Calibration error analysis of inertially stabilized platforms using quaternions and octonions in rotation decomposition

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Abstract
In the calibration process of the inertially stabilized platforms with a high-precision turntable and an autocollimator, significant calibration errors can result from the axis misalignments between the inertially stabilized platforms and the turntable. Based on the relationship between spatial rotations and quaternions or octonions, this article proposes a representation using octonions to realize the decomposition of the rotation axis in two perpendicular axes and subsequently derives the calibration error model. The test results demonstrated that the error is significantly improved after compensation. The azimuth variance is reduced from 0.1379(°)\textsuperscript{2} to 0.0492(°)\textsuperscript{2}, which offers a more accurate set of data for further compensation based on the error model of the platform itself.

Keywords
Rotation, quaternion, octonion, spinor, inertially stabilized platform calibration

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Introduction
As a vector rotates around a straight line in a three-dimensional (3D) space coordinate system, it is easy to obtain the rotated vector through a rotation matrix.\textsuperscript{1–5} Inertially stabilized platforms (ISPs) have been widely utilized to maintain its sensor’s orientation pointing to an accurate direction in many applications, such as vehicles, ships, aircrafts and spacecraft. Its accuracy is technically vital and affects the capture, location and tracking of the target.\textsuperscript{6–9} A two-axial platform is fixed on a high-precision turntable, with an autocollimator showing the rotating angular error,\textsuperscript{10,11} as shown in Figure 1, where \(A_1\) is the azimuth axis of the turntable, \(B_1\) its elevation axis, \(A_2\) the azimuth axis of the platform, \(B_2\) its elevation axis and \(C_2\) the line of sight. At zero positions, \(A_1\) and \(A_2\) are vertical, while \(B_1\) and \(B_2\) are horizontal.

In the calibration process, \(\theta_{PA}\) and \(\theta_{PE}\) are the platform rotation angles in terms of azimuth and elevation, respectively. The azimuth and elevation angles can be similarly defined for the turntable (\(\theta_{TA}\) and \(\theta_{TE}\)) and the autocollimator readouts of axes (\(\delta_A\) and \(\delta_E\)). Generally, it is \(\theta_{TA} = -\theta_{PA}\) and \(\theta_{TE} = -\theta_{PE}\). If \(\theta_{PE} = 0\) and \(\theta_{TE} = 0\), it is obvious that \(\delta_A = \theta_{PA} - \theta_{TA}\). However, if \(\theta_{TE} = -\theta_{PE} \neq 0\), \(A_1\) and \(A_2\) are no longer vertical, which indicates \(\delta_A \neq \theta_{PA} - \theta_{TA}\). It means some calibration error is introduced, and it is obligatory to determine the error to calibrate ISPs accurately.

This article first presents the notation and definitions of quaternions and octonions and then gives a detailed description of the relationship between rotations and quaternions or octonions. It obtains the spinor with octonions as a vector rotates around an axis by means of decomposing the shaft into two perpendicular axes, which indicates octonions can also be used in rotations. The actual test results have demonstrated that the calibration process error can be effectively compensated.

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The norm of the quaternion \( \mathbf{Q} \) is defined as in equation (5). If the norm is 1, the quaternion is called a unit quaternion
\[
\| \mathbf{Q} \| = \sqrt{q_0^2 + |\mathbf{q}|^2} = \sqrt{q_0^2 + a^2 + b^2 + c^2}
\]
(5)

The quaternion \( \mathbf{Q}^* = q_0 - \mathbf{q} \) is called the conjugate of \( \mathbf{Q} \). Hence
\[
\mathbf{Q}^* = \mathbf{Q}^2 = \| \mathbf{Q} \|^2
\]
(6)
\[
\langle \mathbf{Q} \rangle^* = \mathbf{Q}^* \mathbf{P}^*
\]
(7)
\[
\| \mathbf{P} \| = \| \mathbf{P} \| \| \mathbf{Q} \|
\]
(8)

An arbitrary quaternion \( \mathbf{Q} \) can be described as
\[
\mathbf{Q} = \| \mathbf{Q} \| (\cos \theta + \mathbf{n} \sin \theta)
\]
(9)
where \( \mathbf{n} \) is the unit vector, and \( \mathbf{n} = \mathbf{q} / \| \mathbf{q} \| \), \( \cos \theta = q_0 / \| \mathbf{Q} \| \) and \( 0 \leq \theta < \pi \).

