FRACTIONAL SOBOLEV METRICS ON SPACES OF IMMERSED CURVES

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Abstract. Motivated by applications in the field of shape analysis, we study reparametrization invariant, fractional order Sobolev-type metrics on the space of smooth regular curves $\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d)$ and on its Sobolev completions $\mathcal{I}^q(\mathrm{S}^1, \mathbb{R}^d)$. We prove local well-posedness of the geodesic equations both on the Banach manifold $\mathcal{I}^q(\mathrm{S}^1, \mathbb{R}^d)$ and on the Fréchet-manifold $\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d)$ provided the order of the metric is greater or equal to one. In addition we show that the H^s-metric induces a strong Riemannian metric on the Banach manifold $\mathcal{I}^s(\mathrm{S}^1, \mathbb{R}^d)$ of the same order s, provided $s > \frac{3}{2}$. These investigations can be also interpreted as a generalization of the analysis for right invariant metrics on the diffeomorphism group.

Contents

1. Introduction

The interest in Riemannian metrics on infinite-dimensional manifolds is fueled by their connections to mathematical physics and in particular fluid dynamics. It was Arnold who discovered in 1966 that the incompressible Euler equation, which describes the motion of an ideal fluid, has an interpretation as the geodesic equation on an infinite-dimensional manifold; the manifold in question is the group of volume-preserving diffeomorphisms equipped with the L^2-metric. Since then many other PDEs in mathematical physics have been reinterpreted as geodesic equations. Examples include

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Burgers' equation, which is the geodesic equation of the L^2-metric on the group of all diffeomorphisms of the circle, $\operatorname{Diff}(\mathrm{S}^1)$, and the Camassa–Holm equation [11], the geodesic equation of the H^1-metric [23] on the same group. Interestingly, geodesic equations corresponding to fractional orders in the Sobolev scale have also found applications in physics: Wunsch showed that the geodesic equation of the homogenous $\dot{H}^{1/2}$-metric on $\operatorname{Diff}(\mathrm{S}^1)$ is connected to the Constantin–Lax–Majda equation [14, 40, 18], which itself is a simplified model of the vorticity equation.

The geometric interpretation of a PDE as the geodesic equation enables one to show local well-posedness of the PDE. This was done first by Ebin and Marsden [16] for the Euler equation. Using a similar method Constantin and Kolev showed in [13] that the geodesic equation of Sobolev H^n-metrics on $\operatorname{Diff}(\mathrm{S}^1)$ with integer $n \geq 1$ is locally well-posed. In [17] this was extend by Escher and Kolev to the Sobolev H^r-metrics of fractional order $r \geq \frac{1}{2}$. Similar results were shown for the diffeomorphism group of compact manifolds by Shkoller in [33, 34] and by Preston and Misiolek in [31]. Fractional metrics on $\operatorname{Diff}(\mathbb{R}^d)$ have been studied in [5] by Bauer, Escher and Kolev. The local well-posedness of the geodesic equation for fractional order metrics on the diffeomorphism group of a general manifold M remains an open problem.

In this paper we study the local well-posedness of a family of PDEs that arise as geodesic equations on the space $\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d)$ of immersed curves. To be precise $\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d)$ consists of smooth, closed curves with nowhere vanishing derivatives. We can regard the diffeomorphism group

$$\operatorname{Diff}(\mathrm{S}^1) \subseteq \operatorname{Imm}(\mathrm{S}^1, \mathrm{S}^1)$$

as an open subset of the space of immersions. If we replace S^1 on the right hand side by $\mathbb{R}^d$ we obtain the space of curves. The PDE

$$c_{tt} = -\langle D_s c_t, D_s c\rangle c_t - \langle c_t, D_s c_t\rangle D_s c - \frac{1}{2}\left|c_t\right|^2 D_s^2 c\,,$$

where $D_s = \frac{1}{|c'|}\partial_\theta$ and $c = c(t,\theta)$ is a time-dependent curve, is the geodesic equation for the L^2-metric

$$G_c(h,k) = \int_{\mathrm{S}^1} \langle h, k\rangle \left|c'\right| \,\mathrm{d}\theta\,.$$

This is a (weak) Riemannian metric on $\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d)$. The weight $|c'|$ in the integral makes the metric invariant under the natural $\operatorname{Diff}(\mathrm{S}^1)$-action and leads to the appearance of arc length derivatives D_s in the geodesic equation. This PDE can be seen as a generalization of the geodesic equation on $\operatorname{Diff}(\mathrm{S}^1)$,

$$\varphi_{tt} = -2\frac{\varphi_t \varphi_{tx}}{\varphi_x}\,,$$

which becomes, when written in terms of the Eulerian velocity $u = \varphi_t \circ \varphi^{-1}$, Burgers' equation,

$$u_t = -2u_x u\,.$$

While the behaviour of Burgers' equation is well-known, to our knowledge, nothing is known about the local well-posedness of the L^2-geodesic equation on the space of curves.

The situation improves when we add higher derivatives to the Riemannian metric. By doing so, we get the H^n-metric

$$G_c(h,k) = \int_{\mathrm{S}^1} \langle A_c h, k\rangle \left|c'\right| \,\mathrm{d}\theta\,, \qquad A_c = \sum_{j=0}^{n} (-1)^j \alpha_j D_s^{2j}\,,$$

where $\alpha_j \geq 0$ are constants. The corresponding geodesic equation is

$$(A_c c_t)_t = -\langle D_s c_t, D_s c\rangle A_c c_t - \langle A_c c_t, D_s c_t\rangle D_s c - W(c,c_t) D_s^2 c\,,$$

with

$$W(c,c_t) = \frac{1}{2}\left|c_t\right|^2 + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{2j-1} (-1)^{k+1} \langle D_s^{2j-k} c_t, D_s^k c_t\rangle\,.$$

Interestingly, the behaviour of the geodesic equation is better understood in the seemingly more complicated case where $n \geq 1$. For $n = 1$ the PDE is locally but generally not globally well-posed [30], while for $n \geq 2$ the PDE has solutions that are global in time [9].

Inspired by related work [5, 17] on geodesic equations on the diffeomorphism group we consider H^r-metrics of non-integer order r, i.e.

$$G_c(h,k) = \int_{\mathrm{S}^1} \langle A_c h, k\rangle \left|c'\right| \,\mathrm{d}\theta\,,$$

with A_c being, for each fixed curve c, a Fourier multiplier of order $2r$; the precise assumptions made on A_c are described in Section 3. The geodesic equation takes the form

$$(A_c c_t)_t = -\langle D_s c_t, D_s c\rangle A_c c_t - \langle A_c c_t, D_s c_t\rangle D_s c - (w(c,c_t) + w_0(c,c_t) D_s^2 c\,,$$

where $w(c,c_t)$ and $w_0(c,c_t)$ are expressions that are defined in Theorem 4.1. It is a nonlinear evolution equation of second order in t and of order $2s$ in θ; the right hand side is quadratic in c_t and highly nonlinear in c. For non-integer s, both $A_c c_t$ and $w(c,c_t)$ are nonlocal functions of both c and c_t. While not immediately obvious, we show in Example 4.3 that it contains the geodesic equations for the L^2-metric and the H^n-metrics as special cases.

Our contribution is to show in Corollary 6.5 that the geodesic equation for the H^r-metric is locally well-posed in the Sobolev space H^q for $q \geq 2r$ and $r \geq 1$.

Connections to shape analysis. Throughout the article we will assume that the operator A_c defining the metric is equivariant with respect to the action of the diffeomorphism group $\operatorname{Diff}(\mathrm{S}^1)$, i.e.,

$$A_{c\circ\varphi}(h\circ\varphi) = (A_c h)\circ\varphi\,, \quad \forall \varphi \in \operatorname{Diff}(\mathrm{S}^1)\,.$$

This is equivalent to requiring that the Riemannian metric G is invariant under the action of $\operatorname{Diff}(\mathrm{S}^1)$, meaning

$$G_{c\circ\varphi}(h\circ\varphi, k\circ\varphi) = G_c(h,k)\,, \quad \forall \varphi \in \operatorname{Diff}(\mathrm{S}^1)\,.$$

This assumption is necessary in order to apply the class of Sobolev metrics in shape analysis.

