}_k\boldsymbol{\tau}$$

$$T_k^{\pm}(\mathcal{P}\boldsymbol{\rho}, \boldsymbol{V}\boldsymbol{\rho}) = \pm 1/2\rho_k + \mathcal{W}'{}_k\boldsymbol{\rho}.$$

Proof. The proof of the theorem directly follows from relations (2.20) and (2.23) and the analogous jump properties for the counterparts of the operators for the constant coefficient case of $\mu = 1$, see [HsWe08, Lemma 5.6.5]. □

Theorem 2.9. *Let $\boldsymbol{\tau} \in \boldsymbol{H}^{1/2}(S)$. Then, the following jump relation holds:*

$$(\mathcal{L}_k^{\pm} - \widehat{\mathcal{L}}_k)\boldsymbol{\tau} =$$
$$\gamma^{\pm}\left(\mu\left[\partial_i\left(\frac{1}{\mu}\right)\mathring{W}_k(\mu\boldsymbol{\tau}) + \partial_k\left(\frac{1}{\mu}\right)\mathring{W}_i(\mu\boldsymbol{\tau}) - \frac{2}{3}\delta_i^k\partial_j\left(\frac{1}{\mu}\right)\mathring{W}_j(\mu\boldsymbol{\tau})\right]\right)n_i. \tag{2.47}$$

where

$$\widehat{\mathcal{L}}_k(\boldsymbol{\tau}) := \mathring{\mathcal{L}}_k(\mu\boldsymbol{\tau}).$$

Proof. The pair of operators $(\Pi, \boldsymbol{W})$ defines a continuous mapping by virtue of Theorem 2.7. In addition, the co-normal derivative is a continuous operator since it is the composition of a differential operator σ_{ik} and the trace operator, which is continuous by virtue of the Trace Theorem. Consequently, it is only necessary to prove the theorem for functions of $\mathcal{C}^{\infty}(S)$, since this set is dense in $\boldsymbol{H}^{1/2}(S)$. Therefore, let $\tau_i \in \mathcal{C}^{\infty}(S)$,

$$\begin{aligned}
&\mathcal{L}_{ik}^{\pm}\tau_i := T_k^{\pm}(\Pi_i\tau_i, W_{ik}\tau_i) = \gamma^{\pm}\sigma_{ik}(\Pi_i\tau_i, W_{ik}\tau_i)n_i \\
&= \gamma^{\pm}\sigma_{ik}(\mathring{\Pi}_i(\mu\tau_i), \frac{1}{\mu}\mathring{W}_{ik}(\mu\tau_i))n_i \\
&= \gamma^{\pm}\mathring{\sigma}_{ik}(\mathring{\Pi}_i(\mu\tau_i), \mathring{W}_{ik}(\mu\tau_i))n_i \\
&+ \gamma^{\pm}\left(\mu\left[\partial_i\left(\frac{1}{\mu}\right)\mathring{W}_k(\mu\boldsymbol{\tau}) + \partial_k\left(\frac{1}{\mu}\right)\mathring{W}_i(\mu\boldsymbol{\tau}) - \frac{2}{3}\delta_i^k\partial_j\left(\frac{1}{\mu}\right)\mathring{W}_j(\mu\boldsymbol{\tau})\right]\right)n_i \\
&= \mathring{\mathcal{L}}_{ik}^{\pm}(\mu\tau_k) \\
&+ \gamma^{\pm}\left(\mu\left[\partial_i\left(\frac{1}{\mu}\right)\mathring{W}_k(\mu\boldsymbol{\tau}) + \partial_k\left(\frac{1}{\mu}\right)\mathring{W}_i(\mu\boldsymbol{\tau}) - \frac{2}{3}\delta_i^k\partial_j\left(\frac{1}{\mu}\right)\mathring{W}_j(\mu\boldsymbol{\tau})\right]\right)n_i.
\end{aligned}$$

Now, by virtue of the Lyapunov-Tauber theorem, $\mathring{\mathcal{L}}_{ik}^{+}(\mu\tau_k) = \mathring{\mathcal{L}}_{ik}^{-}(\mu\tau_k)$. Hence we can define:

$$\widehat{\mathcal{L}}_k\boldsymbol{\tau} := \mathring{\mathcal{L}}_k^{+}(\mu\boldsymbol{\tau}) = \mathring{\mathcal{L}}_k^{-}(\mu\boldsymbol{\tau}).$$

From which it follows (2.47). □

Corollary 2.10. *Let S_1 be a non empty submanifold of S with smooth boundary. Then, the operators*

$$r_{S_1}\widehat{\boldsymbol{\mathcal{L}}} : \widetilde{\boldsymbol{H}}^{1/2}(S_1) \longrightarrow \boldsymbol{H}^{-1/2}(S),$$
$$r_{S_1}(\boldsymbol{\mathcal{L}}^{\pm} - \widehat{\boldsymbol{\mathcal{L}}}) : \widetilde{\boldsymbol{H}}^{1/2}(S_1) \longrightarrow \boldsymbol{H}^{1/2}(S),$$

are continuous and the operators

$$r_{S_1}(\boldsymbol{\mathcal{L}}^{\pm} - \widehat{\boldsymbol{\mathcal{L}}}) : \widetilde{\boldsymbol{H}}^{1/2}(S_1) \longrightarrow \boldsymbol{H}^{-1/2}(S),$$

are compact.

Proof. The continuity of the operators $r_{S_1}\widehat{\boldsymbol{\mathcal{L}}}$ and $r_{S_1}(\boldsymbol{\mathcal{L}}^{\pm} - \widehat{\boldsymbol{\mathcal{L}}})$ and follows from Theorems 2.9 and 2.5. The compactness of $r_{S_1}(\boldsymbol{\mathcal{L}}^{\pm} - \widehat{\boldsymbol{\mathcal{L}}})$ directly follows from the compact embedding $\boldsymbol{H}^{1/2}(S_1) \subset \boldsymbol{H}^{-1/2}(S_1)$. □

Theorem 2.11. *The following pressure surface potential operators are continuous:*

$$\mathcal{P}_k : H^{s-\frac{3}{2}}(S) \to H^{s-1}(\Omega), \quad s \in \mathbb{R}, \tag{2.48}$$

$$\Pi_k : H^{s-1/2}(S) \to H^{s-1}(\Omega), \quad s \in \mathbb{R}. \tag{2.49}$$

Proof. The proof follows from relations (2.22) and the analogous result [HsWe08, Lemma 5.6.6] for the potentials $\mathring{\mathcal{P}}$ and $\mathring{\Pi}$. □

2.5 The Third Green Identities

Let $B(\boldsymbol{y}, \epsilon) \subset \Omega$ be a ball with a small enough radius ϵ and centre $\boldsymbol{y} \in \Omega$. In this new domain, the integrands of the operators $\boldsymbol{\mathcal{R}}$ and $\mathcal{R}^{\bullet}$ belong to $L^2(\Omega \smallsetminus B(\boldsymbol{y}, \epsilon))$. In addition, the parametrix $(q^k, \boldsymbol{u}^k) \in \boldsymbol{H}^{1,0}(\Omega \smallsetminus B(\boldsymbol{y}, \epsilon); \boldsymbol{\mathcal{A}})$ since we have removed the singularity. Therefore, we can apply the second Green identity (2.6) in the domain $\Omega \setminus B(\boldsymbol{y}, \epsilon)$ to any $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ and to the parametrix $(q^k, \boldsymbol{u}^k)$, keeping in mind the relation (2.12) and applying the standard limiting procedures, i.e., $\epsilon \to 0$, see, e.g. [Mr70], we obtain

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{T}^{+}(p, \boldsymbol{v}) + \boldsymbol{W}\boldsymbol{\gamma}^{+}\boldsymbol{v} = \boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) + \boldsymbol{\mathcal{Q}}(\operatorname{div}(\boldsymbol{v})), \quad \text{in } \Omega. \tag{2.50}$$

Theorem 2.12. *An integral representation formula for the pressure p is given by*

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \mathcal{P}\boldsymbol{T}(p,\boldsymbol{v}) + \Pi\boldsymbol{\gamma}^{+}\boldsymbol{v} = \mathring{\mathcal{Q}}\boldsymbol{\mathcal{A}}(p,\boldsymbol{v}) + \frac{4\mu}{3}div\boldsymbol{v}, \quad in\ \Omega. \tag{2.51}$$

Proof. Multiplying equation (2.1) by the fundamental pressure vector $\mathring{q}_j$, and integrating the result over the domain Ω we obtain

$$\int_\Omega \mathring{q}_j \left[\frac{\partial}{\partial x_i}\left(\mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\delta_i^j \mathrm{div}\boldsymbol{v}\right)\right)\right] dx - \int_\Omega \frac{\partial p}{\partial x_j}\mathring{q}_j dx = \int_\Omega \mathcal{A}_j(p,\boldsymbol{v})\mathring{q}_j\, dx. \tag{2.52}$$

Applying the first Green identity to the first term

$$\begin{aligned}
&\left\langle \mathring{q}_j\,,\, \frac{\partial}{\partial x_i}\left(\mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\delta_i^j \mathrm{div}\boldsymbol{v}\right)\right)\right\rangle_\Omega = \\
&-\left\langle \frac{\partial \mathring{q}_j}{\partial x_i}\,,\, \mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\delta_i^j \mathrm{div}\boldsymbol{v}\right)\right\rangle_\Omega \\
&+\left\langle \mathring{q}_j\,,\, \mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\delta_i^j \mathrm{div}\boldsymbol{v}\right) n_j\right\rangle_S,
\end{aligned} \tag{2.53}$$

and also in the second term

$$\left\langle \mathring{q}_j\,,\, \frac{\partial p}{\partial x_j}\right\rangle_\Omega = -\left\langle \frac{\partial \mathring{q}_j}{\partial x_j}\,,\, p\right\rangle_\Omega + \left\langle \mathring{q}_j\,,\, \frac{\partial p}{\partial x_j} n_j\right\rangle_S. \tag{2.54}$$

The duality brackets $<,>$. in (2.53) and in the remaining part of the proof, emphasise the fact that the kernel of the integral in the second term in (2.53) is strongly singular and hence the integral should be understood in the distribution sense. This integral exists since $\mu \in \mathcal{C}^\infty(\Omega)$ and the remaining part of the integrand belongs to $L^2(\Omega)$. Consequently, the convergence of this integral is guaranteed by the density of $\mathcal{D}(\Omega)$ in Sobolev spaces.

Substituting (2.53) and (2.54) into (2.52) and rearranging terms we get

$$\begin{aligned}
&\left\langle \mathring{q}_j\,,\, \frac{\partial}{\partial x_i}\left(\mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\delta_i^j \mathrm{div}\boldsymbol{v}\right)\right)\right\rangle_\Omega - \left\langle \mathring{q}_j\,,\, \frac{\partial p}{\partial x_j}\right\rangle_\Omega = \\
&-\left\langle \frac{\partial \mathring{q}_j}{\partial x_i}\,,\, \mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\delta_i^j \mathrm{div}\boldsymbol{v}\right)\right\rangle_\Omega \\
&+\left\langle \mu\,,\, \mathring{q}_j\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\delta_i^j \mathrm{div}\boldsymbol{v}\right) n_j\right\rangle_S \\
&+\left\langle \frac{\partial \mathring{q}_j}{\partial x_j}\,,\, p\right\rangle_\Omega - \left\langle \mathring{q}_j\,,\, \frac{\partial p}{\partial x_j} n_j\right\rangle_S = \langle \mathring{q}_j\,,\, \mathcal{A}_j(p,\boldsymbol{v})\rangle_\Omega\,.
\end{aligned} \tag{2.55}$$

Grouping together the integral terms over S from (2.55) we obtain

$$\left\langle \mathring{q}_j \,,\, \mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\delta_i^j \mathrm{div}\boldsymbol{v}\right) n_j \right\rangle_S - \left\langle \mathring{q}_j \,,\, \frac{\partial p}{\partial x_j} n_j \right\rangle_S = \langle \mathring{q}_j \,,\, T_j(p,\boldsymbol{v})\rangle_S \,. \tag{2.56}$$

In addition, from [Bo04, Appendix 3], taking into account that the integration is in the distribution sense

$$\left\langle \frac{\partial \mathring{q}_j}{\partial x_j} \,,\, p \right\rangle_\Omega = \langle \delta, p\rangle = p. \tag{2.57}$$

Let us now simplify the first term in the right hand side of (2.55)

$$\begin{aligned} &\left\langle \frac{\partial \mathring{q}_j}{\partial x_i} \,,\, \mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\delta_i^j \mathrm{div}\boldsymbol{v}\right) \right\rangle = \\ &\left\langle \frac{\partial \mathring{q}_j}{\partial x_i} \,,\, \mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right) \right\rangle_\Omega - \left\langle \frac{\partial \mathring{q}_j}{\partial x_j} \,,\, \frac{2\mu}{3}\mathrm{div}\boldsymbol{v} \right\rangle_\Omega . \end{aligned} \tag{2.58}$$

The first term in (2.58) can be simplified using the symmetry $\dfrac{\partial \mathring{q}_j}{\partial x_i} = \dfrac{\partial \mathring{q}^i}{\partial x_j}$ and the second term can also be simplified in a similar manner as in (2.57). Hence,

$$\left\langle \frac{\partial \mathring{q}_j}{\partial x_i} \,,\, \mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\delta_i^j \mathrm{div}\boldsymbol{v}\right) \right\rangle_\Omega = 2\left\langle \frac{\partial \mathring{q}_j}{\partial x_i} \,,\, \mu\frac{\partial v_i}{\partial x_j} \right\rangle_\Omega - \frac{2\mu}{3}\mathrm{div}\boldsymbol{v} \tag{2.59}$$

Applying the product rule and the first Green identity to the first term in (2.59), we obtain

$$\begin{aligned} \left\langle \frac{\partial \mathring{q}_j}{\partial x_i} \,,\, \mu\frac{\partial v_j}{\partial x_i} \right\rangle_\Omega &= \left\langle dx \,,\, \frac{\partial}{\partial x_i}\left(\mu\frac{\partial \mathring{q}_j}{\partial x_i} v_j\right) \right\rangle_\Omega - \left\langle \frac{\partial}{\partial x_i}\left(\mu\frac{\partial \mathring{q}_j}{\partial x_i}\right) \,,\, v_j \right\rangle_\Omega \\ &= \left\langle \frac{\partial \mathring{q}_j}{\partial x_i} \,,\, \mu v_j n_i \right\rangle_S - \left\langle \frac{\partial \mathring{q}_j}{\partial x_i} \,,\, v_j \frac{\partial \mu}{\partial x_i} \right\rangle_\Omega - \left\langle \frac{\partial^2 \mathring{q}_j}{\partial x_i^2} \,,\, v_j \mu \right\rangle_\Omega . \end{aligned} \tag{2.60}$$

The last term in (2.60) can be simplified further by taking into consideration the harmonic properties of $\mathring{q}^k$.

$$\left\langle \frac{\partial^2 \mathring{q}_j}{\partial x_i^2} \,,\, \mu v_j \right\rangle_\Omega = \left\langle \frac{\partial \delta}{\partial x_j} \,,\, \mu v_j \right\rangle_\Omega = -\frac{\partial(\mu v_j)}{\partial x_j} = -v_j\frac{\partial \mu}{\partial x_j} - \mu \mathrm{div}\boldsymbol{v}. \tag{2.61}$$

Let us now substitute backwards by plugging (2.61) into (2.60)

$$\begin{aligned} \left\langle \frac{\partial \mathring{q}_j}{\partial x_i} \,,\, \mu\frac{\partial v_j}{\partial x_i} \right\rangle_\Omega &= \left\langle \frac{\partial \mathring{q}_j}{\partial x_i} \,,\, \mu v_j n_i \right\rangle_S - \left\langle \frac{\partial \mathring{q}_j}{\partial x_i} \,,\, v_j \frac{\partial \mu}{\partial x_i} \right\rangle_\Omega - \left\langle \frac{\partial^2 \mathring{q}_j}{\partial x_i^2} \,,\, v_j \mu \right\rangle_\Omega \\ &= \left\langle \frac{\partial \mathring{q}_j}{\partial x_i} \,,\, \mu v_j n_i \right\rangle_S - \left\langle \frac{\partial \mathring{q}_j}{\partial x_i} \,,\, v_j \frac{\partial \mu}{\partial x_i} \right\rangle + v_j\frac{\partial \mu}{\partial x_j} + \mu \mathrm{div}\boldsymbol{v}. \end{aligned} \tag{2.62}$$

Now, plug (2.62) into (2.59),

$$\left\langle \frac{\partial \mathring{q}_j}{\partial x_i}\,,\, \mu\left(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}-\frac{2}{3}\delta_i^j \mathrm{div}\boldsymbol{v}\right)\right\rangle_\Omega =$$

$$2\left\langle \frac{\partial \mathring{q}_j}{\partial x_i}\,,\, \mu v_j n_i\right\rangle_S - 2\left\langle \frac{\partial \mathring{q}_j}{\partial x_i}\,,\, v_j\frac{\partial \mu}{\partial x_i}\right\rangle_\Omega + 2v_j\frac{\partial \mu}{\partial x_j} + 2\mu\mathrm{div}\boldsymbol{v} - \frac{2\mu}{3}\mathrm{div}\boldsymbol{v}. \tag{2.63}$$

Now, substitute (2.63) into (2.55), (2.57) into (2.56), and (2.56) into (2.55). As a result, we obtain

$$\left\langle \mathring{q}_j\,,\, \frac{\partial}{\partial x_i}\left(\mu\left(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}-\frac{2}{3}\delta_i^j \mathrm{div}\boldsymbol{v}\right)\right)\right\rangle_\Omega - \left\langle \frac{\partial p}{\partial x_j}\,,\, \mathring{q}_j\right\rangle_\Omega =$$

$$-\left\langle \frac{\partial \mathring{q}_j}{\partial x_i}\,,\, \mu\left(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}-\frac{2}{3}\delta_i^j \mathrm{div}\boldsymbol{v}\right)\right\rangle_\Omega$$

$$+\left\langle \mathring{q}_j\,,\, \mu\left(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}-\frac{2}{3}\delta_i^j \mathrm{div}\boldsymbol{v}\right)n_j\right\rangle_S$$

$$+\left\langle \frac{\partial \mathring{q}_j}{\partial x_j}\,,\, p\right\rangle_\Omega - \left\langle \mathring{q}_j\,,\, \frac{\partial p}{\partial x_j}n_j\right\rangle_S$$

$$= -2\left\langle \frac{\partial \mathring{q}_j}{\partial x_i}\,,\, \mu v_j n_i\right\rangle_S + 2\left\langle \frac{\partial \mathring{q}_j}{\partial x_i}\,,\, v_j\frac{\partial \mu}{\partial x_i}\right\rangle_\Omega - 2v_j\frac{\partial \mu}{\partial x_j} - \frac{4\mu}{3}\mathrm{div}\boldsymbol{v}$$

$$+ p + \langle \mathring{q}_j\,,\, T_j(p,\boldsymbol{v})\rangle_S = \langle \mathring{q}_j\,,\, \mathcal{A}_j(p,\boldsymbol{v})\,\rangle_\Omega\,.$$

Finally, rearranging terms and writing this expression in terms of the potential operators, we obtain the result (2.51). □

If the couple $(p,\boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ is a solution of the Stokes PDEs (2.7a)-(2.7b) with variable coefficient, then (2.50) and (2.51) give

$$p + \mathcal{R}^\bullet \boldsymbol{v} - \mathcal{P}\boldsymbol{T}(p,\boldsymbol{v}) + \Pi\boldsymbol{\gamma}^+\boldsymbol{v} = \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g \quad \text{in} \quad \Omega, \tag{2.64}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{T}^+(p,\boldsymbol{v}) + \boldsymbol{W}\boldsymbol{\gamma}^+\boldsymbol{v} = \boldsymbol{\mathcal{U}}\boldsymbol{f} + \boldsymbol{\mathcal{Q}}g \quad \text{in} \quad \Omega. \tag{2.65}$$

We will also need the trace and traction of the third Green identities for $(p,\boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ on S. We highlight that the traction operator is well defined applied to the third Green identities (2.64)-(2.65) by virtue of Theorem 2.7.

$$1/2\boldsymbol{\gamma}^+\boldsymbol{v} + \gamma^+\boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{\mathcal{V}}\boldsymbol{T}^+(p,\boldsymbol{v}) + \boldsymbol{\mathcal{W}}\gamma^+\boldsymbol{v} = \boldsymbol{\gamma}^+\boldsymbol{\mathcal{U}}\boldsymbol{f} + \gamma^+\boldsymbol{\mathcal{Q}}g, \tag{2.66}$$

$$1/2\boldsymbol{T}^+(p,\boldsymbol{v}) + \boldsymbol{T}^+(\mathcal{R}^\bullet,\boldsymbol{\mathcal{R}})\boldsymbol{v} - \boldsymbol{\mathcal{W}}'\boldsymbol{T}^+(p,\boldsymbol{v}) + \boldsymbol{\mathcal{L}}^+\gamma^+\boldsymbol{v} = \widetilde{\boldsymbol{T}}^+(g,\boldsymbol{f}) \tag{2.67}$$

where

$$\widetilde{\boldsymbol{T}}^{+}(g,\boldsymbol{f}) := \boldsymbol{T}^{+}(\mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g,\ \boldsymbol{\mathcal{U}}\boldsymbol{f} + \boldsymbol{\mathcal{Q}}g). \tag{2.68}$$

One can prove the following three assertions that are instrumental for proving the equivalence of the BDIES and the mixed PDE.

Theorem 2.13. *Let* $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$*,* $p \in L_2(\Omega)$*,* $g \in L_2(\Omega)$*,* $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$*,* $\boldsymbol{\Psi} \in \boldsymbol{H}^{-1/2}(S)$ *and* $\boldsymbol{\Phi} \in \boldsymbol{H}^{1/2}(S)$ *satisfy the equations*

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \mathcal{P}\boldsymbol{\Psi} + \Pi\boldsymbol{\Phi} = \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g \ \ in\ \Omega, \tag{2.69}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{\Psi} + \boldsymbol{W}\boldsymbol{\Phi} = \boldsymbol{\mathcal{U}}\boldsymbol{f} + \boldsymbol{\mathcal{Q}}g \ \ in\ \Omega. \tag{2.70}$$

Then $(p,\boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega,\boldsymbol{\mathcal{A}})$ *and solve the equations* $\boldsymbol{\mathcal{A}}(p,\boldsymbol{v}) = \boldsymbol{f}$ *and* $div(\boldsymbol{v}) = g$*. Moreover, the following relations hold true:*

$$\mathcal{P}(\boldsymbol{\Psi} - \boldsymbol{T}^{+}(p,\boldsymbol{v})) - \Pi(\boldsymbol{\Phi} - \gamma^{+}\boldsymbol{v}) = 0 \quad in\ \Omega, \tag{2.71}$$

$$\boldsymbol{V}(\boldsymbol{\Psi} - \boldsymbol{T}^{+}(p,\boldsymbol{v})) - \boldsymbol{W}(\boldsymbol{\Phi} - \gamma^{+}\boldsymbol{v}) = \boldsymbol{0} \quad in\ \Omega. \tag{2.72}$$

Proof. Firstly, the fact that $(p,\boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega,\boldsymbol{\mathcal{A}})$ is a direct consequence of the Theorem 2.7.

Secondly, let us prove that $(p,\boldsymbol{v})$ solve the PDE and $div(\boldsymbol{v}) = g$. Multiply equation (2.70) by μ and apply relations (2.16)-(2.18) along with relation (2.20) to obtain

$$\boldsymbol{v} = \mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{f} + \mathring{\boldsymbol{\mathcal{Q}}}(\mu g) - \mu\boldsymbol{\mathcal{R}}\boldsymbol{v} + \mathring{\boldsymbol{V}}\boldsymbol{\Psi} - \mathring{\boldsymbol{W}}(\mu\boldsymbol{\Phi}). \tag{2.73}$$

Apply the divergence operator to both sides of (2.73), taking into account relation (2.17) and the fact that the potentials $\mathring{\boldsymbol{\mathcal{U}}}, \mathring{\boldsymbol{V}}$, and $\mathring{\boldsymbol{W}}$ are divergence free. Hence, we obtain

$$\begin{aligned} div(\mu\boldsymbol{v}) &= div\left(\mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{f} + \mathring{\boldsymbol{\mathcal{Q}}}(\mu g) - \mu\boldsymbol{\mathcal{R}}\boldsymbol{v} + \mathring{\boldsymbol{V}}\boldsymbol{\Psi} - \mathring{\boldsymbol{W}}(\mu\boldsymbol{\Phi})\right) = \\ &= div\mathring{\boldsymbol{\mathcal{Q}}}(\mu g) - div(\mu\boldsymbol{\mathcal{R}}\boldsymbol{v}). \end{aligned} \tag{2.74}$$

To work out $div(\mu\boldsymbol{\mathcal{R}}\boldsymbol{v})$ we apply the relation of (2.17) and take into account the divergence free of the operators involved and the harmonic properties of the pressure newtonian potential as follows.

$$div(\mu \boldsymbol{\mathcal{R}} \boldsymbol{v}) = \frac{\partial(\mu \mathcal{R}_k \boldsymbol{v})}{\partial y_k} =$$

$$- \frac{\partial}{\partial y_k} \left(\frac{\partial}{\partial y_j} \mathring{\mathcal{U}}_{ki}(v_j \partial_i \mu) + \frac{\partial}{\partial y_i} \mathring{\mathcal{U}}_{kj}(v_j \partial_i \mu) - \mathring{\mathcal{Q}}_k(v_j \partial_j \mu) \right) =$$

$$\frac{\partial}{\partial y_k} \mathring{\mathcal{Q}}_k(v_j \partial_j \mu) = -\boldsymbol{v} \nabla \mu. \tag{2.75}$$

From (2.74) and (2.75), it immediately follows

$$div(\mu \boldsymbol{v}) = div \mathring{\boldsymbol{\mathcal{Q}}}(\mu g) - div(\mu \boldsymbol{\mathcal{R}} \boldsymbol{v}) = \mu g + \boldsymbol{v} \nabla \mu \Rightarrow \ div(\boldsymbol{v}) = g.$$

Further, to prove that $(p, \boldsymbol{v})$ is a solution of the PDE we use equations (2.64) and (2.65) which we can now use as we have proved that $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$. Then, substract these from equations (2.69) and (2.70) respectively to obtain

$$\Pi \boldsymbol{\Phi}^* - \mathcal{P} \boldsymbol{\Psi}^* = \mathcal{Q}(\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) - \boldsymbol{f}), \tag{2.76}$$

$$\boldsymbol{W} \boldsymbol{\Phi}^* - \boldsymbol{V} \boldsymbol{\Psi}^* = \boldsymbol{\mathcal{U}}(\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) - \boldsymbol{f}). \tag{2.77}$$

where $\boldsymbol{\Psi}^* := \boldsymbol{T}^+(p, \boldsymbol{v}) - \boldsymbol{\Psi}$, and $\boldsymbol{\Phi}^* = \boldsymbol{\gamma}^+ \boldsymbol{v} - \boldsymbol{\Phi}$.

After multiplying (2.77) by the variable viscosity coefficient and apply the potential relation (2.22) along with (2.16) and (2.20), to equations (2.76) and (2.77), we arrive at

$$\mathring{\Pi}(\mu \boldsymbol{\Phi}^*) - \mathring{P} \boldsymbol{\Psi}^* = \mathring{\mathcal{Q}}(\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) - \boldsymbol{f}),$$

$$\mathring{W}(\mu \boldsymbol{\Phi}^*) - \mathring{V} \boldsymbol{\Psi}^* = \mathring{\boldsymbol{\mathcal{U}}}(\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) - \boldsymbol{f}).$$

Applying the Stokes operator with $\mu = 1$, to these two previous equations, taking into account that the right hand side are the newtonian potentials for the velocity and pressure,

$$\mathring{\mathcal{A}}(\mathring{\Pi}(\mu \boldsymbol{\Phi}^*) - \mathring{P}(\boldsymbol{\Psi}^*), \mathring{W}(\mu \boldsymbol{\Phi}^*) - \mathring{V} \boldsymbol{\Psi}^*) = \mathring{\mathcal{A}}(\mathring{\mathcal{Q}}(\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) - \boldsymbol{f}), \mathring{\boldsymbol{\mathcal{U}}}(\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) - \boldsymbol{f}));$$

$$\Rightarrow \quad \boldsymbol{0} = \boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) - \boldsymbol{f} \Rightarrow \boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) = \boldsymbol{f}.$$

Hence, the pair $(p, \boldsymbol{v})$ solves the PDE.

Finally, the relations (2.72) and (2.71) follow from the substitution of

$$\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) - \boldsymbol{f} = \boldsymbol{0}$$

in (2.76) and (2.77). □

Lemma 2.14. *Let* $S = \overline{S}_1 \cup \overline{S}_2$*, where* S_1 *and* S_2 *are open non-empty non-intersecting simply connected submanifolds of* S *with infinitely smooth boundaries. Let* $\mathbf{\Psi}^* \in \widetilde{\boldsymbol{H}}^{-1/2}(S_1)$*,* $\mathbf{\Phi}^* \in \widetilde{\boldsymbol{H}}^{1/2}(S_2)$*. If*

$$\mathcal{P}(\mathbf{\Psi}^*) - \Pi(\mathbf{\Phi}^*) = \mathbf{0}, \qquad \boldsymbol{V}\mathbf{\Psi}^*(\boldsymbol{x}) - \boldsymbol{W}\mathbf{\Phi}^*(\boldsymbol{x}) = \mathbf{0}, \quad \text{in } \Omega, \tag{2.78}$$

then $\mathbf{\Psi}^* = \mathbf{0}$*, and* $\mathbf{\Phi}^* = \mathbf{0}$*, on* S*.*

Proof. Multiplying the second equation in (2.78) by μ and applying the relations (2.20), we obtain

$$\mathring{\boldsymbol{V}}\mathbf{\Psi}^*(\boldsymbol{x}) - \mathring{\boldsymbol{W}}(\mu\mathbf{\Phi}^*(\boldsymbol{x})) = \mathbf{0}. \tag{2.79}$$

Defining the functions $\widehat{\mathbf{\Psi}} = \mathbf{\Psi}^*$ and $\widehat{\mathbf{\Phi}} = \mu\mathbf{\Phi}^*$, we can write the previous equation (2.79) in terms of these new functions:

$$\mathring{\boldsymbol{V}}\widehat{\mathbf{\Psi}}(\boldsymbol{x}) - \mathring{\boldsymbol{W}}\widehat{\mathbf{\Phi}}(\boldsymbol{x}) = \mathbf{0} \quad \text{in } \Omega. \tag{2.80}$$

By keeping in mind the jump relations given in Theorem 2.8, we take the trace of the first equation in (2.80) restricted to S_1 and the traction of both equations in (2.80) restricted to S_2. Consequently, arrive at the following system of equations:

$$\begin{cases} r_{S_1}\mathring{\boldsymbol{\mathcal{V}}}\widehat{\mathbf{\Psi}}(\boldsymbol{x}) - r_{S_1}\mathring{\boldsymbol{\mathcal{W}}}\widehat{\mathbf{\Phi}}(\boldsymbol{x}) = \mathbf{0}, & \text{on } S_1, \\ r_{S_2}\mathring{\boldsymbol{\mathcal{W}}}'\widehat{\mathbf{\Psi}}(\boldsymbol{x}) - r_{S_2}\mathring{\boldsymbol{\mathcal{L}}}\widehat{\mathbf{\Phi}}(\boldsymbol{x}) = \mathbf{0}, & \text{on } S_2, \end{cases}$$

which can be written using matrix notation as follows:

$$\mathring{\mathcal{M}}\mathcal{X} = \mathbf{0},$$

where

$$\mathring{\mathcal{M}} = \begin{bmatrix} r_{S_1}\mathring{\boldsymbol{\mathcal{V}}} & -r_{S_1}\mathring{\boldsymbol{\mathcal{W}}} \\ r_{S_2}\mathring{\boldsymbol{\mathcal{W}}}' & -r_{S_2}\mathring{\boldsymbol{\mathcal{L}}} \end{bmatrix}, \ \mathcal{X} = \begin{bmatrix} \widehat{\mathbf{\Psi}} \\ \widehat{\mathbf{\Phi}} \end{bmatrix}. \tag{2.81}$$

Hence, it will suffice to prove that $\mathring{\mathcal{M}}$ is positive definite. This system has been studied previously in [KoWe06, Theorem 3.10] which concludes that the only possible solution is $\mathcal{X} = \mathbf{0}$. From where it follows the result. □

2.6 BDIES M11

We aim to obtain two different BDIES for mixed BVP (2.7). This is a well known procedure, see [CMN09], [MiPo15-I] and [Mi02] and further references therein.

To this end, let the functions $\mathbf{\Phi}_0 \in \boldsymbol{H}^{1/2}(S)$ and $\mathbf{\Psi}_0 \in \boldsymbol{H}^{-1/2}(S)$ be some continuations of the boundary functions $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(S_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(S_N)$ from (2.7c) and (2.7d). Let us now represent

$$\gamma^+\boldsymbol{v} = \mathbf{\Phi}_0 + \boldsymbol{\varphi}, \quad \boldsymbol{T}^+(p,\boldsymbol{v}) = \mathbf{\Psi}_0 + \boldsymbol{\psi} \text{ on } S, \tag{2.82}$$

where $\boldsymbol{\varphi} \in \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ and $\boldsymbol{\psi} \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D)$ are unknown boundary functions.

Let us now take equations (2.64) and (2.65) in the domain Ω and restrictions of equations (2.66) and (2.67) to the boundary parts S_D and S_N, respectively. Substituting there representations (2.82) and considering further the unknown boundary functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ as formally independent of (segregated from) the unknown domain functions p and $\boldsymbol{v}$, we obtain the following system (M11) consisting of four boundary-domain integral equations for four unknowns, $(p,\boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega,\boldsymbol{\mathcal{A}})$, $\boldsymbol{\varphi} \in \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ and $\boldsymbol{\psi} \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D)$:

$$p + \mathcal{R}^\bullet\boldsymbol{v} - \mathcal{P}\boldsymbol{\psi} + \Pi\boldsymbol{\varphi} = F_0, \quad \text{in } \Omega, \tag{2.83a}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{\psi} + \boldsymbol{W}\boldsymbol{\varphi} = \boldsymbol{F}, \quad \text{in } \Omega, \tag{2.83b}$$

$$r_{S_D}\gamma^+\boldsymbol{\mathcal{R}}\boldsymbol{v} - r_{S_D}\boldsymbol{\mathcal{V}}\boldsymbol{\psi} + r_{S_D}\boldsymbol{\mathcal{W}}\boldsymbol{\varphi} = r_{S_D}\gamma^+\boldsymbol{F} - \boldsymbol{\varphi}_0, \quad \text{on } S_D, \tag{2.83c}$$

$$r_{S_N}\boldsymbol{T}^+(\mathcal{R}^\bullet,\boldsymbol{\mathcal{R}})\boldsymbol{v} - r_{S_N}\boldsymbol{\mathcal{W}}'\boldsymbol{\psi} + r_{S_N}\boldsymbol{\mathcal{L}}^+\boldsymbol{\varphi} = r_{S_N}\boldsymbol{T}^+(F_0,\boldsymbol{F}) - \boldsymbol{\psi}_0, \quad \text{on } S_N, \tag{2.83d}$$

where

$$F_0 = \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4}{3}g\mu + \mathcal{P}\mathbf{\Psi}_0 - \Pi\mathbf{\Phi}_0, \quad \boldsymbol{F} = \boldsymbol{\mathcal{U}}\boldsymbol{f} + \boldsymbol{\mathcal{Q}}g + \boldsymbol{V}\mathbf{\Psi}_0 - \boldsymbol{W}\mathbf{\Phi}_0. \tag{2.84}$$

By virtue of Lemma 2.13, $(F_0,\boldsymbol{F}) \in \boldsymbol{H}^{1,0}(\Omega,\boldsymbol{\mathcal{A}})$ and hence $\boldsymbol{T}(F_0,\boldsymbol{F})$ is well defined.

We denote the right hand side of BDIE system (2.83) as

$$\mathcal{F}^{11}_* := [F_0,\mathcal{F}^{11}] = [F_0,\boldsymbol{F},r_{S_D}\gamma^+\boldsymbol{F} - \boldsymbol{\varphi}_0, r_{S_N}\boldsymbol{T}^+(F_0,\boldsymbol{F}) - \boldsymbol{\psi}_0]^\top, \tag{2.85}$$

which implies $\mathcal{F} \in \boldsymbol{H}^{1,0}(\Omega, \boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{-1/2}(S_N)$.