**Quaternions and rotations**

Let \( S \) be an arbitrary inertial reference frame with \( i, j \) and \( k \) as basic vectors and \( \mathbf{R}_s \) be an arbitrary vector.

The unit quaternion \( \mathbf{Q} \) given in equation (10) can represent a counterclockwise rotation which rotates an angle \( 2\theta \) around the vector \( \mathbf{n} \) in the coordinate system
\[
\mathbf{Q} = \cos \theta + \mathbf{n} \sin \theta
\]
(10)

The resulting vector \( \mathbf{R}'_s \) can be computed through rotation transformation by equation (11)
\[
\mathbf{R}'_s = \mathbf{Q} \mathbf{R}_s \mathbf{Q}^*
\]
(11)
where \( \mathbf{R}'_s \) is the rotated vector.

**Proof.** As shown in Figure 2, the vector \( \mathbf{p} \) rotates around the line \( \mathbf{l} \) at an angle \( 2\theta \) to \( \mathbf{q} \). In order to obtain \( \mathbf{q} \), \( \mathbf{p} \) can be decomposed into \( \mathbf{p}_l \) which is aligned with \( \mathbf{l} \) and \( \mathbf{p}_\perp \) which is perpendicular to \( \mathbf{l} \). \( \mathbf{p}_l \) turns into \( \mathbf{p}'_l \) after rotation, and \( \mathbf{p}_{\perp} \) remains the same, while \( e \) is the unit vector of \( \mathbf{p}_l \). Hence
\[
\begin{align*}
\mathbf{p} &= \mathbf{p}_\perp + \mathbf{p}_l \\
\mathbf{q} &= \mathbf{p}_\perp + \mathbf{p}'_l \\
\end{align*}
\]
(12)

\[
\mathbf{p}'_l \mathbf{p}_\perp = -\mathbf{p}_\perp \mathbf{p}'_l - \mathbf{p}_\perp \mathbf{p}'_\perp = -\| \mathbf{p}_\perp \| |\mathbf{p}'_\perp| \cos 2\theta - e |\mathbf{p}_\perp| |\mathbf{p}'_\perp| \sin 2\theta
\]
(13)

where \( \mathbf{p}_\perp, \mathbf{p}'_\perp \) and \( e \) comply with right-handed coordinate system principle. It then has
\[
\mathbf{p}'_\perp = (\| \mathbf{p}_\perp \| |\mathbf{p}'_\perp| \cos 2\theta - e |\mathbf{p}_\perp| |\mathbf{p}'_\perp| \sin 2\theta) \mathbf{p}_\perp^{-1}
\]
(14)

It is easy to know that
\[
\mathbf{p} = (\mathbf{p} \cdot e) e = -\frac{1}{2} (\mathbf{pe} + e \mathbf{p}) e = \frac{1}{2} (\mathbf{p} - e \mathbf{p})
\]
(15)
Combined with equations (12), (14), (15) and (16), we obtain

\[
q = p_k + p_0 \sin \theta epe + \cos \theta epe
\]

which is in accord with equation (11); thus, quaternions can be used to describe the rotations.

**Definitions of octonions**

Similar to quaternions, the octonions are an eight-dimensional (8D) algebra with bases \(1, i, j, k, il, jl, kl\), with their multiplications given in Table 1, which gives a detailed description of the multiplying results. \(^{16}\)

It is obvious that the octonions are nonassociative since \((ijl = -ilk) = kl \neq ikj\). An arbitrary octonion can be described as in equation (18)

\[
O = a_0 + a_1i + a_2j + a_3k + a_4il + a_5jl + a_6il + a_7kl
\]

where \(a_0 - a_7\) are real numbers.

**Rotations and octonions**

Since the octonions were discovered independently by Arthur Cayley in 1845, one of the most familiar applications is to describe the process whereby an electron emits or absorbs a photon,\(^{16}\) where it can be used to describe the spinor.