The mathematical analysis of shapes has been the focus of intense research in recent years [21, 26, 15, 42, 35] and has found applications in fields such as image analysis, computer vision, biomedical imaging and functional

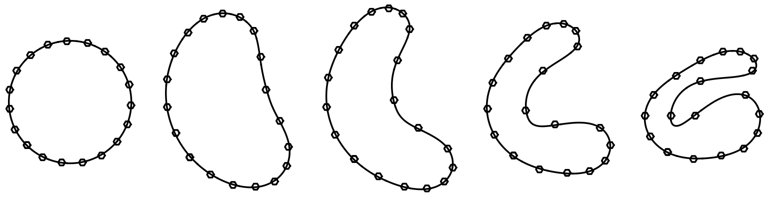

FIGURE 1. Example of a geodesic between two shapes with respect to an H^2-metric [2].

data analysis. An important class of shapes are outlines of planar objects. Mathematically, these shapes can be represented by equivalence classes of parametrized curves modulo the reparametrization group $\mathrm{Diff}(\mathrm{S}^1)$. This yields the following geometric picture,

$$\pi : \mathrm{Imm}(\mathrm{S}^1, \mathbb{R}^d) \to \mathrm{Imm}(\mathrm{S}^1, \mathbb{R}^d)/\mathrm{Diff}(\mathrm{S}^1)\,.$$

A key step in shape analysis is to define an efficiently computable distance function between shapes in order to measure similarity between shapes. This distance function can be the geodesic distance induced by a Riemannian metric. Since it is difficult to work with the quotient $\mathrm{Imm}(\mathrm{S}^1, \mathbb{R}^d)/\mathrm{Diff}(\mathrm{S}^1)$ directly, the standard approach is instead to define a Riemannian metric on $\mathrm{Imm}(\mathrm{S}^1, \mathbb{R}^d)$, that is $\mathrm{Diff}(\mathrm{S}^1)$-invariant.

Such a metric can then induce a Riemannian metric on the quotient space, making π a Riemannian submersion; this metric would be given by the formula

$$G_{\pi(c)}(u, u) = \inf_{T_c\pi.h=u} G_c(h, h)\,.$$

Hence we will only look at $\mathrm{Diff}(\mathrm{S}^1)$-invariant metrics on $\mathrm{Imm}(\mathrm{S}^1, \mathbb{R}^d)$ in this paper.

The arguably simplest invariant metric is the L^2-metric, corresponding to the operator $A_c = \mathrm{Id}$. However, its induced geodesic distance function is identically zero, both, on the space of parametrized curves as well as the quotient space of unparametrized curves; this was shown by Michor and Mumford [28, 29], see also [1]. This means that between any two curves there exists a path of arbitrary short length. This property makes the metric ill-suited for most applications in shape analysis, because a notion of distance between shapes is one of the basic tools there. As a consequence higher order metrics, mostly of integer order, were studied and used successfully in applications; see [30, 22, 36, 41, 39, 37]. For an overview on various metrics on the shape space of parametrized and unparametrized curves see [30, 3, 4]. As an example we have included an optimal deformation between two unparametrized curves with respect to a second order Sobolev metric in Figure 1. Fractional order metrics have been briefly mentioned [27, Section 3]. However, so far, an analysis of the well-posedness of the corresponding geodesic equation was missing. This is the question we will focus on in this paper.

2. Parametrized and unparametrized curves

In this article we consider the space of smooth regular curves with values in $\mathbb{R}^d$

$$\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d) := \left\{ c \in C^\infty(\mathrm{S}^1, \mathbb{R}^d) : \left| c' \right| \neq 0 \right\} .$$

As an open subset of the Fréchet space $C^\infty(\mathrm{S}^1, \mathbb{R}^d)$, it is a Fréchet manifold. Its tangent space at any curve c is the vector space of smooth functions:

$$T_c \operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d) = C^\infty(\mathrm{S}^1, \mathbb{R}^d) .$$

The group of smooth diffeomorphisms of the circle

$$\operatorname{Diff}(\mathrm{S}^1) := \left\{ \varphi \in C^\infty(\mathrm{S}^1, \mathrm{S}^1) : \left| \varphi' \right| > 0 \right\}$$

acts on the space of regular curves via composition from the right:

$$\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d) \times \operatorname{Diff}(\mathrm{S}^1) \to \operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d), \qquad (c, \varphi) \mapsto c \circ \varphi .$$

Taking the quotient with respect to this group action, we obtain the *shape space* of un-parameterized curves

$$B_i(\mathrm{S}^1, \mathbb{R}^d) := \operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d) / \operatorname{Diff}(\mathrm{S}^1) .$$

Note that the action of $\operatorname{Diff}(\mathrm{S}^1)$ on $\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d)$ is *not free*, and thus that the quotient space $B_i(\mathrm{S}^1, \mathbb{R}^d)$ is not a manifold, but only an orbifold with finite isotropy groups. A way to overcome this difficulty is to consider the slightly smaller space of *free immersions* $\operatorname{Imm}_{\mathrm{f}}(\mathrm{S}^1, \mathbb{R}^d)$, i.e., those immersions upon which $\operatorname{Diff}(\mathrm{S}^1)$ acts freely. This space is an open and dense subset of $\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d)$ and the corresponding quotient space

$$B_{i,\mathrm{f}}(\mathrm{S}^1, \mathbb{R}^d) := \operatorname{Imm}_{\mathrm{f}}(\mathrm{S}^1, \mathbb{R}^d) / \operatorname{Diff}(\mathrm{S}^1)$$

is again a Fréchet manifold, see [12].

3. Riemannian metrics on immersions

Let G be a Riemannian metric on $\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d)$. Motivated by applications in the field of shape analysis we require G to be invariant with respect to the diffeomorphism group $\operatorname{Diff}(\mathrm{S}^1)$, i.e.,

$$G_c(h, k) = G_{c \circ \varphi}(h \circ \varphi, k \circ \varphi), \qquad \forall \varphi \in \operatorname{Diff}(\mathrm{S}^1) . \tag{3.1}$$

This invariance is a necessary assumption for G to induce a Riemannian metric on the shape space $B_{i,f}(\mathrm{S}^1, \mathbb{R}^d)$, such that the projection map is a Riemannian submersion.

We assume that the metric is given in the form

$$G_c(h, k) = \int_{\mathrm{S}^1} \langle A_c h, k \rangle \, \mathrm{d}s, \tag{3.2}$$

where $A_c : C^\infty(\mathrm{S}^1, \mathbb{R}^d) \to C^\infty(\mathrm{S}^1, \mathbb{R}^d)$ is a continuous linear operator that depends on the foot point c. Associated to the metric is a map

$$\check{G} : T\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d) \to T^*\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d).$$

In terms of A_c we have $\check{G}_c = A_c \otimes \mathrm{d}s$. If G is $\operatorname{Diff}(\mathrm{S}^1)$-invariant, then the family of operators A_c has to be $\operatorname{Diff}(\mathrm{S}^1)$-equivariant,

$$A_{c \circ \varphi}(h \circ \varphi) = A_c(h) \circ \varphi, \qquad \forall \varphi \in \operatorname{Diff}(\mathrm{S}^1) .$$

Example 3.1 (Integer order Sobolev metrics)**.** Motivated by their applicability in shape analysis [43, 38, 27, 30] important examples are Sobolev metrics with *constant coefficients*,

$$G_c^n(h,k) = \sum_{j=0}^{n} \alpha_j \int_{S^1} \left\langle D_s^j h, D_s^j k \right\rangle \mathrm{d}s\,, \tag{3.3}$$

and *scale-invariant* Sobolev metrics,

$$\widetilde{G}_c^n(h,k) = \sum_{j=0}^{n} \alpha_j \int_{S^1} \ell_c^{2j-3} \left\langle D_s^j h, D_s^j k \right\rangle \mathrm{d}s\,, \tag{3.4}$$

with constants $\alpha_j \geq 0$, for $j = 0, \dots, n$; one requires $\alpha_0, \alpha_n > 0$ and calls n the *order* of the metric. Here, $D_s = \frac{\partial_\theta}{|c'|}$ denotes *differentiation with respect to arc length*, $\mathrm{d}s = |c'|\,\mathrm{d}\theta$ *integration with respect to arc length* and $\ell_c = \int_{S^1} \mathrm{d}s$ the corresponding *curve length*.