Note that BDIE system (2.83) can be split into the BDIE system (M11), of 3 vector equations (2.83b), (2.83c), (2.83d) for 3 vector unknowns, $\boldsymbol{v}$, $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$, and the scalar equation (2.83a) that can be used, after solving the system, to obtain the pressure, p. The system (M11) given by equations (2.83a)-(2.83d) can be written using matrix notation as

$$\mathcal{M}_*^{11}\mathcal{X} = \mathcal{F}_*^{11}, \tag{2.86}$$

where $\mathcal{X}$ represents the vector containing the unknowns of the system

$$\mathcal{X} = (p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})^\top \in L^2(\Omega) \times \boldsymbol{H}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$$

The matrix operator $\mathcal{M}_*^{11}$ is defined by

$$\mathcal{M}_*^{11} = \begin{bmatrix} I & \mathcal{R}^\bullet & -\mathcal{P} & \Pi \\ \mathbf{0} & \boldsymbol{I} + \boldsymbol{\mathcal{R}} & -\boldsymbol{V} & \boldsymbol{W} \\ \mathbf{0} & r_{S_D}\gamma^+\boldsymbol{\mathcal{R}} & -r_{S_D}\boldsymbol{\mathcal{V}} & r_{S_D}\boldsymbol{\mathcal{W}} \\ \mathbf{0} & r_{S_N}\boldsymbol{T}^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) & -r_{S_N}\boldsymbol{\mathcal{W}}' & r_{S_N}\boldsymbol{\mathcal{L}} \end{bmatrix}.$$

We note that the mapping properties of the operators involved in the matrix imply the continuity of the operator

$$\begin{aligned} \mathcal{M}_*^{11} : L^2(\Omega) \times \boldsymbol{H}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N) \\ \longrightarrow L^2(\Omega) \times \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{-1/2}(S_N). \end{aligned}$$

Remark 2.15. *The term $\mathcal{F}_*^{11} = 0$ if and only if $(\boldsymbol{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = \mathbf{0}$.*

Proof. It is evident that $(\boldsymbol{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = 0$ implies $\mathcal{F}_*^{11} = 0$. Hence, we shall only focus on proving $\mathcal{F}_*^{11} = 0 \;\Rightarrow\; (\boldsymbol{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = 0$. Taking into account how the terms $\boldsymbol{F}$ and F_0 are defined, see (2.84), considering that $F_0 = 0$ and $\boldsymbol{F} = 0$, we can deduce by applying Lemma 2.13 to equations (2.84) that $\boldsymbol{f} = 0$, $g = 0$ and that

$$\mathcal{P}\boldsymbol{\Psi}_0 - \Pi\boldsymbol{\Phi}_0 = 0,$$

$$\boldsymbol{V}\boldsymbol{\Psi}_0 - \boldsymbol{W}\boldsymbol{\Phi}_0 = 0.$$

In addition, as $F_0 = 0$ and $\boldsymbol{F} = 0$, we get that

$$r_{S_D}\gamma^+\boldsymbol{F} - r_{S_D}\boldsymbol{\Phi}_0 = 0, \;\Rightarrow\; r_{S_D}\boldsymbol{\Phi}_0 = 0,$$

$$r_{S_N}\boldsymbol{T}^+(F_0, \boldsymbol{F}) - r_{S_N}\boldsymbol{\Psi}_0 = 0 \;\Rightarrow\; r_{S_N}\boldsymbol{\Psi}_0 = 0.$$

Consequently, $\boldsymbol{\Psi}_0 \in \widetilde{H}^{-1/2}(S_N)$ and $\boldsymbol{\Phi}_0 \in \widetilde{H}^{1/2}(S_D)$. Therefore, the hypotheses of Lemma 2.14 are satisfied, we thus obtain that $\boldsymbol{\Psi}_0 = 0$ and $\boldsymbol{\Phi}_0 = 0$ on S. □

Theorem 2.16 (Equivalence Theorem)**.** *Let $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$, $g \in L_2(\Omega)$ and let $\boldsymbol{\Phi}_0 \in \boldsymbol{H}^{-1/2}(S)$ and $\boldsymbol{\Psi}_0 \in \boldsymbol{H}^{-1/2}(S)$ be some fixed extensions of $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(S_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(S_N)$ respectively.*

(i) If some $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ solve the mixed BVP (2.7)*, then*

$$(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N),$$

where

$$\boldsymbol{\varphi} = \gamma^+\boldsymbol{v} - \boldsymbol{\Phi}_0, \quad \boldsymbol{\psi} = \boldsymbol{T}^+(p, \boldsymbol{v}) - \boldsymbol{\Psi}_0 \quad \text{on } S, \tag{2.87}$$

solve BDIE system (2.83)*.*

(ii) If $(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ solve the BDIE system (2.83) *then $(p, \boldsymbol{v})$ solve mixed BVP* (2.7) *and $\boldsymbol{\psi}, \boldsymbol{\varphi}$ satisfy* (2.87)*.*

(iii) The BDIE system (2.83) *is uniquely solvable in $\boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$.*

Proof. (i) Let $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ be a solution of the BVP. Let us define the functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ by (2.87). By the BVP boundary conditions, $\gamma^+\boldsymbol{v} = \boldsymbol{\varphi}_0 = \boldsymbol{\Phi}_0$ on S_D and $\boldsymbol{T}^+(p, \boldsymbol{v}) = \boldsymbol{\psi}_0 = \boldsymbol{\Psi}_0$ on S_N. This implies that $(\boldsymbol{\psi}, \boldsymbol{\varphi}) \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$. Taking into account the third Green identities (2.64)-(2.67), we immediately obtain that $(p, \boldsymbol{v}, \boldsymbol{\varphi}, \boldsymbol{\psi})$ solve system (2.83).

ii) Conversely, let $(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ solve BDIE system (2.83). If we take the trace of (2.83b) restricted to S_D, use the jump relations for the trace of

$\boldsymbol{V}$ and $\boldsymbol{W}$, see Theorem 2.8, and subtract it from (2.83c), we arrive at $r_{S_D}\gamma^+\boldsymbol{v}-\frac{1}{2}r_{S_D}\boldsymbol{\varphi} = \boldsymbol{\varphi}_0$ on S_D. As $\boldsymbol{\varphi}$ vanishes on S_D, therefore the Dirichlet condition of the BVP is satisfied.

Repeating the same procedure but now taking the traction of (2.83a) and (2.83b), restricted to S_N, using the jump relations for the traction of $(\Pi, \boldsymbol{W})$ and subtracting it from (2.83d), we arrive at $r_{S_N}\boldsymbol{T}(p,\boldsymbol{v}) - \frac{1}{2}r_{S_N}\boldsymbol{\psi} = \boldsymbol{\psi}_0$ on S_N. Since $\boldsymbol{\psi}$ vanishes on S_N, the Neumann condition of the BVP is satisfied.

Since $\boldsymbol{\varphi}_0 = \boldsymbol{\Phi}_0$, on S_D; and $\boldsymbol{\psi}_0 = \boldsymbol{\Psi}_0$, on S_N; the conditions (2.87) are satisfied, respectively, on S_D and S_N. Hence, $\boldsymbol{\Psi}^* \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D)$, $\boldsymbol{\Phi}^* \in \widetilde{\boldsymbol{H}}^{1/2}(S_N)$.

Due to relations (2.83a) and (2.83b) the hypotheses of Lemma 2.13 are satisfied with $\boldsymbol{\Psi} = \boldsymbol{\psi}+\boldsymbol{\Psi}_0$ and $\boldsymbol{\Phi} = \boldsymbol{\varphi}+\boldsymbol{\Phi}_0$. As a result, we obtain that $(p,\boldsymbol{v})$ is a solution of $\boldsymbol{\mathcal{A}}(p,\boldsymbol{v}) = \boldsymbol{f}$ satisfying

$$\boldsymbol{V}(\boldsymbol{\Psi}^*) - \boldsymbol{W}(\boldsymbol{\Phi}^*) = \boldsymbol{0}, \quad \mathcal{P}(\boldsymbol{\Psi}^*) - \Pi(\boldsymbol{\Phi}^*) = 0 \quad \text{in } \Omega, \tag{2.88}$$

where

$$\boldsymbol{\Psi}^* = \boldsymbol{\psi} + \boldsymbol{\Psi}_0 - \boldsymbol{T}^+(p,\boldsymbol{v}), \qquad \boldsymbol{\Phi}^* = \boldsymbol{\varphi} + \boldsymbol{\Phi}_0 - \gamma^+\boldsymbol{v}.$$

Since $\boldsymbol{\Psi}^* \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D)$, $\boldsymbol{\Phi}^* \in \widetilde{\boldsymbol{H}}^{1/2}(S_N)$, and (2.88) hold true, then by applying Lemma 2.14 for $S_1 = S_D$, and $S_2 = S_N$, we obtain $\boldsymbol{\Psi}^* = \boldsymbol{\Phi}^* = \boldsymbol{0}$, on S. This implies conditions (2.87).

Finally, item (iii), the unique solvability of the BDIES (2.83) follows from from the unique solvability of the BVP, see Theorem 2.1, and items (i) and (ii). □

Theorem 2.17. *The operator*

$$\begin{aligned}\mathcal{M}_*^{11} : L_2(\Omega) \times \boldsymbol{H}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)\\ \longrightarrow L_2(\Omega) \times \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{-1/2}(S_N)\end{aligned} \tag{2.89}$$

is continuously invertible.

Proof. The operator $\mathcal{M}_*^{11}$ is continuous due the mapping properties of the operators involved, see Theorems 2.2, 2.5 and 2.11.

Let us now prove the invertibility. For this purpose, we define the following operator:

$$\widetilde{\mathcal{M}}^{11} = \begin{bmatrix} I & \mathcal{R}^{\bullet} & -\mathcal{P} & \Pi \\ \mathbf{0} & \boldsymbol{I} & -\boldsymbol{V} & \boldsymbol{W} \\ \mathbf{0} & \mathbf{0} & -r_{S_D}\boldsymbol{\mathcal{V}} & r_{S_D}\boldsymbol{\mathcal{W}} \\ \mathbf{0} & \mathbf{0} & -r_{S_N}\mathring{\boldsymbol{\mathcal{W}}}' & r_{S_N}\widehat{\boldsymbol{\mathcal{L}}} \end{bmatrix},$$

and consider the new system

$$\widetilde{\mathcal{M}}^{11}\widetilde{\mathcal{X}} = \widetilde{\mathcal{F}}^{11} \tag{2.90}$$

where $\widetilde{\mathcal{X}} = [\tilde{p}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\psi}}]^\top$ and $\widetilde{\mathcal{F}} = [\widetilde{\mathcal{F}}_1^{11}, \widetilde{\mathcal{F}}_2^{11}, \widetilde{\mathcal{F}}_3^{11}, \widetilde{\mathcal{F}}_4^{11}]^\top$. In this case, $\widetilde{\mathcal{X}} \in L_2(\Omega) \times \boldsymbol{H}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ and $\widetilde{\mathcal{F}}^{11} \in L_2(\Omega) \times \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{-1/2}(S_N)$.

Consider now, the last two equations of the system (2.90),

$$-r_{S_D}\boldsymbol{\mathcal{V}}\tilde{\boldsymbol{\psi}} + r_{S_D}\boldsymbol{\mathcal{W}}\tilde{\boldsymbol{\phi}} = \widetilde{\mathcal{F}}_3^{11}, \tag{2.91}$$

$$-r_{S_N}\mathring{\boldsymbol{\mathcal{W}}}'\tilde{\boldsymbol{\psi}} + r_{S_N}\widehat{\boldsymbol{\mathcal{L}}}\tilde{\boldsymbol{\phi}} = \widetilde{\mathcal{F}}_4^{11}. \tag{2.92}$$

Multiplying equation (2.91) by μ and apply the relations (2.20) and (2.24) to obtain

$$-r_{S_D}\mathring{\boldsymbol{\mathcal{V}}}\tilde{\boldsymbol{\psi}} + r_{S_D}\mathring{\boldsymbol{\mathcal{W}}}(\mu\tilde{\boldsymbol{\phi}}) = \mu\widetilde{\mathcal{F}}_3^{11}, \tag{2.93}$$

$$-r_{S_N}\mathring{\boldsymbol{\mathcal{W}}}'\tilde{\boldsymbol{\psi}} + r_{S_N}\mathring{\boldsymbol{\mathcal{L}}}(\mu\tilde{\boldsymbol{\phi}}) = \widetilde{\mathcal{F}}_4^{11}. \tag{2.94}$$

This system is uniquely solvable, as the matrix operator of the left hand side is the operator $\mathring{\mathcal{M}}$ from Lemma 2.14 which we already know that is invertible, cf. [KoWe06, Theorem 3.10]. Therefore, $\widetilde{\boldsymbol{\phi}}$ and $\widetilde{\boldsymbol{\psi}}$ are uniquely determined by (2.91) and (2.92). Consequently, $\widetilde{\boldsymbol{v}}$ is uniquely determined from the second equation of the system (2.90) and thus also is p from the first equation. This argument proves the invertibility of the operator $\widetilde{\mathcal{M}}^{11}$ and hence $\widetilde{\mathcal{M}}^{11}$ has Fredholm index 0.

Furthermore, the operator $\mathcal{M}_*^{11} - \widetilde{\mathcal{M}}^{11}$ is a compact perturbation of the operator $\mathcal{M}_*^{11}$ due to the compact mapping properties given by theorems 2.4, 2.6 and 2.10. As a consequence the operator $\mathcal{M}_*^{11}$ has also Fredholm index 0.

By virtue of the Equivalence Theorem 2.16 and Remark 2.15, the homogeneous system (M11) has only the trivial solution, hence $\mathcal{M}^{11}_{*}$ is invertible. □

Theorem 2.18. *The operator*

$$\mathcal{M}^{11}_{*} : \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N) \tag{2.95}$$

$$\longrightarrow \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{-1/2}(S_N) \tag{2.96}$$

is continuously invertible.

Proof. Let us consider the solution $\mathcal{X} = (\mathcal{M}^{11}_{*})^{-1}\mathcal{F}^{11}_{*}$ of the system (2.86). Here, $\mathcal{F}^{11}_{*} \in L^2(\Omega) \times \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{-1/2}(S_N)$ is an arbitrary right hand side and $(\mathcal{M}^{11}_{*})^{-1}$ is the inverse of the operator (2.89) which exists by virtue of Theorem 2.17.

Applying Lemma 2.13 to the first two equations of the system (M11), we get that $\mathcal{X} \in \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ if $\mathcal{F}^{11}_{*} \in \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N)$. Consequently, the operator $(\mathcal{M}^{11}_{*})^{-1}$ is also the continuous inverse of the operator (2.95). □

Corollary 2.19. *Let* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, $g \in L^2(\Omega)$, $\boldsymbol{\phi}_0 \in \boldsymbol{H}^{1/2}(S_D)$ *and* $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(S_N)$ *respectively. Then, the BVP* (2.7) *is uniquely solvable in* $\boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ *and the operator*

$$\boldsymbol{\mathcal{A}}_M : \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \longrightarrow \boldsymbol{L}^2(\Omega) \times L^2(\Omega) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{-1/2}(S_N)$$

is continuously invertible.

Proof. The BDIES (M11) is uniquely solvable and equivalent to the BVP (2.7) by virtue of Theorem 2.16. In addition, as the operator that defines the system (M11) is continuously invertible, see Theorem 2.18,

$$\mathcal{A}_M^{-1}(\boldsymbol{f}, g, r_{S_D}\boldsymbol{\Phi}_0, r_{S_N}\boldsymbol{\Psi}_0) = [c_1\,, c_2]^{\top}$$

where c_1 and c_2 are the first two coordinates of the vector $(\mathcal{M}^{11}_{*})^{-1}\mathcal{F}^{11}_{*}$:

$$c_1 = \pi_1((\mathcal{M}^{11}_{*})^{-1}\mathcal{F}^{11}_{*}) \quad c_2 = \pi_2((\mathcal{M}^{11}_{*})^{-1}\mathcal{F}^{11}_{*}),$$

where the vector $\mathcal{F}_*^{11}$ is given by (2.85). Here, π_1 and π_2 denote the canonical projections into the first component and second component respectively. The term $\mathcal{F}_*^{11}$ can be seen as a continuous function of $(\boldsymbol{f}, g, \boldsymbol{\Psi}_0, \boldsymbol{\Phi}_0)$ due to the mapping properties of the operators involved. The projections are continuous and therefore $\boldsymbol{\mathcal{A}}_M^{-1}$ is a composition of continuous operators, from where the result follows. □

The last three vector equations of the system (M11) are segregated from p. Hence, we can define the new system given by equations (2.83b), (2.83c), (2.83d) which can be written using matrix notation as

$$\mathcal{M}^{11}\mathcal{Y} = \mathcal{F}^{11}, \tag{2.97}$$

where $\mathcal{Y}$ represents the vector containing the unknowns of the system

$$\mathcal{Y} = (\boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})^\top \in \boldsymbol{H}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$$

The matrix operator $\mathcal{M}^{11}$ is defined by

$$\mathcal{M}^{11} = \begin{bmatrix} \boldsymbol{I} + \boldsymbol{\mathcal{R}} & -\boldsymbol{V} & \boldsymbol{W} \\ r_{S_D}\gamma^+\boldsymbol{\mathcal{R}} & -r_{S_D}\boldsymbol{\mathcal{V}} & r_{S_D}\boldsymbol{\mathcal{W}} \\ r_{S_N}\boldsymbol{T}^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) & -r_{S_N}\boldsymbol{\mathcal{W}}' & r_{S_N}\boldsymbol{\mathcal{L}} \end{bmatrix}$$

Corollary 2.20. *The operator*

$$\mathcal{M}^{11} : \boldsymbol{H}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N) \longrightarrow \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{-1/2}(S_N)$$

is continuous and continuously invertible.

Proof. The operator is continuous due to the mapping properties of the operators involved.

Let us assume that $\mathcal{M}^{11}$ is not invertible. Then, the system (2.97) has at least two different solutions $(\boldsymbol{v}_1, \boldsymbol{\psi}_1, \boldsymbol{\phi}_1)$ and $(\boldsymbol{v}_2, \boldsymbol{\psi}_2, \boldsymbol{\phi}_2)$. Then, using equation (2.83a), we can obtain the corresponding pressure for each of the two solutions. Hence, we have two solutions for the system (M11) $(p_1, \boldsymbol{v}_1, \boldsymbol{\psi}_1, \boldsymbol{\phi}_1)$ and $(p_2, \boldsymbol{v}_2, \boldsymbol{\psi}_2, \boldsymbol{\phi}_2)$. However, the BDIES (2.83) is uniquely solvable by virtue of Theorem 2.16. Therefore, both solutions must be the same.

Since the equations (2.83b)-(2.83d) coincide with the equations of the BDIES (2.83), the solution of the latter one given by $\mathcal{X} = \mathcal{M}_*^{11}\mathcal{F}_*^{11}$, where $\mathcal{X} = [p, \boldsymbol{v}, \boldsymbol{\phi}, \boldsymbol{\psi}]^\top$ provides the solution $\mathcal{Y} = [\boldsymbol{v}, \boldsymbol{\phi}, \boldsymbol{\psi}]^\top$ of the system $\mathcal{M}^{11}\mathcal{Y} = \mathcal{F}^{11}$ for any arbitrary right hand side $\mathcal{F}^{11}$ what implies the invertibility of the operator $\mathcal{M}^{11}$. □

2.7 BDIES M22

Let, as before, $\boldsymbol{\Phi}_0 \in \boldsymbol{H}^{1/2}(S)$ and $\boldsymbol{\Psi}_0 \in \boldsymbol{H}^{-1/2}(S)$ be some continuations of the boundary functions $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(S_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(S_N)$ from (2.7c) and (2.7d). Let us now represent

$$\gamma^+\boldsymbol{v} = \boldsymbol{\Phi}_0 + \boldsymbol{\varphi}, \quad \boldsymbol{T}^+(p, \boldsymbol{v}) = \boldsymbol{\Psi}_0 + \boldsymbol{\psi}, \text{ on } S, \tag{2.98}$$

where $\boldsymbol{\varphi} \in \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ and $\boldsymbol{\psi} \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D)$ are unknown boundary functions.

Let us now take equations (2.64) and (2.65) in the domain Ω and restrictions of equations (2.66) and (2.67) to the boundary parts S_N and S_D respectively. Substituting there representations (2.98) and considering further the unknown boundary functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ as formally independent of (segregated from) the unknown domain functions p and $\boldsymbol{v}$, we obtain the following system of BDIEs

$$p + \mathcal{R}^\bullet \boldsymbol{v} - \mathcal{P}\boldsymbol{\psi} + \Pi\boldsymbol{\varphi} = F_0, \quad in \quad \Omega, \tag{2.99a}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{\psi} + \boldsymbol{W}\boldsymbol{\varphi} = \boldsymbol{F}, \quad in \quad \Omega, \tag{2.99b}$$

$$\frac{1}{2}\boldsymbol{\psi} + r_{S_D}\boldsymbol{T}^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}})\boldsymbol{v} - r_{S_D}\boldsymbol{\mathcal{W}}'\boldsymbol{\psi} + r_{S_D}\boldsymbol{\mathcal{L}}^+\boldsymbol{\varphi} = \boldsymbol{F}_D, \quad on \quad S_D, \tag{2.99c}$$

$$\frac{1}{2}\boldsymbol{\varphi} + r_{S_N}\gamma^+\boldsymbol{\mathcal{R}}\boldsymbol{v} - r_{S_N}\boldsymbol{\mathcal{V}}\boldsymbol{\psi} + r_{S_N}\boldsymbol{\mathcal{W}}\boldsymbol{\varphi} = \boldsymbol{F}_N \quad on \quad S_N, \tag{2.99d}$$

whose unknowns are $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$, $\boldsymbol{\varphi} \in \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ and $\boldsymbol{\psi} \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D)$, and where the terms in the right hand side F_0 and $\boldsymbol{F}$ are given by (2.84). On the other hand the terms $\boldsymbol{F}_D$ and $\boldsymbol{F}_N$ are defined as:

$$\boldsymbol{F}_D := r_{S_D}\boldsymbol{T}^+(F_0, \boldsymbol{F}) - r_{S_D}\boldsymbol{\Psi}_0, \qquad \boldsymbol{F}_N := r_{S_N}\gamma^+\boldsymbol{F} - r_{S_N}\boldsymbol{\Phi}_0. \tag{2.100}$$

Note that the BDIE system (2.99a)-(2.99d) can be split into the BDIE system (M22), of 3 vector equations, (2.99b)-(2.99d), for 3 vector unknowns, $\boldsymbol{v}$, $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$, and the separate equation (2.99a) that can be used, after solving the system, to obtain the pressure, p.

However, since the couple $(p, \boldsymbol{v})$ shares the space $\boldsymbol{H}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$, equations (2.99b), (2.99c) and (2.99d) are not completely separate from equation (2.99a).

The system (2.99a)-(2.99d) can be written using matrix notation as follows

$$\mathcal{M}_*^{22}\mathcal{X} = \mathcal{F}_*^{22}, \tag{2.101}$$

where the matrix operator $\mathcal{M}_*^{22}$ is defined by

$$\mathcal{M}_*^{22} = \begin{bmatrix} I & \mathcal{R}^\bullet & -\mathcal{P} & \Pi \\ \mathbf{0} & \boldsymbol{I}+\boldsymbol{\mathcal{R}} & -\boldsymbol{V} & \boldsymbol{W} \\ \mathbf{0} & r_{S_D}\boldsymbol{T}^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) & r_{S_D}\left(\dfrac{1}{2}\boldsymbol{I} - \boldsymbol{\mathcal{W}'}\right) & r_{S_D}\boldsymbol{\mathcal{L}}^+ \\ \mathbf{0} & r_{S_N}\gamma^+\boldsymbol{\mathcal{R}} & -r_{S_N}\boldsymbol{\mathcal{V}} & r_{S_N}\left(\dfrac{1}{2}\boldsymbol{I} + \boldsymbol{\mathcal{W}}\right) \end{bmatrix}, \tag{2.102}$$

the vector $\mathcal{X} = (p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})^\top \in L^2(\Omega) \times \boldsymbol{H}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ represents the unknowns of the system, and the vector

$$\mathcal{F}_*^{22} := [F_0, \mathcal{F}^{22}] = [F_0, \boldsymbol{F}, r_{S_N}\boldsymbol{\gamma}^+\boldsymbol{F} - r_{S_N}\boldsymbol{\Phi}_0, r_{S_D}\boldsymbol{T}^+(F_0, \boldsymbol{F}) - r_{S_D}\boldsymbol{\Psi}_0]^\top$$

is the right hand side and $\mathcal{F}_*^{22} \in L^2(\Omega) \times \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N)$.

Due to the mapping properties involved in (2.102), the operator

$$\mathcal{M}_*^{22} : L^2(\Omega)\times\boldsymbol{H}^1(\Omega)\times\widetilde{\boldsymbol{H}}^{-1/2}(S_D)\times\widetilde{\boldsymbol{H}}^{1/2}(S_N) \longrightarrow L^2(\Omega)\times\boldsymbol{H}^1(\Omega^+)\times\boldsymbol{H}^{-1/2}(S_D)\times\boldsymbol{H}^{1/2}(S_N)$$

is bounded.

Remark 2.21. *The term* $\mathcal{F}_*^{22} := [F_0, \boldsymbol{F}, r_{S_D}\boldsymbol{T}^+(F_0, \boldsymbol{F}) - r_{S_D}\boldsymbol{\Psi}_0, r_{S_N}\gamma^+\boldsymbol{F} - r_{S_N}\boldsymbol{\Phi}_0]^\top = 0$ *if and only if* $(\boldsymbol{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = 0$.

Proof. It is evident that $(\boldsymbol{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = 0$ implies $\mathcal{F}_*^{22} = 0$. Hence, we shall only focus on proving $\mathcal{F}_*^{22} = 0 \Rightarrow (\boldsymbol{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = 0$. Taking into account how the terms $\boldsymbol{F}$ and F_0 are defined, see (2.84), considering that $F_0 = 0$ and $\boldsymbol{F} = 0$, we can deduce by applying Lemma 2.13 that $\boldsymbol{f} = 0$, $g = 0$ and that

$$\mathcal{P}\boldsymbol{\Psi}_0 - \Pi\boldsymbol{\Phi}_0 = 0,$$

$$\boldsymbol{V}\boldsymbol{\Psi}_0 - \boldsymbol{W}\boldsymbol{\Phi}_0 = 0.$$

In addition, since $F_0 = 0$ and $\boldsymbol{F} = 0$, we have

$$r_{S_D}\boldsymbol{T}^+(F_0,\boldsymbol{F}) - r_{S_D}\boldsymbol{\Psi}_0 = 0 \;\Rightarrow\; r_{S_D}\boldsymbol{\Psi}_0 = 0,$$

$$r_{S_N}\gamma^+\boldsymbol{F} - r_{S_N}\boldsymbol{\Phi}_0 = 0, \;\Rightarrow\; r_{S_N}\boldsymbol{\Phi}_0 = 0.$$

Consequently, $\boldsymbol{\Psi}_0 \in \widetilde{H}^{-1/2}(S_N)$ and $\boldsymbol{\Phi}_0 \in \widetilde{H}^{1/2}(S_D)$. Therefore, the hypotheses of Lemma 2.14 are satisfied and by applying it, we thus obtain that $\boldsymbol{\Psi}_0 = 0$ and $\boldsymbol{\Phi}_0 = 0$ on S. □

Theorem 2.22 (Equivalence Theorem)**.** *Let $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$, $g \in L^2(\Omega)$, $\boldsymbol{\Phi}_0 \in \boldsymbol{H}^{-1/2}(S)$ and $\boldsymbol{\Psi}_0 \in \boldsymbol{H}^{-1/2}(S)$ be some fixed extensions of $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(S_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(S_N)$, respectively.*

(i) If some $(p,\boldsymbol{v}) \in L_2(\Omega) \times \boldsymbol{H}^1(\Omega)$ solve the mixed BVP (2.7)*, then $(p,\boldsymbol{v},\boldsymbol{\psi},\boldsymbol{\varphi}) \in \boldsymbol{H}^1(\Omega) \times L_2(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$, where*

$$\boldsymbol{\varphi} = \gamma^+\boldsymbol{v} - \boldsymbol{\Phi}_0, \qquad \boldsymbol{\psi} = \boldsymbol{T}^+(p,\boldsymbol{v}) - \boldsymbol{\Psi}_0, \quad \text{on } S, \tag{2.103}$$

solve the BDIE system (2.99a)-(2.99d)*.*

(ii) If $(p,\boldsymbol{v},\boldsymbol{\psi},\boldsymbol{\varphi}) \in L_2(\Omega) \times \boldsymbol{H}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ solve the BDIE system (2.99a)-(2.99d)*, then $(p,\boldsymbol{v})$ solve the mixed BVP* (2.7) *and the functions $\boldsymbol{\psi},\boldsymbol{\varphi}$ satisfy* (2.103)*.*

(iii) The BDIES (2.99a)-(2.99d) *is uniquely solvable in $L_2(\Omega) \times \boldsymbol{H}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$.*

Proof. i) Let $(p,\boldsymbol{v}) \in L_2(\Omega) \times \boldsymbol{H}^1(\Omega)$ be a solution of the BVP. Let us define the functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ by (2.87). By the BVP boundary conditions, $\gamma^+\boldsymbol{v} = \boldsymbol{\varphi}_0 = \boldsymbol{\Phi}_0$ on S_D and $\boldsymbol{T}^+(p,\boldsymbol{v}) = \boldsymbol{\psi}_0 = \boldsymbol{\Psi}_0$ on S_N. This implies that $(\boldsymbol{\psi},\boldsymbol{\varphi}) \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$. Taking into account the Green identities (2.65)-(2.67), we immediately obtain that $(p,\boldsymbol{v},\boldsymbol{\varphi},\boldsymbol{\psi})$ solves the system (2.102).

ii) Conversely, let $(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in L_2(\Omega) \times \boldsymbol{H}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ solve the BDIE system (2.99a)-(2.99d). Then, the equations (2.84) applied to the BDIEs (2.99a)-(2.99b) allow us to apply Lemma 2.13 with $\boldsymbol{\Psi} = \boldsymbol{\psi} + \boldsymbol{\Psi}_0$ and $\boldsymbol{\Phi} = \boldsymbol{\varphi} + \boldsymbol{\Phi}_0$, to deduce that the pair $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ and solves the system (2.7a)-(2.7b).

As $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$, the (canonical) traction operator is well defined and we can work out the traction of (2.99a) and (2.99b) restricted to S_D and subtract it from (2.99b) to get

$$r_{S_D}\boldsymbol{T}(p, \boldsymbol{v}) - r_{S_D}\boldsymbol{\Psi}_0 = \boldsymbol{\psi}, \quad on \quad S_D. \tag{2.104}$$

Take the trace of (2.99b) restricted to S_N and subtract it from (2.99d) to get

$$r_{S_N}\gamma^+\boldsymbol{v} - r_{S_N}\boldsymbol{\Phi}_0 = \boldsymbol{\varphi}, \quad on \quad S_N. \tag{2.105}$$

Consequently, equations (2.104) and (2.105) imply that conditions (2.103) are satisfied on S_N and S_D respectively.

Furthermore, from Lemma 2.13 we also obtain that the following identities are satisfied

$$\mathcal{P}(\boldsymbol{\Psi}^*) - \Pi(\boldsymbol{\Phi}^*) = 0, \qquad \boldsymbol{V}(\boldsymbol{\Psi}^*) - \boldsymbol{W}(\boldsymbol{\Phi}^*) = \boldsymbol{0}, \quad \text{in } \Omega, \tag{2.106}$$

where

$$\boldsymbol{\Psi}^* := \boldsymbol{\psi} + \boldsymbol{\Psi}_0 - \boldsymbol{T}^+(p, \boldsymbol{v}), \qquad \boldsymbol{\Phi}^* := \boldsymbol{\varphi} + \boldsymbol{\Phi}_0 - \gamma^+\boldsymbol{v}, \qquad on \quad S. \tag{2.107}$$

In addition, $\boldsymbol{\Psi}^* \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D)$ and $\boldsymbol{\Phi}^* \in \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ due to (2.104) and (2.105). Now, we can apply Lemma 2.14 with $S_1 = S_D$ and $S_2 = S_N$, to obtain $\boldsymbol{\Psi}^* = \boldsymbol{\Phi}^* = \boldsymbol{0}$ on S. Subsitute $\boldsymbol{\Psi}^* = \boldsymbol{\Phi}^* = \boldsymbol{0}$ on (2.107) to deduce that relations (2.103) are satisfied in the whole boundary S. Considering that $\text{supp}(\phi) \subset S_D$, $\text{supp}(\psi) \subset S_N$, $r_{S_D}\Phi_0 = \phi_0$ and $r_{S_N}\Psi_0 = \psi_0$; it is easy to deduce that the pair $(p, \boldsymbol{v})$ also satisfy the boundary conditions.

iii) Items (i) and (ii) state that the BDIES (M22) is equivalent to the BVP (2.7a)-(2.7d). Since this BVP has only one solution, then the uniqueness of the solution of the BDIES (M22) follows. $\square$

Lemma 2.23. *Let $S = \bar{S}_1 \cup \bar{S}_2$, where S_1 and S_2 are two non-intersecting simply connected nonempty submanifolds of S with infinitely smooth boundaries. For any vector*

$$\mathcal{F} = (F_0, \boldsymbol{F}, \boldsymbol{\Psi}, \boldsymbol{\Phi})^\top \in \boldsymbol{H}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_1) \times \boldsymbol{H}^{1/2}(S_2)$$

there exists another vector

$$(g_*, \boldsymbol{f}_*, \boldsymbol{\Psi}_*, \boldsymbol{\Phi}_*)^\top = \widetilde{\mathcal{C}}_{S_1,S_2}\mathcal{F} \in L_2(\Omega) \times \boldsymbol{L}_2(\Omega) \times \boldsymbol{H}^{-1/2}(S) \times \boldsymbol{H}^{1/2}(S)$$

which is uniquely determined by $\mathcal{F}$ and such that

$$\mathring{\mathcal{Q}}\boldsymbol{f}_* + \frac{4}{3}\mu g_* + \mathcal{P}\boldsymbol{\Psi}_* - \Pi\boldsymbol{\Phi}_* = F_0, \quad in \quad \Omega, \tag{2.108a}$$

$$\boldsymbol{\mathcal{U}}\boldsymbol{f}_* + \boldsymbol{\mathcal{Q}}g_* + \boldsymbol{V}\boldsymbol{\Psi}_* - \boldsymbol{W}\boldsymbol{\Phi}_* = \boldsymbol{F}, \quad in \quad \Omega, \tag{2.108b}$$

$$r_{S_1}\boldsymbol{\Psi}_* = \boldsymbol{\Psi}, \quad on \quad S_1, \tag{2.108c}$$

$$r_{S_2}\boldsymbol{\Phi}_* = \boldsymbol{\Phi}, \quad on \quad S_2. \tag{2.108d}$$

Furthermore, the operator

$$\widetilde{\mathcal{C}}_{S_1,S_2} : \boldsymbol{H}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_1) \times \boldsymbol{H}^{1/2}(S_2) \longrightarrow L_2(\Omega) \times \boldsymbol{L}_2(\Omega) \times \boldsymbol{H}^{-1/2}(S) \times \boldsymbol{H}^{1/2}(S)$$

is continuous.

Proof. Let $\boldsymbol{\Psi}^0$ be some fixed extension of $\boldsymbol{\Psi}$ from S_1 to the whole boundary S. Likewise, let $\boldsymbol{\Phi}^0$ be some fixed extension of $\boldsymbol{\Phi}$ from S_2 onto S. Assume that such extensions exist and preserve the functions spaces, i.e., $\boldsymbol{\Psi}^0 \in \boldsymbol{H}^{-1/2}(S)$, $\boldsymbol{\Phi}^0 \in \boldsymbol{H}^{1/2}(S)$ and moreover, satisfy

$$\| \boldsymbol{\Psi}^0 \|_{\boldsymbol{H}^{-1/2}(S)} \leq C_0 \| \boldsymbol{\Psi} \|_{\boldsymbol{H}^{-1/2}(S_1)}, \qquad \| \boldsymbol{\Phi}^0 \|_{\boldsymbol{H}^{1/2}(S)} \leq C_0 \| \boldsymbol{\Phi} \|_{\boldsymbol{H}^{1/2}(S_2)}$$

for some C_0 positive constant, independent of $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$, (cf. [Tr78, Subsection 4.2]). Consequently, arbitrary extensions of the functions $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ can be represented as

$$\boldsymbol{\Psi}_* = \boldsymbol{\Psi}^0 + \widetilde{\boldsymbol{\psi}}, \qquad \widetilde{\boldsymbol{\psi}} \in \widetilde{\boldsymbol{H}}^{-1/2}(S_2), \tag{2.109}$$

$$\boldsymbol{\Phi}_* = \boldsymbol{\Phi}^0 + \widetilde{\boldsymbol{\varphi}}, \qquad \widetilde{\boldsymbol{\varphi}} \in \widetilde{\boldsymbol{H}}^{1/2}(S_1). \tag{2.110}$$

The functions $\boldsymbol{\Psi}_*$ and $\boldsymbol{\Phi}_*$, in the form (2.109) and (2.110), satisfy the conditions (2.108c) and (2.108d). Consequently, it is only necessary to show that the functions $g_*, \boldsymbol{f}_*, \widetilde{\boldsymbol{\psi}}$ and $\widetilde{\boldsymbol{\varphi}}$ can be chosen in a particular way such that equations (2.108a)-(2.108b) are satisfied.

Applying the potential relations (2.16)-(2.22) to equations (2.108a)-(2.108b), we obtain

$$\mathring{\mathcal{Q}} f_* + \frac{4}{3}\mu g_* + \mathring{\mathcal{P}}\left(\boldsymbol{\Psi}_0 + \widetilde{\boldsymbol{\psi}}\right) - \mathring{\Pi}\left(\mu\boldsymbol{\Phi}_0 + \mu\boldsymbol{\varphi}\right) = F_0, \tag{2.111}$$

$$\mathring{\boldsymbol{\mathcal{U}}} f_* + \mathring{\boldsymbol{\mathcal{Q}}}(\mu g_*) + \mathring{\boldsymbol{V}}\left(\boldsymbol{\Psi}_0 + \widetilde{\boldsymbol{\psi}}\right) - \mathring{\boldsymbol{W}}\left(\mu\boldsymbol{\Phi}_0 + \mu\boldsymbol{\varphi}\right) = \mu\boldsymbol{F}. \tag{2.112}$$

Apply the Stokes operator with constant viscosity $\mu = 1$, $\mathring{\boldsymbol{\mathcal{A}}}$, to equations (2.111) and (2.112). Then, apply the divergence operator to equation (2.112). As a result, we obtain

$$\boldsymbol{f}_* = \mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F}) \tag{2.113}$$

$$\mu g_* = \operatorname{div}(\mu\boldsymbol{F}) \Rightarrow g_* = \frac{\operatorname{div}(\mu\boldsymbol{F})}{\mu} \tag{2.114}$$

which shows that the function $\boldsymbol{f}_*$ is uniquely determined by F_0 and $\mu\boldsymbol{F}$ and belongs to $\boldsymbol{L}^2(\Omega)$ since $(F_0, \mu\boldsymbol{F}) \in \boldsymbol{H}^{1,0}(\Omega; \mathcal{A})$. In addition, (2.114) shows that g_* is also uniquely determined by $\boldsymbol{F}$ and belongs to $L^2(\Omega)$ due to the fact that $\mu\boldsymbol{F} \in \boldsymbol{H}^1(\Omega)$.