**Octonions in 3D coordinate system**

In Figure 2, the resulting vector \(q = p_k + p_0 \sin \theta epe\) is not affected by the rotation, and it remains unchanged after rotation.

\[
\text{Figure 2. Schematic diagram of rotation.}
\]

Table 1. Octonion multiplication table.

<table>
<thead>
<tr>
<th>l</th>
<th>i</th>
<th>j</th>
<th>k</th>
<th>l</th>
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<tr>
<td>l</td>
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<td>jl</td>
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<tr>
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<td>i</td>
<td>-l</td>
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<td>j</td>
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<td>-il</td>
<td>-j</td>
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<td>kl</td>
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<td>-jl</td>
<td>il</td>
<td>l</td>
<td>-k</td>
<td>-j</td>
<td>i</td>
</tr>
</tbody>
</table>

**Figure 3. Octonion description by employing a mirror.**
In Figure 4, assume X, Y and Z axes are rotated around the Z axis by an angle of $2\theta$, then we obtain

\[ \begin{align*}
X \text{ axis: } O'_x &= Q_zO_xQ_z^* \\
&= (i\cos 2\theta + j\sin 2\theta) + il\cos 2\theta - k\sin 2\theta \\
Y \text{ axis: } O'_y &= Q_zO_yQ_z^* \\
&= (-i\sin 2\theta + j\cos 2\theta) + jl\cos 2\theta - k\sin 2\theta \\
Z \text{ axis: } O'_z &= Q_zO_zQ_z^* = k + kl\cos 2\theta - k\sin 2\theta
\end{align*} \]

where $Q_z = \cos \theta + k\sin \theta$, $Q_z^* = \cos \theta - k\sin \theta$, $O_x = i + il$, $O_y = j + il$ and $O_z = k + kl$.

It is obvious that $(i\cos 2\theta + j\sin 2\theta)$, $(-i\sin 2\theta + j\cos 2\theta)$ and $k\cos 2\theta - k\sin 2\theta$ describe the rotation processes.

Suppose three lines $L_1$, $L_2$ and $L_3$ are in the planes XOY and YOZ and the space OXYZ, respectively, with the same inclined angle $\alpha$ with Y axis as shown in Figure 4. The vectors of the three lines can be described as

\[ \begin{align*}
L_1: V_1 &= i\sin \alpha + j\cos \alpha \\
L_2: V_2 &= j\cos \alpha + k\sin \alpha \\
L_3: V_3 &= -i\sin \alpha \sin \beta + k\sin \alpha \cos \beta + j\cos \alpha
\end{align*} \]

where $\beta$ is the angle between the Z axis and the projection of $L_3$ in the plane XOZ.

Define an octonion

\[ O = O_p + O_r \]

where $O_p = a_0 + a_1i + a_2j + a_3k$, denoting the position vector, $O_r = a_4 + a_5i + a_6j + a_7k$, denoting the rotation angle vector. Since the position vector is the same as the quaternions described above during the rotation, here we only focus on the rotation angle vector.

Suppose that Y axis is rotated 20 around $L_1$, $L_2$ and $L_3$, respectively, then we obtain

\[ \begin{align*}
L_1: O_{11} &= Q_1O_{12}Q_1^* \\
&= -l(\sin 2\theta \cos \alpha + j\cos 2\theta - k\sin 2\theta \sin \alpha) \\
L_2: O_{12} &= Q_2O_{13}Q_2^* \\
&= -l(\sin 2\theta \cos \alpha + j\cos 2\theta + i\sin 2\theta \sin \alpha) \\
O_{13} &= Q_3O_{14}Q_3^* \\
&= -l(\sin 2\theta \cos \alpha + j\cos 2\theta + (i\sin 2\theta \sin \alpha \cos \beta + k\sin 2\theta \sin \alpha \sin \beta))
\end{align*} \]

where $Q_1 = \cos \theta + V_1\sin \theta$, $Q_1^* = \cos \theta - V_1\sin \theta$, $Q_2 = \cos \theta - V_3\sin \theta$, $Q_2^* = \cos \theta + V_3\sin \theta$, $Q_3 = \cos \theta + V_3\sin \theta$, $Q_3^* = \cos \theta - V_3\sin \theta$ and $O_{14} = jl$.