Using integration by parts we obtain a formula for the operator A_c:

$$A_c^n = \sum_{j=0}^{n} (-1)^j \alpha_j\, D_s^{2j} \tag{3.5}$$

for metrics with constant coefficients and

$$\widetilde{A}_c^n = \sum_{j=0}^{n} (-1)^j \alpha_j\, \ell_c^{2j-3}\, D_s^{2j}\,. \tag{3.6}$$

for scale-invariant metrics.

Because it will be important later, we note that if c is a constant speed curve, then the operator for a metric with constant coefficients is

$$A_c^n = \sum_{j=0}^{n} (-1)^j \alpha_j \left(\frac{2\pi}{\ell_c}\right)^{2j} \partial_\theta^{2j}\,;$$

it is a differential operator with constant coefficients and the coefficients depend only on the length of the curve.

Until now local and global well-posedness results have been established only for the Sobolev metrics of integer order [30, 7, 9, 6]. The main goal of this article is to extend these results to *metrics of fractional order*; in particular to metrics, for which the operator A_c is defined using Fourier multipliers of a certain class.

Given a curve $c \in \mathrm{Imm}(S^1, \mathbb{R}^d)$, let $\psi_c \in \mathrm{Diff}(S^1)$ be a diffeomorphism such that $c \circ \psi_c^{-1}$ has constant speed. Reparametrization invariance of the metric implies that

$$G_c(h,k) = G_{c\circ\psi_c^{-1}}(h \circ \psi_c^{-1}, k \circ \psi_c^{-1})\,; \tag{3.7}$$

in other words, G is determined by its behaviour on constant speed curves.

Remark 3.2. The situation is similar to that of right-invariant metrics on diffeomorphism groups, which are determined by their behaviour at the identity diffeomorphism. For curves, the invariance property is weaker and the space of constant speed curves is still quite large.

Let us write the invariance property (3.7) in terms of A_c: by a straightforward calculation we obtain

$$\begin{aligned} G_c(h,k) &= G_{c\circ\psi_c^{-1}}(h\circ\psi_c^{-1}, k\circ\psi_c^{-1}) \\ &= \int_{S^1} \left\langle A_{c\circ\psi_c^{-1}}(h\circ\psi_c^{-1}), k\circ\psi_c^{-1}\right\rangle \frac{\ell_c}{2\pi}\,\mathrm{d}\theta \\ &= \int_{S^1} \left\langle R_{\psi_c}\circ A_{c\circ\psi_c^{-1}}\circ R_{\psi_c^{-1}}(h), k\right\rangle \mathrm{d}s\,, \end{aligned}$$

which implies the identity

$$A_c = R_{\psi_c}\circ A_{c\circ\psi_c^{-1}}\circ R_{\psi_c^{-1}}\,.$$

The class of metrics on constant speed curves is large. To make it more manageable we will restrict the possible dependance of A_c on the curve c.

Assumption. We assume from this point onwards, that the operator $A_{c\circ\psi_c^{-1}}$ depends on c only through its length ℓ_c; in other words

$$A_{c\circ\psi_c^{-1}} = A(\ell_c),$$

where $\lambda\mapsto A(\lambda)$ is a smooth curve with values in the space of linear maps, $L(C^\infty(S^1,\mathbb{R}^d), C^\infty(S^1,\mathbb{R}^d))$[1].

Then, with this assumption,

$$A_c = R_{\psi_c}\circ A(\ell_c)\circ R_{\psi_c^{-1}}\,. \tag{3.8}$$

Requiring this form for the operator A_c imposes a restriction on $A(\ell_c)$. For $\alpha\in S^1$ consider $\varphi_\alpha\in\mathrm{Diff}(S^1)$, defined as $\varphi_\alpha(\theta) = \theta+\alpha \mod 2\pi$. If a curve c has constant speed, then so does $c\circ\varphi_\alpha$ and thus $\ell_c = \ell_{c\circ\varphi_\alpha}$. Therefore (3.8) implies

$$A(\ell_c)\circ R_{\varphi_\alpha} = R_{\varphi_\alpha}\circ A(\ell_c)\,,$$

or equivalently after differentiating with respect to α,

$$A(\ell_c)\circ\partial_\theta = \partial_\theta\circ A(\ell_c)\,.$$

This means that $A(\ell_c)$ has to be a *Fourier multiplier*, i.e.,

$$A(\ell_c).u(\theta) = \sum_{m\in\mathbb{Z}} \mathbf{a}(\ell_c,m).\hat{u}(m)\exp(im\theta)\,,$$

where $\mathbf{a}(\ell_c,\cdot):\mathbb{Z}\to\mathcal{L}(\mathbb{C}^d)$ is called the *symbol* of $A(\ell_c)$. We will write $A(\ell_c) = \mathbf{a}(\ell_c, D)$ or $A(\ell_c) = \mathbf{op}\,(\mathbf{a}(\ell_c,\cdot))$. For now we make no assumptions on the symbol apart from requiring that the map $\ell_c\mapsto A(\ell_c)$ is smooth. More control on the symbol will be necessary in order to prove well-posedness of the geodesic equation.

Example 3.3 (Integer order Sobolev metrics)**.** For Sobolev metrics with constant coefficients the symbol of the operator $A^n(\ell_c)$ is

$$\mathbf{a}^n(\ell_c,m) = \left(\sum_{j=0}^n (-1)^j\alpha_j\left(\frac{2\pi m}{\ell_c}\right)^{2j}\right)\mathbf{I}_d\,,$$

[1] This is equivalent to requiring that the map $\mathbb{R}_+\times C^\infty(S^1,\mathbb{R}^d)\to C^\infty(S^1,\mathbb{R}^d)$ given by $(\lambda,h)\mapsto A(\lambda)h$ is smooth.

with $\mathbf{I}_d \in \mathbb{C}^{d\times d}$ the identity matrix, and for scale-invariant metrics it is

$$\widetilde{\mathbf{a}}^n(\ell_c, m) = \left(\ell_c^{-3} \sum_{j=0}^{n} (-1)^j \alpha_j \, (2\pi m)^{2j} \right) \mathbf{I}_d \,,$$

Remark 3.4. Even though all known situations correspond to Fourier multipliers of the form $A(\ell_c) = \mathbf{a}(\ell_c, D)$ with $\mathbf{a}(\ell_c, m) = a(\ell_c, m)\mathbf{I}_d$, a multiple of the identity matrix, we will treat in this article general matrix-valued symbols, because doing so introduces no additional difficulties.

4. The geodesic equation

Geodesics between two curves $c_0, c_1 \in \operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d)$ are critical points of the *energy functional*

$$E(c) = \frac{1}{2} \int_0^1 \int_{\mathrm{S}^1} \langle A_c c_t, c_t \rangle \, \mathrm{d}s \, \mathrm{d}t \,,$$

where $c = c(t, \theta)$ is a path in $\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d)$ joining c_0 and c_1. The geodesic equation is obtained by computing the derivative of E on paths with fixed endpoints. We will use the following notations

$$c_t := \partial_t c \qquad c' := \partial_\theta c \qquad v := \frac{c'}{|c'|} = D_s c \,,$$

and

$$A_c = R_{\psi_c} \circ A(\ell_c) \circ R_{\psi_c}^{-1} \qquad A'_c := R_{\psi_c} \circ A'(\ell_c) \circ R_{\psi_c}^{-1} \,,$$

where $A'(\ell_c)$ is the derivative of $A(\ell_c)$ with respect to the parameter ℓ_c and

$$\psi_c(\theta) = \frac{2\pi}{\ell_c} \int_0^\theta \left| c' \right| \, \mathrm{d}\sigma \,.$$

We have the following commutation rules for D_s and R_{ψ_c},

$$D_s \circ R_{\psi_c} = \frac{2\pi}{\ell_c} R_{\psi_c} \circ \partial_\theta \qquad \partial_\theta \circ R_{\psi_c^{-1}} = \frac{\ell_c}{2\pi} R_{\psi_c^{-1}} \circ D_s \,,$$

which imply that because $A(\ell_c)$ commutes with ∂_θ, that the operator A_c commutes with D_s,

$$D_s \circ A_c = A_c \circ D_s \,.$$

This will be used when computing the geodesic equation.