Let us substitute now (2.113) and (2.114) into equations (2.111)-(2.112) and move each term which is not depending on either $\widetilde{\boldsymbol{\psi}}$ or $\widetilde{\boldsymbol{\phi}}$ to the right hand side

$$\mathring{\mathcal{P}}\widetilde{\boldsymbol{\psi}} - \mathring{\Pi}(\mu\widetilde{\boldsymbol{\varphi}}) = F_0 - \frac{4}{3}\operatorname{div}(\mu\boldsymbol{F}) - \mathring{\mathcal{Q}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F})\right) - \mathring{\mathcal{P}}(\boldsymbol{\Psi}^0) + \mathring{\Pi}(\mu\boldsymbol{\Phi}^0), \quad \text{in } \Omega, \tag{2.115}$$

$$\mathring{\boldsymbol{V}}\widetilde{\boldsymbol{\psi}} - \mathring{\boldsymbol{W}}(\mu\widetilde{\boldsymbol{\varphi}}) = \mu\boldsymbol{F} - \mathring{\boldsymbol{\mathcal{U}}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F})\right) - \mathring{\boldsymbol{\mathcal{Q}}}(\mu\boldsymbol{F}) - \mathring{\boldsymbol{V}}(\boldsymbol{\Psi}^0) + \mathring{\boldsymbol{W}}(\mu\boldsymbol{\Phi}^0), \quad \text{in } \Omega. \tag{2.116}$$

Let us denote with $J = (J_0, \boldsymbol{J})$ the right hand side of (2.115)-(2.116)

$$J_0 := \left(F_0 - \frac{4}{3}\operatorname{div}(\mu\boldsymbol{F}) - \mathring{\mathcal{Q}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F})\right) - \mathring{\mathcal{P}}(\boldsymbol{\Psi}^0) + \mathring{\Pi}(\mu\boldsymbol{\Phi}^0)\right),$$

$$\boldsymbol{J} := \left(\mu\boldsymbol{F} - \mathring{\boldsymbol{\mathcal{U}}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F})\right) - \mathring{\boldsymbol{\mathcal{Q}}}\operatorname{div}(\mu\boldsymbol{F}) - \mathring{\boldsymbol{V}}(\boldsymbol{\Psi}^0) + \mathring{\boldsymbol{W}}(\mu\boldsymbol{\Phi}^0)\right).$$

It is easy to check that J satisfies the incompressible homogeneous Stokes system with $\mu = 1$: $\mathring{\boldsymbol{\mathcal{A}}}(J_0, \boldsymbol{J}) = \boldsymbol{0}$ and $\operatorname{div}\boldsymbol{J} = 0$.

If the functions $\widetilde{\boldsymbol{\psi}}$ and $\widetilde{\boldsymbol{\varphi}}$ satisfy (2.115)-(2.116). Then, they will also satisfy the following system:

$$r_{S_2}\gamma^+\left(\mathring{\boldsymbol{V}}\widetilde{\boldsymbol{\psi}} - \mathring{\boldsymbol{W}}(\mu\widetilde{\boldsymbol{\varphi}})\right) = r_{S_2}\left(\gamma^+\boldsymbol{J}\right), \tag{2.117}$$

$$r_{S_1}\left[\mathring{\boldsymbol{T}}^+\left(\mathring{\mathcal{P}}(\widetilde{\psi}) - \mathring{\Pi}(\mu\widetilde{\boldsymbol{\varphi}}), \mathring{\boldsymbol{V}}\widetilde{\psi} - \mathring{\boldsymbol{W}}(\mu\widetilde{\boldsymbol{\varphi}})\right)\right] = r_{S_1}\left(\mathring{\boldsymbol{T}}^+(J_0, \boldsymbol{J})\right). \tag{2.118}$$

The system (2.117)-(2.118) can be written using matrix notation as follows

$$\begin{bmatrix} r_{S_2}\mathring{\boldsymbol{\mathcal{V}}} & r_{S_2}\gamma^+\mathring{\boldsymbol{W}} \\ r_{S_1}\mathring{\boldsymbol{W}}' & r_{S_1}\mathring{\boldsymbol{\mathcal{L}}} \end{bmatrix}\begin{bmatrix} \widetilde{\boldsymbol{\psi}} \\ \mu\widetilde{\boldsymbol{\varphi}} \end{bmatrix} = \begin{bmatrix} r_{S_2}\left(\gamma^+\boldsymbol{J}\right) \\ r_{S_1}\left(\mathring{\boldsymbol{T}}^+(J_0, \boldsymbol{J})\right) \end{bmatrix}. \tag{2.119}$$

The matrix operator given by the lefthand side of the equations (2.117)-(2.118) is an isomorphism between the spaces $\widetilde{\boldsymbol{H}}^{-1/2}(S_2) \times \widetilde{\boldsymbol{H}}^{1/2}(S_1)$ onto $\boldsymbol{H}^{1/2}(S_2) \times \boldsymbol{H}^{-1/2}(S_1)$ (see, [KoWe06, Theorem 3.10]). Therefore, the simultaneous equations (2.117) and (2.118) are uniquely solvable with respect to $\widetilde{\boldsymbol{\varphi}}$ and $\widetilde{\boldsymbol{\psi}}$. We denote the solution of (2.117)-(2.118) by $\widetilde{\boldsymbol{\psi}}^0$ and $\widetilde{\boldsymbol{\varphi}}^0$.

Substitute now $\widetilde{\boldsymbol{\psi}}^0$ and $\widetilde{\boldsymbol{\varphi}}^0$ into (2.111)-(2.112)

$$\mathring{\mathcal{P}}\widetilde{\boldsymbol{\psi}}^0 - \mathring{\Pi}(\mu\widetilde{\boldsymbol{\varphi}}^0) = F_0 - \frac{4}{3}\mu\mathrm{div}(\mu\boldsymbol{F}) - \mathring{\mathcal{Q}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F})\right) - \mathring{\mathcal{P}}(\boldsymbol{\Psi}^0) + \mathring{\Pi}(\mu\boldsymbol{\Phi}^0), \quad \text{in } \Omega, \tag{2.120}$$

$$\mathring{\boldsymbol{V}}\widetilde{\boldsymbol{\psi}}^0 - \mathring{\boldsymbol{W}}(\mu\widetilde{\boldsymbol{\varphi}}^0) = \mu\boldsymbol{F} - \mathring{\boldsymbol{\mathcal{U}}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F})\right) - \mathring{\boldsymbol{\mathcal{Q}}}\mathrm{div}(\mu\boldsymbol{F}) - \mathring{\boldsymbol{V}}(\boldsymbol{\Psi}^0) + \mathring{\boldsymbol{W}}(\mu\boldsymbol{\Phi}^0), \quad \text{in } \Omega. \tag{2.121}$$

Let us rewrite equations (2.120) and (2.121) in terms of the parametrix-based potential operators by applying (2.16)-(2.22)

$$\mathcal{P}(\boldsymbol{\Psi}^0 + \widetilde{\boldsymbol{\psi}}^0) - \Pi(\boldsymbol{\Phi}^0 + \widetilde{\boldsymbol{\varphi}}^0) + \mathcal{Q}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F})\right) + \frac{4}{3}\mu\,\mathrm{div}(\mu\boldsymbol{F}) = F_0, \ \text{ in } \Omega,$$
$$\boldsymbol{V}(\boldsymbol{\Psi}^0 + \widetilde{\boldsymbol{\psi}}^0) - \boldsymbol{W}(\boldsymbol{\Phi}^0 + \widetilde{\boldsymbol{\varphi}}^0) + \boldsymbol{\mathcal{U}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F})\right) + \boldsymbol{\mathcal{Q}}\,\mathrm{div}(\mu\boldsymbol{F}) = \boldsymbol{F}, \ \text{ in } \Omega.$$

Hence, $\boldsymbol{\Psi}_* = \boldsymbol{\Psi}_0 + \widetilde{\boldsymbol{\psi}}^0$ and $\boldsymbol{\Phi}_* = \boldsymbol{\Phi}_0 + \widetilde{\boldsymbol{\varphi}}^0$ are uniquely determined by virtue of the uniqueness of solution of the mixed problem for the Stokes system with $\mu = 1$. Additionally, g_* and $\boldsymbol{f}_*$ are uniquely determined by conditions (2.113) and (2.114).

The continuity and linearity of the operator $\widetilde{\mathcal{C}}_{S_1,S_2}$ is owed to the linearity and continuity of the operators involved. $\square$

Corollary 2.24. *For any*

$$\mathcal{F} = ((F_0, \boldsymbol{F}), \boldsymbol{\mathcal{F}}_2, \boldsymbol{\mathcal{F}}_3)^\top \in \boldsymbol{H}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_1) \times \boldsymbol{H}^{1/2}(S_2),$$

there exists a unique four-tuple

$$(g_*, \boldsymbol{f}_*, \boldsymbol{\Psi}_*, \boldsymbol{\Phi}_*)^\top = \mathcal{C}_{S_1,S_2}\mathcal{F} \in L^2(\Omega) \times \boldsymbol{L}_2(\Omega) \times \boldsymbol{H}^{-1/2}(S) \times \boldsymbol{H}^{1/2}(S),$$

such that

$$\mathring{\mathcal{Q}}\boldsymbol{f}_* + \frac{4}{3}\mu g_* + \mathcal{P}\boldsymbol{\Psi}_* - \Pi\boldsymbol{\Phi}_* = F_0, \text{ in } \Omega, \tag{2.122}$$

$$\boldsymbol{\mathcal{Q}} g_* + \boldsymbol{\mathcal{U}}\boldsymbol{f}_* + \boldsymbol{V}\boldsymbol{\Psi}_* - \boldsymbol{W}\boldsymbol{\Phi}_* = \boldsymbol{F}, \text{ in } \Omega, \tag{2.123}$$

$$r_{S_1}(\boldsymbol{T}^+(\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1) - \boldsymbol{\Psi}_*) = \boldsymbol{\mathcal{F}}_2, \text{ on } S_1 \tag{2.124}$$

$$r_{S_2}(\boldsymbol{\gamma}^+\boldsymbol{\mathcal{F}}_1 - \boldsymbol{\Phi}_*) = \boldsymbol{\mathcal{F}}_3, \text{ on } S_2. \tag{2.125}$$

Furthermore, the operator

$$\mathcal{C}_{S_1,S_2} : \boldsymbol{H}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_1) \times \boldsymbol{H}^{1/2}(S_2) \longrightarrow L^2(\Omega) \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{H}^{-1/2}(S) \times \boldsymbol{H}^{1/2}(S)$$

is continuous.

Proof. Take $\boldsymbol{\Psi} := r_{S_1}\boldsymbol{T}^+(\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1) - \boldsymbol{\mathcal{F}}_2$. Let us check, $\boldsymbol{\Psi} \in \boldsymbol{H}^{-1/2}(S_1)$. Firstly, $\boldsymbol{\mathcal{F}}_2 \in \boldsymbol{H}^{-1/2}(S_1)$. Secondly, $(\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1) \in \boldsymbol{H}^{1,0}(\Omega; \mathcal{A})$, then $\boldsymbol{T}^+(\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1) \in \boldsymbol{H}^{-1/2}(S)$ and hence $r_{S_1}\boldsymbol{T}^+(\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1) \in \boldsymbol{H}^{-1/2}(S_1)$. Therefore $\boldsymbol{\Psi} \in \boldsymbol{H}^{-1/2}(S_1)$.

In a similar fashion we take $\boldsymbol{\Phi} := r_{S_2}\gamma^+\boldsymbol{\mathcal{F}}_1 - \boldsymbol{\mathcal{F}}_3$. It is easy to see by applying the trace theorem that $r_{S_2}\gamma^+\boldsymbol{\mathcal{F}}_1 \in \boldsymbol{H}^{1/2}(S_2)$ and therefore $\boldsymbol{\Phi} \in \boldsymbol{H}^{1/2}(S_2)$. The Corollary follows from applying Lemma 2.23 with $\boldsymbol{\Psi} := r_{S_1}\boldsymbol{T}^+(\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1) - \boldsymbol{\mathcal{F}}_2$ and $\boldsymbol{\Phi} := r_{S_2}\gamma^+\boldsymbol{\mathcal{F}}_1 - \boldsymbol{\mathcal{F}}_3$. $\square$

Theorem 2.25. *The operator*

$$\mathcal{M}^{22}_* : \boldsymbol{H}^{1,0}(\Omega; \mathcal{A}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N) \longrightarrow \boldsymbol{H}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N) \tag{2.126}$$

is continuous and continuously invertible.

Proof. Let us consider an arbitrary right hand side to the system (2.101),

$$\mathcal{F}_*^{22} \in \boldsymbol{H}^{1,0}(\Omega;\mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N)$$

. By virtue of the Corollary 2.24, the right hand side $\mathcal{F}_*^{22}$ can be written in the form (2.122)-(2.125) with $S_1 = S_D$ and $S_2 = S_N$. In addition, $(g_*, \boldsymbol{f}_*, \boldsymbol{\Psi}_*, \boldsymbol{\Phi}_*)^\top = \mathcal{C}_{S_D,S_N}\mathcal{F}^{22}$ where the operator $\mathcal{C}_{S_D,S_N}$ is bounded and has the following mapping property

$$\begin{aligned}\mathcal{C}_{S_D,S_N} : \boldsymbol{H}^{1,0}(\Omega;\mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N)\\ \longrightarrow L^2(\Omega) \times \boldsymbol{L}_2(\Omega) \times \boldsymbol{H}^{-1/2}(S) \times \boldsymbol{H}^{1/2}(S).\end{aligned}$$

By virtue of Corollary 2.19 and the equivalence theorem of the system (M22), Theorem 2.22, there exists a solution of the equation $\mathcal{M}_*^{22}\mathcal{X} = \mathcal{F}_*^{22}$. This solution can be represented as

$$\mathcal{X} = [p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\phi}]^\top = (\mathcal{M}_*^{22})^{-1}\mathcal{F}_*^{22},$$

where the operator

$$\begin{aligned}(\mathcal{M}_*^{22})^{-1} : \boldsymbol{H}^{1,0}(\Omega;\mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N)\\ \longrightarrow \boldsymbol{H}^{1,0}(\Omega;\mathcal{A}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N).\end{aligned}$$

is given by

$$(p, \boldsymbol{v}) = \boldsymbol{\mathcal{A}}_M^{-1}[g_*, \boldsymbol{f}_*, r_{S_D}\boldsymbol{\Psi}_*, r_{S_N}\boldsymbol{\Phi}_*]^\top, \;\; \boldsymbol{\psi} = \boldsymbol{T}^+(p, \boldsymbol{v}) - \boldsymbol{\Psi}_*, \;\; \boldsymbol{\phi} = \gamma^+\boldsymbol{v} - \boldsymbol{\Phi}_*,$$

where the $\boldsymbol{\mathcal{A}}_M^{-1}$. Consequently, the operator $(\mathcal{M}_*^{22})^{-1}$ is a right inverse of the operator (2.126). In addition, $(\mathcal{M}_*^{22})^{-1}$ is also the double sided inverse due to the injectivity of (2.126) given by the Theorem 2.22. □

Particularly, when $\mu = 1$, the operator $\boldsymbol{\mathcal{A}}$ becomes $\mathring{\boldsymbol{\mathcal{A}}}$ and $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{R}}^\bullet \equiv 0$. Consequently, the boundary-domain integral equations system (2.102) can be reduced to a boundary integral equation system (BIES) consisting of 2 vector equations

$$r_{S_D}\left(\frac{1}{2}\boldsymbol{\psi} - \mathring{\boldsymbol{\mathcal{W}}}'\boldsymbol{\psi} + \mathring{\boldsymbol{\mathcal{L}}}\boldsymbol{\varphi}\right) = r_{S_D}\boldsymbol{T}^+(F_0, \boldsymbol{F}) - r_{S_D}\boldsymbol{\Psi}_0 \;\; on \;\, S_D, \tag{2.127}$$

$$r_{S_N}\left(\frac{1}{2}\boldsymbol{\varphi} - \mathring{\mathcal{V}}\boldsymbol{\psi} + \mathring{\boldsymbol{\mathcal{W}}}\boldsymbol{\varphi}\right) = r_{S_N}\gamma^+\boldsymbol{F} - r_{S_N}\boldsymbol{\Phi}_0 \;\; on \;\, S_N. \tag{2.128}$$

and a (BDIES) consisting of a scalar equation and a vector equation

$$p = F_0 + \mathring{\mathcal{P}}\boldsymbol{\psi} - \mathring{\Pi}\boldsymbol{\varphi} \, in \, \Omega, \tag{2.129}$$

$$\boldsymbol{v} = \boldsymbol{F} + \mathring{\boldsymbol{V}}\boldsymbol{\psi} - \mathring{\boldsymbol{W}}\boldsymbol{\varphi} \, in \, \Omega, \tag{2.130}$$

where the terms F_0 and $\boldsymbol{F}$ are given by (2.84). The theorem of equivalence between the BVP and BDIES, Theorem 2.22 leads to the following result of equivalence for the constant coefficient case

Corollary 2.26. *Let $\mu = 1$ in Ω, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $g \in L^2(\Omega)$. Moreover, let $\boldsymbol{\Phi}_0 \in \boldsymbol{H}^{1/2}(S)$ and $\boldsymbol{\Psi}_0 \in \boldsymbol{H}^{-1/2}(S)$ be some extensions of $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(S_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(S_N)$, respectively.*

i) If some $(p, \boldsymbol{v}) \in L_2(\Omega) \times \boldsymbol{H}^1(\Omega)$ solves the mixed BVP (2.7a)-(2.7d), then the solution is unique, the couple $(\boldsymbol{\psi}, \boldsymbol{\varphi}) \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ given by

$$\boldsymbol{\varphi} = \gamma^+ \boldsymbol{v} - \boldsymbol{\Phi}_0, \quad \boldsymbol{\psi} = \boldsymbol{T}^+(p, \boldsymbol{v}) - \boldsymbol{\Psi}_0 \quad on \; S, \tag{2.131}$$

solves the BIE system (2.127)-(2.128) and $(p, \boldsymbol{v})$ satisfies (2.129)(2.130).

ii) If $(\boldsymbol{\psi}, \boldsymbol{\varphi}) \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ solves the BIES (2.127)-(2.128), then $(p, \boldsymbol{v})$ given by (2.129)-(2.130) solves the BVP (2.7a)-(2.7d) and the relations (2.131) hold. Moreover, the system (2.127)-(2.128) is uniquely solvable in $\widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$.

The system (2.127)-(2.128) can be expressed using matrix notation as follows

$$\mathring{\mathcal{M}}^{22} \mathring{\mathcal{X}} = \mathring{\mathcal{F}}^{22} \tag{2.132}$$

where $\mathring{\mathcal{X}} = (\boldsymbol{\psi}, \boldsymbol{\varphi})^\top \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$; the operator

$$\mathring{\mathcal{M}}^{22} : \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N) \longrightarrow \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N),$$

defined by

$$\mathring{\mathcal{M}}^{22} = \begin{bmatrix} r_{S_D}\left(\frac{1}{2}\boldsymbol{I} - \mathring{\boldsymbol{\mathcal{W}}}'\right) & r_{S_D}\mathring{\boldsymbol{\mathcal{L}}} \\ -r_{S_N}\mathring{\boldsymbol{\mathcal{V}}} & r_{S_N}\left(\frac{1}{2}\boldsymbol{I} + \mathring{\boldsymbol{\mathcal{W}}}\right) \end{bmatrix}, \tag{2.133}$$

and the right hand side $\mathring{\mathcal{F}}^{22} \in \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N)$ is given by

$$\mathring{\mathcal{F}}^{22} = \begin{bmatrix} r_{S_D}\left(\mathring{T}^{+}\boldsymbol{F} - \boldsymbol{\Psi}_0\right) \\ r_{S_N}\left(\gamma^{+}\boldsymbol{F} - \boldsymbol{\Phi}_0\right) \end{bmatrix}. \tag{2.134}$$

The operator $\mathring{\mathcal{M}}^{22}$ is evidently continuous. Moreover, in virtue of Corollary 2.26, the operator $\mathring{\mathcal{M}}^{22}$ is also injective.

Theorem 2.27. *The operator*

$$\mathring{\mathcal{M}}^{22} : \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N) \longrightarrow \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N)$$

is invertible.

Proof. A solution of the system (2.132) with an arbitrary right hand side

$$\mathring{\mathcal{F}}^{22} = [\widehat{\boldsymbol{\mathcal{F}}}_2^{22}, \widehat{\boldsymbol{\mathcal{F}}}_3^{22}]^{\top} \in \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N) \tag{2.135}$$

is given by the pair $(\boldsymbol{\psi}, \boldsymbol{\varphi})$ which satisfies the following extended system:

$$\widehat{\mathcal{M}}^{22}\mathcal{X} = \widehat{\mathcal{F}}^{22}, \tag{2.136}$$

where $\mathcal{X} = (p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})^{\top}$, $\widehat{\mathcal{F}}^{22} = (0, \boldsymbol{0}, \mathring{\boldsymbol{\mathcal{F}}}_2^{22}, \mathring{\boldsymbol{\mathcal{F}}}_3^{22})^{\top}$ and

$$\widehat{\mathcal{M}}^{22} = \begin{bmatrix} I & 0 & -\mathring{\mathcal{P}} & \mathring{\Pi} \\ 0 & \boldsymbol{I} & -\mathring{\boldsymbol{V}} & \mathring{\boldsymbol{W}} \\ 0 & 0 & r_{S_D}\left(\frac{1}{2}\boldsymbol{I} - \mathring{\boldsymbol{\mathcal{W}}}'\right) & r_{S_D}\mathring{\boldsymbol{\mathcal{L}}} \\ 0 & 0 & -r_{S_N}\mathring{\boldsymbol{\mathcal{V}}} & r_{S_N}\left(\frac{1}{2}\boldsymbol{I} + \mathring{\boldsymbol{\mathcal{W}}}\right) \end{bmatrix} \tag{2.137}$$

In virtue of Theorem 2.25 with $\mu = 1$, the operator $\widehat{\mathcal{M}}^{22}$ has a bounded inverse. □

Theorem 2.28. *The operator*

$$\mathcal{M}_*^{22} : L^2(\Omega)\times\boldsymbol{H}^1(\Omega)\times\widetilde{\boldsymbol{H}}^{-1/2}(S_D)\times\widetilde{\boldsymbol{H}}^{1/2}(S_N) \longrightarrow L^2(\Omega)\times\boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^{-1/2}(S_D)\times\boldsymbol{H}^{1/2}(S_N),$$

is continuously invertible.

Proof. Let us consider the following operator

$$\widetilde{\mathcal{M}}^{22} = \begin{bmatrix} I & 0 & -\mathcal{P} & \Pi \\ 0 & \boldsymbol{I} & -\boldsymbol{V} & \boldsymbol{W} \\ 0 & 0 & r_{S_D}\left(\frac{1}{2}\boldsymbol{I} - \mathring{\boldsymbol{\mathcal{W}}}'\right) & r_{S_D}\widehat{\boldsymbol{\mathcal{L}}} \\ 0 & 0 & -r_{S_N}\boldsymbol{\mathcal{V}} & r_{S_N}\left(\frac{1}{2}\boldsymbol{I} + \boldsymbol{\mathcal{W}}\right) \end{bmatrix} \tag{2.138}$$

The operator $\widetilde{\mathcal{M}}^{22}$ is a compact perturbation of the operator $\mathcal{M}^{22}_{*}$ due to the compact properties given by Theorem 2.6, Theorem 2.4 and Corollary 2.10. Using the relations (2.23) and (2.20), we can express the operator $\widetilde{\mathcal{M}}^{22}$ in the form

$$\widetilde{\mathcal{M}}^{22} = diag\left(1, \frac{1}{\mu}\boldsymbol{I}, \boldsymbol{I}, \frac{1}{\mu}\boldsymbol{I}\right)\widehat{\mathcal{M}}^{22} diag(1, \mu\boldsymbol{I}, \boldsymbol{I}, \mu\boldsymbol{I}) \tag{2.139}$$

where $diag(a, b\boldsymbol{I}, c\boldsymbol{I}, d\boldsymbol{I})$ represents a 10 by 10 diagonal matrix

$$diag(a, b\boldsymbol{I}, c\boldsymbol{I}, d\boldsymbol{I}) = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & \boldsymbol{b} & 0 & 0 \\ 0 & 0 & \boldsymbol{c} & 0 \\ 0 & 0 & 0 & \boldsymbol{d} \end{bmatrix}. \tag{2.140}$$

The operator $\widehat{\mathcal{M}}^{22}$ defined by (2.137) can be understood as a triangular block matrix with the three following diagonal operators

$$\begin{aligned} I :&\ L^2(\Omega^+) \longrightarrow L^2(\Omega^+), \\ I :&\ \boldsymbol{H}^1(\Omega^+) \longrightarrow \boldsymbol{H}^1(\Omega^+), \\ \mathring{\mathcal{M}}^{22} :&\ \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N) \longrightarrow \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N). \end{aligned}$$

In virtue of Theorem 2.27, the operator $\mathring{\mathcal{M}}^{22}$ is invertible. Consequently, $\widehat{\mathcal{M}}^{22}$ is an invertible operator as well. As μ is strictly positive, the diagonal matrices are invertible and the operator $\widetilde{\mathcal{M}}^{22}$ is also invertible. Thus, the operator $\mathcal{M}^{22}_{*}$ is a zero index Fredholm operator.

The invertibility of the operator simply follows from the injectivity of the operator $\mathcal{M}^{22}_{*}$ derived from Theorem 2.22 (iii). □

The last three vector equations of the system (M22) are segregated from p. Therefore, we can define the new system given by equations (2.99b)-(2.99d) which can be written using matrix notation as

$$\mathcal{M}^{22}\mathcal{Y} = \mathcal{F}^{22}, \tag{2.141}$$

where $\mathcal{Y}$ represents the vector containing the unknowns of the system

$$\mathcal{Y} = (\boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\phi}) \in \boldsymbol{H}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N),$$

and the matrix operator $\mathcal{M}^{22}$ is given by

$$\mathcal{M}^{22} := \begin{bmatrix} \boldsymbol{I} + \boldsymbol{\mathcal{R}} & -\boldsymbol{V} & \boldsymbol{W} \\ r_{S_D}\boldsymbol{T}^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) & r_{S_D}\left(\frac{1}{2}\boldsymbol{I} - \boldsymbol{\mathcal{W}}'\right) & r_{S_D}\boldsymbol{\mathcal{L}}^+ \\ r_{S_N}\gamma^+\boldsymbol{\mathcal{R}} & -r_{S_N}\boldsymbol{\mathcal{V}} & r_{S_N}\left(\frac{1}{2}\boldsymbol{I} + \boldsymbol{\mathcal{W}}\right) \end{bmatrix}.$$

Theorem 2.29. *The operator* $\mathcal{M}^{22}$

$$\mathcal{M}^{22} : \boldsymbol{H}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N) \longrightarrow \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N),$$

is continuous and continuously invertible.

Proof. The operator is continuous due to the mapping properties of the operators involved.

Let us assume that $\mathcal{M}^{22}$ is not invertible. Then, the system (2.141) has at least two different solutions $(\boldsymbol{v}_1, \boldsymbol{\psi}_1, \boldsymbol{\phi}_1)$ and $(\boldsymbol{v}_2, \boldsymbol{\psi}_2, \boldsymbol{\phi}_2)$. Then, using equation (2.99a), we can obtain the corresponding pressure for each of the two solutions. Hence, we have two solutions for the system (M22) $(p_1, \boldsymbol{v}_1, \boldsymbol{\psi}_1, \boldsymbol{\phi}_1)$ and $(p_2, \boldsymbol{v}_2, \boldsymbol{\psi}_2, \boldsymbol{\phi}_2)$. However, the BDIES (2.99a)-(2.99d) is uniquely solvable by virtue of Theorem 2.22. Therefore, both solutions must be the same what implies the invertibility of the operator $\mathcal{M}^{22}$.

Since the equations (2.99b)-(2.99d) coincide with the equations of the BDIES (2.101), the solution of the latter one given by $\mathcal{X} = \mathcal{M}^{22}_* \mathcal{F}^{22}_*$, where $\mathcal{X} = [p, \boldsymbol{v}, \boldsymbol{\phi}, \boldsymbol{\psi}]^\top$ provides the solution $\mathcal{Y} = [\boldsymbol{v}, \boldsymbol{\phi}, \boldsymbol{\psi}]^\top$ of the system $\mathcal{M}^{22}\mathcal{Y} = \mathcal{F}^{22}$ for any arbitrary right hand side $\mathcal{F}^{22}$ what implies the invertibility of the operator $\mathcal{M}^{22}$. □

Chapter 3

BDIES for the compressible Stokes system in exterior domains

3.1 Introduction

In this chapter, we derived two BDIES equivalent to the original mixed compressible Stokes system defined on a exterior domain. Furthermore, mapping properties of the hydrodynamic surface and volume potentials are shown in weighted Sobolev spaces. The main results are the equivalence theorems and the invertibility theorems of the operators defined by the BDIES.

3.2 Preliminaries

Let $\Omega := \Omega^+$ be a unbounded (exterior) simply connected domain and let $\Omega^- := \mathbb{R}^3 \smallsetminus \overline{\Omega}^+$ be the complementary (bounded) subset of Ω. The boundary $S := \partial\Omega$ is simply connected, closed and infinitely differentiable, $S \in \mathcal{C}^\infty$. Furthermore, $S := \overline{S}_N \cup \overline{S}_D$ where both S_N and S_D are non-empty, connected disjoint manifolds of S. In addition, the border of these two submanifolds is also infinitely differentiable, $\partial S_N = \partial S_D \in \mathcal{C}^\infty$.

To ensure uniquely solvability of the BVPs in exterior domains, we will use *weighted Sobolev spaces* with weight

$$\omega(\boldsymbol{x}) = (1 + |\boldsymbol{x}|^2)^{1/2},$$

(see e.g., [CMN13, Ha71, Ne03, Gr87, Gr78, LiMa73, NePl73]. Let

$$L^2(\omega^{-1}; \Omega) = \{g : \omega^{-1} g \in L^2(\Omega)\},$$

be the weighted Lebesgue space and $\mathcal{H}^1(\Omega)$ the following weighted Sobolev (Beppo-Levi) space constructed using the $L^2(\omega^{-1};\Omega)$ space

$$\mathcal{H}^1(\Omega) := \{g \in L^2(\omega^{-1};\Omega) : \nabla g \in L^2(\Omega)\}$$

endowed with the corresponding norm

$$\| g \|^2_{\mathcal{H}^1(\Omega)} := \| \omega^{-1} g \|^2_{L^2(\Omega)} + \| \nabla g \|^2_{L^2(\Omega)} .$$

The analogous vector counterpart of $\mathcal{H}^1(\Omega)$ reads

$$\boldsymbol{\mathcal{H}}^1(\Omega) := \{\boldsymbol{g} \in \boldsymbol{L}^2(\omega^{-1};\Omega) : \partial_j g_i \in L^2(\Omega)\}, \quad i,j \in \{1,2,3\}.$$

Taking into account that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^1(\Omega)$ it is easy to prove that $\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{\mathcal{H}}^1(\Omega)$. For further details, cf. [CMN13, p.3] and more references therein.

If Ω is unbounded, then the seminorm

$$|\boldsymbol{g}|_{\boldsymbol{\mathcal{H}}^1(\Omega)} := \| \nabla \boldsymbol{g} \|_{\boldsymbol{L}^2(\Omega)},$$

is equivalent to the norm $\| \boldsymbol{g} \|_{\boldsymbol{\mathcal{H}}^1(\Omega)}$ in $\boldsymbol{\mathcal{H}}^1(\Omega)$ [LiMa73, Chapter XI, Part B, §1]. On the contrary, if Ω^- is bounded, then $\boldsymbol{\mathcal{H}}^1(\Omega^-) = \boldsymbol{H}^1(\Omega^-)$. If Ω' is a bounded subdomain of an unbounded domain Ω and $\boldsymbol{g} \in \boldsymbol{\mathcal{H}}^1(\Omega)$, then $\boldsymbol{g} \in \boldsymbol{H}^1(\Omega')$.

Let us introduce $\widetilde{\boldsymbol{\mathcal{H}}}^1(\Omega)$ as the completion of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\boldsymbol{\mathcal{H}}^1(\mathbb{R}^3)$; let $\widetilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega) := [\boldsymbol{\mathcal{H}}^1(\Omega)]^*$ and $\boldsymbol{\mathcal{H}}^{-1}(\Omega) := [\widetilde{\boldsymbol{\mathcal{H}}}^1(\Omega)]^*$ be the corresponding dual spaces. Evidently, the space $\boldsymbol{L}^2(\omega;\Omega) \subset \boldsymbol{\mathcal{H}}^{-1}(\Omega)$.

For any generalised function $\boldsymbol{g}$ in $\widetilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$, we have the following representation property (see ansatz (2.5.129) in [Ne03]), $g_j = \partial_i g_{ij} + g_j^0, \quad g_{ij} \in L^2(\mathbb{R}^3)$ and are zero outside the domain Ω, whereas $g_j^0 \in L^2(\omega;\Omega)$. Consequently, $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\widetilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$ and $\boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ is dense in $\boldsymbol{\mathcal{H}}^{-1}(\mathbb{R}^3)$.

Condition 3.1. *The remainder includes first order derivatives of the variable coefficient μ. For this reason, we will assume that $\mu \in \mathcal{C}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ as well as $\omega\nabla\mu \in L^\infty(\mathbb{R}^3)$. In addition, we will assume that there exist constants C_1 and C_2 such that*

$$0 < C_1 < \mu(\boldsymbol{x}) < C_2. \tag{3.1}$$

The operator $\boldsymbol{\mathcal{A}}$ acting on $\boldsymbol{v} \in \boldsymbol{\mathcal{H}}^1(\Omega)$ and $p \in L^2(\Omega)$ is well defined in the weak sense as long as the variable coefficient $\mu(\boldsymbol{x})$ is bounded, i.e. $\mu \in L^\infty(\Omega)$, as

$$\langle \boldsymbol{\mathcal{A}}(p, \boldsymbol{v}), \boldsymbol{u} \rangle_\Omega = -\mathcal{E}((p, \boldsymbol{v}), \boldsymbol{u}), \quad \forall \boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\Omega), \tag{3.2}$$

where $\mathcal{E}\left((p, \boldsymbol{v}), \boldsymbol{u}\right)(\boldsymbol{x})$ and $E\left((p, \boldsymbol{v}), \boldsymbol{u}\right)$ are defined as in (2.2) and (2.3) respectively.

$$\begin{aligned} E\left((p, \boldsymbol{v}), \boldsymbol{u}\right)(\boldsymbol{x}) := \; & \frac{1}{2}\mu(\boldsymbol{x}) \left(\frac{\partial u_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial u_j(\boldsymbol{x})}{\partial x_i} \right) \left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i} \right) \\ & - \frac{2}{3}\mu(\boldsymbol{x}) \operatorname{div} \boldsymbol{v}(\boldsymbol{x}) \operatorname{div} \boldsymbol{u}(\boldsymbol{x}) - p(\boldsymbol{x}) \operatorname{div} \boldsymbol{u}(\boldsymbol{x}). \end{aligned}$$

The bilinear functional

$$\mathcal{E}\left((p, \boldsymbol{v}), \boldsymbol{u}\right) : \left(L^2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega)\right) \times \widetilde{\boldsymbol{\mathcal{H}}}^1(\Omega) \longrightarrow \mathbb{R}$$

is bounded. Thus, by density of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\widetilde{\boldsymbol{\mathcal{H}}}^1(\Omega)$, the operator

$$\boldsymbol{\mathcal{A}} : L^2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \longrightarrow \boldsymbol{\mathcal{H}}^{-1}(\Omega)$$

is also bounded and gives the weak form of the operator (2.1).

We will also make use of the following space, (cf. e.g., [Co88] [CMN09]),

$$\boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) := \{(p, \boldsymbol{v}) \in L^2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) : \boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) \in \boldsymbol{L}^2(\omega; \Omega)\},$$

endowed with the norm, $\| \cdot \|_{\boldsymbol{\mathcal{H}}^{1,0}(\Omega; L)}$, where

$$\|(p, \boldsymbol{v})\|_{\boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})} := \left(\|p\|^2_{L^2(\Omega)} + \|\boldsymbol{v}\|^2_{\boldsymbol{\mathcal{H}}^1(\Omega)} + \|\omega \boldsymbol{\mathcal{A}}(p, \boldsymbol{v})\|^2_{\boldsymbol{L}_2(\Omega)} \right)^{1/2}.$$

For sufficiently smooth functions $\boldsymbol{v}$ and p in $\Omega^\pm$, we can write the classical traction operators on the boundary S as (2.4), in which $n_j(\boldsymbol{x})$ denote the components of the unit normal vector $\boldsymbol{n}(\boldsymbol{x})$ to the boundary S *directed outwards the exterior domain* Ω. Moreover, $\boldsymbol{\gamma}^\pm$ denote the trace operators from inside and outside Ω which according to the trace theorem satisfy the mapping property $\gamma^\pm : \boldsymbol{\mathcal{H}}^1(\Omega) \longrightarrow \boldsymbol{H}^{1/2}(S)$, [CMN13, p.4].

Traction operators (2.4) can be continuously extended to the *canonical* traction operators $\boldsymbol{T}^{\pm} : \boldsymbol{\mathcal{H}}^{1,0}(\Omega^{\pm}, \boldsymbol{\mathcal{A}}) \to \boldsymbol{H}^{-1/2}(S)$ defined in the weak form similar to [Co88, Mi11, CMN13] as

$$\langle \boldsymbol{T}^{+}(p, \boldsymbol{v}), \boldsymbol{w} \rangle_S := \int_{\Omega^{\pm}} \left[\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) \boldsymbol{\gamma}^{+}_{-1} \boldsymbol{w} + E\left((p, \boldsymbol{v}), \boldsymbol{\gamma}^{+}_{-1} \boldsymbol{w} \right) \right] dx$$
$$\forall\, (p, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega^{\pm}, \boldsymbol{\mathcal{A}}),\ \forall\, \boldsymbol{w} \in \boldsymbol{H}^{1/2}(S),$$

where the operator $\boldsymbol{\gamma}^{+}_{-1} : \boldsymbol{H}^{1/2}(S) \to \boldsymbol{\mathcal{H}}^{1}(\Omega)$ denotes a continuous right inverse of the trace operator $\boldsymbol{\gamma}^{+} : \boldsymbol{\mathcal{H}}^{1}(\Omega) \to \boldsymbol{H}^{1/2}(S)$.