The results indicate that $(\sin 2\theta \cos \alpha)$ is a constant; $(j\cos 2\theta)$ is the segment related to the rotated vector (Y axis); $(-k\sin 2\theta \sin \alpha)$, $(i\sin 2\theta \sin \alpha)$ and $[(i\cos \beta + k\sin \beta)\sin 2\theta \sin \alpha]$ are the segments describing the perpendicular vector that complies with the left-hand coordinate system with the rotation axis and Y axis. Combined with equations (20), (21) and (22), the rotation process of a vector around an axis can be denoted by two rotation segments of an octonion, that is, the rotating component around a perpendicular axis and the spinor, the coefficients are invariant in different coordinate systems and are determined by the rotation quaternion.

**Arbitrary nonorthogonal rotation**

An arbitrary nonorthogonal rotation can be shown in Figure 5. $OA$ makes an inclined angle $\alpha$ $(0 < \alpha < \pi/2)$ with OP and rotates 20 $(0 \leq 20 < \pi)$ around OP to OB. Define the original vector OA as OY1 axis; OY0, OY1, OZ0 and OZ1 are in the same plane; OZ2 is perpendicular to OA and OB; and OX0 $\perp$ OY0 $\perp$ OZ0. OX0 $\perp$ OY1 $\perp$ OZ1, OX2 $\perp$ OY1 $\perp$ OZ2 and $\perp$ OZ0 $\perp$ OX0. $\perp$ OZ0, $\perp$ OZ0 could be used to denote the spinning angle of $O A$.

In the coordinate system OXY1Z1, the rotation is same as the rotation around $L_2$ in Figure 4. Take $O A$ and $O B$ as unit vectors, the result can be described as

\[ O_{AB} = Q_{OP}O_{OA}Q_{OP}^* = -l(\sin 2\theta \cos \alpha + j\cos 2\theta + il\sin 2\theta \sin \alpha) \]

where $Q_{OP} = \cos \theta + (j\cos \alpha + k\sin \alpha)\sin \theta$ and $Q_{OP}^* = \cos \theta - (j\cos \alpha + k\sin \alpha)\sin \theta$ and $O_{OA} = jl$.

Meanwhile, in the coordinate system OXY1Z2, the rotation can be divided into rotating around OZ2 which represents the position rotation process and spinning around itself. As $O A$ rotates 20 around $O P$, based on two rotation quaternions, $(\cos \gamma + k\sin \gamma)$ and $(\cos \delta + j\sin \delta)$, the combined quaternion vector is defined as
where $2\gamma$ is the rotating angle (around $OZ_2$) and $2\delta$ is the spinning angle (around $OY_1$) in the coordinate system $OX_2Y_1Z_2$. It can be shown $\sin^2\gamma + \sin^2\delta = \sin^2\theta$. Then, we obtain

$$OOB_2 = Q_0OOAQ_0 = l_2\sin\delta\cos\theta + jl\cos 2\theta + jl_2\sin\gamma\cos\theta$$

(32)

From $O_{OBI} = O_{OBB}$, we obtain

$$\begin{cases} 2\delta = 2\sin^{-1}(\sin \theta \cos \alpha) & (0 \leq \alpha \leq \frac{\pi}{2}, 0 \leq \theta, \gamma < \pi) \\ 2\gamma = 2\sin^{-1}(\sin \theta \sin \alpha) \end{cases}$$

(33)

According to Figure 5, it is facile to find that $2\gamma$ is $\angle AOB$, and $2\delta$ is the rotation angle of $AM$ around $OA$; since $OZ_2$ is not parallel to $AM$, in the rotating process of $O A$ around $OZ_2$, $AM$ would produce minor rotation around $OA$, which means $2\delta$ is not the real spinning angle of $O A$. However, as the inclined angle of $AM$ and $OZ_2$ is minute, this article chooses $2\delta$ as the spinning angle of $O A$, and the minor error is hence neglected.