Theorem 4.1. *Assume that, for each $\lambda \in \mathbb{R}^+$, the operator*

$$A(\lambda) : C^\infty(\mathrm{S}^1, \mathbb{R}^d) \to C^\infty(\mathrm{S}^1, \mathbb{R}^d)$$

is invertible with a continuous inverse. Then the weak Riemannian metric (3.2) on $\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d)$ has a geodesic spray which is given by

$$F : (c, h) \mapsto (h, S_c(h)) \,, \qquad T\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d) \to TT\operatorname{Imm}(\mathrm{S}^1, \mathbb{R}^d) \tag{4.1}$$

where

$$S_c(h) = -A_c^{-1} \left\{ (D_{c,h} A_c)h + \langle D_s h, v \rangle A_c h + \langle A_c h, D_s h \rangle v + (w + w_0) D_s v \right\} \,. \tag{4.2}$$

with

$$w(c,h) = \int_{S^1} \langle A_c h, D_s h\rangle \,\mathrm{d}s$$

and

$$w_0(c,h) = \int_{S^1} \frac{1}{2\pi}\langle A_c h, \psi_c D_s h\rangle + \frac{1}{2}\langle (\ell_c^{-1} A_c + A_c')h, h\rangle \,\mathrm{d}s\,.$$

The geodesic equation is

$$(A_c c_t)_t = -\langle D_s c_t, v\rangle A_c c_t - \langle A_c c_t, D_s c_t\rangle v - (w(c,c_t) + w_0(c,c_t)) D_s v\,. \tag{4.3}$$

To compute these equations, we will first derive a variational formula for the reparametrization function ψ_c.

Lemma 4.2. *The derivative of the map $c \mapsto \psi_c$ is given by*

$$D_{c,h}\psi_c(\theta) = \frac{2\pi}{\ell_c}\int_0^\theta \langle D_s h, v\rangle \,\mathrm{d}\tilde{s} - \frac{1}{\ell_c}\left(\int_{S^1} \langle D_s h, v\rangle \,\mathrm{d}s\right)\psi_c(\theta)\,.$$

Proof. Using the variational formulas for $|c'|$ and ℓ_c,

$$D_{c,h}\left|c'\right| = \langle D_s h, v\rangle \left|c'\right| \qquad\qquad D_{c,h}\ell_c = \int_{S^1} \langle D_s h, v\rangle \,\mathrm{d}s\,,$$

the lemma follows directly from

$$\psi_c(\theta) = \frac{2\pi}{\ell_c}\int_0^\theta \left|c'\right| \,\mathrm{d}\sigma\,. \qquad \square$$

Proof of Theorem 4.1. We can write the energy of a path $c(t,\theta)$ as

$$E(c) = \frac{1}{2}\int_0^1 \frac{\ell_c}{2\pi}\int_{S^1} \left\langle A(\ell_c)(c_t\circ\psi_c^{-1}), c_t\circ\psi_c^{-1}\right\rangle \,\mathrm{d}\theta\,\mathrm{d}t\,.$$

The variation of the energy in direction h is then given by

$$\begin{aligned}
D_{c,h}E = &\int_0^1 \frac{\ell_c}{2\pi}\int_{S^1} \left\langle A(\ell_c)(c_t\circ\psi_c^{-1}), h_t\circ\psi_c^{-1} + \left(c_t'\circ\psi_c^{-1}\right) D_{c,h}\,\psi_c^{-1}\right\rangle \,\mathrm{d}\theta \\
&\qquad + \frac{1}{2}\frac{\ell_c}{2\pi} D_{c,h}\ell_c \int_{S^1} \left\langle \left(\ell_c^{-1}A(\ell_c) + A'(\ell_c)\right)(c_t\circ\psi_c^{-1}), c_t\circ\psi_c^{-1}\right\rangle \,\mathrm{d}\theta\,\mathrm{d}t \\
= &\int_0^1 \int_{S^1} \left\langle A_c c_t, h_t + c_t'\left(D_{c,h}\psi_c^{-1}\right)\circ\psi_c\right\rangle \,\mathrm{d}s \\
&\qquad + \frac{1}{2} D_{c,h}\ell_c \int_{S^1} \left\langle (\ell_c^{-1}A_c + A_c')c_t, c_t\right\rangle \,\mathrm{d}s\,\mathrm{d}t\,.
\end{aligned}$$

Let us start with the term involving h_t. After integrating by parts we get

$$\begin{aligned}
\int_0^1\int_{S^1} \langle A_c c_t, h_t\rangle \,\mathrm{d}s\,\mathrm{d}t &= -\int_0^1\int_{S^1} \left\langle (A_c c_t)_t + (A_c c_t)\frac{\partial_t(|c'|)}{|c'|}, h\right\rangle \mathrm{d}s\,\mathrm{d}t \\
&= -\int_0^1\int_{S^1} \langle (A_c c_t)_t + \langle D_s c_t, v\rangle A_c c_t, h\rangle \,\mathrm{d}s\,\mathrm{d}t\,.
\end{aligned}$$

Next we deal with ψ_c^{-1}. Using the formula

$$D_{c,h}\,\psi_c^{-1} = -\left(\frac{1}{\psi_c'} D_{c,h}\,\psi_c\right)\circ\psi_c^{-1}\,,$$

and Lemma 4.2 we obtain a formula for the variation of ψ_c^{-1},

$$\left(D_{c,h}\,\psi_c^{-1}\right)\circ\psi_c(\theta) = -\frac{1}{|c'|}\left(\int_0^\theta \langle D_s h, v\rangle\,\mathrm{d}\tilde{s} - \frac{1}{2\pi}\left(\int_{S^1}\langle D_s h, v\rangle\,\mathrm{d}s\right)\psi_c(\theta)\right).$$

Therefore we have

$$\begin{aligned}
&\int_{S^1}\left\langle A_c c_t, c_t'\left(D_{c,h}\psi_c^{-1}\right)\circ\psi_c\right\rangle\,\mathrm{d}s\\
&\quad= -\int_{S^1}\langle A_c c_t, D_s c_t\rangle\left(\int_0^\theta\langle D_s h, v\rangle\,\mathrm{d}\tilde{s} - \frac{1}{2\pi}\left(\int_{S^1}\langle D_s h, v\rangle\,\mathrm{d}\tilde{s}\right)\psi_c\right)\mathrm{d}s\\
&\quad= -\int_{S^1}\langle A_c c_t, D_s c_t\rangle\int_0^\theta\langle D_s h, v\rangle\,\mathrm{d}\tilde{s}\,\mathrm{d}s\\
&\qquad\quad+\frac{1}{2\pi}\int_{S^1}\langle D_s h, v\rangle\,\mathrm{d}s\int_{S^1}\langle A_c c_t, \psi_c D_s c_t\rangle\,\mathrm{d}s
\end{aligned}$$

Because A_c is symmetric and commutes with D_s,

$$\int_{S^1}\langle A_c c_t, D_s c_t\rangle\,\mathrm{d}s = 0\,,$$

and thus the function

$$w(\theta) = \int_0^\theta\langle A_c c_t, D_s c_t\rangle\,\mathrm{d}s\,,$$

is periodic with $D_s w = \langle A_c c_t, D_s c_t\rangle$. Next we integrate by parts,

$$\begin{aligned}
&\int_{S^1}\left\langle A_c c_t, c_t'\left(D_{c,h}\psi_c^{-1}\right)\circ\psi_c\right\rangle\,\mathrm{d}s\\
&\quad= \int_{S^1} w\langle D_s h, v\rangle\,\mathrm{d}s + \frac{1}{2\pi}\int_{S^1}\langle D_s h, v\rangle\,\mathrm{d}s\int_{S^1}\langle A_c c_t, \psi_c D_s c_t\rangle\,\mathrm{d}s\\
&\quad= -\int_{S^1}\langle D_s(wv), h\rangle\,\mathrm{d}s - \frac{1}{2\pi}\int_{S^1}\langle D_s v, h\rangle\,\mathrm{d}s\int_{S^1}\langle A_c c_t, \psi_c D_s c_t\rangle\,\mathrm{d}s\,.
\end{aligned}$$