Furthermore, if $(p, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$ and $\boldsymbol{u} \in \boldsymbol{\mathcal{H}}^{1}(\Omega)$, the first Green identity (2.5) holds.

Applying the identity (2.5) to the pairs $(p, \boldsymbol{v}), (q, \boldsymbol{u}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$ with exchanged roles and subtracting the one from the other, we arrive at the second Green identity, cf. [McL00, Mi11],

$$\int_S \left(T_j(p, \boldsymbol{v}) u_j - T_j(q, \boldsymbol{u}) v_j \right) \; dS(\boldsymbol{x}) =$$
$$\int_{\Omega} \Big[\; \mathcal{A}_j(p, \boldsymbol{v}) u_j - \mathcal{A}_j(q, \boldsymbol{u}) v_j + q \operatorname{div} \boldsymbol{v} - p \operatorname{div} \boldsymbol{u} \Big] \; dx. \tag{3.3}$$

Mixed problem For $\boldsymbol{f} \in \boldsymbol{L}_2(\omega, \Omega)$, $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(S_D)$, $g \in L_2(\omega, \Omega)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(S_N)$, find $(p, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$ such that:

$$\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) = \boldsymbol{f}, \quad \mathit{in}\ \Omega, \tag{3.4a}$$

$$\operatorname{div}(\boldsymbol{v})(\boldsymbol{x}) = g, \ \mathit{in}\ \Omega, \tag{3.4b}$$

$$\boldsymbol{\gamma}^{+} \boldsymbol{v} = \boldsymbol{\varphi}_0, \ \mathit{on}\ S_D, \tag{3.4c}$$

$$\boldsymbol{T}^{+}(p, \boldsymbol{v}) = \boldsymbol{\psi}_0, \ \mathit{on}\ S_N. \tag{3.4d}$$

Theorem 3.2. *The mixed BVP* (3.4) *has at most one solution in the space* $\boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$.

Proof. Let us suppose that there are two possible solutions: $(p_1, \boldsymbol{v}_1)$ and $(p_2, \boldsymbol{v}_2)$ belonging to the space $\boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$, that satisfy the BVP (3.4). Then, the pair $(p, \boldsymbol{v}) := (p_2, \boldsymbol{v}_2) -$

$(p_1, \boldsymbol{v}_1) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ satisfies the homogeneous mixed BVP

$$\boldsymbol{\mathcal{A}}(p, \boldsymbol{v})(\boldsymbol{x}) = \boldsymbol{0}, \ \ \boldsymbol{x} \in \Omega, \tag{3.5a}$$

$$\operatorname{div}(\boldsymbol{v})(\boldsymbol{x}) = 0, \ \ \boldsymbol{x} \in \Omega, \tag{3.5b}$$

$$r_{S_D} \boldsymbol{\gamma}^+ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{0}, \ \boldsymbol{x} \in S_D, \tag{3.5c}$$

$$r_{S_N} \boldsymbol{T}^+(p, \boldsymbol{v})(\boldsymbol{x}) = \boldsymbol{0}, \ \boldsymbol{x} \in S_N. \tag{3.5d}$$

Substituting $(p, \boldsymbol{v})$ in the first Green identity (2.5), since this one holds for any $(q, \boldsymbol{u}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$, in particular $(q, \boldsymbol{u}) \in \boldsymbol{\mathcal{H}}^1_{S_D, div}(\Omega; \boldsymbol{\mathcal{A}})$ where

$$\boldsymbol{\mathcal{H}}^1_{S_D, div}(\Omega; \boldsymbol{\mathcal{A}}) := \{(q, \boldsymbol{u}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) : \boldsymbol{\gamma}^+_{S_D} \boldsymbol{u} = \boldsymbol{0}, \ \operatorname{div} \boldsymbol{u} = 0 \ \ in \ \Omega\}.$$

The first Green identity applied to any $(q, \boldsymbol{u}) \in \boldsymbol{\mathcal{H}}^1_{S_D, div}(\Omega; \boldsymbol{\mathcal{A}})$ and $(p, \boldsymbol{v})$ results in (2.9).

In particular, one could choose $(q, \boldsymbol{u}) := (p, \boldsymbol{v})$ with $(p, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^1_{S_D, div}(\Omega; \boldsymbol{\mathcal{A}})$. Then, the first Green identity (2.5) yields and the rest of the uniqueness arguments repeats as in the end of the proof of Theorem 2.1. □

This BVP (3.4) can be represented by the following operator:

$$\boldsymbol{\mathcal{A}}_M : \boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}}) \longrightarrow \boldsymbol{L}^2(\omega; \Omega) \times \boldsymbol{L}^2(\omega, \Omega) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{-1/2}(S_N). \tag{3.6}$$

3.3 Parametrix and Remainder

We recall from the previous chapter that the pair of functions $(q^k, \boldsymbol{u}^k)_{k=1,2,3}$ defined by (2.10) and (2.11) is a parametrix of the operator $\boldsymbol{\mathcal{A}}$, i.e.,

$$\mathcal{A}_j(x; q^k, \boldsymbol{u}^k)(\boldsymbol{x}, \boldsymbol{y}) = \delta^k_j \delta(\boldsymbol{x} - \boldsymbol{y}) + R_{kj}(\boldsymbol{x}, \boldsymbol{y}). \tag{3.7}$$

Remark 3.3. *Let μ satisfy condition 3.1. Then $\mu(\boldsymbol{x})$ and $\dfrac{1}{\mu(\boldsymbol{x})}$ are multipliers in the space $\boldsymbol{\mathcal{H}}^1(\Omega)$.*

3.4 Hydrodynamic potentials

As the parametrix that we are considering is the same as in the previous chapter the hydrodynamic potentials defined in Section 2.4 remain the same. In addition, we will introduce the operators $\mathbf{U}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{R}^{\bullet}$ whose definitions coincide, respectively, with the definition of the operators $\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{R}}^{\bullet}$ with the sole difference that $\Omega = \mathbb{R}^3$.

Condition 3.4. *Apart from the condition 3.1, we will sometimes also assume the following:*

$$\mu \in \mathcal{C}^2(\mathbb{R}^3); \qquad \omega^2 \partial_j \partial_i \mu \in L^\infty(\mathbb{R}^3). \tag{3.8}$$

3.4.1 Mapping properties

The following assertions have been recently studied by [KLMW] for the constant coefficient case. Then by relations (2.16)-(2.24) we obtain their counterparts for the variable-coefficient case. Let us highlight that the operators $\boldsymbol{U}, \boldsymbol{Q}, Q, \boldsymbol{R}, R^{\bullet}$ are defined in the same way as $\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}, \mathcal{Q}, \boldsymbol{\mathcal{R}}$ and $\mathcal{R}^{\bullet}$ with the particularity that $\Omega = \mathbb{R}^3$.

Theorem 3.5. *The following vector operators are continuous under condition 3.1,*

$$\boldsymbol{U} : \boldsymbol{\mathcal{H}}^{-1}(\mathbb{R}^3) \longrightarrow \boldsymbol{\mathcal{H}}^1(\mathbb{R}^3), \tag{3.9}$$

$$\boldsymbol{\mathcal{U}} : \widetilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega) \longrightarrow \boldsymbol{\mathcal{H}}^1(\Omega), \tag{3.10}$$

$$\boldsymbol{Q} : L^2(\mathbb{R}^3) \longrightarrow \boldsymbol{\mathcal{H}}^1(\mathbb{R}^3), \tag{3.11}$$

$$\boldsymbol{\mathcal{Q}} : L^2(\Omega) \longrightarrow \boldsymbol{\mathcal{H}}^1(\Omega), \tag{3.12}$$

$$Q : \boldsymbol{\mathcal{H}}^{-1}(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^3), \tag{3.13}$$

$$\mathcal{Q} : \widetilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega) \longrightarrow L^2(\Omega), \tag{3.14}$$

$$\boldsymbol{R} : \boldsymbol{L}^2(\omega^{-1}; \mathbb{R}^3) \longrightarrow \boldsymbol{\mathcal{H}}^1(\mathbb{R}^3), \tag{3.15}$$

$$\boldsymbol{\mathcal{R}} : \boldsymbol{L}^2(\omega^{-1}; \Omega) \longrightarrow \boldsymbol{\mathcal{H}}^1(\Omega), \tag{3.16}$$

$$R^{\bullet} : \boldsymbol{L}^2(\omega^{-1}; \mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^3), \tag{3.17}$$

$$\mathcal{R}^{\bullet} : \boldsymbol{L}^2(\omega^{-1}; \Omega) \longrightarrow L^2(\Omega). \tag{3.18}$$

Proof. Let us consider relations (2.16) and (2.18). Then, the continuity of the operators $\boldsymbol{U}$, $\boldsymbol{\mathcal{U}}$, Q and $\mathcal{Q}$ then follows from the continuity of the operators $\mathring{\boldsymbol{\mathcal{U}}}$, $\mathring{\boldsymbol{U}}$, $\mathring{\mathcal{Q}}$ and $\mathring{Q}$ proved in [KLMW, Lemma 3.1].

Let us prove now the continuity of the operator (3.15). Taking into consideration (2.17), we need only need to prove the continuity of the terms $\partial_j \mathring{\mathcal{U}}_{ki}(g_j \partial_i \mu)$ and $\mathring{\mathcal{Q}}_k(g_j \partial_j \mu)$.

First, let us note that by condition 3.1, μ and $\dfrac{1}{\mu}$ are bounded and act as multipliers in the space $\boldsymbol{\mathcal{H}}^1(\Omega)$. In addition, condition 3.1 states that $\omega \partial_i \mu \in L^\infty(\mathbb{R}^3)$. Consequently, for any function $g_j \in L^2(\omega^{-1}; \mathbb{R}^3)$, we have that $g_j \partial_i \mu \in L^2(\mathbb{R}^3)$, see [CMN13, Theorem 4.1].

Let us prove continuity of the term $\partial_j \mathring{\mathcal{U}}_{ki}(g_j \partial_i \mu)$ in (2.17). For this purpose, we consider a function g_j in $\mathcal{D}(\mathbb{R}^3)$. Then

$$\partial_j \mathring{\mathcal{U}}_{ki}(g_j \partial_i \mu) = -\mathring{\mathcal{U}}_{ki} \partial_j (g_j \partial_i \mu). \tag{3.19}$$

Considering the density of $\mathcal{D}(\mathbb{R}^3)$ in $\mathcal{H}^{-1}(\mathbb{R}^3)$, we can extend the relation (3.19) from $\mathcal{D}(\mathbb{R}^3)$ to $\mathcal{H}^{-1}(\mathbb{R}^3)$. Hence, we only need to prove that $\partial_j(g_j \partial_i \mu) \in \mathcal{H}^{-1}(\Omega)$. To do this, we will use again the density of $\mathcal{D}(\mathbb{R}^3)$ in $\mathcal{H}^{-1}(\mathbb{R}^3)$. Let $\rho_n \in \mathcal{D}(\mathbb{R}^3)$ converging to $g_j \partial_i \mu \in L^2(\mathbb{R}^3)$. Then, $\partial_j(\rho_n)$ will converge to $\partial_j(g_j \partial_i \mu)$ in $H^{-1}(\mathbb{R}^3) \subset \mathcal{H}^{-1}(\mathbb{R}^3)$. Then, the continuity of the operator $\partial_j \mathring{\mathcal{U}}_{ki}(g_j \partial_i \mu)$ follows from the continuity of the operator (3.9).

The continuity of the operators $\partial_i \mathring{\mathcal{U}}_{kj}(g_j \partial_i \mu)$ and $\mathring{\mathcal{Q}}_k(g_j \partial_j \mu)$ can be proved in a similar way. Consequently, the operator (3.15) is continuous.

Continuity of the operator (3.15) implies the continuity of the operator (3.16).

Let us prove now the continuity of the operator (3.17). Taking into account (2.19), the continuity of the operator (3.17) will follow from the continuity of the operator $\partial_j \mathcal{Q}(\partial_i \mu g_j)$. Applying a similar density argument as for the previous proof, we can deduce $\partial_j \mathcal{Q}(\partial_i \mu g_j) = -\mathcal{Q} \partial_j(\partial_i \mu g_j)$. Since, $\partial_j(\partial_i \mu g_j) \in \mathcal{H}^{-1}(\mathbb{R}^3)$, the continuity of $\partial_j \mathcal{Q}(\partial_i \mu g_j)$ directly follows now from the continuity of the operator (3.14).

Continuity of the operator (3.18) is implied by the continuity of the operator (3.17). The mapping property of the operator (3.11)-(3.12) differs from the vector operators Q and $\mathcal{Q}$ and need to be proven.

Let us consider $\phi \in \mathcal{D}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$. Note the following property

$$\mathring{q}^k = \frac{\partial E_\Delta}{\partial x_k}. \tag{3.20}$$

where

$$E_\Delta(x, y) = \frac{-1}{4\pi|x-y|},$$

is the fundamental solution of the Laplace equation. The newtonian volume potential is defined as

$$P_\Delta \phi(y) = \int_{\mathbb{R}^3} E_\Delta(x, y)\, \phi(x)\, dx, \tag{3.21}$$

and solves the Poisson equation $\Delta\omega = \phi$ in $\mathbb{R}^3$. It is well known that the Laplace operator $\Delta : \mathcal{H}^1(\mathbb{R}^3) \longrightarrow \mathcal{H}^{-1}(\mathbb{R}^3)$ has a continuous inverse, $\Delta^{-1} : \mathcal{H}^{-1}(\mathbb{R}^3) \longrightarrow \mathcal{H}^1(\mathbb{R}^3)$. The inverse operator, Δ^{-1} can be seen as a continuous extension of the operator P_Δ due to the density of $\mathcal{D}(\mathbb{R}^3)$ within $\mathcal{H}^{-1}(\mathbb{R}^3)$. Take into account (3.20), we can deduce

$$\mathring{\mathcal{Q}}_k \phi = \int_{\mathbb{R}^3} \mathring{q}^k(x, y)\phi(x)\, dx = \int_{\mathbb{R}^3} \frac{\partial E_\Delta}{\partial x_k}(x, y)\phi(x)\, dx = -\int_{\mathbb{R}^3} E_\Delta(x, y) \frac{\partial \phi(x)}{\partial x_k}\, dx.$$

Hence, $\mathring{\mathcal{Q}}_k \phi = -P_\Delta(\partial_k \phi)$.

Let $\phi \in L^2(\mathbb{R}^3)$, then for any function $g \in \mathcal{D}(\Omega)$, we have that

$$\langle \phi\, ,\, \partial_k g \rangle_{\mathbb{R}^3} = -\langle \partial_k \phi\, ,\, g \rangle_{\mathbb{R}^3} \tag{3.22}$$

Hence, as the first term in (3.22) is the scalar product in $L^2(\mathbb{R}^3)$ which is well defined since the integrand belongs to $L^2(\mathbb{R}^3)$ by virtue of the Cauchy-Schwartz inequality. Therefore, also is the second term. By density, we can extend (3.22) from $g \in \mathcal{D}(\mathbb{R}^3)$ to $g \in H^1(\mathbb{R}^3)$ by density, and hence to $\mathcal{H}^1(\mathbb{R}^3)$.

Now, the second term could thus be interpreted as $\partial_k \phi \in H^{-1}(\mathbb{R}^3) \subset \mathcal{H}^1(\mathbb{R}^3)$ since $\mathcal{D}(\mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3)$, see [Bn10, Corollary 10.4.11].

As a result, $\mathring{\mathcal{Q}}_k \phi = -P_\Delta(\partial_k \phi) \in \mathcal{H}^1(\mathbb{R}^3)$. Consequently, by (2.18), the mapping property (3.11) follows and thus also does (3.12).

□

Theorem 3.6. *The following operators are continuous under condition 3.1*

$$\boldsymbol{V} : \boldsymbol{H}^{-1/2}(S) \longrightarrow \boldsymbol{\mathcal{H}}^{1}(\Omega), \tag{3.23}$$

$$\mathcal{P} : \boldsymbol{H}^{-1/2}(S) \longrightarrow L^{2}(\Omega), \tag{3.24}$$

$$\boldsymbol{W} : \boldsymbol{H}^{1/2}(S) \longrightarrow \boldsymbol{\mathcal{H}}^{1}(\Omega), \tag{3.25}$$

$$\Pi : \boldsymbol{H}^{1/2}(S) \longrightarrow L^{2}(\Omega). \tag{3.26}$$

Proof. Let us consider relations (2.20) and (2.22). The continuity of the operators $\boldsymbol{V}$, $\mathcal{P}$, $\boldsymbol{W}$ and $\boldsymbol{\Pi}$ then follows from the continuity of the operators $\mathring{\boldsymbol{V}}$, $\mathring{\boldsymbol{W}}$, $\mathring{\mathcal{P}}$ and $\mathring{\Pi}$ which has already being proved in [KLMW, Lemma 3.3]. □

Corollary 3.7. *The following operators are continuous under conditions 3.1 and condition 3.4,*

$$(\mathcal{P}, \boldsymbol{V}) : \boldsymbol{H}^{-1/2}(S) \longrightarrow \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}), \tag{3.27}$$

$$(\Pi, \boldsymbol{W}) : \boldsymbol{H}^{1/2}(S) \longrightarrow \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}), \tag{3.28}$$

$$(\mathcal{Q}, \boldsymbol{\mathcal{U}}) : \boldsymbol{L}^{2}(\omega; \Omega) \longrightarrow \boldsymbol{\mathcal{H}}^{1,0}(\mathbb{R}^3; \boldsymbol{\mathcal{A}}), \tag{3.29}$$

$$(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}}) : \boldsymbol{\mathcal{H}}^{1}(\Omega) \longrightarrow \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}), \tag{3.30}$$

$$\left(\frac{4}{3}\mu I, \boldsymbol{\mathcal{Q}}\right) : L^{2}(\Omega) \longrightarrow \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}). \tag{3.31}$$

Proof. Let us consider first the single layer potentials $(\mathcal{P}\boldsymbol{g}, \boldsymbol{V}\boldsymbol{g}) \in \boldsymbol{\mathcal{H}}^{1}(\Omega) \times L^{2}(\Omega)$ for $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(S)$. Let us apply the operator $\boldsymbol{\mathcal{A}}$ taking into consideration (2.20) and (2.22)

$$\begin{aligned}\boldsymbol{\mathcal{A}}(\mathcal{P}\boldsymbol{g}, \boldsymbol{V}\boldsymbol{g}) &= \boldsymbol{\mathcal{A}}\left(\mathring{\mathcal{P}}\boldsymbol{g}, \frac{1}{\mu}\mathring{\boldsymbol{V}}\boldsymbol{g}\right) \\ &= \mathring{\mathcal{A}}_j\left(\mathring{\mathcal{P}}\boldsymbol{g}, \mathring{V}_k\boldsymbol{g}\right) + \partial_k\left(\mu\left[\partial_j(1/\mu)\mathring{V}_k\boldsymbol{g} + \partial_k(1/\mu)\mathring{V}_j\boldsymbol{g} - \frac{2}{3}\delta_j^k\partial_i(1/\mu)\mathring{V}_i\boldsymbol{g}\right]\right).\end{aligned}$$

Now, the term $\mathring{\mathcal{A}}_j\left(\mathring{\mathcal{P}}\boldsymbol{g}, \mathring{V}_k\boldsymbol{g}\right)$ vanishes and the second term belongs to $\boldsymbol{L}^{2}(\omega; \Omega)$ since $\boldsymbol{V}\boldsymbol{g} \in \boldsymbol{\mathcal{H}}^{1}(\Omega)$. The same argument works for the double layer potential $(\boldsymbol{W}, \Pi)\boldsymbol{g}$ with $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(S)$. In addition it works for the Newtonian potentials $(\boldsymbol{\mathcal{U}}, \mathcal{Q})$ with the sole difference that $\mathring{\mathcal{A}}_j\left(\mathring{\mathcal{Q}}\boldsymbol{g}, \mathring{\mathcal{U}}_k\boldsymbol{g}\right) = g_j$ and $\boldsymbol{g} \in \boldsymbol{L}^{2}(\omega; \Omega)$.

For the remainder operators $g \in \boldsymbol{\mathcal{H}}^1(\Omega)$ and hence $(\mathcal{R}^\bullet \boldsymbol{g}, \boldsymbol{\mathcal{R}}\boldsymbol{g}) \in \boldsymbol{\mathcal{H}}^1(\Omega) \times \boldsymbol{\mathcal{H}}^2(\Omega)$. Consequently, $\boldsymbol{\mathcal{A}}(\mathcal{R}^\bullet \boldsymbol{g}, \boldsymbol{\mathcal{R}}\boldsymbol{g}) \in \boldsymbol{\mathcal{H}}^1(\Omega) \subset \boldsymbol{L}^2(\omega;\Omega)$.

For the operator $(\frac{4}{3}g\mu, \mathring{\boldsymbol{\mathcal{Q}}}(\mu g))$, we proceed on a similar manner as before

$$\boldsymbol{\mathcal{A}}\left(\frac{4}{3}g\mu, \boldsymbol{\mathcal{Q}}g\right) = \boldsymbol{\mathcal{A}}\left(\frac{4}{3}g\mu, \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{Q}}}(\mu g)\right)$$
$$= \mathring{\mathcal{A}}_j\left(\frac{4}{3}g\mu, \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{Q}}}(\mu g)\right) + \partial_k\left(\partial_j(1/\mu)\mathring{\mathcal{Q}}_k(\mu g) + \partial_k(1/\mu)\mathring{\mathcal{Q}}_j(\mu g) - \frac{4}{3}\delta_{jk}\partial_i(1/\mu)\mathring{\mathcal{Q}}_k(\mu g)\right).$$

Since $\mathring{\mathcal{Q}}_j(\mu g) \in \mathcal{H}^1(\Omega)$ by virtue of Theorem 3.5, the whole second term belongs to $L^2(\omega;\Omega)$. Hence, we only need to prove that $\mathring{\mathcal{A}}_j(\frac{4}{3}g\mu, \mathring{\boldsymbol{\mathcal{Q}}}(\mu g)) \in L^2(\omega;\Omega)$. We have that (2.46) holds. Hence, $(\frac{4}{3}g\mu, \mathring{\boldsymbol{\mathcal{Q}}}(\mu g)) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$, as in the proof of the analogous property (2.44) for bounded domains. □

3.5 The Third Green Identities

Third Green identities for $(p,\boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ and to the parametrix $(q^k, \boldsymbol{u}^k)$ can be obtained following a similar approach as in the previous chapter, see Theorem 2.12 and identities (2.50).

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{T}^+(p,\boldsymbol{v}) + \boldsymbol{W}\boldsymbol{\gamma}^+\boldsymbol{v} = \boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{A}}(p,\boldsymbol{v}) + \boldsymbol{\mathcal{Q}}div(\boldsymbol{v}) \quad \text{in } \Omega, \tag{3.32}$$

$$p + \mathcal{R}^\bullet\boldsymbol{v} - \mathcal{P}\boldsymbol{T}(p,\boldsymbol{v}) + \Pi\boldsymbol{\gamma}^+\boldsymbol{v} = \mathcal{Q}\boldsymbol{\mathcal{A}}(p,\boldsymbol{v}) + \frac{4\mu}{3}\text{div}(\boldsymbol{v}) \quad \text{in } \Omega. \tag{3.33}$$

If the couple $(p,\boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ is a solution of the Stokes PDE (3.4a) with variable coefficient, then (3.32) and (3.33) give

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{T}^+(p,\boldsymbol{v}) + \boldsymbol{W}\boldsymbol{\gamma}^+\boldsymbol{v} = \boldsymbol{\mathcal{U}}\boldsymbol{f} + \boldsymbol{\mathcal{Q}}g, \tag{3.34}$$

$$p + \mathcal{R}^\bullet\boldsymbol{v} - \mathcal{P}\boldsymbol{T}(p,\boldsymbol{v}) + \Pi\boldsymbol{\gamma}^+\boldsymbol{v} = \mathcal{Q}\boldsymbol{f} + \frac{4\mu}{3}g \quad \text{in } \Omega. \tag{3.35}$$

Let us recall that the traction operator is well defined due to the mapping properties provided by the Theorem 3.7. Consequently we can obtain the trace and traction of the third Green identities for $(p,\boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ on S.

$$1/2\boldsymbol{\gamma}^+\boldsymbol{v} + \boldsymbol{\gamma}^+\boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{\mathcal{V}}\boldsymbol{T}^+(p,\boldsymbol{v}) + \boldsymbol{\mathcal{W}}\boldsymbol{\gamma}^+\boldsymbol{v} = \boldsymbol{\gamma}^+\boldsymbol{\mathcal{U}}\boldsymbol{f} + \boldsymbol{\gamma}^+\boldsymbol{\mathcal{Q}}g, \tag{3.36}$$

$$1/2\boldsymbol{T}^{+}(p,\boldsymbol{v})+\boldsymbol{T}^{+}(\mathcal{R}^{\bullet},\boldsymbol{\mathcal{R}})\boldsymbol{v}-\boldsymbol{\mathcal{W}}'\boldsymbol{T}^{+}(p,\boldsymbol{v})+\boldsymbol{\mathcal{L}}^{+}\boldsymbol{\gamma}^{+}\boldsymbol{v}=\widetilde{\boldsymbol{T}}^{+}(\boldsymbol{f},g), \tag{3.37}$$

where

$$\widetilde{\boldsymbol{T}}^{+}(\boldsymbol{f},g):=\boldsymbol{T}^{+}(\mathcal{Q}\boldsymbol{f}+\frac{4\mu}{3}g,\boldsymbol{\mathcal{U}}\boldsymbol{f}+\boldsymbol{\mathcal{Q}}g) \tag{3.38}$$

One can prove the following two assertions that are instrumental for proof of equivalence of the BDIES and the BVPs.

Lemma 3.8. *Let conditions 3.1 and 3.4 hold. Let* $\boldsymbol{v}\in\boldsymbol{\mathcal{H}}^{1}(\Omega)$*,* $p\in L_2(\Omega)$*,* $g\in L_2(\Omega)$*,* $\boldsymbol{f}\in\boldsymbol{L}_2(\omega;\Omega)$*,* $\boldsymbol{\Psi}\in\boldsymbol{H}^{-1/2}(S)$ *and* $\boldsymbol{\Phi}\in\boldsymbol{H}^{1/2}(S)$ *satisfy the equations*

$$p+\mathcal{R}^{\bullet}\boldsymbol{v}-\mathcal{P}\boldsymbol{\Psi}+\Pi\boldsymbol{\Phi}=\mathring{\mathcal{Q}}\boldsymbol{f}+\frac{4\mu}{3}g,\quad in\quad\Omega \tag{3.39}$$

$$\boldsymbol{v}+\boldsymbol{\mathcal{R}}\boldsymbol{v}-\boldsymbol{V}\boldsymbol{\Psi}+\boldsymbol{W}\boldsymbol{\Phi}=\boldsymbol{\mathcal{U}}\boldsymbol{f}+\boldsymbol{\mathcal{Q}}g,\quad in\quad\Omega. \tag{3.40}$$

Then $(p,\boldsymbol{v})\in\boldsymbol{\mathcal{H}}^{1,0}(\Omega,\boldsymbol{\mathcal{A}})$ *and solve the equations* $\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})=\boldsymbol{f}$ *and* $\operatorname{div}\boldsymbol{v}=g$*. Moreover, the following relations hold true:*

$$\mathcal{P}(\boldsymbol{\Psi}-\boldsymbol{T}^{+}(p,\boldsymbol{v}))-\Pi(\boldsymbol{\Phi}-\boldsymbol{\gamma}^{+}\boldsymbol{v})=0,\quad in\quad\Omega, \tag{3.41}$$

$$\boldsymbol{V}(\boldsymbol{\Psi}-\boldsymbol{T}^{+}(p,\boldsymbol{v}))-\boldsymbol{W}(\boldsymbol{\Phi}-\boldsymbol{\gamma}^{+}\boldsymbol{v})=\boldsymbol{0},\quad in\quad\Omega. \tag{3.42}$$

Proof. By virtue of Corollary 3.7, it is easy to deduce that $(p,\boldsymbol{v})\in\boldsymbol{\mathcal{H}}^{1,0}(\Omega,\boldsymbol{\mathcal{A}})$. The remaining part of the proof follows word by word from Theorem 2.13. □

Lemma 3.9. *Let* $S=\overline{S}_1\cup\overline{S}_2$*, where* S_1 *and* S_2 *are open non-empty non-intersecting simply connected submanifolds of* S *with infinitely smooth boundaries. Let* $\boldsymbol{\Psi}^{*}\in\widetilde{\boldsymbol{H}}^{-1/2}(S_1)$*,* $\boldsymbol{\Phi}^{*}\in\widetilde{\boldsymbol{H}}^{1/2}(S_2)$*. If*

$$\mathcal{P}(\boldsymbol{\Psi}^{*})-\Pi(\boldsymbol{\Phi}^{*})=\boldsymbol{0},\qquad\boldsymbol{V}\boldsymbol{\Psi}^{*}(\boldsymbol{x})-\boldsymbol{W}\boldsymbol{\Phi}^{*}(\boldsymbol{x})=\boldsymbol{0},\quad in\ \Omega, \tag{3.43}$$

then $\boldsymbol{\Psi}^{*}=\boldsymbol{0}$*, and* $\boldsymbol{\Phi}^{*}=\boldsymbol{0}$*, on* S*.*

Proof. Multiply the second equation in (3.43) by μ and apply relations (2.20)

$$\mathring{\boldsymbol{V}}\boldsymbol{\Psi}^{*}-\mathring{\boldsymbol{W}}(\mu\boldsymbol{\Phi}^{*})=\boldsymbol{0}\quad\text{on }\Omega. \tag{3.44}$$

Then apply the trace operator in both sides taking into account the jump relations given by Theorem 2.8,

$$\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\Psi}^*(\boldsymbol{x}) + \frac{1}{2}(\mu\boldsymbol{\Phi}^*) - \mathring{\boldsymbol{\mathcal{W}}}(\mu\boldsymbol{\Phi}^*) = \mathbf{0} \quad \text{in } S. \tag{3.45}$$

Apply the potential relations (2.22) to the first equation in (3.43) to obtain

$$\mathring{\mathcal{P}}\boldsymbol{\Psi}^* - \mathring{\Pi}(\mu\boldsymbol{\Phi}^*) = 0 \tag{3.46}$$

Now apply the traction operator at both sides of equations (3.45)-(3.46) taking as pressure equation (3.46) and as velocity (3.45), keeping in mind the jump relations given by Theorem 2.8 to obtain

$$\frac{1}{2}\boldsymbol{\Psi}^* + \mathring{\boldsymbol{\mathcal{W}}}'\boldsymbol{\Psi}^* - \mathring{\boldsymbol{\mathcal{L}}}(\mu\boldsymbol{\Phi}^*) = \mathbf{0} \quad \text{on} S. \tag{3.47}$$

To ease the notation, let $\widehat{\Phi} := (\mu\boldsymbol{\Phi}^*)$ and $\widehat{\boldsymbol{\Psi}} := \boldsymbol{\Psi}^*$. We consider now the system with equations (3.45) and (3.47) which can be written using matrix notation as follows:

$$\mathring{\mathcal{C}}_{\Omega^+}\mathcal{X} = \mathbf{0}, \tag{3.48}$$

where,

$$\mathring{\mathcal{C}}_{\Omega^+} = \begin{bmatrix} \frac{1}{2}\boldsymbol{I} - \mathring{\boldsymbol{\mathcal{W}}} & \mathring{\boldsymbol{\mathcal{V}}} \\ -\mathring{\boldsymbol{\mathcal{L}}} & \frac{1}{2}\boldsymbol{I} + \mathring{\boldsymbol{\mathcal{W}}}' \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix} \widehat{\boldsymbol{\Phi}} \\ \widehat{\boldsymbol{\Psi}} \end{bmatrix}. \tag{3.49}$$

Here, the matrix operator $\mathring{\mathcal{C}}_{\Omega^+}$ denotes the so-called Calderon projector. We can relate the Calderon projector of the exterior domain with the corresponding projector for the bounded domain by the relation $\mathring{\mathcal{C}}_{\Omega^+} = I - \mathring{\mathcal{C}}_{\Omega^-}$, see [HsWe08, Formula 2.3.28]. Therefore, we only need to focus on showing that the system $[I - \mathring{\mathcal{C}}_{\Omega^-}]\mathcal{X} = \mathbf{0}$ is uniquely solvable.

The system $[I - \mathring{\mathcal{C}}_{\Omega^-}]\mathcal{X} = \mathbf{0}$ reads:

$$-\mathring{\boldsymbol{\mathcal{V}}}\widehat{\boldsymbol{\Psi}} - \frac{1}{2}\widehat{\boldsymbol{\Phi}} + \mathring{\boldsymbol{\mathcal{W}}}\widehat{\boldsymbol{\Phi}} = \mathbf{0}, \quad \text{in } S, \tag{3.50}$$

$$\frac{1}{2}\widehat{\boldsymbol{\Psi}} - \mathring{\boldsymbol{\mathcal{W}}}'\widehat{\boldsymbol{\Psi}} + \mathring{\boldsymbol{\mathcal{L}}}\widehat{\boldsymbol{\Phi}} = \mathbf{0}, \quad \text{in } S. \tag{3.51}$$

Take the restriction to S_D of (3.50) and the restriction of (3.51) to S_N, we obtain

$$\mathring{\mathcal{M}} = \begin{bmatrix} -r_{S_1}\mathring{\boldsymbol{\mathcal{V}}} & +r_{S_1}\mathring{\boldsymbol{\mathcal{W}}} \\ -r_{S_2}\mathring{\boldsymbol{\mathcal{W}}}' & +r_{S_2}\mathring{\boldsymbol{\mathcal{L}}} \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix} \widehat{\boldsymbol{\Psi}} \\ \widehat{\boldsymbol{\Phi}} \end{bmatrix}. \tag{3.52}$$

The system (3.52) can be written using matrix notation as

$$-\overset{\circ}{\mathcal{M}}\mathcal{X} = \mathbf{0}.$$

The operator $\overset{\circ}{\mathcal{M}}$ was already studied in Lemma 2.14, and is positive definite, from where it follows the unique solvability of (3.48). □

3.6 BDIES

We aim to obtain two different segregated BDIES for the mixed BVP (3.4). This is a well known procedure as shown in [CMN09], [MiPo15-I] and [Mi02] and further references therein.

To this end, let the functions $\mathbf{\Phi}_0 \in \boldsymbol{H}^{1/2}(S)$ and $\mathbf{\Psi}_0 \in \boldsymbol{H}^{-1/2}(S)$ be respective continuations of the boundary functions $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(S_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(S_N)$ from (3.4c) and (3.4d).

where $\boldsymbol{\varphi} \in \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ and $\boldsymbol{\psi} \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D)$ are unknown boundary functions.

3.6.1 BDIES - M11

Let us now take equations (3.34) and (3.35) in the domain Ω and restrictions of equations (3.36) and (3.37) to the boundary parts S_D and S_N, respectively. Substituting there representations (2.82) and considering further the unknown boundary functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ as formally independent of (segregated from) the unknown domain functions p and $\boldsymbol{v}$, we obtain the following system of four boundary-domain integral equations for four unknowns, $(p, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$, $\boldsymbol{\varphi} \in \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ and $\boldsymbol{\psi} \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D)$,:

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \mathcal{P}\boldsymbol{\psi} + \Pi\boldsymbol{\varphi} = F_0 \quad \text{in } \Omega, \tag{3.53a}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{\psi} + \boldsymbol{W}\boldsymbol{\varphi} = \boldsymbol{F} \quad \text{in } \Omega, \tag{3.53b}$$

$$r_{S_D}\gamma^+\boldsymbol{\mathcal{R}}\boldsymbol{v} - r_{S_D}\boldsymbol{\mathcal{V}}\boldsymbol{\psi} + r_{S_D}\boldsymbol{\mathcal{W}}\boldsymbol{\varphi} = r_{S_D}\gamma^+\boldsymbol{F} - \boldsymbol{\varphi}_0 \quad \text{on } S_D, \tag{3.53c}$$

$$r_{S_N}\boldsymbol{T}^+(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}})\boldsymbol{v} - r_{S_N}\boldsymbol{\mathcal{W}}'\boldsymbol{\psi} + r_{S_N}\boldsymbol{\mathcal{L}}^+\boldsymbol{\varphi} = r_{S_N}\boldsymbol{T}^+(F_0, \boldsymbol{F}) - \boldsymbol{\psi}_0 \quad \text{on } S_N, \tag{3.53d}$$

where

$$F_0 = \mathcal{Q}\boldsymbol{f} - \frac{2}{3}\mu g + \mathcal{P}\mathbf{\Psi}_0 - \Pi\mathbf{\Phi}_0, \quad \boldsymbol{F} = \boldsymbol{\mathcal{U}}\boldsymbol{f} + \boldsymbol{\mathcal{Q}}g + \boldsymbol{V}\mathbf{\Psi}_0 - \boldsymbol{W}\mathbf{\Phi}_0. \tag{3.54}$$

Applying Lemma 3.8 to (3.54), keeping in mind the equations (3.53a) and (3.53b), and taking into account the mapping properties delivered by Theorem 3.5, Theorem 3.6 and Theorem 3.7, we obtain that $(F_0, \boldsymbol{F}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$.

We denote the right hand side of BDIE system (3.53) as

$$\mathcal{F}_*^{11} := [F_0, \mathcal{F}^{11}] = [F_0, \boldsymbol{F}, r_{S_D}\boldsymbol{\gamma}^+\boldsymbol{F} - \boldsymbol{\varphi}_0, r_{S_N}\boldsymbol{T}^+(\boldsymbol{F}, F) - \boldsymbol{\psi}_0]^\top, \tag{3.55}$$

which implies $\mathcal{F}_*^{11} \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{-1/2}(S_N)$.

Note that BDIE system (3.53) can be split into the BDIE system (M11), of 3 vector equations (3.53b), (3.53c), (3.53d) for 3 vector unknowns, $\boldsymbol{v}$, $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$, and the separate equation (3.53a) that can be used, after solving the system, to obtain the pressure, p. However since the couple $(p, \boldsymbol{v})$ shares the space $\boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$, equations (3.53b), (3.53c), (3.53d) are not completely separate from equation (3.53a).