### Calibration error analysis

In section “Rotations and octonions,” the rotation has been realized by means of decomposing the axes, compared to the decomposition of the rotated vector described in section “Quaternions and octonions.” In Figure 6, $M_N$ is the surface axis (surface normal) of the reflecting mirror, $M_H$ the horizontal axis and $M_V$ the vertical axis. $P_A$ and $P_E$ are the azimuth axis and the elevation axis of the platform, respectively. $T_A$ and $T_E$ are the medium axis and the outer axis of the turntable, respectively. At zero positions, $P_A$ and $T_A$ are vertical, and $P_E$ and $T_E$ are horizontal.

As described in section “Introduction,” the azimuth and elevation angles are defined for the platform ($\theta_{PA}$ and $\theta_{PE}$), the turntable ($\theta_{TA}$ and $\theta_{TE}$) and the autocollimator readouts of axes ($\delta_A$ and $\delta_E$). When $\theta_{TE} = -\theta_{PE} = \alpha \neq 0$, if the turntable rotates $\Omega$ around $T_A$, the mirror will rotate at the same angle around $M_V$, if the platform continues rotating $-\Omega$ around $P_A$, according to equation (33), the mirror will only rotate $-[2\sin^{-1}(\sin (\Omega/2) \cos \alpha)]$ around $M_V$ with a spinor of $-\sin^{-1}(\sin (\Omega/2) \sin \alpha)$ around $M_H$. The spinning of the mirror will result in the coupling error in the azimuth and elevation, but the angle is minute in this calibration process. The rotation angle will introduce an angular error $\eta$ in the azimuth

$$\eta = \Omega - \left[2\sin^{-1}\left(\frac{\Omega}{2}\cos \alpha\right)\right]$$

(34)

With $\Omega$ changing from $-40^\circ$ to $40^\circ$ and $\alpha$ from $-20^\circ$ to $20^\circ$, the calibration error in simulation is shown in Figure 7. Since the calibration error significantly increases with $\Omega$ and $\alpha$, it is necessary to separate the error.

### Experimental results

The pointing errors have been acquired with a high-precision turntable and an autocollimator at the
azimuth of $-20^\circ$, $-15^\circ$, $-10^\circ$, $-5^\circ$, $0^\circ$, $5^\circ$, $10^\circ$, $15^\circ$ and $20^\circ$ and at the elevation of $-20^\circ$, $-15^\circ$, $-10^\circ$, $-5^\circ$, $0^\circ$, $5^\circ$ and $10^\circ$. The compensation results calculated by equation (34) (the elevation angle is $\alpha$, and the azimuth angle is the rotated angle $\Omega$) are shown in Figure 8 and Table 2, which have shown that the calibration error has been compensated effectively. However, in order to improve its pointing accuracy further, it is necessary to analyze errors that result from the platform’s misalignment error, nonperpendicularity, initialization error and so on, and then relevant compensation models can be adopted. The detailed information has been presented and reported in Hong et al.\textsuperscript{9} and Tang et al.\textsuperscript{17}.

**Table 2.** Compensation result comparisons.

<table>
<thead>
<tr>
<th></th>
<th>Azimuth (original error)</th>
<th>Azimuth (compensated)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean value</td>
<td>$-0.1703^\circ$</td>
<td>$-0.1674^\circ$</td>
</tr>
<tr>
<td>Variance</td>
<td>$0.1379(^\circ)^2$</td>
<td>$0.0492(^\circ)^2$</td>
</tr>
<tr>
<td>Maximum value</td>
<td>$0.6935^\circ$</td>
<td>$0.2496^\circ$</td>
</tr>
<tr>
<td>Minimum value</td>
<td>$-1.5610^\circ$</td>
<td>$-0.3880^\circ$</td>
</tr>
</tbody>
</table>

Figure 8. Azimuth error before and after compensation.

**Declaration of conflicting interests**

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**Conclusion**

With respect to the ISPs, calibration process with a high-precision turntable and an autocollimator, this article has developed a novel approach to the rotation decomposition to determine the calibration errors. It has shown that octonions can represent the rotation process completely, including the rotation around a space vector and the spinning around itself. It has realized a new representation of the rotation by means of decomposing the rotation axis into two perpendicular axes. The error in the calibration process is then derived. According to the test results, the accuracy after error correction is significantly improved with the variance in azimuth decreased from $0.1379(^\circ)^2$ to $0.0492(^\circ)^2$, which offers a more accurate set of data for further compensation based on the error model of the platform itself.

**References**