Finally we consider the term involving $D_{c,h}\ell_c$. Since

$$D_{c,h}\ell_c = \int_{S^1}\langle D_s h, v\rangle\,\mathrm{d}s = -\int_{S^1}\langle D_s v, h\rangle\,\mathrm{d}s\,,$$

we have

$$\begin{aligned}
&\frac{1}{2}D_{c,h}\ell_c\int_{S^1}\langle(\ell_c^{-1}A_c + A_c')c_t, c_t\rangle\,\mathrm{d}s =\\
&\qquad\qquad = -\frac{1}{2}\int_{S^1}\langle D_s v, h\rangle\,\mathrm{d}s\int_{S^1}\langle(\ell_c^{-1}A_c + A_c')c_t, c_t\rangle\,\mathrm{d}s\,.
\end{aligned}$$

Grouping all these expressions together, we can express $D_{c,h}E$ as

$$\begin{aligned}
&D_{c,h}E(c) =\\
&\quad\int_0^1\int_{S^1}\left\langle -(A_c c_t)_t - \langle D_s c_t, v\rangle A_c c_t - D_s(wv) - w_0 D_s v, h\right\rangle\,\mathrm{d}s\,\mathrm{d}t\,,
\end{aligned}\tag{4.4}$$

where

$$w_0 = \int_{S^1}\left(\frac{1}{2\pi}\langle A_c c_t, \psi_c D_s c_t\rangle + \frac{1}{2}\langle(\ell_c^{-1}A_c + A_c')c_t, c_t\rangle\right)\mathrm{d}s\,.$$

Thus we obtain the geodesic equation

$$(A_c c_t)_t = -\langle D_s c_t, v\rangle A_c c_t - D_s(wv) - w_0 D_s v\,.$$

The existence of the geodesic spray follows from (4.4), which can be rewritten as

$$D_{c,h}E(c) = \int_0^1 G_c(-c_{tt} + S_c(c_t), h)\,\mathrm{d}t\,.$$

Note that in this last step we used the invertibility of A_c on $C^\infty(\mathrm{S}^1,\mathbb{R}^d)$. □

Next we will see how the geodesic equation simplifies for Sobolev metrics with constant coefficients and scale-invariant metrics. First note, that using integration by parts we can write

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathrm{S}^1}\langle A_c c_t, \psi_c D_s c_t\rangle\,\mathrm{d}s &= \frac{1}{\ell_c}\int_{\mathrm{S}^1}\langle A_c c_t, D_s c_t\rangle \int_0^\theta \left|c'\right|\,\mathrm{d}\sigma\,\mathrm{d}s\\ &= -\frac{1}{\ell_c}\int_{\mathrm{S}^1}\int_0^\theta \langle A_c c_t, D_s c_t\rangle\,\mathrm{d}\tilde{s}\,\mathrm{d}s\,.\end{aligned}$$

We shall also make use of the following identity, valid for $j \geq 1$ and $h \in C^\infty(\mathrm{S}^1,\mathbb{R}^d)$,

$$\langle D_s^{2j}h, D_s h\rangle = D_s\left(\frac{1}{2}\sum_{k=1}^{2j-1}(-1)^{k+1}\langle D_s^{2j-k}h, D_s^k h\rangle\right)\,.$$

Setting

$$W_j = \frac{1}{2}\sum_{k=1}^{2j-1}(-1)^{k+1}\langle D_s^{2j-k}c_t, D_s^k c_t\rangle \;\text{ for } j\geq 1\,,\qquad W_0 = \frac{1}{2}\left|c_t\right|^2\,,$$

we obtain

$$\int_0^\theta \langle D_s^{2j}c_t, D_s c_t\rangle\,\mathrm{d}\tilde{s} = W_j(\theta) - W_j(0)\,,$$

as well as

$$\int_{\mathrm{S}^1} W_j(\theta)\,\mathrm{d}s = \frac{1}{2}(1-2j)\int_{\mathrm{S}^1}\langle D_s^{2j}c_t, c_t\rangle\,\mathrm{d}s\,.$$

Example 4.3 (Metrics with constant coefficients)**.** For metrics with constant coefficients we have

$$A_c^n = \sum_{j=0}^n (-1)^j \alpha_j D_s^{2j},\qquad A^n(\ell_c) = \sum_{j=0}^n (-1)^j\alpha_j\left(\frac{2\pi}{\ell_c}\right)^{2j}\partial_\theta^{2j}\,,$$

and thus their ℓ_c-derivatives are

$$(A_c^n)' = -2\ell_c^{-1}\sum_{j=0}^n(-1)^j j\alpha_j D_s^{2j},\quad A'(\ell_c) = -2\ell_c^{-1}\sum_{j=0}^n(-1)^j j\alpha_j\left(\frac{2\pi}{\ell_c}\right)^{2j}\partial_\theta^{2j}\,.$$

Therefore

$$\ell_c^{-1}A_c^n + (A_c^n)' = \ell_c^{-1}\sum_{j=0}^n(-1)^j(1-2j)\alpha_j D_s^{2j}\,,$$

and

$$\begin{aligned}w_0 &= \frac{1}{\ell_c}\sum_{j=0}^n (-1)^j \alpha_j \left[-\int_{S^1} W_j(\theta) - W_j(0)\,\mathrm{d}s + \frac{1}{2}(1-2j)\int_{S^1} \langle D_s^{2j} c_t, c_t\rangle\,\mathrm{d}s\right] \\ &= \sum_{j=0}^n (-1)^j \alpha_j W_j(0)\,.\end{aligned}$$

Since

$$w = \sum_{j=0}^n (-1)^j \alpha_j \int_0^\theta \langle D_s^{2j} c_t, D_s c_t\rangle\,\mathrm{d}\tilde{s} = \sum_{j=0}^n (-1)^j \alpha_j \left(W_j(\theta) - W_j(0)\right)\,,$$

it follows that

$$w(\theta) + w_0 = \sum_{j=0}^n (-1)^j \alpha_j W_j(\theta)\,.$$

Thus the geodesic equation has the form

$$(A_c c_t)_t = -\langle D_s c_t, v\rangle A_c c_t - \langle A_c c_t, D_s c_t\rangle v - \left(\sum_{j=0}^n (-1)^j \alpha_j W_j\right) D_s v\,.$$

Thus we have regained the formula from [30].

Example 4.4 (Scale-invariant metrics)**.** For scale-invariant metrics we have

$$\tilde{A}_c^n = \sum_{j=0}^n (-1)^j \alpha_j \ell_c^{2j-3} D_s^{2j}, \qquad \tilde{A}^n(\ell_c) = \ell_c^{-3}\sum_{j=0}^n (-1)^j \alpha_j (2\pi)^{2j} \partial_\theta^{2j}\,,$$

and thus their ℓ_c-derivatives are

$$(A_c^n)' = -3\ell_c^{-1}\sum_{j=0}^n (-1)^j \alpha_j \ell_c^{2j-3} D_s^{2j}, \quad \tilde{A}'(\ell_c) = -3\ell_c^{-4}\sum_{j=0}^n (-1)^j \alpha_j \left(2\pi\right)^{2j} \partial_\theta^{2j}\,.$$