The system (M11) given by equations (3.53b), (3.53c), (3.53d) can be written using matrix notation as

$$\mathcal{M}_*^{11}\mathcal{X} = \mathcal{F}_*^{11}, \tag{3.56}$$

where $\mathcal{X}$ represents the vector containing the unknowns of the system

$$\mathcal{X} = (p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})^\top \in L^2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$$

The matrix operator $\mathcal{M}^{11}$ is defined by

$$\mathcal{M}_*^{11} = \begin{bmatrix} I & \mathcal{R}^\bullet & -\mathcal{P} & \Pi \\ 0 & \boldsymbol{I} + \boldsymbol{\mathcal{R}} & -\boldsymbol{V} & \boldsymbol{W} \\ 0 & r_{S_D}\boldsymbol{\gamma}^+\boldsymbol{\mathcal{R}} & -r_{S_D}\boldsymbol{\mathcal{V}} & r_{S_D}\boldsymbol{\mathcal{W}} \\ 0 & r_{S_N}\boldsymbol{T}^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) & -r_{S_N}\boldsymbol{\mathcal{W}}' & r_{S_N}\boldsymbol{\mathcal{L}} \end{bmatrix}.$$

We note that the mapping properties of the operators involved in the matrix imply the continuity of the operator

$$\begin{aligned} \mathcal{M}_*^{11} : L^2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N) \\ \longrightarrow L^2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N). \end{aligned}$$

Remark 3.10. *The term* $\mathcal{F}_*^{11} = 0$ *if and only if* $(\boldsymbol{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = \boldsymbol{0}$.

Proof. Lemmas 3.8 and 3.9 are the analogous version of Theorem 2.13 and Lemma 2.14 for weighted Sobolev spaces. Therefore, the same argument used to prove Remark 2.15 for bounded domains can be applied here. Hence, the result follows. □

3.6.2 BDIES - M22

Let us now take equations (3.34) and (3.35) in the domain Ω and restrictions of equations (3.36) and (3.37) to the boundary parts S_N and S_D respectively. Substituting there representations (2.82) and considering further the unknown boundary functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ as formally independent of (segregated from) the unknown domain functions $\boldsymbol{v}$ and p, we obtain the following system of four boundary-domain integral equations for four unknowns, $(p, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$, $\boldsymbol{\varphi} \in \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ and $\boldsymbol{\psi} \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D)$:

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \mathcal{P}\boldsymbol{\psi} + \Pi\boldsymbol{\varphi} = F_0, \tag{3.57a}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{\psi} + \boldsymbol{W}\boldsymbol{\varphi} = \boldsymbol{F}, \tag{3.57b}$$

$$\frac{1}{2}\boldsymbol{\psi} + r_{S_D}\boldsymbol{T}^{+}(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}})\boldsymbol{v} - r_{S_D}\boldsymbol{\mathcal{W}}'\boldsymbol{\psi} + r_{S_D}\boldsymbol{\mathcal{L}}^{+}\boldsymbol{\varphi} = r_{S_D}\boldsymbol{T}^{+}(F_0, \boldsymbol{F}) - r_{S_D}\boldsymbol{\Psi}_0, \tag{3.57c}$$

$$\frac{1}{2}\boldsymbol{\varphi} + r_{S_N}\gamma^{+}\boldsymbol{\mathcal{R}}\boldsymbol{v} - r_{S_N}\boldsymbol{\mathcal{V}}\boldsymbol{\psi} + r_{S_N}\boldsymbol{\mathcal{W}}\boldsymbol{\varphi} = r_{S_N}\gamma^{+}\boldsymbol{F} - r_{S_N}\boldsymbol{\Phi}_0. \tag{3.57d}$$

where the terms in the right hand sides F_0 and $\boldsymbol{F}$ are given by (3.54).

We remark that the first two equations, (3.57a) and (3.57b), are defined inside of the domain Ω; the third equation (3.57c) is defined on S_D and the forth equation (3.57d) on S_N.

Note that BDIE system (3.57a)-(3.57d) can be splitted into the BDIE system (M22), of 3 vector equations, (3.57b)-(3.57d), for 3 vector unknowns, $\boldsymbol{v}$, $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$, and the separate equation (3.57a) that can be used, after solving the system, to obtain the pressure, p. However, since the couple $(p, \boldsymbol{v})$ shares the space $\boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$, equations (3.57b), (3.57c) and (3.57d) are not completely separate from equation (3.57a).

The system can be written using matrix notation as follows

$$\mathcal{M}^{22}_{*}\mathcal{X} = \mathcal{F}^{22}_{*}, \tag{3.58}$$

where the matrix operator $\mathcal{M}^{22}$ is defined by

$$\mathcal{M}^{22}_{*} = \begin{bmatrix} I & \mathcal{R}^{\bullet} & -\mathcal{P} & \Pi \\ \mathbf{0} & \boldsymbol{I}+\boldsymbol{\mathcal{R}} & -\boldsymbol{V} & \boldsymbol{W} \\ \mathbf{0} & r_{S_D}\boldsymbol{T}^{+}(\mathcal{R}^{\bullet},\boldsymbol{\mathcal{R}}) & r_{S_D}\left(\frac{1}{2}\boldsymbol{I}-\boldsymbol{\mathcal{W}}'\right) & r_{S_D}\boldsymbol{\mathcal{L}}^{+} \\ \mathbf{0} & r_{S_N}\gamma^{+}\boldsymbol{\mathcal{R}} & -r_{S_N}\boldsymbol{\mathcal{V}} & r_{S_N}\left(\frac{1}{2}\boldsymbol{I}+\boldsymbol{\mathcal{W}}\right) \end{bmatrix}, \tag{3.59}$$

the vector $\mathcal{X} = (p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})^{\top} \in L^2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ represents the unknowns of the system, and the vector

$$\mathcal{F}^{22}_{*} = [F_0, \boldsymbol{F}, r_{S_D}\boldsymbol{T}^{+}(F_0, \boldsymbol{F}) - r_{S_D}\boldsymbol{\Psi}_0, r_{S_N}\gamma^{+}\boldsymbol{F} - r_{S_N}\boldsymbol{\Phi}_0]^{\top}$$

is the right hand side and $\mathcal{F}^{22}_{*} \in L^2(\Omega) \times \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N)$.

Due to the mapping properties of the operators involved in (3.59), we have the continuous mapping

$$\mathcal{M}^{22}_{*} : L^2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$$
$$\longrightarrow L^2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N).$$

Remark 3.11. *The term* $\mathcal{F}^{22}_{*} := [F_0, \boldsymbol{F}, r_{S_D}\boldsymbol{T}^{+}(F_0, \boldsymbol{F}) - r_{S_D}\boldsymbol{\Psi}_0, r_{S_N}\gamma^{+}\boldsymbol{F} - r_{S_N}\boldsymbol{\Phi}_0]^{\top} = 0$ *if and only if* $(\boldsymbol{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = 0$.

Proof. Lemma 3.8 and Lemma 3.9 are the analogous version of Theorem 2.13 and Lemma 2.14 for weighted Sobolev spaces respectively. Therefore, the same argument used to prove Remark 2.21 for bounded domains can be applied here. Hence, the result follows. □

3.7 Equivalence and Invertibility theorems

3.7.1 Equivalence theorem

This result is analogous to the equivalence theorems proven for bounded domain in the previous chapter.

Theorem 3.12 (Equivalence Theorem)**.** *Let* $\boldsymbol{f} \in \boldsymbol{L}_2(\omega; \Omega)$, $g \in L^2(\Omega)$ *and let* $\boldsymbol{\Phi}_0 \in \boldsymbol{H}^{-1/2}(S)$ *and* $\boldsymbol{\Psi}_0 \in \boldsymbol{H}^{-1/2}(S)$ *be some fixed extensions of* $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(S_D)$ *and* $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(S_N)$ *respectively. Let conditions 3.1 and 3.4 hold.*

(i) *If some* $(p, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ *solves the mixed BVP* (3.4)*, then*

$$(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N),$$

where

$$\boldsymbol{\varphi} = \gamma^{+}\boldsymbol{v} - \boldsymbol{\Phi}_0, \quad \boldsymbol{\psi} = \boldsymbol{T}^{+}(p, \boldsymbol{v}) - \boldsymbol{\Psi}_0 \quad \text{on } S, \tag{3.60}$$

solve the BDIES (M11) and (M22).

(ii) *If* $(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ *solves one of the BDIES, (M11) or (M22), then it solves all the BDIES. Furthermore, the pair* $(p, \boldsymbol{v})$ *solves the mixed BVP* (3.4) *and the functions* $\boldsymbol{\psi}, \boldsymbol{\varphi}$ *satisfy* (3.60)*.*

(iii) *The BDIES: (M11) and (M22) have at most one solution in the space* $\boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$*.*

Proof. The proof of item (i) follows from the derivation of the BDIES (M11) and (M22) in a similar way as in Theorems 2.16 and 2.22 for the corresponding results for bounded domains.

Item (ii) in bounded domains is proven by applying Theorems 2.13 and 2.14. As we have proven an analogous result for unbounded domains, see Lemmas 3.8 and 3.9 which also holds for bounded domains. Therefore, a similar argument as in Theorems 2.16 and 2.22 for the corresponding results for bounded domains can be applied here from where the result follows.

Finally, item (iii) follows from the fact that the BVP (3.4) has at most one solution. As the BVP is equivalent to the BDIES (M11) and (M22) then, these latter ones can have only up to one solution. □

3.7.2 Invertibility results for the system (M11)

To prove the more general results of invertibility using wider spaces: $\boldsymbol{\mathcal{H}}^{1}(\Omega) \times L^2(\omega; \Omega)$ instead of $\boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$, in bounded domains required compactness of the operator $\boldsymbol{\mathcal{R}}$ and $\mathcal{R}^{\bullet}$ which is obtained in virtue of the compact embedding properties of Sobolev spaces, see

[Br84]. However, the compact embeddings provided by the Rellich compactness theorem do not hold for exterior (unbounded) domains. To overcome this issue, it is possible to split the operators $\boldsymbol{\mathcal{R}}$ and $\mathcal{R}^{\bullet}$ in the sum of two operators, one operator whose norm can be made arbitrarily small whilst the other possesses the compact property, as in [CMN13, Lemma 7.4].

To make the decomposition of the remainder operators we will require the following condition.

Condition 3.13. *In the following theorems, we will require the following condition:*

$$\lim_{|x|\to\infty} \omega(\boldsymbol{x})\nabla\mu(\boldsymbol{x}) = 0. \tag{3.61}$$

The proof of the following Lemma follows a similar argument as in [CMN13, Lemma 7.4] for the corresponding scalar case.

Lemma 3.14. *Let conditions 3.1 and 3.13 hold. Then, for any $\varepsilon > 0$ the operator $\boldsymbol{\mathcal{R}}$ can be represented as $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{R}}_s + \boldsymbol{\mathcal{R}}_c$, where $\| \boldsymbol{\mathcal{R}}_s \|_{\boldsymbol{\mathcal{H}}^1(\Omega)} < \varepsilon$, while $\boldsymbol{\mathcal{R}}_c : \boldsymbol{\mathcal{H}}^1(\Omega) \longrightarrow \boldsymbol{\mathcal{H}}^1(\Omega)$ is compact.*

Proof. Let $B(\mathbf{0}, \varepsilon)$ be a ball with centre in $\mathbf{0} \in \mathbb{R}^3$ and radius $\varepsilon > 0$ big enough such that $S \subseteq B(\mathbf{0}, \varepsilon)$. Consider a cut-ff function χ such that $0 \le \chi(\boldsymbol{y}) \le 1$ in $\mathbb{R}^3$ with the particular property that $\chi(\boldsymbol{y}) = 1$, if $\boldsymbol{y} \in B(\mathbf{0}, \varepsilon)$, and $\chi(\boldsymbol{y}) = 0$, if $\boldsymbol{y} \in \mathbb{R}^3 \setminus B(\mathbf{0}, 2\varepsilon)$. Let us now introduce the two following operators:

$$\boldsymbol{\mathcal{R}}_c \boldsymbol{g} := \boldsymbol{\mathcal{R}}(\chi \boldsymbol{g}), \quad \boldsymbol{\mathcal{R}}_s \boldsymbol{g} := \boldsymbol{\mathcal{R}}((1-\chi)\boldsymbol{g}), \quad \textit{for } \boldsymbol{g} \in \boldsymbol{\mathcal{H}}^1(\Omega). \tag{3.62}$$

Taking into account the relations (2.16) and (2.17), we can obtain the following inequality

$$\begin{aligned}
\|\boldsymbol{\mathcal{R}}_s \boldsymbol{g}\|_{\mathcal{H}^1(\Omega)} &= \|\boldsymbol{\mathcal{R}}((1-\chi)\boldsymbol{g})\|_{\mathcal{H}^1(\Omega)} \\
&= \| \frac{-2}{\mu} \partial_j \mathring{\mathcal{U}}_{ki} h_{ij} - \frac{2}{\mu} \partial_i \mathring{\mathcal{U}}_{kj} h_{ij} - \frac{1}{\mu} \mathring{\mathcal{Q}}_k h_{jj} \|_{\mathcal{H}^1(\Omega)} \\
&= \| \frac{2}{\mu} \mathring{\mathcal{U}}_{ki} [\partial_j h_{ij}] + \frac{2}{\mu} \mathring{\mathcal{U}}_{kj} [\partial_i h_{ij}] - \frac{1}{\mu} \mathring{\mathcal{Q}}_k h_{jj} \|_{\mathcal{H}^1(\Omega)} \\
&\le \| \frac{4}{C_1} \mathcal{U}_{kj} [\partial_i h_{ij}] - \frac{1}{\mu} \mathring{\mathcal{Q}}_k h_{jj} \|_{\mathcal{H}^1(\Omega)} \\
&\le k_1 \|\mathring{\mathcal{U}}\|_{\widetilde{\mathcal{H}}^{-1}(\Omega) \to \mathcal{H}^1(\Omega)} + k_2 \|\mathring{\mathcal{Q}}\|_{L^2(\omega;\Omega) \to \mathcal{H}^1(\Omega)}
\end{aligned}$$

where $h_{ij} := (1-\chi)g_j\partial_i\mu$.

Let us find estimates for k_1 and k_2. On one hand,

$$\begin{aligned}k_1 &:= 4\|\partial_i h_{ij}\|_{\widetilde{\mathcal{H}}^{-1}(\Omega)} = 4\|\partial_i[(1-\chi)g_j\partial_i\mu]\|_{\widetilde{\mathcal{H}}^{-1}(\Omega)} \leq 4\|[(1-\chi)g_j\partial_i\mu]\|_{L^2(\Omega)} \\ &\leq 12\,\|g_j\|_{L^2(\omega^{-1};\Omega)}\|\omega\partial_i\mu\|_{L^\infty(\mathbb{R}^3\smallsetminus B(\mathbf{0},\varepsilon))} \leq 12\|g_j\|_{\mathcal{H}^1(\Omega)}\|\omega\partial_i\mu\|_{L^\infty(\mathbb{R}^3\smallsetminus B(\mathbf{0},\varepsilon))}.\end{aligned}$$

On the other hand,

$$\begin{aligned}k_2 &:= \|h_{jj}\|_{L^2(\Omega)} = \|[(1-\chi)g_j\partial_j\mu]\|_{L^2(\Omega)} \\ &\leq \|g_j\|_{L^2(\omega^{-1};\Omega)}\|\omega\partial_j\mu\|_{L^\infty(\mathbb{R}^3\smallsetminus B(\mathbf{0},\varepsilon))} \leq \|g_j\|_{\mathcal{H}^1(\Omega)}\|\omega\partial_i\mu\|_{L^\infty(\mathbb{R}^3\smallsetminus B(\mathbf{0},\varepsilon))}.\end{aligned}$$

Using the estimates for k_1 and k_2, we obtain the following estimate for the norm of $\boldsymbol{\mathcal{R}}_s\boldsymbol{g}$

$$\begin{aligned}\|\boldsymbol{\mathcal{R}}_s\boldsymbol{g}\|_{\mathcal{H}^1(\Omega)} &\leq k_1\|\mathring{\mathcal{U}}\|_{\widetilde{\mathcal{H}}^{-1}(\Omega)\to\mathcal{H}^1(\Omega)} + k_2\|\mathring{\mathcal{Q}}\|_{L^2(\omega;\Omega)\to\mathcal{H}^1(\Omega)} \\ &\leq 12\|g_j\|_{\mathcal{H}^1(\Omega)}\|\omega\partial_i\mu\|_{L^\infty(\mathbb{R}^3\smallsetminus B(\mathbf{0},\varepsilon))}\|\mathring{\mathcal{U}}\|_{\widetilde{\mathcal{H}}^{-1}(\Omega)\to\mathcal{H}^1(\Omega)} \\ &+ \|g_j\|_{\mathcal{H}^1(\Omega)}\|\omega\partial_i\mu\|_{L^\infty(\mathbb{R}^3\smallsetminus B(\mathbf{0},\varepsilon))}\|\mathring{\mathcal{Q}}\|_{L^2(\omega;\Omega)\to\mathcal{H}^1(\Omega)}. \end{aligned} \tag{3.63}$$

Taking the limit as $\varepsilon \to +\infty$ in (3.63), the term $\|\omega\partial_i\mu\|_{L^\infty(\mathbb{R}^3\smallsetminus B(\mathbf{0},\varepsilon))} \to 0$ by virtue of condition 3.13. Therefore, as $\varepsilon \to +\infty$, $\|\boldsymbol{\mathcal{R}}_s\boldsymbol{g}\|_{\mathcal{H}^1(\Omega)} \to 0$, what completes the proof for the operator $\boldsymbol{\mathcal{R}}_s\boldsymbol{g}$.

To prove now that $\boldsymbol{\mathcal{R}}_c\boldsymbol{g}$ is compact, we still consider the same cut-off function χ. By the definition of χ, we know that $\chi(\boldsymbol{y}) > 0$ if $\boldsymbol{y}$ is in the closure of $B(\mathbf{0}, 2\varepsilon)$. Consequently, the operator

$$\boldsymbol{\mathcal{R}}_c : \boldsymbol{H}^1(\Omega) \longrightarrow \boldsymbol{H}^1(\Omega),$$

satisfies the following relation $\boldsymbol{\mathcal{R}}_c\boldsymbol{g} = \boldsymbol{\mathcal{R}}_{\Omega_{2\varepsilon}}(\chi\boldsymbol{g}|_{\Omega_{2\varepsilon}})$ where $\Omega_{2\varepsilon} := \Omega \cap B(\mathbf{0}, 2\varepsilon)$ and

$$(\boldsymbol{\mathcal{R}}_{\Omega_{2\varepsilon}})_k\boldsymbol{\rho}(\boldsymbol{y}) := \int_{\Omega_{2\varepsilon}} R_{kj}(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})dx, \quad \boldsymbol{y} \in \mathbb{R}^3.$$

Note that $\Omega_{2\varepsilon}$ is a bounded domain and hence, the operator $\boldsymbol{\mathcal{R}}_c : \boldsymbol{L}^2(\Omega_{2\varepsilon}) \longrightarrow \boldsymbol{\mathcal{H}}^1(\Omega)$ is continuous by virtue of Theorem 3.5. Furthermore, the restriction operator $|_{\Omega_{2\varepsilon}}$ is continuous and has the following mapping property $|_{\Omega_{2\varepsilon}} : \boldsymbol{\mathcal{H}}^1(\Omega) \longrightarrow \boldsymbol{H}^1(\Omega_{2\varepsilon})$. Now, as the domain $\Omega_{2\varepsilon}$ is bounded we can apply the Rellich compact embedding theorem to show that the embedding $\boldsymbol{H}^1(\Omega_{2\varepsilon}) \subset \boldsymbol{L}^2(\Omega_{2\varepsilon})$ is compact from where it follows that the operator $\boldsymbol{\mathcal{R}}_c : \boldsymbol{\mathcal{H}}^1(\Omega) \longrightarrow \boldsymbol{\mathcal{H}}^1(\Omega)$ is compact. $\square$

Reasoning very similarly we can obtain the equivalent result for the remainder operator resulting from the pressure terms.

Lemma 3.15. *Let conditions 3.1 and 3.13 hold. Then, for any $\varepsilon > 0$ the operator $\mathcal{R}^\bullet$ can be represented as $\mathcal{R}^\bullet = \mathcal{R}_s^\bullet + \mathcal{R}_c^\bullet$, where $\| \mathcal{R}_s^\bullet \|_{\boldsymbol{\mathcal{H}}^1(\Omega)} < \varepsilon$, while $\mathcal{R}_c^\bullet : \boldsymbol{\mathcal{H}}^1(\Omega) \longrightarrow L^2(\Omega)$ is compact.*

Proof. Let $B(\mathbf{0}, \varepsilon)$ be a ball with centre in $\mathbf{0} \in \mathbb{R}^3$ and radius $\varepsilon > 0$ big enough such that $S \subseteq B(\mathbf{0}, \varepsilon)$. Consider a cut-ff function χ such that $0 \le \chi(\boldsymbol{y}) \le 1$ in $\mathbb{R}^3$ with the particular property that $\chi(\boldsymbol{y}) = 1$, if $\boldsymbol{y} \in B(\mathbf{0}, \varepsilon)$, and $\chi(\boldsymbol{y}) = 0$, if $\boldsymbol{y} \in \mathbb{R}^3 \setminus B(\mathbf{0}, 2\varepsilon)$. Let us now introduce the two following operators:

$$\mathcal{R}_c^\bullet \boldsymbol{g} := \mathcal{R}^\bullet(\chi \boldsymbol{g}), \quad \mathcal{R}_s^\bullet \boldsymbol{g} := \mathcal{R}^\bullet((1-\chi)\boldsymbol{g}), \quad \text{for } \boldsymbol{g} \in \boldsymbol{\mathcal{H}}^1(\Omega). \tag{3.64}$$

Taking into account the relations (2.19), we can obtain the following inequality

$$\|\mathcal{R}_s^\bullet \boldsymbol{g}\|_{\mathcal{H}^1(\Omega)} = k_3 \|\mathring{Q}\|_{L^2(\omega;\Omega) \to \mathcal{H}^1(\Omega)} + k_4. \tag{3.65}$$

On one hand

$$k_3 := \|2\partial_i h_{ij}\|_{\widetilde{\mathcal{H}}^{-1}(\Omega)} \le 6 \|g_j\|_{\mathcal{H}^1(\Omega)} \|\omega \partial_i \mu\|_{L^\infty(\mathbb{R}^3 \setminus B(\mathbf{0},\varepsilon))}.$$

where $h_{ij} := (1-\chi) g_j \partial_i \mu$. On the other hand

$$k_4 := \frac{4}{3} \|h_{jj}\|_{\mathcal{H}^1(\Omega)} \le \|g_j\|_{\mathcal{H}^1(\Omega)} \|\omega \partial_i \mu\|_{L^\infty(\mathbb{R}^3 \setminus B(\mathbf{0},\varepsilon))}.$$

Substituting k_3 and k_4 into (3.65), and taking the limit $\varepsilon \to +\infty$ the result follows for the operator $\mathcal{R}_s^\bullet$. For the operator $\mathcal{R}_c^\bullet$ we follow a word by word argument to deduce that the operator $\mathcal{R}_c^\bullet|_{\Omega_{2\varepsilon}} : \boldsymbol{\mathcal{H}}^1(\Omega) \longrightarrow H^1(\Omega_{2\varepsilon})$ is continuous, see Theorem 3.5, and therefore by virtue of the Rellich compactness theorem, the operator $\mathcal{R}_c^\bullet|_{\Omega_{2\varepsilon}} : \boldsymbol{\mathcal{H}}^1(\Omega) \longrightarrow L^2(\Omega_{2\varepsilon})$ is compact. $\square$

Corollary 3.16. *Let conditions 3.1 and 3.61 hold. Then, the operator*

$$\boldsymbol{I} + \boldsymbol{\mathcal{R}} : \boldsymbol{\mathcal{H}}^1(\Omega) \longrightarrow \boldsymbol{\mathcal{H}}^1(\Omega),$$

is Fredholm with zero index.

Proof. Applying the Lemma 3.14, we have $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{R}}_s + \boldsymbol{\mathcal{R}}_c$ so $\| \boldsymbol{\mathcal{R}}_s \|_{\boldsymbol{\mathcal{H}}^1(\Omega)} < 1$ hence $\boldsymbol{I} + \boldsymbol{\mathcal{R}}_s$ is invertible. On the other hand $\boldsymbol{\mathcal{R}}_c$ is compact and hence $I + \boldsymbol{\mathcal{R}}_s$ is invertible and a compact perturbation of the operator $\boldsymbol{I} + \boldsymbol{\mathcal{R}}$, from where follows the result. □

To simplify the notation we will consider the following notation:

$$\mathbb{X}^{11} := \boldsymbol{\mathcal{H}}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N), \quad \mathbb{X}^{11}_* := L^2(\Omega) \times \mathbb{X}^{11},$$

$$\mathbb{Y}^{11} := \boldsymbol{\mathcal{H}}^1(\Omega) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{-1/2}(S_N), \quad \mathbb{Y}^{11}_* := L^2(\Omega) \times \mathbb{Y}^{11},$$

$$\mathbb{Y}^{22} := \boldsymbol{\mathcal{H}}^1(\Omega) \times \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N), \quad \mathbb{Y}^{22}_* = L^2(\Omega) \times \mathbb{Y}^{22}.$$

Theorem 3.17. *Let conditions 3.1, 3.4 and 3.13 hold. Then, the operator*

$$\mathcal{M}^{11}_* : \mathbb{X}_* \longrightarrow \mathbb{Y}^{11}_* \tag{3.66}$$

is invertible.

Proof. Let $\widetilde{\mathcal{M}}^{11}_* : \mathbb{X}_* \to \mathbb{Y}^{11}_*$ be the operator defined by the following matrix:

$$\widetilde{\mathcal{M}}^{11}_* := \begin{bmatrix} I & \mathcal{R}^{\bullet} & -\mathcal{P} & \Pi \\ \mathbf{0} & \boldsymbol{I} & -\boldsymbol{V} & \boldsymbol{W} \\ \mathbf{0} & \mathbf{0} & -r_{S_D}\boldsymbol{\mathcal{V}} & r_{S_D}\boldsymbol{\mathcal{W}} \\ \mathbf{0} & \mathbf{0} & -r_{S_N}\mathring{\boldsymbol{\mathcal{W}}}' & r_{S_N}\widehat{\boldsymbol{\mathcal{L}}} \end{bmatrix}$$

Note that the operator $\widetilde{\mathcal{M}}^{11}_*$ is a block diagonal upper triangular matrix operator. The first two blocks are given by the identity operators I and $\boldsymbol{I}$ whereas the third block is given by the corresponding terms of the third and forth rows and columns.

Taking into account [KoWe06, Theorem 3.10] applied to the last two rows of the matrix operator $\widetilde{\mathcal{M}}^{11}_*$ along with the mapping properties appearing in Theorems 3.6 and 3.7, we can conclude that the operator $\widetilde{\mathcal{M}}^{11}_*$ is continuously invertible operator and hence Fredholm with $index(\widetilde{\mathcal{M}}^{11}_*) = 0$.

The operator $\mathcal{M}_*^{11} - \widetilde{\mathcal{M}}_*^{11}$ has the form:

$$\mathcal{M}_*^{11} - \widetilde{\mathcal{M}}_*^{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbf{0} & \boldsymbol{\mathcal{R}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r_{S_D}\boldsymbol{\gamma}^+\boldsymbol{\mathcal{R}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r_{S_N}\boldsymbol{T}^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) & r_{S_N}(\boldsymbol{\mathcal{W}}' - \mathring{\boldsymbol{\mathcal{W}}}') & r_{S_N}(\boldsymbol{\mathcal{L}}^+ - \widehat{\boldsymbol{\mathcal{L}}}) \end{bmatrix}.$$

By virtue of Lemma 3.14, we can obtain the following decomposition for the operator $\mathcal{M}_*^{11} - \widetilde{\mathcal{M}}_*^{11}$

$$\mathcal{M}_*^{11} - \widetilde{\mathcal{M}}_*^{11} = \mathcal{M}_{*s}^{11} + \mathcal{M}_{*c}^{11},$$

where

$$\mathcal{M}_s^{11} := \begin{bmatrix} 0 & \mathcal{R}_s^\bullet & 0 & 0 \\ \mathbf{0} & \boldsymbol{\mathcal{R}}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r_{S_D}\boldsymbol{\gamma}^+\boldsymbol{\mathcal{R}}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r_{S_N}\boldsymbol{T}^+(\mathcal{R}_s^\bullet, \boldsymbol{\mathcal{R}}_s) & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

and

$$\mathcal{M}_{*c}^{11} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbf{0} & \boldsymbol{\mathcal{R}}_c & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r_{S_D}\boldsymbol{\gamma}^+\boldsymbol{\mathcal{R}}_c & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r_{S_N}\boldsymbol{T}^+(\mathcal{R}_c^\bullet, \boldsymbol{\mathcal{R}}_c) & r_{S_N}(\boldsymbol{\mathcal{W}}' - \mathring{\boldsymbol{\mathcal{W}}}') & r_{S_N}(\boldsymbol{\mathcal{L}}^+ - \widehat{\boldsymbol{\mathcal{L}}}) \end{bmatrix}.$$

Using the Lemma 3.14, we know that $\parallel \mathcal{R}_s \parallel$ and $\parallel \mathcal{R}_s^\bullet \parallel$ can be made aribitrarly small, and hence small enough to satisfy the inequality $\parallel \mathcal{M}_{*s}^{11} \parallel_{\mathbb{X}_* \to \mathbb{Y}_*^{11}} < 1/ \parallel (\widetilde{\mathcal{M}}_*^{11})^{-1} \parallel_{\mathbb{X}_* \to \mathbb{Y}_*^{11}}$. Consequently, the operator $\widetilde{\mathcal{M}}_*^{11} + \mathcal{M}_{*s}^{11} : \mathbb{X}_* \to \mathbb{Y}_*^{11}$ is continuously invertible.

Furthermore, the operator $\mathcal{M}_c^{11}$ is compact since $\mathcal{R}_c$ is compact due to Lemma 3.14 and the mapping properties of the operators $\mathcal{W}'$ and $\mathcal{L}^+ - \widehat{\mathcal{L}}$ given by Theorem 2.6 and Corollary 2.10.

Therefore, the operator $\mathcal{M}_*^{11} : \mathbb{X}_* \to \mathbb{Y}_*^{11}$ is Fredholm with zero index. Moreover, this operator is also injective in virtue of Theorem 3.12 from where follows its invertibility. □

Theorem 3.18. *Let conditions 3.1, 3.4 and 3.13 hold. Then the operator*

$$\begin{aligned} \mathcal{M}_*^{11} : \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N) \\ \longrightarrow \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{-1/2}(S_N) \end{aligned} \tag{3.67}$$

is continuous and continuously invertible.

Proof. Let us consider the solution $\mathcal{X} = (\mathcal{M}_*^{11})^{-1}\mathcal{F}_*^{11}$ of the system (3.56). Here, $\mathcal{F}_*^{11} \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{-1/2}(S_N)$ is an arbitrary right hand side and $(\mathcal{M}_*^{11})^{-1}$ is the inverse of the operator (3.66) which exists by virtue of Theorem 3.17.

Applying Lemma 3.8 to the first two equations of the system (M11), we get that $\mathcal{X} \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ if $\mathcal{F}_*^{11} \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N)$. Consequently, the operator $(\mathcal{M}_*^{11})^{-1}$ is also the continuous inverse of the operator (3.67). □

The following corollary is the analogous of the corollary 2.19 for bounded domains.

Corollary 3.19. *Let* $\boldsymbol{f} \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$*,* $g \in L^2(\omega;\Omega)$*,* $\boldsymbol{\phi}_0 \in \boldsymbol{H}^{1/2}(S_D)$ *and* $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(S_N)$*. In addition, let conditions 3.1, 3.4 and 3.13 hold. Then, the BVP* (3.4) *is uniquely solvable in* $\boldsymbol{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$*. Furthermore, the operator*

$$\boldsymbol{\mathcal{A}}_M : \boldsymbol{\mathcal{H}}^{1,0}(\Omega,\boldsymbol{\mathcal{A}}) \longrightarrow \boldsymbol{L}^2(\omega;\Omega) \times \boldsymbol{L}^2(\omega,\Omega) \times \boldsymbol{H}^{1/2}(S_D) \times \boldsymbol{H}^{-1/2}(S_N), \tag{3.68}$$

is continuously invertible.

Proof. Let $\boldsymbol{\Phi}_0 \in \boldsymbol{H}^{1/2}(S)$ and $\boldsymbol{\Psi}_0 \in \boldsymbol{H}^{-1/2}(S)$ be some extensions of $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(S_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(S_N)$, respectively.

The BDIES (M11) is uniquely solvable and equivalent to the BVP (3.4) by virtue of Theorem 3.12. In addition, as the operator that defines the system (M11) is continuously invertible, see Theorem 3.18. The remaining part of the proof is similar is as in Corollary 2.19. □

When $\mu = 1$, the operator $\boldsymbol{\mathcal{A}}$ becomes $\mathring{\boldsymbol{\mathcal{A}}}$, $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{R}}^\bullet \equiv 0$ and the boundary-domain integral equations system (3.57a)-(3.57d) becomes a BIES with 6 equations and 6 unknowns, namely,

$$r_{S_D}\left(\frac{1}{2}\boldsymbol{\psi} - \mathring{\boldsymbol{\mathcal{W}}}'\boldsymbol{\psi} + \mathring{\boldsymbol{\mathcal{L}}}\boldsymbol{\varphi}\right) = r_{S_D}\boldsymbol{T}^+(F_0,\boldsymbol{F}) - r_{S_D}\boldsymbol{\Psi}_0, \ \ on\ S_D, \tag{3.69}$$

$$r_{S_N}\left(\frac{1}{2}\boldsymbol{\varphi} - \mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\psi} + \mathring{\boldsymbol{\mathcal{W}}}\boldsymbol{\varphi}\right) = r_{S_N}\boldsymbol{\gamma}^+\boldsymbol{F} - r_{S_N}\boldsymbol{\Phi}_0, \ \ on\ S_N. \tag{3.70}$$

and a BDIES with 4 equations and 4 unknowns, namely,

$$p = F_0 + \mathring{\mathcal{P}}\boldsymbol{\psi} - \mathring{\Pi}\boldsymbol{\varphi},\ in\,\Omega, \tag{3.71}$$

$$\boldsymbol{v} = \boldsymbol{F} + \mathring{\boldsymbol{V}}\boldsymbol{\psi} - \mathring{\boldsymbol{W}}\boldsymbol{\varphi},\ in\,\Omega. \tag{3.72}$$

where the terms F_0 and $\boldsymbol{F}$ are given by (3.54).

By considering $\mu = 1$ in Theorem 3.17 and Corollary 3.19, we obtain the following corollary

Corollary 3.20. *Let $\mu = 1$ in Ω, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $g \in L^2(\omega;\Omega)$. Moreover, let $\boldsymbol{\Phi}_0 \in \boldsymbol{H}^{1/2}(S)$ and $\boldsymbol{\Psi}_0 \in \boldsymbol{H}^{-1/2}(S)$ be some extensions of $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(S_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(S_N)$, respectively. Furthermore, let conditions 3.1, 3.4 and 3.13 hold.*

i) If some $(p, \boldsymbol{v}) \in L^2(\omega;\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega)$ solves the mixed BVP (3.4a)-(3.4d), then the solution is unique, the couple $(\boldsymbol{\psi}, \boldsymbol{\varphi}) \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$ given by (3.60) solves the BIE system (3.69)-(3.70) and $(p, \boldsymbol{v})$ satisfies (3.71)-(3.72).

ii) If

$$(\boldsymbol{\psi}, \boldsymbol{\varphi}) \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$$

solves the BIES (3.69)-(3.70), then $(p, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ given by (3.71)-(3.72) solves the BVP (3.4a)-(3.4d) and the relations (3.60) hold. Moreover, the BDIE solution is unique $\widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$.

3.7.3 Invertibility results for the system (M22)

In this section we present some lemmas which deal with integral representation formulae for the corresponding right hand sides of the BDIES given. These theorems are analogous of those for bounded domains presented in the previous chapter, see Lemma 2.23 and Corollary 2.24.

Lemma 3.21. *Let $S = \overline{S}_1 \cup \overline{S}_2$, where S_1 and S_2 are two non-intersecting simply connected nonempty submanifolds of S with infinitely smooth boundaries. For any vector*

$$\mathcal{F} = (F_0, \boldsymbol{F}, \boldsymbol{\Psi}, \boldsymbol{\Phi})^\top \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_1) \times \boldsymbol{H}^{1/2}(S_2)$$

there exists another vector

$$(g_*, \boldsymbol{f}_*, \boldsymbol{\Psi}_*, \boldsymbol{\Phi}_*)^\top = \widetilde{\mathcal{C}}_{S_1,S_2}\mathcal{F} \in L_2(\omega; \Omega) \times \boldsymbol{L}_2(\omega; \Omega) \times \boldsymbol{H}^{-1/2}(S) \times \boldsymbol{H}^{1/2}(S)$$

which is uniquely determined by $\mathcal{F}$ *and such that*

$$\mathcal{Q}\boldsymbol{f}_* + \frac{4}{3}\mu g_* + \mathcal{P}\boldsymbol{\Psi}_* - \Pi\boldsymbol{\Phi}_* = F_0, \quad in \quad \Omega, \tag{3.73a}$$

$$\boldsymbol{\mathcal{U}}\boldsymbol{f}_* + \mathring{\boldsymbol{\mathcal{Q}}}g_* + \boldsymbol{\mathcal{V}}\boldsymbol{\Psi}_* - \boldsymbol{\mathcal{W}}\boldsymbol{\Phi}_* = \boldsymbol{F}, \quad in \quad \Omega, \tag{3.73b}$$

$$r_{S_1}\boldsymbol{\Psi}_* = \boldsymbol{\Psi}, \quad on \quad S_1, \tag{3.73c}$$

$$r_{S_2}\boldsymbol{\Phi}_* = \boldsymbol{\Phi}, \quad on \quad S_2. \tag{3.73d}$$

Furthermore, the operator

$$\widetilde{\mathcal{C}}_{S_1,S_2} : \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_1) \times \boldsymbol{H}^{1/2}(S_2) \longrightarrow L_2(\Omega) \times \boldsymbol{L}_2(\Omega) \times \boldsymbol{H}^{-1/2}(S) \times \boldsymbol{H}^{1/2}(S)$$

is continuous.