Therefore

$$\ell_c^{-1}\tilde{A}_c^n + (\tilde{A}_c^n)' = -2\ell_c^{-1}\tilde{A}_c^n\,.$$

A similar calculation as before gives

$$\begin{aligned}w_0 &= \sum_{j=0}^n (-1)^j \alpha_j \ell_c^{2j-3}\left(W_j(0) + \frac{1}{\ell_c}\left(j+\frac{1}{2}\right)\int_{S^1}\langle D_s^{2j} c_t, c_t\rangle\,\mathrm{d}s\right) \\ w(\theta) &= \sum_{j=0}^n (-1)^j \alpha_j \ell_c^{2j-3}\left(W_j(\theta) - W_j(0)\right)\,,\end{aligned}$$

and therefore

$$w(\theta) + w_0 = \sum_{j=0}^n (-1)^j \alpha_j \ell_c^{2j-3}\left(W_j(\theta) + \frac{1}{\ell_c}\left(j+\frac{1}{2}\right)\int_{S^1}\langle D_s^{2j} c_t, c_t\rangle\,\mathrm{d}s\right)\,.$$

5. Smoothness of the metric

For any $q > \frac{3}{2}$ we can consider the Sobolev completion $\mathcal{I}^q = \mathcal{I}^q(\mathrm{S}^1, \mathbb{R}^d)$, which is an open set of the Hilbert vector space $H^q(\mathrm{S}^1, \mathbb{R}^d)$. It is the aim of this section to show that the metric G_c extends to a smooth weak metric on $\mathcal{I}^q$, for high enough q. It turns out that the smoothness of the metric reduces to the smoothness of the mapping

$$c \mapsto A_c = R_{\psi_c} \circ A(\ell_c) \circ R_{\psi_c^{-1}}.$$

We will begin our investigations with this question.

It is well-known (see for instance [16, Appendix A]) that, given a *differential operator* A of order r with *smooth coefficients* and a diffeomorphism ψ, the conjugate operator $A_\psi = R_\psi \circ A \circ R_{\psi^{-1}}$ is again a differential operator of order r, whose coefficients are polynomial expressions in ψ, its derivatives and $1/\psi'$; in particular, the mapping

$$\psi \mapsto A_\psi = R_\psi \circ A \circ R_{\psi^{-1}}, \quad \mathcal{D}^q(\mathrm{S}^1) \to \mathcal{L}(H^q(\mathrm{S}^1, \mathbb{R}^d), H^{q-r}(\mathrm{S}^1, \mathbb{R}^d))$$

is smooth for $q > 3/2$ and $q \geq r \geq 1$, where $\mathcal{D}^q(\mathrm{S}^1)$ is group of H^q-diffeomorphisms of S^1. In [17] Escher and Kolev extended this result to *Fourier multipliers* by showing that the map $\psi \mapsto A_\psi$ remains smooth when A is a Fourier multiplier of class $\mathcal{S}^r(\mathbb{Z})$ (to be defined below).

The proof that the mapping $c \mapsto A_c$ is smooth that we present here is inspired by [17, 5], but in the present case we have to deal with Fourier multipliers that depend (in a nice way) on a parameter λ.

First we introduce the classes $\mathcal{S}^r(\mathbb{Z})$ and $\mathcal{S}^r_\lambda(\mathbb{Z})$ of Fourier multipliers (for further details see [32] for instance). A Fourier multiplier $\mathbf{a}(D)$ acts on a function u via

$$\mathbf{a}(D).u(\theta) = \sum_{m \in \mathbb{Z}} \mathbf{a}(m).\hat{u}(m) \exp(im\theta)\,, \tag{5.1}$$

and its symbol is a function $\mathbf{a} : \mathbb{Z} \to \mathcal{L}(\mathbb{C}^d)$.

Definition 5.1. Given $r \in \mathbb{R}$, the Fourier multiplier $\mathbf{a}(D)$ belongs to the class $\mathcal{S}^r(\mathbb{Z})$, if $\mathbf{a}(m)$ satisfies

$$\|\Delta^\alpha \mathbf{a}(m)\| \leq C_\alpha \langle m \rangle^{r-\alpha},$$

for each $\alpha \in \mathbb{N}$, where $\langle m \rangle := (1 + |m|^2)^{1/2}$.

Here we define the difference operator Δ^α via

$$\Delta \mathbf{a}(m) = \mathbf{a}(m+1) - \mathbf{a}(m), \qquad \Delta^\alpha = \Delta \circ \Delta^{\alpha-1}\,.$$

We equip the space $\mathcal{S}^r(\mathbb{Z})$ with the topology induced by the seminorms

$$p_\alpha(\mathbf{a}(D)) = \sup_{m \in \mathbb{Z}} \|\Delta^\alpha \mathbf{a}(m)\| \, \langle m \rangle^{-(r-\alpha)}\,,$$

where $\|\cdot\|$ is some norm on $\mathcal{L}(\mathbb{C}^d)$. With this topology $\mathcal{S}^r(\mathbb{Z})$ is a Fréchet space. The class $\mathcal{S}^r(\mathbb{Z})$ coincides with the one defined in [17]; the equivalence is shown in [32].

Example 5.2. Any linear differential operator of order r with constant coefficients belongs to $\mathcal{S}^r(\mathbb{Z})$. Furthermore the operator $\Lambda^{2r} = \mathbf{op}\left(\langle m\rangle^{2r}\right)$, which defines the H^r-norm via

$$\|u\|_{H^r} = \int_{\mathrm{S}^1} \Lambda^{2r} u \cdot u \, \mathrm{d}\theta,$$

belongs to $\mathcal{S}^{2r}(\mathbb{Z})$.

Remark 5.3. A Fourier multiplier $\mathbf{a}(D)$ of class $\mathcal{S}^r(\mathbb{Z})$ extends for any $q \in \mathbb{R}$ to a *bounded linear operator*

$$\mathbf{a}(D) : H^q(\mathrm{S}^1, \mathbb{R}^d) \to H^{q-r}(\mathrm{S}^1, \mathbb{R}^d)\,,$$

and the linear embedding

$$\mathcal{S}^r(\mathbb{Z}) \to \mathcal{L}(H^q(\mathrm{S}^1, \mathbb{R}^d), H^{q-r}(\mathrm{S}^1, \mathbb{R}^d))$$

is continuous.

Now we introduce the class of parameter-dependent symbols.

Definition 5.4. Given $r \in \mathbb{R}$, the class $\mathcal{S}^r_\lambda(\mathbb{Z}) = C^\infty(\mathbb{R}^+, \mathcal{S}^r(\mathbb{Z}))$ consists of smooth curves $\mathbb{R}^+ \to \mathcal{S}^r(\mathbb{Z})$, $\lambda \mapsto \mathbf{a}(\lambda, D)$ of Fourier multipliers. We will call such a curve a *λ-symbol*.

We will often write $\mathbf{a}_\lambda(D)$ for $\mathbf{a}(\lambda, D)$ or, by slight abuse of notation, to denote the whole curve $\lambda \mapsto \mathbf{a}(\lambda, D)$. Using the material in Appendix A, we can give an alternative description of the space $\mathcal{S}^r_\lambda(\mathbb{Z})$ of one-parameter families of symbols.

Lemma 5.5. *Given $r \in \mathbb{R}$, a family of smooth curves $\mathbf{a}(\cdot, m) \in C^\infty(\mathbb{R}^+, \mathcal{L}(\mathbb{C}^d))$ with $m \in \mathbb{Z}$ defines an element $\mathbf{a}_\lambda(D)$ in the class $\mathcal{S}^r_\lambda(\mathbb{Z})$ if and only if for each $\alpha, \beta \in \mathbb{N}$,*

$$\left\| \partial_\lambda^\beta \Delta^\alpha \mathbf{a}(\lambda, m) \right\| \leq C_{\alpha,\beta} \langle m\rangle^{r-\alpha}\,, \tag{5.2}$$

holds locally uniformly in $\lambda \in \mathbb{R}^+$.

Proof. Let $\mathbf{a}_\lambda(D) \in \mathcal{S}^r_\lambda(\mathbb{Z})$. Since all derivatives of a smooth curve are locally bounded, it follows that for all $\alpha, \beta \in \mathbb{N}$, the expression $p_\alpha\left(\partial_\lambda^\beta \mathbf{a}_\lambda(D)\right)$ is bounded locally uniformly in λ and hence $\mathbf{a}(\lambda, m)$ satisfies (5.2) for some constants $C_{\alpha,\beta}$.