Proof. Let $\boldsymbol{\Psi}^0$, $\boldsymbol{\Phi}^0$ be some fixed extensions of $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ from S_1 to the whole boundary S and from S_2 onto S respectively. Let us choose this extensions in such a way that they preserve the functions spaces, i.e., $\boldsymbol{\Psi}^0 \in \boldsymbol{H}^{-1/2}(S)$, $\boldsymbol{\Phi}^0 \in \boldsymbol{H}^{1/2}(S)$ (cf. [Tr78, Subsection 4.2]). Consequently, arbitrary extensions of the functions $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ can be represented as

$$\boldsymbol{\Psi}_* = \boldsymbol{\Psi}^0 + \widetilde{\boldsymbol{\psi}}, \qquad \widetilde{\boldsymbol{\psi}} \in \widetilde{\boldsymbol{H}}^{-1/2}(S_2), \tag{3.74}$$

$$\boldsymbol{\Phi}_* = \boldsymbol{\Phi}^0 + \widetilde{\boldsymbol{\varphi}}, \qquad \widetilde{\boldsymbol{\varphi}} \in \widetilde{\boldsymbol{H}}^{1/2}(S_1). \tag{3.75}$$

The functions $\boldsymbol{\Psi}_*$ and $\boldsymbol{\Phi}_*$, in the form (3.74) and (3.75), satisfy the conditions (3.73c) and (3.73d). Consequently, it is only necessary to show that the functions $g_*, \boldsymbol{f}_*, \widetilde{\boldsymbol{\psi}}$ and $\widetilde{\boldsymbol{\varphi}}$ can be chosen in a particular way such that equations (3.73a)-(3.73b) are satisfied.

Applying the potential relations (2.16)-(2.22) to equations (3.73a)-(3.73b), we obtain

$$\mathring{\mathcal{Q}}f_* + \frac{4}{3}\mu g + \mathring{\mathcal{P}}\left(\boldsymbol{\Psi}_0 + \widetilde{\boldsymbol{\psi}}\right) - \mathring{\Pi}\left(\mu\boldsymbol{\Phi}_0 + \mu\boldsymbol{\varphi}\right) = F_0, \tag{3.76}$$

$$\mathring{\boldsymbol{\mathcal{U}}}f_* + \mathring{\boldsymbol{\mathcal{Q}}}(\mu g_*) + \mathring{\boldsymbol{V}}\left(\boldsymbol{\Psi}_0 + \widetilde{\boldsymbol{\psi}}\right) - \mathring{\boldsymbol{W}}\left(\mu\boldsymbol{\Phi}_0 + \mu\boldsymbol{\varphi}\right) = \mu\boldsymbol{F}. \tag{3.77}$$

Apply the Stokes operator with constant viscosity $\mu = 1$, $\mathring{\boldsymbol{\mathcal{A}}}$, to equations (3.76) and (3.77). Then, apply the divergence operator to equation (3.77). As a result, we obtain

$$\boldsymbol{f}_* = \mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F}) \tag{3.78}$$

$$\mu g_* = div(\mu\boldsymbol{F}) \Rightarrow g_* = \frac{div(\mu\boldsymbol{F})}{\mu} \tag{3.79}$$

which shows that the function $\boldsymbol{f}_*$ is uniquely determined by $F_0 \in L^2(\omega;\Omega)$ and ,$\mu\boldsymbol{F}$ and belongs to $\boldsymbol{L}^2(\omega;\Omega)$ since $(F_0, \mu\boldsymbol{F}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\mathcal{A})$ by virtue of the mapping properties given by Theorem 2.7. In addition, (3.79) shows that g_* is also uniquely determined by $\boldsymbol{F}$ and belongs to $L^2(\omega;\Omega)$ due to the fact that $\mu\boldsymbol{F} \in \boldsymbol{\mathcal{H}}^1(\Omega)$.

Let us substitute now (3.78) and (3.79) into equations (3.76)-(3.77) and move each term which is not depending on either $\widetilde{\boldsymbol{\psi}}$ or $\widetilde{\boldsymbol{\phi}}$ to the right hand side

$$\mathring{\mathcal{P}}\widetilde{\boldsymbol{\psi}} - \mathring{\Pi}(\mu\widetilde{\boldsymbol{\varphi}}) = F_0 - \frac{4}{3}div(\mu\boldsymbol{F}) - \mathring{\mathcal{Q}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F})\right) - \mathring{\mathcal{P}}(\boldsymbol{\Psi}^0) + \mathring{\Pi}(\mu\boldsymbol{\Phi}^0), \quad \text{in } \Omega, \tag{3.80}$$

$$\mathring{\boldsymbol{V}}\widetilde{\boldsymbol{\psi}} - \mathring{\boldsymbol{W}}(\mu\widetilde{\boldsymbol{\varphi}}) = \mu\boldsymbol{F} - \mathring{\boldsymbol{\mathcal{U}}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F})\right) - \mathring{\boldsymbol{\mathcal{Q}}}(\mu\boldsymbol{F}) - \mathring{\boldsymbol{V}}(\boldsymbol{\Psi}^0) + \mathring{\boldsymbol{W}}(\mu\boldsymbol{\Phi}^0), \quad \text{in } \Omega. \tag{3.81}$$

Let us denote with $J = (J_0, \boldsymbol{J})$ the right hand side of (3.80)-(3.81)

$$J_0 := \left(F_0 - \frac{4}{3}div(\mu\boldsymbol{F}) - \mathring{\mathcal{Q}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F})\right) - \mathring{\mathcal{P}}(\boldsymbol{\Psi}^0) + \mathring{\Pi}(\mu\boldsymbol{\Phi}^0)\right),$$
$$\boldsymbol{J} := \left(\mu\boldsymbol{F} - \mathring{\boldsymbol{\mathcal{U}}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F})\right) - \mathring{\boldsymbol{\mathcal{Q}}}div(\mu\boldsymbol{F}) - \mathring{\boldsymbol{V}}(\boldsymbol{\Psi}^0) + \mathring{\boldsymbol{W}}(\mu\boldsymbol{\Phi}^0)\right).$$

If the functions that we are looking for satisfy $\widetilde{\boldsymbol{\psi}}$ and $\widetilde{\boldsymbol{\varphi}}$ satisfy (3.80)-(3.81). Then, they will satisfy as well the following system

$$r_{S_2}\gamma^+\left(\mathring{\boldsymbol{V}}\widetilde{\boldsymbol{\psi}} - \mathring{\boldsymbol{W}}(\mu\widetilde{\boldsymbol{\varphi}})\right) = r_{S_2}\left(\gamma^+\boldsymbol{J}\right), \tag{3.82}$$

$$r_{S_1}\left[\mathring{\boldsymbol{T}}^+\left(\mathring{\mathcal{P}}(\widetilde{\psi}) - \mathring{\Pi}(\mu\widetilde{\boldsymbol{\varphi}}), \mathring{\boldsymbol{V}}\widetilde{\psi} - \mathring{\boldsymbol{W}}(\mu\widetilde{\boldsymbol{\varphi}})\right)\right] = r_{S_1}\left(\mathring{\boldsymbol{T}}^+(J_0, \boldsymbol{J})\right). \tag{3.83}$$

The system (3.82)-(3.83) can be written using matrix notation as follows

$$\begin{bmatrix} r_{S_2}\mathring{\boldsymbol{\mathcal{V}}} & r_{S_2}\gamma^+\mathring{\boldsymbol{\mathcal{W}}} \\ r_{S_1}\boldsymbol{\mathcal{W}}' & r_{S_1}\mathring{\boldsymbol{\mathcal{L}}} \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{\psi}} \\ \mu\widetilde{\boldsymbol{\varphi}} \end{bmatrix} = \begin{bmatrix} r_{S_2}\left(\gamma^+\boldsymbol{J}\right) \\ r_{S_1}\left(\mathring{\boldsymbol{T}}^+(J_0,\boldsymbol{J})\right) \end{bmatrix}. \tag{3.84}$$

The matrix operator given by the lefthand side of the equations (3.82)-(3.83) is an isomorphism between the spaces $\widetilde{\boldsymbol{H}}^{-1/2}(S_2)\times\widetilde{\boldsymbol{H}}^{1/2}(S_1)$ onto $\boldsymbol{H}^{1/2}(S_2)\times\boldsymbol{H}^{-1/2}(S_1)$ (see, [KoWe06, Theorem 3.10]). Note that this system depends only in the boundary and hence the same argument works for both bounded and unbounded domains.

Therefore, the simultaneous equations (3.82) and (3.83) are uniquely solvable with respect to $\widetilde{\boldsymbol{\varphi}}$ and $\widetilde{\boldsymbol{\psi}}$. We denote the solution of (3.82)-(3.83) by $\widetilde{\boldsymbol{\psi}}^0$ and $\widetilde{\boldsymbol{\varphi}}^0$.

Substitute now $\widetilde{\boldsymbol{\psi}}^0$ and $\widetilde{\boldsymbol{\varphi}}^0$ into (3.76)-(3.77)

$$\mathring{\mathcal{P}}\widetilde{\boldsymbol{\psi}}^0 - \mathring{\Pi}(\mu\widetilde{\boldsymbol{\varphi}}^0) = F_0 - \frac{4}{3}\mu div(\mu\boldsymbol{F}) - \mathring{\mathcal{Q}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0,\mu\boldsymbol{F})\right) - \mathring{\mathcal{P}}(\boldsymbol{\Psi}^0) + \mathring{\Pi}(\mu\boldsymbol{\Phi}^0), \quad \text{in}\,\Omega, \tag{3.85}$$

$$\mathring{\boldsymbol{V}}\widetilde{\boldsymbol{\psi}}^0 - \mathring{\boldsymbol{W}}(\mu\widetilde{\boldsymbol{\varphi}}^0) = \mu\boldsymbol{F} - \mathring{\boldsymbol{\mathcal{U}}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0,\mu\boldsymbol{F})\right) - \mathring{\boldsymbol{\mathcal{Q}}}div(\mu\boldsymbol{F}) - \mathring{\boldsymbol{V}}(\boldsymbol{\Psi}^0) + \mathring{\boldsymbol{W}}(\mu\boldsymbol{\Phi}^0), \quad \text{in}\,\Omega. \tag{3.86}$$

Let us rewrite equations (3.85) and (3.86) in terms of the parametrix-based potential operators by applying (2.16)-(2.22)

$$\mathcal{P}(\boldsymbol{\Psi}^0 + \widetilde{\boldsymbol{\psi}}^0) - \Pi(\boldsymbol{\Phi}^0 + \widetilde{\boldsymbol{\varphi}}^0) + \mathcal{Q}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0,\mu\boldsymbol{F})\right) + \frac{4}{3}\mu\, div(\mu\boldsymbol{F}) = F_0, \ \text{in}\,\Omega,$$
$$\boldsymbol{V}(\boldsymbol{\Psi}^0 + \widetilde{\boldsymbol{\psi}}^0) - \boldsymbol{W}(\boldsymbol{\Phi}^0 + \widetilde{\boldsymbol{\varphi}}^0) + \boldsymbol{\mathcal{U}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0,\mu\boldsymbol{F})\right) + \boldsymbol{\mathcal{Q}}\, div(\mu\boldsymbol{F}) = \boldsymbol{F}, \ \text{in}\,\Omega.$$

Hence, $\boldsymbol{\Psi}_* = \boldsymbol{\Psi}_0 + \widetilde{\boldsymbol{\psi}}$ and $\boldsymbol{\Phi}_* = \boldsymbol{\Phi}_0 + \widetilde{\boldsymbol{\varphi}}$ are uniquely determined by virtue of the uniqueness of solution of the mixed problem for the Stokes system with $\mu = 1$. Additionally, g_* and $\boldsymbol{f}_*$ are uniquely determined by conditions(3.78) and (3.79).

Therefore, we have found a vector $(g_*, \boldsymbol{f}_*, \boldsymbol{\Psi}_*, \boldsymbol{\Phi}_*)$ which is uniquely determined by $(F_0, \boldsymbol{F}, \boldsymbol{\Psi}, \boldsymbol{\Phi})$ and such that it satisfies (3.73a)-(3.73d). The uniqueness follows from the system (3.84). Making this system homogeneous by considering $\boldsymbol{F} = \boldsymbol{\Psi} = \boldsymbol{\Phi} = \boldsymbol{0}$ and

$F_0 = 0$ then $\boldsymbol{f}_* = \mathbf{0}$ which leads to:

$$\mathcal{P}(\boldsymbol{\Psi}_*) - \Pi(\boldsymbol{\Phi}_*) = 0,$$

$$\boldsymbol{V}(\boldsymbol{\Psi}_*) - \boldsymbol{W}(\boldsymbol{\Phi}_*) = \mathbf{0}.$$

with $\boldsymbol{\Psi}_* \in \widetilde{\boldsymbol{H}}^{-1/2}(S_2)$ and $\boldsymbol{\Phi}_* \in \widetilde{\boldsymbol{H}}^{1/2}(S_1)$. Hence, we conclude that $\boldsymbol{\Psi}_* = 0$ and $\boldsymbol{\Phi}_* = 0$ in virtue of Lemma 2.14.

The continuity and linearity of the operator $\widetilde{\mathcal{C}}_{S_1,S_2}$ is owed to the linearity and continuity of the operators involved. □

Corollary 3.22. *For any*

$$\mathcal{F} = ((\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1), \boldsymbol{\mathcal{F}}_2, \boldsymbol{\mathcal{F}}_3)^\top \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_1) \times \boldsymbol{H}^{1/2}(S_2),$$

there exists a unique four-tuple

$$(g_*, \boldsymbol{f}_*, \boldsymbol{\Psi}_*, \boldsymbol{\Phi}_*)^\top = \mathcal{C}_{S_1,S_2}\mathcal{F} \in L^2(\omega;\Omega) \times \boldsymbol{L}_2(\omega;\Omega) \times \boldsymbol{H}^{-1/2}(S) \times \boldsymbol{H}^{1/2}(S),$$

such that

$$\mathcal{Q}\boldsymbol{f}_* + \frac{4}{3}\mu g_* + \mathcal{P}\boldsymbol{\Psi}_* - \Pi\boldsymbol{\Phi}_* = \mathcal{F}_0, \ \textit{in}\ \Omega, \tag{3.87}$$

$$\boldsymbol{\mathcal{U}}\boldsymbol{f}_* + \mathring{\boldsymbol{\mathcal{Q}}}g_* + \boldsymbol{\mathcal{V}}\boldsymbol{\Psi}_* - \boldsymbol{\mathcal{W}}\boldsymbol{\Phi}_* = \boldsymbol{\mathcal{F}}_1, \ \textit{in}\ \Omega, \tag{3.88}$$

$$r_{S_1}(\boldsymbol{T}^+(\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1) - \boldsymbol{\Psi}_*) = \boldsymbol{\mathcal{F}}_2, \ \textit{on}\ S_1 \tag{3.89}$$

$$r_{S_2}(\gamma^+\boldsymbol{\mathcal{F}}_1 - \boldsymbol{\Phi}_*) = \boldsymbol{\mathcal{F}}_3, \ \textit{on}\ S_2. \tag{3.90}$$

Furthermore, the operator

$$\mathcal{C}_{S_1,S_2} : \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_1) \times \boldsymbol{H}^{1/2}(S_2) \longrightarrow L^2(\omega;\Omega) \times \boldsymbol{L}^2(\omega;\Omega) \times \boldsymbol{H}^{-1/2}(S) \times \boldsymbol{H}^{1/2}(S)$$

is continuous.

Proof. Take $\boldsymbol{\Psi} := r_{S_1}\boldsymbol{T}^+(\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1) - \boldsymbol{\mathcal{F}}_2$. Let us check, $\boldsymbol{\Psi} \in \boldsymbol{H}^{-1/2}(S_1)$. Firstly, $\boldsymbol{\mathcal{F}}_2 \in \boldsymbol{H}^{-1/2}(S_1)$. Secondly, $(\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\mathcal{A})$, then $\boldsymbol{T}^+(\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1) \in \boldsymbol{H}^{-1/2}(S)$ and hence $r_{S_1}\boldsymbol{T}^+(\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1) \in \boldsymbol{H}^{-1/2}(S_1)$. Therefore $\boldsymbol{\Psi} \in \boldsymbol{H}^{-1/2}(S_1)$.

In a similar fashion, let $\mathbf{\Phi} := r_{S_2}\gamma^+\boldsymbol{\mathcal{F}}_1 - \boldsymbol{\mathcal{F}}_3$. It is easy to see by applying the trace theorem that $r_{S_2}\gamma^+\boldsymbol{\mathcal{F}}_1 \in \boldsymbol{H}^{1/2}(S_2)$ and therefore $\mathbf{\Phi} \in \boldsymbol{H}^{1/2}(S_2)$. The Corollary follows from applying Lemma 3.21 with $\mathbf{\Psi} := r_{S_1}\boldsymbol{T}^+(\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1) - \boldsymbol{\mathcal{F}}_2$ and $\mathbf{\Phi} := r_{S_2}\gamma^+\boldsymbol{\mathcal{F}}_1 - \boldsymbol{\mathcal{F}}_3$. □

Theorem 3.23. *Let conditions 3.1, 3.4 and 3.13 hold. Then, the operator*

$$\mathcal{M}^{22}_* : \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\mathcal{A}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N) \longrightarrow \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N), \tag{3.91}$$

is continuously invertible.

Proof. Let us consider an arbitrary right hand side to the system (3.58)

$$\mathcal{F}^{22} \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N).$$

By virtue of the Corollary 3.22, the right hand side $\mathcal{F}^{22}$ can be written in the form (3.87)-(3.90) with $S_1 = S_D$ and $S_2 = S_N$. In addition, $(g_*, \boldsymbol{f}_*, \mathbf{\Psi}_*, \mathbf{\Phi}_*)^\top = \mathcal{C}_{S_D,S_N}\mathcal{F}^{22}$ where the operator $\mathcal{C}_{S_D,S_N}$ is bounded and has the following mapping property

$$\begin{aligned}\mathcal{C}_{S_D,S_N} : \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N)&\\ \longrightarrow L^2(\omega;\Omega) \times \boldsymbol{L}_2(\omega;\Omega) \times \boldsymbol{H}^{-1/2}(S) \times \boldsymbol{H}^{1/2}(S).&\end{aligned}$$

By virtue of Corollary 2.19 and the equivalence theorem of the system (M22), Theorem 3.12, there exists a solution of the equation $\mathcal{M}^{22}_*\mathcal{X} = \mathcal{F}^{22}_*$. This solution can be represented as $\mathcal{X} = (\mathcal{M}^{22}_*)^{-1}\mathcal{F}^{22}_*$ where the operator

$$\begin{aligned}(\mathcal{M}^{22}_*)^{-1} : \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N)&\\ \longrightarrow \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\mathcal{A}) \times \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N),&\end{aligned}$$

which is given by

$$(p, \boldsymbol{v}) = \boldsymbol{\mathcal{A}}_M^{-1}[g_*, \boldsymbol{f}_*, r_{S_D}\mathbf{\Psi}_*, r_{S_N}\mathbf{\Phi}_*]^\top, \;\; \boldsymbol{\psi} = \boldsymbol{T}^+(p, \boldsymbol{v}) - \mathbf{\Psi}_*, \;\; \boldsymbol{\phi} = \gamma^+\boldsymbol{v} - \mathbf{\Phi}_*,$$

where the $\boldsymbol{\mathcal{A}}_M^{-1}$ is continuous, see Corollary 3.19. Consequently, the operator $(\mathcal{M}^{22}_*)^{-1}$ is a right inverse of the operator (3.91). In addition, $(\mathcal{M}^{22}_*)^{-1}$ is also the double sided inverse due to the injectivity of (3.91) given by the Theorem 3.12. □

The system (3.69)-(3.70) can be expressed using matrix notation as follows

$$\mathring{\mathcal{M}}^{22}\mathring{\mathcal{X}} = \mathring{\mathcal{F}}^{22} \tag{3.92}$$

where $\mathring{\mathcal{X}} = (\boldsymbol{\psi}, \boldsymbol{\varphi})^\top \in \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N)$; the operator

$$\mathring{\mathcal{M}}^{22} : \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N) \longrightarrow \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N),$$

is defined by

$$\mathring{\mathcal{M}}^{22} = \begin{bmatrix} r_{S_D}\left(\dfrac{1}{2}\boldsymbol{I} - \mathring{\boldsymbol{\mathcal{W}}}'\right) & r_{S_D}\mathring{\boldsymbol{\mathcal{L}}} \\ -r_{S_N}\mathring{\boldsymbol{\mathcal{V}}} & r_{S_N}\left(\dfrac{1}{2}\boldsymbol{I} + \mathring{\boldsymbol{\mathcal{W}}}\right) \end{bmatrix}, \tag{3.93}$$

and the right hand side $\mathring{\mathcal{F}}^{22} \in \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N)$ is given by

$$\mathring{\mathcal{F}}^{22} = \begin{bmatrix} r_{S_D}\left(\mathring{T}^{+}(F_0, \boldsymbol{F}) - \boldsymbol{\Psi}_0\right) \\ r_{S_N}\left(\gamma^{+}\boldsymbol{F} - \boldsymbol{\Phi}_0\right) \end{bmatrix}. \tag{3.94}$$

The operator $\mathring{\mathcal{M}}^{22}$ is evidently continuous. Moreover, in virtue of Corollary 3.20(ii), the operator $\mathring{\mathcal{M}}^{22}$ is also injective.

Theorem 3.24. *Let conditions 3.1, 3.4 and 3.13 hold. Then, the operator*

$$\mathring{\mathcal{M}}^{22} : \widetilde{\boldsymbol{H}}^{-1/2}(S_D) \times \widetilde{\boldsymbol{H}}^{1/2}(S_N) \longrightarrow \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N)$$

is continuously invertible.

Proof. A solution of the system (3.92) with an arbitrary right hand side

$$\mathring{\mathcal{F}}^{22} = [\mathring{\widehat{\boldsymbol{\mathcal{F}}}}{}_2^{22}, \mathring{\widehat{\boldsymbol{\mathcal{F}}}}{}_3^{22}]^\top \in \boldsymbol{H}^{-1/2}(S_D) \times \boldsymbol{H}^{1/2}(S_N) \tag{3.95}$$

is given by the pair $(\boldsymbol{\psi}, \boldsymbol{\varphi})$ which satisfies the following extended system

$$\widehat{\mathcal{M}}^{22}\mathcal{X} = \widehat{\mathcal{F}}^{22} \tag{3.96}$$

where $\mathcal{X} = (p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})^\top$, $\widehat{\mathcal{F}}^{22} = (0, \boldsymbol{0}, \mathring{\boldsymbol{\mathcal{F}}}{}_2^{22}, \mathring{\boldsymbol{\mathcal{F}}}{}_3^{22})^\top$ and

$$\widehat{\mathcal{M}}^{22} = \begin{bmatrix} I & 0 & -\mathring{\mathcal{P}} & \mathring{\Pi} \\ 0 & \boldsymbol{I} & -\mathring{\boldsymbol{V}} & \mathring{\boldsymbol{W}} \\ 0 & 0 & r_{S_D}\left(\dfrac{1}{2}\boldsymbol{I} - \mathring{\boldsymbol{\mathcal{W}}}'\right) & r_{S_D}\mathring{\boldsymbol{\mathcal{L}}} \\ 0 & 0 & -r_{S_N}\mathring{\boldsymbol{\mathcal{V}}} & r_{S_N}\left(\dfrac{1}{2}\boldsymbol{I} + \mathring{\boldsymbol{\mathcal{W}}}\right) \end{bmatrix} \tag{3.97}$$

In virtue of Theorem 3.23 with $\mu = 1$, the operator $\widehat{\mathcal{M}}^{22}$ has a bounded right inverse which is also the left inverse of $\widehat{\mathcal{M}}^{22}$ since the latter one is injective. $\square$

Theorem 3.25. *Let conditions 3.1, 3.4 and 3.13 hold. Then, the operator*

$$\mathcal{M}_*^{22} : \mathbb{X}_* \longrightarrow \mathbb{Y}_*^{22} \tag{3.98}$$

is invertible.

Proof. Let $\widetilde{\mathcal{M}}_*^{22} = \widetilde{\mathcal{M}}^{22}$ given by (2.138). As in the proof of Theorem 2.28 we obtain that the operator $\widetilde{\mathcal{M}}_*^{22}$ is invertible.

By virtue of Lemma 3.14, we have

$$\mathcal{M}^{22} - \widetilde{\mathcal{M}}_*^{22} = \mathcal{M}_{*s}^{22} + \mathcal{M}_{*c}^{22},$$

where

$$\mathcal{M}_s^{22} := \begin{bmatrix} 0 & \mathcal{R}_s^\bullet & 0 & 0 \\ \mathbf{0} & \boldsymbol{\mathcal{R}}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r_{S_D}\gamma^+\boldsymbol{\mathcal{R}}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r_{S_N}\boldsymbol{T}^+(\mathcal{R}_s^\bullet, \boldsymbol{\mathcal{R}}_s) & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Using the Lemma 3.14, we know that $\| \mathcal{R}_s \|$ and $\| \mathcal{R}_S^\bullet \|$ can be made aribitrarly small, and hence small enough to satisfy the inequality $\| \mathcal{M}_{*s}^{22} \|_{\mathbb{X}_* \to \mathbb{Y}_*^{22}} < 1/ \| (\widetilde{\mathcal{M}}_*^{22})^{-1} \|_{\mathbb{X}_* \to \mathbb{Y}_*^{22}}$. Consequently, the operator $\widetilde{\mathcal{M}}_*^{22} + \mathcal{M}_{*s}^{22} : \mathbb{X}_* \to \mathbb{Y}_*^{22}$ is continuously invertible.

Furthermore, the operator $\mathcal{M}_c^{22}$ is compact since $\mathcal{R}_c$ is compact due to Lemma 3.14 and the mapping properties of the operators $\mathcal{W}'$ and $\mathcal{L}^+ - \widehat{\mathcal{L}}$ given by Theorem 2.6 and Corollary 2.10.

Therefore, the operator $\mathcal{M}_*^{22} : \mathbb{X}_* \to \mathbb{Y}_*^{22}$ is Fredholm with zero index. Moreover, this operator is also injective in virtue of Theorem 3.12 and the Remark 3.11, from where follows its invertibility. $\square$

Let us consider the BDIES (M11) and (M22). Since the unknown p only appears in the first equation, then we can focus on solving the simplified system containing the remaining vector equations with unknowns $\mathcal{X} = (\boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\phi}) \in \mathbb{X}$. Then the operators that define the simplified systems (M11) and (M22) are given by

$$\mathcal{M}^{11} = \begin{bmatrix} \boldsymbol{I} + \boldsymbol{\mathcal{R}} & -\boldsymbol{V} & \boldsymbol{W} \\ r_{S_D}\gamma^+\boldsymbol{\mathcal{R}} & -r_{S_D}\boldsymbol{\mathcal{V}} & r_{S_D}\boldsymbol{\mathcal{W}} \\ r_{S_N}\boldsymbol{T}^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) & -r_{S_N}\boldsymbol{\mathcal{W}}' & r_{S_N}\boldsymbol{\mathcal{L}} \end{bmatrix},$$

$$\mathcal{M}^{22} = \begin{bmatrix} \boldsymbol{I} + \boldsymbol{\mathcal{R}} & -\boldsymbol{V} & \boldsymbol{W} \\ r_{S_D}\boldsymbol{T}^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) & r_{S_D}\left(\frac{1}{2}\boldsymbol{I} - \boldsymbol{\mathcal{W}}'\right) & r_{S_D}\boldsymbol{\mathcal{L}}^+ \\ r_{S_N}\gamma^+\boldsymbol{\mathcal{R}} & -r_{S_N}\boldsymbol{\mathcal{V}} & r_{S_N}\left(\frac{1}{2}\boldsymbol{I} + \boldsymbol{\mathcal{W}}\right) \end{bmatrix}.$$

The corresponding right hand sides are given by

$$\mathcal{F}^{11} := [\boldsymbol{F}, r_{S_D}\gamma^+\boldsymbol{F} - \boldsymbol{\varphi}_0, r_{S_N}\boldsymbol{T}^+(\boldsymbol{F}, F) - \boldsymbol{\psi}_0]^\top \in \mathbb{Y}^{11},$$

$$\mathcal{F}^{22} := [\boldsymbol{F}, r_{S_D}\boldsymbol{T}^+(F_0, \boldsymbol{F}) - r_{S_D}\boldsymbol{\Psi}_0, r_{S_N}\gamma^+\boldsymbol{F} - r_{S_N}\boldsymbol{\Phi}_0]^\top \in \mathbb{Y}^{22}.$$

Consequently, we can write the systems (M11) and (M22) as

$$\mathcal{M}^{11}\mathcal{X} = \mathcal{F}^{11}, \qquad \mathcal{M}^{22}\mathcal{X} = \mathcal{F}^{22}.$$

Since the pressure unknown only appears on the first equation of the BDIES (M11) and (M22), the invertibility of the operators $\mathcal{M}^{11}$ and $\mathcal{M}^{22}$ is implied by the invertibility of the operators $\mathcal{M}^{11}_*$ and $\mathcal{M}^{22}_*$.

Corollary 3.26. *The operators*

$$\mathcal{M}^{11} : \mathbb{X} \longrightarrow \mathbb{Y}^{11}, \qquad \mathcal{M}^{22} : \mathbb{X} \longrightarrow \mathbb{Y}^{22},$$

are continuous and continuously invertible.

Chapter 4

A new family of BDIES for a scalar mixed elliptic interior BVP

4.1 Introduction

Boundary-Domain Integral Equations for the *scalar* equation with variable coefficient have been obtained using parametrix. This chapter concentrates on the idea that there is not only one appropriate parametrix for a PDE (or system) that works.

For this scalar equation, a family of weakly singular parametrix of the form $P^x(x, y; a(x))$ for the particular operator:

$$A(x, \partial_x; a(x))u := \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right),$$

has been studied in [CMN09, CMN10, CMN13].

The new family of parametrices of the form $P^y(x, y; a(y))$ has not been studied yet and we analyse this scenario for a mixed elliptic boundary value problem in both bounded and unbounded domains. Mapping properties of the corresponding P^y-based potentials are proved in bounded domains and appropriate Sobolev spaces.

The main difference from considering a parametrix depending on the same variable or different from the PDE operator stems from the fact that the relations between the parametrix based potentials with their counterparts for constant coefficients become more difficult to deal with. Notwithstanding, the same mapping properties in Sobolev-Bessel potential spaces still hold. Therefore, it is still possible to prove equivalence and invertibility for a BDIES derived from the original BVP. These results are published in [MiPo15-II].

4.2 Preliminaries and the BVP

The domains. Let $\Omega = \Omega^+$ be a bounded simply connected domain, $\Omega^- := \mathbb{R}^3 \smallsetminus \bar{\Omega}^+$ the complementary (unbounded) subset of Ω. The boundary $S := \partial\Omega$ is simply connected, closed and infinitely differentiable, $S \in \mathcal{C}^\infty$. Furthermore, $S := \overline{S}_N \cup \overline{S}_D$ where both S_N and S_D are non-empty, connected disjoint manifolds of S. The border of these two submanifolds is also infinitely differentiable, $\partial S_N = \partial S_D \in \mathcal{C}^\infty$.

PDE. Let us introduce the following partial differential equation whith variable smooth positive coefficient $a(x) \in \mathcal{C}^\infty(\overline{\Omega})$:

$$\mathcal{A}u(x) := \mathcal{A}(x)[u(x)] := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u(x)}{\partial x_i}\right) = f(x), \qquad x \in \Omega, \tag{4.1}$$

where $u(x)$ is an unknown function and f is a given function on Ω. It is easy to see that if $a \equiv 1$ then, the operator $\mathcal{A}$ becomes Δ, the Laplace operator.

Function spaces. We will make use of the space, see e.g. [Co88, CMN09],

$$H^{1,0}(\Omega;\mathcal{A}) := \{u \in H^1(\Omega) : \mathcal{A}u \in L^2(\Omega)\}$$

which is a Hilbert space with the norm defined by

$$\| u \|^2_{H^{1,0}(\Omega;\mathcal{A})} := \| u \|^2_{H^1(\Omega)} + \| \mathcal{A}u \|^2_{L^2(\Omega)}.$$

Traces and conormal derivatives. For a scalar function $w \in H^s(\Omega^\pm)$, $s > 1/2$, the trace operator $\gamma^\pm(\,\cdot\,) := \gamma^\pm_S(\,\cdot\,)$, acting on w is well defined and $\gamma^\pm w \in H^{s-1/2}(S)$ (see, e.g., [McL00, Mi11]). For $u \in H^s(\Omega)$, $s > 3/2$, we can define on S the conormal derivative operator, $T^\pm$, in the classical (trace) sense:

$$T^\pm[u(x)] := T^\pm_x u = \sum_{i=1}^{3} a(x)n_i(x)\left(\frac{\partial u}{\partial x_i}\right)^\pm = a(x)\left(\frac{\partial u(x)}{\partial n(x)}\right)^\pm,$$

where $n(x)$ is the exterior unit normal vector directed *outwards* the interior domain domain Ω at a point $x \in S$. Note the subscript x refers to the variable of differentiation in the conormal derivative.

Moreover, for any function $u \in H^{1,0}(\Omega;\mathcal{A})$, the *canonical* conormal derivative $T^{\pm}u \in H^{-1/2}(\Omega)$ is well defined, cf. [Co88, McL00, Mi11],

$$\langle T^{\pm}u, w\rangle_S := \pm\int_{\Omega^{\pm}}[(\gamma^{-1}\omega)\mathcal{A}u + E(u,\gamma^{-1}w)]dx, \text{ for all } w \in H^{1/2}(S), \tag{4.2}$$

where $\gamma^{-1} : H^{1/2}(S) \longrightarrow H^1_K(\mathbb{R}^3)$ is a continuous right inverse to the trace operator whereas the function E is defined as

$$E(u,v)(x) := \sum_{i=1}^{n} a(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i},$$

and $\langle\,\cdot\,,\,\cdot\,\rangle_S$ represents the L^2-based dual form on S.

Boundary value problem We aim to derive boundary-domain integral equation systems for the following *mixed* boundary value problem. Given $f \in L^2(\Omega)$, $\phi_0 \in H^{1/2}(S_D)$ and $\psi_0 \in H^{-1/2}(S_N)$, we seek a function $u \in H^1(\Omega)$ such that

$$\mathcal{A}u = f, \quad \text{in} \quad \Omega; \tag{4.3a}$$

$$r_{S_D}\gamma^{+}u = \phi_0, \quad \text{on} \quad S_D; \tag{4.3b}$$

$$r_{S_N}T^{+}u = \psi_0, \quad \text{on} \quad S_N; \tag{4.3c}$$

where equation (4.3a) is understood in the weak sense, the Dirichlet condition (4.3b) is understood in the trace sense and the Neumann condition (4.3c) is understood in the functional sense (4.2).

By Lemma 3.4 of [Co88] (cf. also Theorem 3.9 in [Mi11]), the first Green identity holds for any $u \in H^{1,0}(\Omega;\mathcal{A})$ and $v \in H^1(\Omega)$,

$$\langle T^{\pm}u, \gamma^{+}v\rangle_S := \pm\int_{\Omega}[v\mathcal{A}u + E(u,v)]dx. \tag{4.4}$$

The following assertion is well known and can be proved, e.g., using the Lax-Milgram lemma as in [Ste07, Chapter 4].

Theorem 4.1. *The BVP (4.3) has one and only one solution.*

4.3 Parametrices and remainders

For a given operator $\mathcal{A}$, the parametrix is not unique. For example, the parametrix

$$P^y(x,y) = \frac{1}{a(y)} P_\Delta(x-y), \quad x,y \in \mathbb{R}^3,$$

was employed in [Mi02, CMN09], for the operator $\mathcal{A}$ defined in (4.1). The remainder corresponding to the parametrix P^y is

$$R^y(x,y) = \sum_{i=1}^{3} \frac{1}{a(y)} \frac{\partial a(x)}{\partial x_i} \frac{\partial}{\partial x_i} P_\Delta(x-y)\,, \quad x,y \in \mathbb{R}^3. \tag{4.5}$$

In this chapter, for the same operator $\mathcal{A}$ defined in (4.1), we will use another parametrix,

$$P(x,y) := P^x(x,y) = \frac{1}{a(x)} P_\Delta(x-y), \quad x,y \in \mathbb{R}^3, \tag{4.6}$$

which leads to the corresponding remainder

$$\begin{aligned} R(x,y) = R^x(x,y) &= -\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\frac{1}{a(x)} \frac{\partial a(x)}{\partial x_i} P_\Delta(x,y) \right) \\ &= -\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\frac{\partial (\ln a(x))}{\partial x_i} P_\Delta(x,y) \right), \quad x,y \in \mathbb{R}^3. \end{aligned}$$

Note that the both remainders R_x and R_y are weakly singular, i.e.,

$$R^x(x,y),\, R^y(x,y) \in \mathcal{O}(|x-y|^{-2}).$$

This is due to the smoothness of the variable coefficient a.

4.4 Volume and surface potentials

The volume parametrix-based Newton-type potential and the remainder potential are respectively defined, for $y \in \mathbb{R}^3$, as

$$\mathcal{P}\rho(y) := \int_\Omega P(x,y)\rho(x)\,dx, \qquad \mathcal{R}\rho(y) := \int_\Omega R(x,y)\rho(x)\,dx.$$

The parametrix-based single layer and double layer surface potentials are defined for $y \in \mathbb{R}^3 : y \notin S$, as

$$V\rho(y) := -\int_S P(x,y)\rho(x)\, dS(x), \qquad W\rho(y) := -\int_S T_x^+ P(x,y)\rho(x)\, dS(x).$$

We also define the following pseudo-differential operators associated with direct values of the single and double layer potentials and with their conormal derivatives, for $y \in S$,

$$\mathcal{V}\rho(y) := -\int_S P(x,y)\rho(x)\, dS(x), \qquad \mathcal{W}\rho(y) := -\int_S T_x P(x,y)\rho(x)\, dS(x),$$
$$\mathcal{W}'\rho(y) := -\int_S T_y P(x,y)\rho(x)\, dS(x), \qquad \mathcal{L}^{\pm}\rho(y) := T_y^{\pm} W\rho(y).$$

The operators $\mathcal{P}, \mathcal{R}, V, W, \mathcal{V}, \mathcal{W}, \mathcal{W}'$ and $\mathcal{L}$ can be expressed in terms the volume and surface potentials and operators associated with the Laplace operator, as follows

$$\mathcal{P}\rho = \mathcal{P}_\Delta \left(\frac{\rho}{a}\right), \tag{4.7}$$

$$\mathcal{R}\rho = -\nabla \cdot \left[\mathcal{P}_\Delta(\rho)\nabla \ln a\right], \tag{4.8}$$

$$V\rho = V_\Delta \left(\frac{\rho}{a}\right), \tag{4.9}$$

$$\mathcal{V}\rho = \mathcal{V}_\Delta \left(\frac{\rho}{a}\right), \tag{4.10}$$

$$W\rho = W_\Delta\rho - V_\Delta \left(\rho\frac{\partial(\ln a)}{\partial n}\right), \tag{4.11}$$

$$\mathcal{W}\rho = \mathcal{W}_\Delta\rho - \mathcal{V}_\Delta \left(\rho\frac{\partial(\ln a)}{\partial n}\right), \tag{4.12}$$

$$\mathcal{W}'\rho = a\mathcal{W}'_\Delta \left(\frac{\rho}{a}\right), \tag{4.13}$$

$$\mathcal{L}^{\pm}\rho = \widehat{\mathcal{L}}\rho - aT_\Delta^{\pm}V_\Delta \left(\rho\frac{\partial(\ln a)}{\partial n}\right), \tag{4.14}$$

$$\widehat{\mathcal{L}}\rho := a\mathcal{L}_\Delta\rho. \tag{4.15}$$

The symbols with the subscript Δ denote the analogous surface potentials for the constant coefficient case, $a \equiv 1$. Furthermore, by the Liapunov-Tauber theorem, $\mathcal{L}_\Delta^+\rho = \mathcal{L}_\Delta^-\rho = \mathcal{L}_\Delta\rho$.

Using relations (4.7)-(4.15) it is now rather simple to obtain, similar to [CMN09], the mapping properties, jump relations and invertibility results for the parametrix-based surface and volume potentials, provided in theorems/corollary 4.2-4.8, from the well-known properties of their constant-coefficient counterparts (associated with the Laplace equation).

Theorem 4.2. *Let $s \in \mathbb{R}$. Then, the following operators are continuous:*

$$\mathcal{P} : \widetilde{H}^s(\Omega) \longrightarrow H^{s+2}(\Omega), \quad s \in \mathbb{R}, \tag{4.16}$$

$$\mathcal{P} : H^s(\Omega) \longrightarrow H^{s+2}(\Omega), \quad s > -\frac{1}{2}, \tag{4.17}$$

$$\mathcal{R} : \widetilde{H}^s(\Omega) \longrightarrow H^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{4.18}$$

$$\mathcal{R} : H^s(\Omega) \longrightarrow H^{s+1}(\Omega), \quad s > -\frac{1}{2}\,. \tag{4.19}$$

Corollary 4.3. *Let $s > 1/2$, let S_1 be a non-empty submanifold of S with smooth boundary. Then, the following operators are compact:*

$$\mathcal{R} : H^s(\Omega) \longrightarrow H^s(\Omega),$$

$$r_{S_1}\gamma^+\mathcal{R} : H^s(\Omega) \longrightarrow H^{s-1/2}(S_1),$$

$$r_{S_1}T^+\mathcal{R} : H^s(\Omega) \longrightarrow H^{s-3/2}(S_1).$$

Theorem 4.4. *Let $s \in \mathbb{R}$. Then, the following operators are continuous:*

$$V : H^s(S) \longrightarrow H^{s+3/2}(\Omega), \qquad W : H^s(S) \longrightarrow H^{s+1/2}(\Omega).$$

Theorem 4.5. *Let $s \in \mathbb{R}$. Then, the following operators are continuous:*

$$\mathcal{V} : H^s(S) \longrightarrow H^{s+1}(S), \qquad \mathcal{W} : H^s(S) \longrightarrow H^{s+1}(S),$$

$$\mathcal{W}' : H^s(S) \longrightarrow H^{s+1}(S), \qquad \mathcal{L}^{\pm} : H^s(S) \longrightarrow H^{s-1}(S).$$

Theorem 4.6. *Let $\rho \in H^{-1/2}(S)$, $\tau \in H^{1/2}(S)$. Then the following operators jump relations hold:*

$$\gamma^{\pm}V\rho = \mathcal{V}\rho, \qquad \gamma^{\pm}W\tau = \mp\frac{1}{2}\tau + \mathcal{W}\tau, \qquad T^{\pm}V\rho = \pm\frac{1}{2}\rho + \mathcal{W}'\rho.$$

Theorem 4.7. *Let $s \in \mathbb{R}$, let S_1 and S_2 be two non-empty manifolds with smooth boundaries, ∂S_1 and ∂S_2, respectively. Then, the following operators*

$$r_{S_2}\mathcal{V} : \widetilde{H}^s(S_1) \longrightarrow H^s(S_2),$$

$$r_{S_2}\mathcal{W} : \widetilde{H}^s(S_1) \longrightarrow H^s(S_2),$$

$$r_{S_2}\mathcal{W}' : \widetilde{H}^s(S_1) \longrightarrow H^s(S_2).$$

are compact.

Theorem 4.8. *Let S_1 be a non-empty simply connected submanifold of S with infinitely smooth boundary curve, and $0 < s < 1$. Then, the operators*

$$r_{S_1}\mathcal{V} : \widetilde{H}^{s-1}(S_1) \longrightarrow H^s(S_1), \qquad \mathcal{V} : H^{s-1}(S) \longrightarrow H^s(S),$$

are invertible.

Proof. Relation (4.9) gives $\mathcal{V}g = \mathcal{V}_\Delta g^*$, where $g = g^*/a$. The invertibility of $\mathcal{V}$ then follows from the invertibility of $\mathcal{V}_\Delta$, see references [CoSt87, Theorem 2.4] and [CMN10, Theorem 3.5]. □

Theorem 4.9. *Let S_1 be a non-empty simply connected submanifold of S with infinitely smooth boundary curve, and $0 < s < 1$. Then, the operator*

$$r_{S_1}\hat{\mathcal{L}} : \widetilde{H}^s(S_1) \longrightarrow H^{s-1}(S_1),$$

is invertible whilst the operators

$$r_{S_1}(\mathcal{L}^\pm - \hat{\mathcal{L}}) : \widetilde{H}^s(S_1) \longrightarrow H^{s-1}(S_1),$$

are compact.

Proof. Relation (4.14) gives

$$\hat{\mathcal{L}}\rho = \mathcal{L}^\pm\rho + aT_\Delta^+ V_\Delta\left(\rho\frac{\partial(\ln a)}{\partial n}\right) = \mathcal{L}^\pm\rho + aT_\Delta^- V_\Delta\left(\rho\frac{\partial(\ln a)}{\partial n}\right).$$

Take into account $\hat{\mathcal{L}}\rho := a\mathcal{L}_\Delta\rho$ and the invertibility of the operator $\mathcal{L}_\Delta$, see references [CoSt87, Theorem 2.4] and [CMN10, Theorem 3.6]; we deduce the invertibility of the operator $\hat{\mathcal{L}}$. To prove the compactness properties, we consider the identity:

$$\mathcal{L}^\pm\rho - \hat{\mathcal{L}}\rho = a\left(\mp\frac{1}{2}I - \mathcal{W}'_\Delta\right)\left(\rho\frac{\partial(\ln a)}{\partial n}\right).$$

Since $\rho \in \widetilde{H}^s(S_1)$, due to the mapping properties of the operator $\mathcal{W}'$, $\mathcal{L}^\pm - \rho\hat{\mathcal{L}}\rho \in H^s$. Then, immediately follows from the compact embedding $H^s(S) \subset H^{s-1}(S)$, that the operators

$$r_{S_1}(\mathcal{L}^\pm - \hat{\mathcal{L}}) : \widetilde{H}^s(S_1) \longrightarrow H^{s-1}(S_1),$$

are compact. □

4.5 Third Green identities and integral relations

In this section we provide the results similar to the ones in [CMN09] but for our, different, parametrix (4.6).

Let $u, v \in H^{1,0}(\Omega; \mathcal{A})$. Subtracting from the first Green identity (4.4) its counterpart with the swapped u and v, we arrive at the second Green identity, see e.g. [McL00],

$$\int_\Omega [u\,\mathcal{A}v - v\,\mathcal{A}u]\,dx = \int_S \left[u\,T^+v \;-\; v\,T^+u\right] dS(x). \tag{4.20}$$

Taking now $v(x) := P(x, y)$, we obtain from (4.20) by the standard limiting procedures (cf. [Mr70]) the third Green identity for any function $u \in H^{1,0}(\Omega; \mathcal{A})$:

$$u + \mathcal{R}u - VT^+u + W\gamma^+u = \mathcal{P}\mathcal{A}u, \quad \text{in}\,\Omega. \tag{4.21}$$

If $u \in H^{1,0}(\Omega; \mathcal{A})$ is a solution of the partial differential equation (4.3a), then, from (4.21) we obtain:

$$u + \mathcal{R}u - VT^+u + W\gamma^+u = \mathcal{P}f, \;\; in\,\Omega; \tag{4.22}$$

$$\frac{1}{2}\gamma^+u + \gamma^+\mathcal{R}u - \mathcal{V}T^+u + \mathcal{W}\gamma^+u = \gamma^+\mathcal{P}f, \;\; on\,S; \tag{4.23}$$

$$\frac{1}{2}T^+u + T^+\mathcal{R}u - \mathcal{W}'T^+u + \mathcal{L}^+\gamma^+u = T^+\mathcal{P}f, \;\; on\,S. \tag{4.24}$$

For some distributions f, Ψ and Φ, we consider a more general, indirect integral relation associated with the third Green identity (4.22):

$$u + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}f, \;\; \text{in}\ \Omega. \tag{4.25}$$

Lemma 4.10. *Let $u \in H^1(\Omega)$, $f \in L_2(\Omega)$, $\Psi \in H^{-1/2}(S)$ and $\Phi \in H^{1/2}(S)$ satisfying the relation (4.25). Then u belongs to $H^{1,0}(\Omega, \mathcal{A})$; solves the equation $\mathcal{A}u = f$ in Ω, and the following identity is satisfied,*

$$V(\Psi - T^+u) - W(\Phi - \gamma^+v) = 0 \;\; in\ \Omega. \tag{4.26}$$

Proof. First, let us prove that $u \in H^{1,0}(\Omega; \mathcal{A})$. Since $u \in H^1(\Omega)$ it suffices to prove that $\mathcal{A}u \in L^2(\Omega)$. Therefore, take equation (4.25) and apply the relations (4.7), (4.9) and (4.11) to obtain

$$\begin{aligned} u =& \mathcal{P}f - \mathcal{R}u + V\Psi - W\Phi \\ =& \mathcal{P}_\Delta\left(\frac{f}{a}\right) - \mathcal{R}u + V_\Delta\left(\frac{\Psi}{a}\right) - W_\Delta\Phi + V_\Delta\left(\frac{\partial(\ln a)}{\partial n}\,\Phi\right). \end{aligned} \tag{4.27}$$

We note that $\mathcal{R}u \in H^2(\Omega)$ due to the mapping properties (4.19). Moreover, V_Δ and W_Δ in (4.27) are harmonic potentials, while $\mathcal{P}_\Delta$ is the Newtonian potential for the Laplacian, i.e., $\Delta\mathcal{P}_\Delta\left(\frac{f}{a}\right) = \frac{f}{a}$. Consequently, $\Delta u = \frac{f}{a} - \Delta\mathcal{R}u \in L^2(\Omega)$. Hence, $\mathcal{A}u \in L^2(\Omega)$ and $u \in H^{1,0}(\Omega; \mathcal{A})$.

Since $u \in H^{1,0}(\Omega; \mathcal{A})$, the third Green identity (4.22) is valid for the function u, and we proceed subtracting (4.22) from (4.25) to obtain

$$W(\gamma^+u - \Phi) - V(T^+u - \Psi) = \mathcal{P}(\mathcal{A}u - f). \tag{4.28}$$

Let us apply relations (4.7), (4.9) and (4.11) to (4.28), and then, apply the Laplace operator to both sides. Hence, we obtain

$$\mathcal{A}u - f = 0, \tag{4.29}$$

i.e., u solves (4.3a). Finally, substituting (4.29) into (4.28), we prove (4.26). □

Lemma 4.11. *Let* $\Psi^* \in H^{-1/2}(S)$. *If*

$$V\Psi^*(y) = 0, \qquad y \in \Omega \tag{4.30}$$

then $\Psi^*(y) = 0$.

Proof. Taking the trace of (4.30)gives:

$$\mathcal{V}\Psi^*(y) = \mathcal{V}_\triangle\left(\frac{\Psi^*}{a}\right)(y) = 0, \qquad y \in \Omega,$$

from where the result follows due to the invertibility of the operator $\mathcal{V}_\triangle$ (cf. Theorem 4.8). □

4.6 BDIE system for the mixed problem

We aim to obtain a segregated boundary-domain integral equation system for mixed BVP (4.3). To this end, let the functions $\Phi_0 \in H^{1/2}(S)$ and $\Psi_0 \in H^{-1/2}(S)$ be respective continuations of the boundary functions $\phi_0 \in H^{1/2}(S_D)$ and $\psi_0 \in H^{-1/2}(S_N)$ to the whole S. Let us now represent

$$\gamma^+ u = \Phi_0 + \phi, \qquad T^+ u = \Psi_0 + \psi, \quad \text{on} \ \ S, \tag{4.31}$$

where $\phi \in \widetilde{H}^{1/2}(S_N)$ and $\psi \in \widetilde{H}^{-1/2}(S_D)$ are unknown boundary functions.

To obtain one of the possible boundary-domain integral equation systems we employ identity (4.22) in the domain Ω, and identity (4.23) on S, substituting there $\gamma^+ u = \Phi_0 + \phi$ and $T^+ u = \Psi_0 + \psi$ and further considering the unknown functions ϕ and ψ as formally independent (segregated) of u in Ω. Consequently, we obtain the following system (M12) of two equations for three unknown functions,

$$u + \mathcal{R}u - V\psi + W\phi = F_0 \qquad \text{in} \ \ \Omega, \tag{4.32a}$$

$$\frac{1}{2}\phi + \gamma^+ \mathcal{R}u - \mathcal{V}\psi + \mathcal{W}\phi = \gamma^+ F_0 - \Phi_0 \qquad \text{on} \ \ S, \tag{4.32b}$$

where

$$F_0 = \mathcal{P}f + V\Psi_0 - W\Phi_0. \tag{4.33}$$

We remark that F_0 belongs to the space $H^1(\Omega)$ in virtue of the mapping properties of the surface and volume potentials, see Theorems 4.2 and 4.4.

The system (M12), given by (4.32a)-(4.32b) can be written in matrix notation as

$$\mathcal{M}^{12}\mathcal{X} = \mathcal{F}^{12}, \tag{4.34}$$

where $\mathcal{X}$ represents the vector containing the unknowns of the system,

$$\mathcal{X} = (u, \psi, \phi)^\top \in H^1(\Omega) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N), \tag{4.35}$$

the right hand side vector is

$$\mathcal{F}^{12} := [F_0, \gamma^+ F_0 - \Psi_0]^\top \in H^1(\Omega) \times H^{1/2}(S),$$

and the matrix operator $\mathcal{M}^{12}$ is defined by:

$$\mathcal{M}^{12} = \begin{bmatrix} I + \mathcal{R} & -V & W \\ \gamma^+\mathcal{R} & -\mathcal{V} & \frac{1}{2}I + \mathcal{W} \end{bmatrix}. \tag{4.36}$$

We note that the mapping properties of the operators involved in the matrix imply the continuity of the operator

$$\mathcal{M}^{12} : H^1(\Omega) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N) \longrightarrow H^1(\Omega) \times H^{1/2}(S).$$

Theorem 4.12. *Let $f \in L_2(\Omega)$. Let $\Phi_0 \in H^{1/2}(S)$ and $\Psi_0 \in H^{-1/2}(S)$ be some fixed extensions of $\phi_0 \in H^{1/2}(S_D)$ and $\psi_0 \in H^{-1/2}(S_N)$ respectively.*

i) If some $u \in H^1(\Omega)$ solves the BVP (4.3)*, then the triple $(u, \psi, \phi)^\top \in H^1(\Omega) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$, where*

$$\phi = \gamma^+u - \Phi_0, \qquad \psi = T^+u - \Psi_0, \qquad \textit{on } S, \tag{4.37}$$

solves the BDIE system (M12).

ii) If a triple $(u, \psi, \phi)^\top \in H^1(\Omega) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ solves the BDIE system then u solves the BVP and the functions ψ, ϕ satisfy (4.37)*.*

iii) The system (M12) is uniquely solvable.

Proof. First, let us prove item *i*). Let $u \in H^1(\Omega)$ be a solution of the boundary value problem (4.3) and let ϕ, ψ be defined by (4.37). Then, due to (4.3b) and (4.3c), we have

$$(\psi, \phi) \in \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N).$$

Then, it immediately follows from the third Green identities (4.22) and (4.23) that the triple (u, ϕ, ψ) solves BDIE system (M12).

Let us prove now item *ii*). Let the triple $(u, \psi, \phi)^\top \in H^1(\Omega) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ solve the BDIE system (M12). Taking the trace of the equation (4.32a) and substract it from the equation (4.32b), we obtain

$$\phi = \gamma^+u - \Phi_0, \quad \text{on } S. \tag{4.38}$$

This means that the first condition in (4.37) is satisfied. Now, restricting equation (4.38) to S_D, we observe that ϕ vanishes as $supp(\phi) \subset S_N$. Hence, $\phi_0 = \Phi_0 = \gamma^+ u$ on S_D and consequently, the Dirichlet condition of the BVP (4.3b) is satisfied.

We proceed using the Lemma 4.10 in the first equation of the system (M12), (4.32a), with $\Psi = \psi + \Psi_0$ and $\Phi = \phi + \Phi_0$ which implies that u is a solution of the equation (4.3a) and also the following equality:

$$V(\Psi_0 + \psi - T^+ u) - W(\Phi_0 + \phi - \gamma^+ u) = 0 \text{ in } \Omega. \tag{4.39}$$

In virtue of (4.38), the second term of the previous equation vanishes. Hence,

$$V(\Psi_0 + \psi - T^+ u) = 0, \quad \text{ in } \Omega. \tag{4.40}$$

Now, in virtue of Lemma 4.11 we obtain

$$\Psi_0 + \psi - T^+ u = 0, \quad \text{ on } S. \tag{4.41}$$

Since ψ vanishes on S_N, we can conclude that $\Psi_0 = \psi_0$ on S_N. Consequently, equation (4.41) implies that u satisfies the Neumann condition (4.3c).

Item *iii)* immediately follows from the uniqueness of the solution of the mixed boundary value problem 4.1. □

Lemma 4.13. $(F_0, \gamma^+ F_0 - \Phi_0) = 0$ *if and only if* $(f, \Phi_0, \Psi_0) = 0$.

Proof. It is trivial that if $(f, \Phi_0, \Psi_0) = 0$ then $(F_0, \gamma^+ F_0 - \Phi_0) = 0$. Conversely, supposing that $(F_0, \gamma^+ F_0 - \Phi_0) = 0$, then taking into account equation (4.33) and applying Lemma 4.10 with $F_0 = 0$ as u, we deduce that $f = 0$ and $V\Psi_0 - W\Phi_0 = 0$ in Ω. Now, the second equality, $\gamma^+ F_0 - \Phi_0 = 0$, implies that $\Phi_0 = 0$ on S and applying Lemma 4.11 gives $\Psi_0 = 0$ on S. □

Theorem 4.14. *The operator*

$$\mathcal{M}^{12} : H^1(\Omega) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N) \longrightarrow H^1(\Omega) \times H^{1/2}(S),$$

is invertible.

Proof. Let $\mathcal{M}_0^{12}$ be the matrix operator defined by

$$\mathcal{M}_0^{12} := \begin{bmatrix} I & -V & W \\ 0 & -\mathcal{V} & \frac{1}{2}I \end{bmatrix}. \tag{4.42}$$

The operator $\mathcal{M}_0^{12}$ is also bounded due to the mapping properties of the operators involved. Furthermore, the operator

$$\mathcal{M}^{12} - \mathcal{M}_0^{12} : H^1(\Omega) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N) \longrightarrow H^1(\Omega) \times H^{1/2}(S)$$

is compact due to the compact mapping properties of the operators $\mathcal{R}$ and $\mathcal{W}$, (cf. Theorems 4.3 and 4.7).

Let us prove that the operator $\mathcal{M}_0^{12}$ is invertible. For this purpose, we consider the following system with arbitrary right hand side $\widetilde{F} = [\widetilde{F_1}, \widetilde{F_2}]^\top \in H^1(\Omega) \times H^{1/2}(S)$ and let $\mathcal{X} = (u, \psi, \phi)^\top \in H^1(\Omega) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ be the vector of unknowns

$$\mathcal{M}_0^{12}\mathcal{X} = \widetilde{F}. \tag{4.43}$$

Writing (4.43) component-wise,

$$u - V\psi + W\phi = \widetilde{F_1}, \qquad \text{in } \Omega, \tag{4.44a}$$

$$1/2\phi - \mathcal{V}\psi = \widetilde{F_2}, \qquad \text{on } S. \tag{4.44b}$$

Equation (4.44b) restricted to S_D gives:

$$-r_{S_D}\mathcal{V}\psi = r_{S_D}\widetilde{F_2}. \tag{4.45}$$

Due to the invertibility of the operator $\mathcal{V}$ (cf. Lemma 4.8), equation (4.45) is uniquely solvable on S_D. Equation (4.45) means that $(\mathcal{V}\psi + \widetilde{F_2}) \in \widetilde{H}^{1/2}(S_N)$. Thus, the unique solvability of (4.45) implies that ϕ is also uniquely determined by the equation

$$\phi = (2\mathcal{V}\psi + 2\widetilde{F_2}) \in \widetilde{H}^{1/2}(S_N). \tag{4.46}$$

Consequently, u also is uniquely determined by the first equation (4.44a) of the system. Furthermore, since $V\psi$, $W\phi \in H^1(\Omega)$, we have $u \in H^1(\Omega)$.

Thus, the operator $\mathcal{M}_0^{12}$ is invertible and the operator $\mathcal{M}^{12}$ is a zero index Fredholm operator due to the compactness of the operator $\mathcal{M}^{12} - \mathcal{M}_0^{12}$. Hence the Fredholm property and the injectivity of the operator $\mathcal{M}^{12}$, provided by item *iii)* of Theorem 4.13, imply the invertibility of operator $\mathcal{M}^{12}$. $\square$

Chapter 5

A new family of BDIES for a scalar mixed elliptic exterior BVP

5.1 Introduction

Unlike for the case of bounded domains, the Rellich compactness embeding theorem is not available for Sobolev spaces defined over unbounded domains. Nevertheless, we present a lemma to reduce the remainder operator to two operators: one invertible and one compact. Therefore, we can still benefit from the Fredholm Alternative theory to prove uniqueness of the solution.

5.2 Preliminaries

Let $\Omega = \Omega^+$ be a unbounded exterior connected domain, $\Omega^- := \mathbb{R}^3 \smallsetminus \overline{\Omega}^+$ the complementary (bounded) subset of Ω. The boundary $S := \partial\Omega$ is simply connected, closed and infinitely differentiable, $S \in \mathcal{C}^\infty$. Furthermore, $S := \overline{S}_N \cup \overline{S}_D$ where both S_N and S_D are non-empty, connected disjoint manifolds of S. The border of these two submanifolds is also infinitely differentiable: $\partial S_N = \partial S_D \in \mathcal{C}^\infty$.

We consider the following PDE:

$$\mathcal{A}u := \mathcal{A}(x)[u(x)] := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u(x)}{\partial x_i}\right) = f(x), \qquad x \in \Omega, \tag{5.1}$$

where $u(x)$ is the unknown function, $a(x) \in \mathcal{C}^\infty$, $a(x) > 0$, is the variable coefficient and f is a given function on Ω. It is easy to see that if $a \equiv 1$ then, the operator $\mathcal{A}$ becomes the Laplace operator Δ.

We will also make use of some the Sobolev weighted spaces the weight $\omega(x) = (1+|x|^2)^{1/2}$ as introduced in Chapter 4. We recall here those that are particularly relevant for this chapter.

The operator $\mathcal{A}$ acting on $u \in \mathcal{H}^1(\Omega)$ is well defined in the weak sense as long as the variable coefficient $a(x)$ is bounded, i.e. $a \in L^\infty(\Omega)$, as

$$\langle \mathcal{A}u, v\rangle = -\langle a\nabla u, \nabla v\rangle = -\mathcal{E}(u,v) \quad \forall v \in \mathcal{D}(\Omega), \tag{5.2}$$

where

$$\mathcal{E}(u,v) := \int_\Omega E(u,v)(x)dx, \qquad E(u,v)(x) := a(x)\nabla u(x)\nabla v(x). \tag{5.3}$$

Note that the functional $\mathcal{E}(u,v) : \mathcal{H}^1(\Omega) \times \widetilde{\mathcal{H}}^1(\Omega) \longrightarrow \mathbb{R}$ is continuous, thus by the density of $\mathcal{D}(\Omega)$ in $\widetilde{\mathcal{H}}^1(\Omega)$, also is the operator $\mathcal{A} : \mathcal{H}^1(\Omega) \longrightarrow \mathcal{H}^{-1}(\Omega)$ in (5.2) which gives the distributional form of the operator $\mathcal{A}$ given in (5.1).

From now on, we will assume $a(x) \in L^\infty(\Omega)$ and that there exist two positive constants, C_1 and C_2, such that:

$$0 < C_1 < a(x) < C_2. \tag{5.4}$$

For a scalar function $w \in H^1(\Omega)$ in virtue of the trace theorem it follows that $\gamma^\pm w \in H^{1/2}(S)$ where the trace operator on S from $\Omega^\pm$ are denoted by $\gamma^\pm$. Consequently, if $w \in H^1(\Omega)$, then $w \in \mathcal{H}^1(\Omega)$ and it follows that $\gamma^\pm w \in H^{1/2}(S)$, (see, e.g., [McL00, Mi11]).

For $u \in H^s(\Omega); s > 3/2$, we can define by $T^\pm$ the conormal derivative operator acting on S understood in the classical sense:

$$T^\pm[u(x)] := \sum_{i=1}^{3} a(x)n_i(x)\gamma^\pm\left(\frac{\partial u}{\partial x_i}\right) = a(x)\gamma^\pm\left(\frac{\partial u(x)}{\partial n(x)}\right), \tag{5.5}$$

where $n(x)$ is the exterior unit normal vector to the domain Ω at a point $x \in S$.

However, for $u \in \mathcal{H}^1(\Omega)$ (as well as for $u \in H^1(\Omega)$), the classical co-normal derivative operator may not exist on the trace sense. We can overcome this difficulty by introducing the following function space for the operator $\mathcal{A}$, (cf. [CMN13, Gr78])

$$\mathcal{H}^{1,0}(\Omega; \mathcal{A}) := \{g \in \mathcal{H}^1(\Omega) : \mathcal{A}g \in L^2(\omega; \Omega)\} \tag{5.6}$$

endowed with the norm

$$\| g \|^2_{\mathcal{H}^{1,0}(\Omega;\mathcal{A})} := \| g \|^2_{\mathcal{H}^1(\Omega)} + \| \omega \mathcal{A} g \|^2_{L^2(\Omega)} .$$

Now, if a distribution $u \in \mathcal{H}^{1,0}(\Omega;\mathcal{A})$ we can define the conormal derivative $T^+u \in H^{-1/2}(S)$ using the Green's formula, cf. [McL00, CMN13],

$$\langle T^+u, w\rangle_S := \pm \int_{\Omega^\pm} [(\gamma^+_{-1}\omega)\mathcal{A}u + E(u, \gamma^+_{-1}w)]\, dx, \text{ for all } w \in H^{1/2}(S), \tag{5.7}$$

where $\gamma^+_{-1} : H^{1/2}(S) \to \mathcal{H}^1(\Omega)$ is a continuous right inverse to the trace operator $\gamma^+ : \mathcal{H}^1(\Omega) \longrightarrow H^{1/2}(S)$. whereas the brackets $\langle u, v\rangle_S$ represent the duality brackets of the spaces $H^{1/2}(S)$ and $H^{-1/2}(S)$ which coincide with the scalar product in $L^2(S)$ when $u, v \in L^2(S)$.

The operator $T^+ : \mathcal{H}^{1,0}(\Omega;\mathcal{A}) \longrightarrow H^{-1/2}(S)$ is bounded and gives a continuous extension on $\mathcal{H}^{1,0}(\Omega;\mathcal{A})$ of the classical co-normal derivative operator (5.5). We remark that when $a \equiv 1$, the operator T^+ becomes $T^+_\Delta = \delta_n := n \cdot \nabla$, which is the continuous extension on $\mathcal{H}^{1,0}(\Omega;\Delta)$ of the classical normal derivative operator.

In a similar manner as in the proof [McL00, Lemma 4.3] or [Co88, Lemma 3.2], the first Green identity holds for a distribution $u \in \mathcal{H}^{1,0}(\Omega;\mathcal{A})$

$$\langle T^+u, \gamma^+v\rangle_S = \int_\Omega [v\mathcal{A}u + E(u, v)]dx, \quad \forall v \in \mathcal{H}^1(\Omega). \tag{5.8}$$

Applying the identity (5.8) to $u, v \in \mathcal{H}^{1,0}(\Omega;\mathcal{A})$ and then exhanging roles and subtracting the one from the other, we arrive to the following second Green identity, see e.g. [McL00]

$$\int_\Omega [v\mathcal{A}u - u\mathcal{A}v]\, dx = \int_S \left[\gamma^+v\, T^+u - \gamma^+u\, T^+v\right] dS(x). \tag{5.9}$$

5.3 Boundary Value Problem

We aim to derive a BDIES equivalent to the following BVP defined in an exterior domain for further investigation of existence and uniqueness of solution. In the following, $S := \overline{S}_N \cup \overline{S}_D$ where both S_N and S_D are non-empty, connected disjoint manifolds of S.

Mixed problem Let $S := \overline{S}_N \cup \overline{S}_D$, where both S_N and S_D are non-empty, connected disjoint manifolds of S. Find $v \in \mathcal{H}^{1,0}(\Omega; \mathcal{A})$ such that:

$$\mathcal{A}u = f, \quad \text{in } \Omega; \tag{5.10}$$

$$r_{S_D}\gamma^+ u = \phi_0, \quad \text{on } S_D; \tag{5.11}$$

$$r_{S_N}T^+ u = \psi_0, \quad \text{on } S_N; \tag{5.12}$$

where equation (5.10) is understood in the distributional sense and $f \in L^2(\omega, \Omega)$, the second equation (5.11) is understood in the trace sense and $\phi_0 \in H^{1/2}(S_D)$ and the third equation (5.12) is understood in the functional sense and $\psi_0 \in H^{-1/2}(S_N)$. The boundary of Ω, $S = \bar{S_D} \cup \bar{S_N}$.

Each of these systems can be represented by the three following operators:

$$\mathcal{A}_M : \mathcal{H}^{1,0}(\Omega; \mathcal{A}) \longrightarrow L^2(\omega, \Omega) \times H^{1/2}(S_D) \times H^{-1/2}(S_N);$$

The following result is well known and it has been proven using variational settings and the Lax Milgram lemma, see [CMN13, Appendix A] and also [Ha71, Nt65, Gr87, Gr78] and more references therein.

Theorem 5.1. *If $a(x) \in L^\infty(\Omega^+)$ and $a(x) > 0$, the mixed, Dirichlet and Neumann problems are uniquely solvable in $\mathcal{H}^{1,0}(\Omega^+; \mathcal{A})$ and the corresponding inverse operators are continuous*

$$\mathcal{A}_M^{-1} : L^2(\omega, \Omega) \times H^{1/2}(S_D) \times H^{-1/2}(S_N) \longrightarrow \mathcal{H}^{1,0}(\Omega; \mathcal{A});$$

5.4 Parametrices and remainders

In this chapter we will use the same parametrix and remainder as in the previous chapter

$$P^y(x, y) = \frac{1}{a(y)} P_\Delta(x - y), \quad x, y \in \mathbb{R}^3,$$

whose corresponding remainder is

$$R^y(x,y) = \sum_{i=1}^{3} \frac{1}{a(y)} \frac{\partial a(x)}{\partial x_i} \frac{\partial}{\partial x_i} P_\Delta(x-y)\,, \quad x,y \in \mathbb{R}^3. \tag{5.13}$$

Condition 5.2. *To obtain BDIES, we will assume the following condition further on unless stated otherwise:*

$$a \in \mathcal{C}^1(\mathbb{R}^3) \quad \textit{and} \quad \omega \nabla a \in L^\infty(\mathbb{R}^3). \tag{5.14}$$

Remark 5.3. *If a satisfies* (5.4) *and* (5.14), *then* $\parallel ga \parallel_{\mathcal{H}^1(\Omega)} \leq k_1 \parallel g \parallel_{\mathcal{H}^1(\Omega)}$, $\parallel g/a \parallel_{\mathcal{H}^1(\Omega)} \leq k_2 \parallel g \parallel_{\mathcal{H}^1(\Omega)}$ *where the constants* k_1 *and* k_2 *do not depend on* $g \in \mathcal{H}^1(\Omega)$, *i.e., the functions* a *and* $1/a$ *are multipliers in the space* $\mathcal{H}^1(\Omega)$. *Furthermore, as long as* $a \in \mathcal{C}^1(S)$, *then* $\dfrac{\partial a}{\partial n}$ *is also a multiplier.*

5.5 Surface and volume potentials

Since we are using the same parametrix as in the previous chapter, all the notations, relations and mapping properties remain valid. We shall only focus on those mapping properties which are different, especially the mapping properties in weighted Sobolev spaces.

One of the main differences with respect the bounded domain case is that the integrands of the operators V, W, $\mathcal{P}$ and $\mathcal{R}$ and their corresponding direct values and conormal derivatives do not always belong to L^1. In these cases, the integrals should be understood as the corresponding duality forms (or their their limits of these forms for the infinitely smooth functions, existing due to the density in corresponding Sobolev spaces).

The stationary diffusion equation with variable coefficient preserves a strong relation with the Laplace equation which can be exploited to obtain mapping properties of the surface and volume potentials and its jump relations. Mapping properties for slightly different parametrix based potential type operators in weighted Sobolev spaces are analysed in [CMN13].

Condition 5.4. *In addition to conditions* (5.4) *and* (5.14), *we will also sometimes assume the following condition:*

$$\omega^2 \Delta a \in L^\infty(\Omega). \tag{5.15}$$

Remark 5.5. *Note as well that due to the boundedness of the function* a *and the continuity of the function* $\ln a$*, the components of* $\nabla(\ln a)$ *and* $\Delta(\ln a)$ *will be bounded as well.*

Theorem 5.6. *The following operators are continuous under condition* (5.14)*:*

$$V : H^{-1/2}(S) \longrightarrow \mathcal{H}^1(\Omega), \qquad W : H^{1/2}(S) \longrightarrow \mathcal{H}^1(\Omega).$$

Proof. Let us consider a function $g \in H^{-1/2}(S)$, then $\frac{g}{a}$ also belongs to $H^{-1/2}(S)$ in virtue of Remark 5.3. Then, relation (4.9) along with the mapping property $V_\Delta : H^{-1/2}(S) \longrightarrow \mathcal{H}^1(\Omega;\Delta)$, it is clear that $Vg = V_\Delta\left(\frac{g}{a}\right) \in \mathcal{H}^1(\Omega;\Delta)$ what implies $Vg \in \mathcal{H}^1(\Omega)$.

Let us consider a function $g \in H^{1/2}(S)$, then $\frac{\partial(\ln a)}{\partial n} g$ also belongs to $H^{1/2}(S)$ in virtue of Remark 5.3. In addition, by virtue of the Rellich embedding theorem $H^{-1/2}(S) \subset H^{1/2}(S)$. Then, relation (4.11) along with the mapping properties $V_\Delta : H^{-1/2}(S) \longrightarrow \mathcal{H}^1(\Omega;\Delta)$ and $W_\Delta : H^{1/2}(S) \longrightarrow \mathcal{H}^1(\Omega;\Delta)$ it is clear that $Wg \in \mathcal{H}^1(\Omega;\Delta)$ what implies $Wg \in \mathcal{H}^1(\Omega)$.

□

Corollary 5.7. *The following operators are continuous under condition* (5.14) *and* (5.15)*,*

$$V : H^{-1/2}(S) \longrightarrow \mathcal{H}^{1,0}(\Omega;\mathcal{A}), \tag{5.16}$$

$$W : H^{1/2}(S) \longrightarrow \mathcal{H}^{1,0}(\Omega;\mathcal{A}). \tag{5.17}$$

Proof.

$$\mathcal{A}g = \nabla a \nabla g + a\Delta g. \tag{5.18}$$

From Theorem 5.6, we have that $Vg \in \mathcal{H}^1(\Omega)$ for some $g \in H^{1/2}(S)$. Hence, it suffices to prove that $Vg \in L^2(\omega;\Omega)$.

Taking into account relation (4.9) and applying (5.18) to we get

$$\mathcal{A}V_\Delta\left(\frac{g}{a}\right) = \sum_{i=1}^{3} \frac{\partial a}{\partial y_i}\frac{\partial V_\Delta}{\partial y_i}\left(\frac{g}{a}\right) + a\Delta V_\Delta\left(\frac{g}{a}\right) = \sum_{i=1}^{3} \frac{\partial a}{\partial y_i}\frac{\partial V_\Delta}{\partial y_i}\left(\frac{g}{a}\right). \tag{5.19}$$

By virtue of the mapping property for the operator V provided by Theorem 5.6, the last term belongs to $L^2(\omega;\Omega)$ since (5.4) is satisfied. This, completes the proof for the operator V.