Conversely, let a family $\mathbf{a}(\lambda, m)$ of smooth curves satisfying the above estimates be given. Define the curves $\mathbf{a}^\beta(\lambda, m) = \partial_\lambda^\beta \mathbf{a}(\lambda, m)$. The estimate (5.2) shows that the curves $\lambda \mapsto a^\beta(\lambda, \cdot)$ are locally bounded in $\mathcal{S}^r(\mathbb{Z})$ and hence by Lemma A.1 the curve $\lambda \mapsto \mathbf{a}_\lambda(D)$ is an element of $\mathcal{S}^r_\lambda(\mathbb{Z})$. ☐

The following is the main theorem that will be used to show the smoothness of the metric. Set

$$A_c = R_{\psi_c} \circ A(\ell_c) \circ R_{\psi_c^{-1}}\,,$$

with $\psi_c(\theta) = \frac{2\pi}{\ell_c} \int_0^\theta |c'| \, \mathrm{d}\sigma$.

Proposition 5.6. *Let $r \geq 1$ and $A(\lambda) = \mathbf{a}_\lambda(D)$ belong to the class $\mathcal{S}^r_\lambda(\mathbb{Z})$. Then the map*

$$c \mapsto A_c\,, \qquad \mathcal{I}^q(\mathrm{S}^1, \mathbb{R}^d) \to \mathcal{L}(H^q(\mathrm{S}^1, \mathbb{R}^d), H^{q-r}(\mathrm{S}^1, \mathbb{R}^d))\,,$$

is smooth provided $q > \frac{3}{2}$ and $q \geq r$.

Proof. It was established in [17, Thm. 3.7] that for $A \in \mathcal{S}^r(\mathbb{Z})$ the mapping

$$\psi \mapsto A_\psi := R_\psi \circ A \circ R_{\psi^{-1}}\,, \quad \mathcal{D}^q \to \mathcal{L}(H^q, H^{q-r})$$

is smooth. Using the uniform boundedness principle [24, 5.18] it follows that the map

$$\psi \mapsto (A \mapsto A_\psi)\,, \quad \mathcal{D}^q \to \mathcal{L}(\mathcal{S}^r(\mathbb{Z}), \mathcal{L}(H^q, H^{q-r}))$$

is smooth[2]. Because the inclusion

$$\mathcal{L}(\mathcal{S}^r(\mathbb{Z}), \mathcal{L}(H^q, H^{q-r})) \subset C^\infty(\mathcal{S}^r(\mathbb{Z}), \mathcal{L}(H^q, H^{q-r})\,,$$

is bounded [24, 5.3], the following map is smooth

$$\psi \mapsto (A \mapsto A_\psi)\,, \quad \mathcal{D}^q \to C^\infty(\mathcal{S}^r(\mathbb{Z}), \mathcal{L}(H^q, H^{q-r}))\,;$$

via the exponential law [24, 3.12] this is equivalent to the smoothness of the joint map

$$(A, \psi) \mapsto A_\psi\,, \quad \mathcal{S}^r(\mathbb{Z}) \times \mathcal{D}^q \to \mathcal{L}(H^q, H^{q-r}))\,.$$

Next we note that the maps

$$c \mapsto \ell_c\,, \quad \mathcal{I}^q \to \mathbb{R}^+ \qquad \text{and} \qquad c \mapsto \psi_c\,, \quad \mathcal{I}^q \to \mathcal{D}^q$$

are smooth – this can be seen from their definitions – and since $\mathcal{S}^r_\lambda(\mathbb{Z}) = C^\infty(\mathbb{R}^+, \mathcal{S}^r(\mathbb{Z}))$ the composition

$$c \mapsto A(\ell_c)\,, \quad \mathcal{I}^q \to \mathcal{S}^r(\mathbb{Z})$$

is smooth as well. To conclude the proof we note that $c \mapsto A_c$ can be written as the composition $c \mapsto (A(\ell_c), \psi_c) \mapsto A(\ell_c)_{\psi_c}$. □

Corollary 5.7. *Let $r \geq \frac{1}{2}$ and $A(\lambda) = \mathbf{a}_\lambda(D)$ belong to the class $\mathcal{S}^{2r}_\lambda(\mathbb{Z})$. Then the bilinear form*

$$G_c(h, k) = \int_{\mathrm{S}^1} \langle A_c h, k \rangle \,\mathrm{d}s$$

extends smoothly to $\mathcal{I}^q(\mathrm{S}^1, \mathbb{R}^d)$ provided $q > \frac{3}{2}$ and $q \geq 2r$.

Proof. It is enough to show that

$$c \mapsto \check{G}_c = M_{|c'|} \circ A_c, \quad \mathcal{I}^q \to \mathcal{L}(H^q, H^{q-r}),$$

is smooth, where $M_{|c'|}$ denotes pointwise multiplication by $|c'|$. Now

$$c \mapsto M_{|c'|}, \quad \mathcal{I}^q \to \mathcal{L}(H^\rho, H^\rho),$$

is smooth for $0 \leq \rho \leq q - 1$ and the conclusion follows by Proposition 5.6 and the fact that composition of continuous linear mappings between Banach spaces is smooth. □

[2]When E, F are Fréchet spaces or more generally convenient vector spaces, $\mathcal{L}(E, F)$ is the space of bounded linear maps equipped with the topology of uniform convergence on bounded sets; see [24, 5.3] for details.

6. Smoothness of the spray

In order to prove the existence and smoothness of the spray, we will require moreover an *ellipticity condition* on λ-symbols. For the purpose of this article, we will adopt the following definition.

Definition 6.1. An element $\mathbf{a}_\lambda(D)$ of the class $\mathcal{S}^r_\lambda(\mathbb{Z})$ is called *locally uniformly elliptic*, if $\mathbf{a}(\lambda, m) \in GL(\mathbb{C}^d)$ for all $(\lambda, m) \in \mathbb{R}^+ \times \mathbb{Z}$ and

$$\left\| \mathbf{a}(\lambda, m)^{-1} \right\| \leq C \langle m \rangle^{-r}$$

holds locally uniformly in λ.

Remark 6.2. A Fourier multiplier $\mathbf{a}(D)$ of class $\mathcal{S}^r(\mathbb{Z})$ is *elliptic*, if $\mathbf{a}(m) \in GL(\mathbb{C}^d)$ for all $m \in \mathbb{Z}$ and

$$\left\| \mathbf{a}(m)^{-1} \right\| \leq C \langle m \rangle^{-r} ,$$

holds for all m. Such an $\mathbf{a}(D)$ induces a *bounded isomorphism* between $H^q(\mathrm{S}^1, \mathbb{R}^d)$ and $H^{q-r}(\mathrm{S}^1, \mathbb{R}^d)$, for all $q \in \mathbb{R}$.

We summarize our considerations by introducing the following class of operators which will be denoted $\mathcal{E}^r_\lambda(\mathbb{Z})$.

Definition 6.3. A family $\mathbf{a}_\lambda(D)$ of Fourier multipliers is an element of the class $\mathcal{E}^r_\lambda(\mathbb{Z})$, if

(1) $\mathbf{a}_\lambda(D)$ is in the class $\mathcal{S}^r_\lambda(\mathbb{Z})$,
(2) $\mathbf{a}(\lambda, m)$ is a positive Hermitian matrix for all $(\lambda, m) \in \mathbb{R}^+ \times \mathbb{Z}$ and
(3) $\mathbf{a}_\lambda(D)$ is locally uniformly elliptic.

Theorem 6.4. *Let $r \geq 1$, $q \geq 2r$ and $A(\lambda) = \mathbf{a}_\lambda(D)$ belong to $\mathcal{E}^{2r}_\lambda(\mathbb{Z})$. Then*

$$G_c(h, k) = \int_{\mathrm{S}^1} \langle A_c h, k \rangle \,\mathrm{d}s$$

defines a smooth weak Riemannian metric of order r on $\mathcal{I}^q(\mathrm{S}^1, \mathbb{R}^d)$ with a smooth geodesic spray.