The proof for the operator W follows from a similar argument. □

Theorem 5.8. *The following operators are continuous under condition* (5.14)*,*

$$\boldsymbol{P} : \mathcal{H}^{-1}(\mathbb{R}^3) \longrightarrow \mathcal{H}^1(\mathbb{R}^3), \tag{5.20}$$

$$\boldsymbol{R} : L^2(\omega^{-1}; \mathbb{R}^3) \longrightarrow \mathcal{H}^1(\mathbb{R}^3), \tag{5.21}$$

$$\mathcal{P} : \widetilde{\mathcal{H}}^{-1}(\Omega) \longrightarrow \mathcal{H}^1(\mathbb{R}^3). \tag{5.22}$$

Proof. Let $g \in \mathcal{H}^{-1}(\mathbb{R}^3)$. Then, by virtue of the relation (4.7) $\mathbf{P}g = \mathbf{P}_\Delta(g/a)$. Since Condition 5.14 holds, $(g/a) \in \mathcal{H}^{-1}(\mathbb{R}^3)$ and thefore the continuity of the operator $\mathbf{P}$ follows from the continuity of $\mathbf{P}_\Delta : \mathcal{H}^{-1}(\mathbb{R}^3) \longrightarrow \mathcal{H}^1(\mathbb{R}^3)$, which at the same time implies the continuity of the operator (5.22), see [CMN13, Theorem 4.1] and more references therein.

Let us prove now the continuity of the operator $\mathbf{R}$. Due to the second condition in (5.14), $\nabla a \in L^2(\mathbb{R}^3)$ is a multiplier in the space $L^2(\omega^{-1}; \mathbb{R}^3)$. Let $g \in L^2(\omega^{-1}; \mathbb{R}^3)$, then the relation (4.8) applies and gives

$$\begin{aligned}\mathbf{R}g(y) = -\nabla \cdot \mathbf{P}_\Delta(g \cdot \nabla(\ln a))(y) &= -\sum_{i=1}^{3} \frac{\partial}{\partial y_i} \mathbf{P}_\Delta \left(g \cdot \frac{\partial(\ln a)}{\partial x_i} \right)(y) \\ &= -\sum_{i=1}^{3} \mathbf{P}_\Delta \left[\frac{\partial}{\partial x_i} \left(g \cdot \frac{\partial(\ln a)}{\partial x_i} \right) \right](y) := -\mathbf{P}_\Delta g^*. \end{aligned} \tag{5.23}$$

In this case, $g^* \in \mathcal{H}^{-1}(\mathbb{R}^3)$ as a result of a similar argument as in Theorem 3.5 to prove the property (3.11). Here $\Delta \ln a$ and $\nabla \ln a$ are multipliers under conditions (5.14) and (5.4) respectively in the space $\mathcal{H}^{-1}(\mathbb{R}^3)$ as well as in $\mathcal{H}^1(\mathbb{R}^3)$.

As the operator

$$\mathbf{P}_\Delta : \mathcal{H}^{-1}(\mathbb{R}^3) \longrightarrow \mathcal{H}^1(\mathbb{R}^3)$$

is continuous, the operator $\mathbf{R} : L^2(\omega^{-1}; \mathbb{R}^3) \longrightarrow \mathcal{H}^1(\mathbb{R}^3)$ is also continuous. □

Theorem 5.9. *The following operators are continuous under condition* (5.14) *and* (5.15)*,*

$$\mathcal{P} : L^2(\omega; \Omega) \longrightarrow \mathcal{H}^{1,0}(\mathbb{R}^3; \mathcal{A}), \tag{5.24}$$

$$\mathcal{R} : \mathcal{H}^1(\Omega) \longrightarrow \mathcal{H}^{1,0}(\Omega; \mathcal{A}). \tag{5.25}$$

Proof. To prove the continuity of the operator (5.24), we consider a function $g \in L^2(\omega;\Omega)$ and its extension by zero to $\mathbb{R}^3$ which we denote by $\widetilde{g}$. Clearly, $\widetilde{g} \in L^2(\omega;\mathbb{R}^3) \subset \mathcal{H}^{-1}(\mathbb{R}^3)$ and then $\mathcal{P}_\Delta g = \mathbf{P}_\Delta \widetilde{g} \in \mathcal{H}^1(\mathbb{R}^3)$. Bearing in mind that

$$\mathcal{A}(y)[\mathcal{P}g(y)] = g(y) + \sum_{i=1}^{3} \frac{\partial a(y)}{\partial y_i} \frac{\partial \mathcal{P}_\Delta}{\partial y_i} \left(\frac{g}{a}\right)(y),$$

under conditions (5.14) and (5.15), we conclude that $\mathcal{A}(y)[\mathcal{P}g(y)] \in L^2(\omega,\Omega)$ and therefore $\mathcal{P}g \in \mathcal{H}^{1,0}(\Omega,\mathcal{A})$.

Finally, let us prove the continuity of the operator (5.25). The continuity of the operator $\mathcal{R} : \mathcal{H}^1(\Omega) \longrightarrow \mathcal{H}^1(\Omega)$ follows from the continuous embedding $\mathcal{H}^1(\Omega) \subset L^2(\omega^{-1};\Omega)$ and the continuity of the operator (5.21). Hence, we only need to prove that $\mathcal{A}(y)[\mathcal{R}g(y)] \in L^2(\omega;\Omega)$. For $g \in \mathcal{H}^1(\Omega)$ we have

$$\mathcal{A}(y)[\mathcal{R}g(y)] = \frac{\partial a(y)}{\partial y_i} \frac{\partial \mathcal{R}g}{\partial y_i} + a(y)\Delta\mathcal{R}g(y).$$

As $\mathcal{R}g \in \mathcal{H}^1(\Omega)$, we only need to prove that $\Delta\mathcal{R}g(y) \in L^2(\omega;\Omega)$. Using the relation (4.8)

$$\Delta\mathcal{R}g(y) = \Delta\left[-\nabla \cdot \mathcal{P}_\Delta(g\nabla(\ln a))\right] = -\nabla \cdot \Delta\mathcal{P}_\Delta(g\nabla(\ln a)) = -\nabla \cdot (g\nabla(\ln a)),$$

since $g \in \mathcal{H}^1(\Omega)$, then $g \in L^2(\omega,\Omega)$. $\nabla(\ln a)$ is a multiplier in the space $\mathcal{H}^1(\Omega)$ by virtue of the second condition in (5.14), then $(g\nabla \ln a) \in \mathcal{H}^1(\Omega)$. Consequently, $-\nabla \cdot (g\nabla \ln a) \in \mathcal{H}^1(\Omega)$. The rest of the proof follows from condition (5.15) and Theorem 5.8 which imply the continuity of the operator $\mathcal{A}\mathcal{R} : \mathcal{H}^1(\Omega) \longrightarrow L^2(\omega;\Omega)$ and hence the continuity of the operator (5.25). □

5.6 Third Green identities and integral relations

Let $B_\epsilon(y)$ be the ball centered at $y \in \Omega$ with radius ϵ sufficiently small. Then, $R(\cdot,y) \in L^2(\omega;\Omega \smallsetminus B_\epsilon(y))$ and thus $P(\cdot,y) \in \mathcal{H}^{1,0}(\Omega \smallsetminus B_\epsilon(y))$. Applying the second Green (5.9) identity with $v(y) = P(y,\cdot)$ and any distribution $u \in \mathcal{H}^{1,0}(\Omega;\mathcal{A})$ in $\Omega \smallsetminus B_\epsilon(y)$ with $v = P(y,\cdot)$ and using standard limiting procedures $\epsilon \to 0$ (cf. [Mr70]) we obtain the third Green identity (*integral representation formula*) for the function $u \in \mathcal{H}^{1,0}(\Omega;\mathcal{A})$:

$$u + \mathcal{R}u - VT^+u + W\gamma^+u = \mathcal{P}\mathcal{A}u, \quad \text{in } \Omega. \tag{5.26}$$

If $u \in H^{1,0}(\Omega; \mathcal{A})$ is a solution of the PDE (5.10), then, from (5.26), we obtain

$$u + \mathcal{R}u - VT^+(u) + W\gamma^+ u = \mathcal{P}f, \text{ in } \Omega. \tag{5.27}$$

Taking the trace and the conormal derivative of (5.27), we obtain integral representation formulae for the trace and traction of u respectively:

$$\frac{1}{2}\gamma^+ u + \gamma^+\mathcal{R}u - \mathcal{V}T^+ u + \mathcal{W}\gamma^+ u = \gamma^+\mathcal{P}f, \text{ on } S, \tag{5.28}$$

$$\frac{1}{2}T^+ u + T^+\mathcal{R}u - \mathcal{W}'T^+ u + \mathcal{L}^+\gamma^+ u = T^+\mathcal{P}f, \text{ on } S. \tag{5.29}$$

For some distributions f, Ψ and Ψ, we consider a more indirect integral relation associated with the third Green identity (5.27)

$$u + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}f, \text{ in } \Omega. \tag{5.30}$$

Lemma 5.10. *Let $u \in \mathcal{H}^1(\Omega)$, $f \in L_2(\omega; \Omega)$, $\Psi \in H^{-1/2}(S)$ and $\Phi \in H^{1/2}(S)$, satisfying the relation* (5.30)*. Let conditions* (5.14) *and* (5.15) *hold. Then $u \in \mathcal{H}^{1,0}(\Omega, \mathcal{A})$, solves the equation $\mathcal{A}u = f$ in Ω and the following identity is satisfied*

$$V(\Psi - T^+ u) - W(\Phi - \gamma^+ v) = 0, \text{ in } \Omega. \tag{5.31}$$

Proof. To prove that $u \in \mathcal{H}^{1,0}(\Omega; \mathcal{A})$, taking into account that by hypothesis $u \in \mathcal{H}^1(\Omega)$, so there is only left to prove that $\mathcal{A}u \in L^2(\omega; \Omega)$. Firstly we write the operator $\mathcal{A}$ as follows:

$$\mathcal{A}(x)[u(x)] = \Delta(au)(x) - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(u\left(\frac{\partial a(x)}{\partial x_i}\right)\right).$$

It is easy to see that the second term belongs to $L^2(\omega; \Omega)$. Keeping in mind Remark 5.3 and the fact that $u \in \mathcal{H}^1(\Omega)$, then we can conclude that the term $u\nabla a \in \mathcal{H}^1(\Omega)$ since due to the second condition in (5.14) ∇a is a multiplier in the space $\mathcal{H}^1(\Omega)$ and therefore $\nabla(u\nabla a) \in L^2(\omega; \Omega)$.

Now, we only need to prove that $\Delta(au) \in L^2(\omega; \Omega)$. To prove this we look at the relation (5.30) and we put u as the subject of the formula. Then, we use the potential relations (4.7), (4.9) and (4.11)

$$u = \mathcal{P}f - \mathcal{R}u + V\Psi - W\Phi = \mathcal{P}_\Delta\left(\frac{f}{a}\right) - \mathcal{R}u + V_\Delta\left(\frac{\Psi}{a}\right) - W_\Delta\Phi + V_\Delta\left(\frac{\partial(\ln(a))}{\partial n}\Phi\right) \tag{5.32}$$

In virtue of the Theorem 5.9, $\mathcal{R}u \in L^2(\omega;\Omega)$. Moreover, the terms in previous expression depending on V_Δ or W_Δ are harmonic functions and $\mathcal{P}_\Delta$ is the newtonian potential for the Laplacian, i.e. $\Delta\mathcal{P}_\Delta\left(\frac{f}{a}\right) = \frac{f}{a}$. Consequently, applying the Laplacian operator in both sides of (5.32), we obtain:

$$\Delta u = \frac{f}{a} - \Delta\mathcal{R}u. \tag{5.33}$$

Thus, $\Delta u \in L^2(\omega;\Omega)$ from where it immediately follows that $\Delta(au) \in L^2(\omega;\Omega)$. Hence $u \in \mathcal{H}^{1,0}(\Omega;\mathcal{A})$. The rest of the proof is equivalent to Lemma 4.10. □

The proof of the following statement is the counterpart of Lemma 4.11 for exterior domains. The proof follows from the invertibility of the operator $\mathcal{V}_\Delta$, see [McL00, Corollary 8.13].

Lemma 5.11. *Let* $\Psi^* \in H^{-1/2}(S)$. *If*

$$V\Psi^*(y) = 0, \;\; y \in \Omega, \tag{5.34}$$

then $\Psi^*(y) = 0$.

Proof. Take the trace of (5.34) and relation (4.9), to obtain

$$\mathcal{V}\Psi^*(y) = \mathcal{V}_\Delta\left(\frac{\Psi^*}{a}\right)(y) = 0, \;\; y \in \; S. \tag{5.35}$$

Then, applying [McL00, Corollary 8.13], we obtain that the equation (5.35) is uniquely solvable. Hence, $\Psi^*(y) = 0$. □

5.7 BDIES

Let the functions $\Phi_0 \in H^{1/2}(S)$ and $\Psi_0 \in H^{-1/2}(S)$ be continuous fixed extensions to S of the functions $\phi_0 \in H^{1/2}(S_D)$ and $\psi_0 \in H^{1/2}(S_N)$. Moreover, let $\phi \in \widetilde{H}^{1/2}(S_N)$ and $\psi \in \widetilde{H}^{-1/2}(S_D)$ be arbitrary functions formally segregated from u, cf. [CMN09, CMN13, MiPo15-II].

We will derive a system of BDIEs for the BVP (5.10)-(5.12) substituting the following functions:

$$\gamma^+ u = \Phi_0 + \phi, \qquad T^+ u = \Psi_0 + \psi, \quad \text{on} \quad S; \tag{5.36}$$

in the third Green identities (5.27)-(5.29).

In what follows, we will denote by $\mathcal{X}$ the vector of unknown functions

$$\mathcal{X} = (u, \psi, \phi)^\top \in \mathbb{H} := \mathcal{H}^{1,0}(\Omega; \mathcal{A}) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N) \subset \mathbb{X}$$

where

$$\mathbb{X} := \mathcal{H}^1(\Omega) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N). \tag{5.37}$$

M12 We substitute the functions (5.36) in (5.27) and (5.28) to obtain the following BDIES (M12)

$$u + \mathcal{R}u - V\psi + W\phi = F_0, \;\; in \;\; \Omega, \tag{5.38a}$$

$$\frac{1}{2}\phi + \gamma^+\mathcal{R}u - \mathcal{V}\psi + \mathcal{W}\phi = \gamma^+ F_0 - \Phi_0, \;\; on \;\; S. \tag{5.38b}$$

We denote by $\mathcal{M}^{12}$ the matrix operator that defines the system $(M12)$:

$$\mathcal{M}^{12} = \begin{bmatrix} I + \mathcal{R} & -V & +W \\ \gamma^+\mathcal{R} & -\mathcal{V} & \frac{1}{2}I + \mathcal{W} \end{bmatrix}, \tag{5.39}$$

and by $\mathcal{F}^{12}$ the right hand side of the system

$$\mathcal{F}^{12} = [\, F_0, \; \gamma^+ F_0 - \Phi_0 \,]^\top.$$

The systems (M12) can be expressed in terms of matrix notation as

$$\mathcal{M}^{12}\mathcal{X} = \mathcal{F}^{12} \tag{5.40}$$

If the conditions (5.14) and (5.15) hold, then due to the mapping properties of the potentials, $\mathcal{F}^{12} \in \mathbb{F}^{12} \subset \mathbb{Y}^{12}$, while operators $\mathcal{M}^{12} : \mathbb{H} \to \mathbb{F}^{12}$ and $\mathcal{M}^{12} : \mathbb{X} \to \mathbb{Y}^{12}$ are continuous. Here, we denote

$$\mathbb{F}^{12} := \mathcal{H}^{1,0}(\Omega, \mathcal{A}) \times H^{1/2}(S), \qquad \mathbb{Y}^{12} := \mathcal{H}^1(\Omega) \times H^{1/2}(S).$$

Theorem 5.12. *[Equivalence BDIES - BVP] Let* $f \in L_2(\omega;\Omega)$*, let* $\Phi_0 \in H^{-1/2}(S)$ *and let* $\Psi_0 \in H^{-1/2}(S)$ *be some fixed extensions of* $\phi_0 \in H^{1/2}(S_D)$ *and* $\psi_0 \in H^{-1/2}(S_N)$*, respectively. Let conditions* (5.14) *and* (5.15) *hold.*

i) If some $u \in \mathcal{H}^{1,0}(\Omega;\mathcal{A})$ *solves the BVP* (5.10)-(5.12)*, then the triplet* $(u,\psi,\phi)^\top \in \mathcal{H}^{1,0}(\Omega;\mathcal{A}) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ *where*

$$\phi = \gamma^+ u - \Phi_0, \qquad \psi = T^+ u - \Psi_0, \qquad \text{on } S,$$

solves the BDIES (M12).

ii) If a triple $(u,\psi,\phi)^\top \in \mathcal{H}^{1,0}(\Omega;\mathcal{A}) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ *solves the BDIES (M12), then this solution is unique. Furthermore,* u *solves the BVP* (5.10)-(5.12) *and the functions* ψ, ϕ *satisfy*

$$\phi = \gamma^+ u - \Phi_0, \qquad \psi = T^+ u - \Psi_0, \qquad \text{on } S. \tag{5.41}$$

Proof. The proof of item *i*) automatically follows from the derivation of the BDIES (M12).

Let us prove now item *ii*). Let the triple $(u,\psi,\phi)^\top \in \mathcal{H}^1(\Omega) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ solve the BDIE system. Taking the trace of the equation (5.38a) and substract it from the equation (5.38b), we obtain

$$\phi = \gamma^+ u - \Phi_0, \quad \text{on } S. \tag{5.42}$$

This means that the first condition in (5.41) is satisfied. Now, restricting equation (5.42) to S_D, we observe that ϕ vanishes as *supp*(ϕ) $\subset S_N$. Hence, $\phi_0 = \Phi_0 = \gamma^+ u$ on S_D and consequently, the Dirichlet condition of the BVP (5.11) is satisfied.

We proceed using the Lemma 5.10 in equation (5.38a), with $\Psi = \psi + \Psi_0$ and $\Phi = \phi + \Phi_0$ which implies that u is a solution of the equation (5.10) and also the following equality:

$$V(\Psi_0 + \psi - T^+ u) - W(\Phi_0 + \phi - \gamma^+ u) = 0 \text{ in } \Omega. \tag{5.43}$$

In virtue of (5.42), the second term of the previous equation vanishes. Hence,

$$V(\Psi_0 + \psi - T^+ u) = 0, \quad \text{in } \Omega. \tag{5.44}$$

Now, in virtue of Lemma 5.11 we obtain

$$\Psi_0 + \psi - T^+ u = 0, \quad \text{on } S. \tag{5.45}$$

Since ψ vanishes on S_N, we can conclude $\Psi_0 = \psi_0$ on S_N. Consequently, equation (5.45) implies that u satisfies the Neumann condition (5.12).

□

5.8 Representation Theorems and Invertibility

In this section, we aim to prove the invertibility of the operator $\mathcal{M}^{12} : \mathbb{H} \to \mathbb{F}^{12}$ by showing first that the arbitrary right hand side $\mathbb{F}^{12}$ from the respective spaces can be represented in terms of the parametrix-based potentials and using then the equivalence theorems.

The following result is the counterpart of [CMN13, Lemma 7.1]. The analogous result for bounded domains can be found in [CMN09, Lemma 3.5].

Lemma 5.13. *For any function $\mathcal{F}_* \in \mathcal{H}^{1,0}(\Omega; \mathcal{A})$, there exists a unique couple $(f_*, \Psi_*) = \mathcal{C}\mathcal{F}_* \in L^2(\omega; \Omega) \times H^{-1/2}(S)$ such that*

$$\mathcal{F}_*(y) = \mathcal{P} f_*(y) + V\Psi_*(y), \quad y \in \Omega, \tag{5.46}$$

where $\mathcal{C} : \mathcal{H}^{1,0}(\Omega; \mathcal{A}) \to L^2(\omega; \Omega) \times H^{-1/2}(S)$ is a linear continuous operator.

Proof. Let us assume that such functions f_* and Ψ_*, satisfying (5.46), exist. Then, we aim to find expressions of these functions in terms of $\mathcal{F}_*$. Applying the potential relations (4.9), (4.7) to the equation (5.46), we obtain

$$\mathcal{F}_*(y) = \mathcal{P}_\Delta \left(\frac{f_*}{a} \right)(y) + V_\Delta \left(\frac{\Psi_*}{a} \right)(y), \quad y \in \Omega. \tag{5.47}$$

Applying the Laplace operator at both sides of the equation (5.47), we get

$$f_* = a\Delta \mathcal{F}_*. \tag{5.48}$$

On the other hand, we can rewrite equation (5.47) as

$$V_\Delta \left(\frac{\Psi_*}{a} \right)(y) = Q(y), \quad y \in \Omega, \tag{5.49}$$

where

$$Q(y) := \mathcal{F}_*(y) - \mathcal{P}_\Delta\left(\Delta \mathcal{F}_*\right). \tag{5.50}$$

Now, we take the trace of (5.49)

$$\mathcal{V}_\Delta\left(\frac{\Psi_*}{a}\right)(y) = \gamma^+ Q(y), \quad y \in S. \tag{5.51}$$

It is well known that the direct value operator of the single layer potential for the Laplace equation $\mathcal{V}_\Delta : H^{-1/2}(S) \longrightarrow H^{1/2}(S)$ is invertible (cf. e.g. [McL00, Corollary 8.13]). Hence, we obtain the following expresion for Ψ_*:

$$\Psi_*(y) = a\mathcal{V}_\Delta^{-1}\gamma^+ Q(y), \quad y \in S. \tag{5.52}$$

Relations (5.48) and (5.52) imply the uniqueness of the couple (f_*, Ψ_*).

Now, we just simply need to prove that the pair (f_*, Ψ_*) given by (5.52) and (5.48) satisfies (5.46). For this purpose, let us note that the single layer potential operator, $V_\Delta(\Psi_*/a)$ with Ψ_* given by (5.52), as well as $Q(y)$ given by (5.50) are both harmonic functions. Since $Q(y)$ and $V_\Delta(\Psi_*/a)$ are two harmonic functions that coincide on the boundary due to (5.51), then they must be identical in the whole Ω due to the uniqueness of solution to the Dirichlet problem for the Laplace equation, see [CMN13, Theorem 3.1]. As a consequence, (5.49) is true which implies (5.46). Thus, relations (5.48), (5.50) and (5.52) give

$$(f_*, \Psi_*) = \mathcal{C}\mathcal{F}_* := (a\Delta\mathcal{F}_*, a\mathcal{V}_\Delta^{-1}\gamma^+[\mathcal{F}_* - \mathcal{P}_\Delta(a\Delta\mathcal{F}_*)]). \tag{5.53}$$

Since all the operators involved in the definition (5.53) of the operator $\mathcal{C}$ are continuous and linear, the operator $\mathcal{C}$ is also continuous and linear. □

Corollary 5.14. *Let*

$$(\mathcal{F}_0, \mathcal{F}_1) \in \mathcal{H}^{1,0}(\Omega; \mathcal{A}) \times H^{1/2}(\partial\Omega).$$

Then there exists a unique triplet (f_*, Ψ_*, Φ_*) *such that* $(f_*, \Psi_*, \Phi_*) = \mathcal{C}_*(\mathcal{F}_0, \mathcal{F}_1)^\top$*, where* $\mathcal{C}_* : \mathcal{H}^{1,0}(\Omega, \mathcal{A}) \times H^{1/2}(S) \to L^2(\omega; \Omega) \times H^{-1/2}(S) \times H^{1/2}(S)$ *is a linear an bounded operator* *and* $(\mathcal{F}_0, \mathcal{F}_1)$ *are given by*

$$\mathcal{F}_0 = \mathcal{P}f_* + V\Psi_* - W\Phi_* \quad in\ \Omega \tag{5.54}$$

$$\mathcal{F}_1 = \gamma^+\mathcal{F}_0 - \Phi_* \quad on\ \partial\Omega \tag{5.55}$$

Proof. Taking $\Phi_* = \gamma^+\mathcal{F}_0 - \mathcal{F}_1$ and applying the previous lemma to $\mathcal{F}_* = \mathcal{F}_0 + W\Phi_*$ we prove existence of the representation (5.54) and (5.55). The uniqueness follows from the homogenenous case when $\mathcal{F}_0 = \mathcal{F}_1 = 0$. Then, (5.55) implies $\Phi_* = 0$ and consequently, by (5.54) and Lemma 5.13, we get $\Psi_* = 0$ and $f_* = 0$. □

We are ready to prove one of the main results for the invertibility of the matrix operator of the BDIES (M12).

Theorem 5.15. *If conditions* (5.14) *and* (5.15) *hold, then the following operator is continuous and continuously invertible:*

$$\mathcal{M}^{12} : \mathbb{H} \to \mathbb{F}^{12} \tag{5.56}$$

Proof. In order to prove the invertibility of the operator $\mathcal{M}^{12} : \mathbb{H} \longrightarrow \mathbb{F}^{12}$, we apply the Corollary 5.14 to any right-hand side $\mathcal{F}^{12} \in \mathbb{F}^{12}$ of the equation $\mathcal{M}^{12}\mathcal{U} = \mathcal{F}^{12}$. Thus, $\mathcal{F}^{12}$ can be uniquely represented as $(f_*, \Psi_*, \Phi_*)^\top = \mathcal{C}_*\mathcal{F}^{12}$ as in (5.54)-(5.55) where $\mathcal{C}_* : \mathbb{F}^{12} \longrightarrow L^2(\omega;\Omega) \times H^{-1/2}(S) \times H^{1/2}(S)$ is continuous.

In virtue of the equivalence theorem for the system (M12), Theorem 5.12, and the invertibility theorem for the boundary value problem with mixed boundary conditions, Theorem 5.1, the matrix equation $\mathcal{M}^{12}\mathcal{U} = \mathcal{F}^{12}$ has a solution $\mathcal{U} = (\mathcal{M}^{12})^{-1}\mathcal{F}^{12}$ where the operator $(\mathcal{M}^{12})^{-1}$, is given by expressions

$$u = \mathcal{A}_M^{-1}[f_*, r_{S_D}\Phi_*, r_{S_N}\Psi_*], \quad \psi = T^+u - \Psi_*, \quad \phi = \gamma^+u - \Phi_*, \tag{5.57}$$

where $(f_*, \Psi_*, \Phi_*)^\top = \mathcal{C}_*\mathcal{F}^{12}$. Consequently, the operator $(\mathcal{M}^{12})^{-1}$ is a continuous right inverse to the operator (5.56). Moreover, the operator $(\mathcal{M}^{12})^{-1}$ results to be a double sided inverse in virtue of the injectivity implied by Theorem 5.12. □

5.9 Fredholm properties and Invertibility

In this section, similar to [CMN13, Section 7.2], we are going to benefit from the compactness properties of the operator $\mathcal{R}$ to prove invertibility of the operator $\mathcal{M}^{12} : \mathbb{X} \to \mathbb{Y}^{12}$. For this we will have to split the operator $\mathcal{R}$ into two parts , one which can be made arbitrarily small and the other part will be contact. Then, we shall simply make use of the Fredholm alternative to prove the invertibility of these operators. However, we can only split the operator $\mathcal{R}$ if the PDE satisfies the additional condition

$$\lim_{|x|\to\infty} \omega(x)\nabla a(x) = 0. \tag{5.58}$$

Lemma 5.16. *Let conditions* (5.14) *and* (5.58) *hold. Then, for any* $\epsilon > 0$ *the operator* $\mathcal{R}$ *can be represented as* $\mathcal{R} = \mathcal{R}_s + \mathcal{R}_c$*, where* $\| \mathcal{R}_s \|_{\mathcal{H}^1(\Omega)} < \epsilon$*, while* $\mathcal{R}_c : \mathcal{H}^1(\Omega) \to \mathcal{H}^1(\Omega)$ *is compact.*

Proof. Let $B(0,r)$ be the ball centered at 0 with radius r big enough such that $S \subset B_r$. Furthermore, let $\chi \in \mathcal{D}(\mathbb{R}^3)$ be a cut-off function such that $\chi = 1$ in $S \subset B_r$, $\chi = 0$ in $\mathbb{R}^3 \setminus B_{2r}$ and $0 \le \chi(x) \le 1$ in $\mathbb{R}^3$. Let us define by $\mathcal{R}_c g := \mathcal{R}(\chi g)$, $\mathcal{R}_s g := \mathcal{R}((1-\chi)g)$.

We will prove first that the norm of $\mathcal{R}_s$ can be made infinitely small. Let $g \in \mathcal{H}^1(\Omega)$, then

$$\begin{aligned}
&\| \mathcal{R}_s g \|_{\mathcal{H}^1(\Omega)} = \| \sum_{i=1}^{3} \mathcal{P}_\Delta \left[\frac{\partial}{\partial x_i} \left(\sum_{i=1}^{3} \frac{\partial(\ln a)}{\partial x_i} (1-\chi) g \right) \right] \|_{\mathcal{H}^1(\Omega)} \le k \| \mathcal{P}_\Delta \|_{\widetilde{\mathcal{H}}^{-1}(\Omega)},\\
&\quad \text{with} \quad k := \sum_{i=1}^{3} \| \frac{\partial}{\partial x_i} \left(\sum_{i=1}^{3} \frac{\partial(\ln a)}{\partial x_i} (1-\chi) g \right) \|_{\widetilde{\mathcal{H}}^{-1}(\Omega)} \le \\
&\quad \sum_{i=1}^{3} \| \frac{\partial(\ln a)}{\partial x_i} (1-\chi) g \|_{L^2(\Omega)} \le 3 \| g \|_{L^2(\omega^{-1};\Omega)} \| \omega \nabla a \|_{L^\infty(\mathbb{R}^3 \setminus B_r)} \le \\
&\quad 3 \| g \|_{\mathcal{H}^1(\Omega)} \| \omega \nabla a \|_{L^\infty(\mathbb{R}^3 \setminus B_r)} .
\end{aligned}$$

Consequently, we have the following estimate:

$$\| \mathcal{R}_s g \|_{\mathcal{H}^1(\Omega)} = 3 \| g \|_{\mathcal{H}^1(\Omega)} \| \omega \nabla a \|_{L^\infty(\mathbb{R}^3 \setminus B_r)} \| \mathcal{P}_\Delta \|_{\widetilde{\mathcal{H}}^{-1}(\Omega)} .$$

Using the previous estimate is easy to see that when $\epsilon \to +\infty$ the norm $\| \mathcal{R}_s g \|_{\mathcal{H}^1(\Omega)}$ tends to 0. Hence, the norm of the operator $\mathcal{R}_s$ can be made arbitrarily small.

To prove the compactness of the operator $\mathcal{R}_c g := \mathcal{R}(\chi g)$, we recall that $supp(\chi) \subset \bar{B}(0, 2r)$. Then, one can express $\mathcal{R}_c g := \mathcal{R}_{\Omega_r}([\chi g|_{\Omega_r}])$ where the operator $\mathcal{R}$ is defined now over $\Omega_r := \Omega \cap B_{2r}$ which is a bounded domain. As the restriction operator $|_{\Omega_r} : \mathcal{H}^1(\Omega) \longrightarrow \mathcal{H}^1(\Omega_r)$ is continuous, in virtue of Theorem 5.8, the operator $\mathcal{R}_c g : L^2(\Omega_r) \longrightarrow \mathcal{H}^1(\Omega_r)$ is also continuous. Due to the boundedness of Ω_r, we have $\mathcal{H}^1(\Omega_r) = H^1(\Omega_r)$ and thus the compactness of $\mathcal{R}_c g$ follows from the Rellich Theorem applied to the embedding $L^2(\Omega_r) \subset H^1(\Omega_r)$. □

Corollary 5.17. *Let conditions* (5.14) *and* (5.58) *hold. Then, the operator* $I+\mathcal{R} : \mathcal{H}^1(\Omega) \to \mathcal{H}^1(\Omega)$ *is Fredholm with zero index.*

Proof. Using the previous Lemma, we have $\mathcal{R} = \mathcal{R}_s + \mathcal{R}_c$ so $\| \mathcal{R}_s \| < 1$ hence $I + \mathcal{R}_s$ is invertible. On the other hand $\mathcal{R}_c$ is compact and hence $I + \mathcal{R}_s$ a compact perturbation of the operator $I + \mathcal{R}$, from where it follows the result. □

Theorem 5.18. *If conditions* (5.14), (5.15) *and* (5.58) *hold, then the operator*

$$\mathcal{M}^{12} : \mathbb{X} \to \mathbb{Y}^{12}, \tag{5.59}$$

is continuously invertible.

Proof. Let

$$\mathcal{M}_0^{12} = \begin{bmatrix} I & -V & W \\ 0 & -\mathcal{V} & \frac{1}{2}I \end{bmatrix}.$$

Let $\mathcal{U} = (u, \psi, \phi) \in \mathbb{X}$ be a solution of the equation $\mathcal{M}_0^{12}\mathcal{U} = \mathcal{F}$, where $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) \in \mathcal{H}^1(\Omega) \times H^{1/2}(S)$. Then, $\mathcal{U}$ will also solve the following extended system

$$\begin{aligned} u + W\phi - V\psi &= \mathcal{F}_1 \quad \text{in } \Omega, \\ \frac{1}{2}\phi - \mathcal{V}\psi &= \mathcal{F}_2 \quad \text{on } S, \\ -r_{S_D}\mathcal{V}\psi &= r_{S_D}\mathcal{F}_2 \quad \text{on } S_D. \end{aligned} \tag{5.60}$$

Furthermore, every solution of the system (5.60) will solve the equation $\mathcal{M}_0^{12}\mathcal{U} = \mathcal{F}$.

The system (5.60) can be written also in matrix form as $\widetilde{\mathcal{M}}_0^{12}\mathcal{U} = \widetilde{\mathcal{F}}$ where $\widetilde{\mathcal{F}}$ denotes the right hand side and $\widetilde{\mathcal{M}}_0^{12}$ is defined as

$$\widetilde{\mathcal{M}}_0^{12} := \begin{bmatrix} I & W & -V \\ 0 & \frac{1}{2}I & -\mathcal{V} \\ 0 & 0 & -r_{S_D}\mathcal{V} \end{bmatrix}.$$

We note that the three diagonal operators:

$$I : \mathcal{H}^1(\Omega) \longrightarrow \mathcal{H}^1(\Omega),$$
$$\frac{1}{2}I : H^{1/2}(S) \longrightarrow H^{1/2}(S),$$
$$-r_{S_D}\mathcal{V} : \widetilde{H}^{-1/2}(S_D) \longrightarrow H^{1/2}(S_D)$$

are invertible, cf. Theorem 4.8. Hence, the operator $\widetilde{\mathcal{M}}_0^{12}$ which defines the system (5.60) is invertible.

Now, let $\psi \in \widetilde{H}^{-1/2}(S_D)$ such that the third equation in the system (5.60) is satisfied. Then, solving ϕ from the second equation of the system, we get $\phi = 2(\mathcal{V}\psi + \mathcal{F}_2) \in \widetilde{H}^{1/2}(S_N)$ from where the invertibility of the operator $\mathcal{M}_0^{12}$ follows.

Now, we decompose $\mathcal{M}^{12} - \mathcal{M}_0^{12} = \mathcal{M}_s^{12} + \mathcal{M}_c^{12}$ and we prove that $\mathcal{M}_0^{12} + \mathcal{M}_s^{12}$ is a compact perturbation of $\mathcal{M}^{12}$. Consequently, $\mathcal{M}^{12}$ is Fredholm with index zero. In addition, as the operator $\mathcal{M}^{12}$ is one to one, we conclude that it is also continuously invertible. □

Chapter 6

Conclusions and Further Work

6.1 Conclusions

A parametrix for the Stokes system with variable viscosity has been obtained in Chapters 2 and 3. This parametrix has allowed us to establish relationships between the hydrodinamic potentials for the constant coefficient case, $\mu = 1$ and the variable viscosity case. As a result, multiple mapping properties regarding the compactness and boundedness of the hydrodynamical surface and volume operators in appropriate Sobolev spaces have been proved.

Furthermore, we have obtained integral representation formulae for the solution of the mixed BVP for the Stokes system, in the interior case and in the exterior case. These formulae have allowed us to construct BDIES equivalent to the original mixed BVP with variable coefficient.

The existence and uniqueness of solution of the BDIES have been proved as well as mapping properties of the matrix operators that defined these systems, such us boundedness and invertibility on the usual Sobolev spaces for the interior domain case and also on weighted Sobolev spaces for the case of exterior domains.

Moreover, the second part of the thesis, Chapters 4 and 5, has concentrated on the idea that there is more than one appropriate parametrix for a PDE (or system) that works. In particular, the family of parametrices of the form $P^x(x, y; a(x))$, different to the family $P^y(x, y; a(y))$ already analysided in [CMN09]. This new family of parametrices has not yet been studied and we have analysed this scenario for a mixed elliptic BVP in both bounded and unbounded domains. Mapping properties of the corresponding P^x-based potentials are proved in both bounded and unbounded domains in Sobolev and weighted Sobolev spaces, respectively.

Using this new family of parametrices, we have been able to deduce a BDIES. Furthermore, we have proven that this BDIES is equivalent to the original BVP in both interior and exterior domains. As a result, the uniqueness is automatically proved once the uniqueness of the BVP has been proved.

Moreover, continuouity and invertibility of the matrix operator that defines the BDIES has been proved by applying the compactness properties and the Fredholm alternative theory.

6.2 Further Work

- One of the main ideas for further work is studying the systems analogous (M12) and (M21) of [CMN09] for the compressible Stokes system in both bounded and unbounded domains. These two systems will have the feature that they are not uniquely solvable due to the degenerate case of the operators $\mathring{\mathcal{V}}$ and $\mathring{\mathcal{L}}$, see [ReSt03].

- Derive the corresponding BDIES for the compressible Stokes system for the Dirichlet and Neumann problems in both bounded and unbounded domains.

- Reproduce the analogous results as in the two previous items for the 2D case.

- Generalise the previous results for Lipschitz domains.

- Develop and implement a numerical scheme to solve the BDIES.

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