Proof. It is shown in Corollary 5.7 that G extends smoothly to $\mathcal{I}^q(\mathrm{S}^1, \mathbb{R}^d)$. We can write

$$G_c(h, h) = 2\pi \sum_{m \in \mathbb{Z}} \langle \mathbf{a}(\ell_c, m) \hat{h}(m), \hat{h}(m) \rangle ,$$

and because $\mathbf{a}(\ell_c, m)$ are positive Hermitian matrices, $G_c(h, h) = 0$ only for $h = 0$. Thus G is a Riemannian metric. It is a weak Riemannian metric, because the inner product $G_c(\cdot, \cdot)$ induces on each tangent space $T_c\mathcal{I}^q(\mathrm{S}^1, \mathbb{R}^d)$ the H^r-topology, while the manifold $\mathcal{I}^q(\mathrm{S}^1, \mathbb{R}^d)$ itself carries the H^q-topology and by our assumptions $q > r$.

It remains to show that the geodesic spray of G exists and is smooth. For each $\lambda \in \mathbb{R}^+$, the operator $A(\lambda)$ is an elliptic Fourier multiplier and thus induces a bi-bounded linear isomorphism on $C^\infty(\mathrm{S}^1, \mathbb{R}^d)$. Thus we can apply Theorem 4.1, which shows that the geodesic spray exists on the space $\mathrm{Imm}(\mathrm{S}^1, \mathbb{R}^d)$ of smooth immersions. We will show that the map $S_c(h)$ extends smoothly to the Sobolev completion $\mathcal{I}^q(\mathrm{S}^1, \mathbb{R}^d)$. Then $F(c, h) = (h, S_c(h))$ is necessarily the geodesic spray of G on $\mathcal{I}^q(\mathrm{S}^1, \mathbb{R}^d)$.

We have the following formula for $S_c(h)$,

$$S_c(h) = -A_c^{-1}\left\{(D_{c,h}A_c)h + \langle D_s h, v\rangle A_c h + \langle A_c h, D_s h\rangle v + (w + w_0)D_s v\right\}.$$

The map

$$A \mapsto A^{-1}, \quad \mathcal{U} \subset \mathcal{L}(H^q, H^{q-2r}) \to \mathcal{L}(H^{q-2r}, H^q),$$

defined on the open subset $\mathcal{U}$ of invertible operators is smooth and so is $c \mapsto A_c$ by Proposition 5.6; therefore so is the composition $c \mapsto A_c^{-1} : \mathcal{I}^q \to \mathcal{L}(H^{q-2r}, H^q)$. Thus we need to show that

$$(c, h) \mapsto (D_{c,h}A_c)h + \langle D_s h, v\rangle A_c h + \langle A_c h, D_s h\rangle v + (w + w_0)D_s v,$$

is smooth between $\mathcal{I}^q \times H^q \to H^{q-2r}$. The first term $(D_{c,h}A_c)h$ is the derivative of $(c, h) \mapsto A_c h$ with respect to the first variable and hence the map $(c, h) \mapsto (D_{c,h}A_c)h$ is smooth between $\mathcal{I}^q \times H^q \to H^{q-2r}$. Arc-length derivation

$$(c, u) \mapsto D_s u, \quad \mathcal{I}^q \times H^\rho \to H^{\rho-1}, \quad 0 \le \rho \le q,$$

is smooth and pointwise multiplication in $C^\infty(\mathrm{S}^1, \mathbb{R})$ extends to a continuous bilinear mapping

$$H^\sigma \times H^\rho \to H^\rho, \quad \sigma > \frac{1}{2},\ 0 \le \rho \le \sigma.$$

Therefore, noting that $v = D_s c$ and $q - 1 > \frac{1}{2}$, the expression $\langle D_s h, v\rangle A_c h + \langle A_c h, D_s h\rangle v$ is a smooth map $\mathcal{I}^q \times H^q \to H^{q-2r}$. Next we use the fact that the antiderivative

$$u \mapsto \left(\theta \mapsto \int_0^\theta u(\sigma)\,\mathrm{d}\sigma\right), \quad H^\rho \to H^{\rho+1}, \quad \rho \ge 0,$$

is a bounded linear mapping. Thus $(c, h) \mapsto w(c, h)$ maps smoothly $\mathcal{I}^q \times H^q \to H^{q-2r+1}$ and because $q - 2r + 1 > \frac{1}{2}$ we have that $(c, h) \mapsto w(c, h)D_s v$ is smooth between $\mathcal{I}^q \times H^q \to H^{q-2r}$. The last term is $w_0(c, h)$. First we note that the map $c \mapsto \psi_c$ between $\mathcal{I}^q \to \mathcal{D}^q$ is smooth. A term-by-term inspection shows that the map $(c, h) \mapsto f(c, h)$, where $f(c, h)$ is the integrand in the definition of $w_0(c, h)$ is a smooth map $\mathcal{I}^q \times H^q \to L^1$ and thus $(c, h) \mapsto w_0(c, h)$ is smooth as well. Thus the extension of G to $\mathcal{I}^q(\mathrm{S}^1, \mathbb{R}^d)$ has a smooth spray. □

As a corollary, using the Cauchy–Lipschitz theorem, we obtain the local existence of geodesics on the Hilbert manifold $\mathcal{I}^q(\mathrm{S}^1, \mathbb{R}^d)$.

Corollary 6.5. *Let $A_\lambda = \mathbf{a}_\lambda(D)$ belong to the class $\mathcal{E}_\lambda^{2r}(\mathbb{Z})$, where $r \ge 1$, and let $q \ge 2r$. Consider the geodesic flow on the tangent bundle $T\mathcal{I}^q$ induced by the Fourier multiplier A_λ. Then, given any $(c_0, h_0) \in T\mathcal{I}^q$, there exists a unique non-extendable geodesic*

$$(c, h) \in C^\infty(J, T\mathcal{I}^q)$$

with $c(0) = c_0$ and $h(0) = h_0$ on the maximal interval of existence J, which is open and contains 0.

As a consequence of a no-loss-no-gain argument, we also obtain local well-posedness on the space of smooth curves.

Corollary 6.6. *Let $A_\lambda = \mathbf{a}_\lambda(D)$ belong to the class $\mathcal{E}_\lambda^{2r}(\mathbb{Z})$, where $r \geq 1$. Consider the geodesic flow on the tangent bundle $T\mathrm{Imm}(\mathrm{S}^1, \mathbb{R}^d)$ induced by the Fourier multiplier A_λ. Then, given any $(c_0, h_0) \in T\mathrm{Imm}(\mathrm{S}^1, \mathbb{R}^d)$, there exists a unique non-extendable geodesic*

$$(c, h) \in C^\infty(J, T\mathrm{Imm}(\mathrm{S}^1, \mathbb{R}^d))$$

with $c(0) = c_0$ and $h(0) = h_0$ on the maximal interval of existence J, which is open and contains 0.

Proof. Since the metric is invariant by re-parametrization, it is invariant in particular by translations of the parameter. A similar observation has been used in [17, Section 4] to show that local existence of geodesic still holds in the smooth category. The proof is based on a *no loss-no gain result* in spatial regularity, initially formulated in [16] (see also [8]). The same argument is still true here and leads to the local existence of the geodesics on the Fréchet manifold $\mathrm{Imm}(\mathrm{S}^1, \mathbb{R}^d)$. □

7. Strong Riemannian metrics

The goal of this section is to show, that metrics of order $s > \frac{3}{2}$ (induced by a Fourier multiplier A_λ in the class $\mathcal{E}_\lambda^{2s}(\mathbb{Z})$), induce *strong* smooth Riemannian metrics on the Sobolev completion $\mathcal{I}^s$ of the same order as the metric. Let $A(\lambda)$ be of class $\mathcal{E}_\lambda^{2s}(\mathbb{Z})$ with $s > \frac{3}{2}$. In Section 6 it was shown that the corresponding Riemannian metric

$$G_c(h, k) = \int_{\mathrm{S}^1} \langle A_c h, k \rangle \, \mathrm{d}s \,, \tag{7.1}$$

can be extended smoothly to $\mathcal{I}^q(\mathrm{S}^1, \mathbb{R}^d)$ for $q \geq 2s$. Now we want to improve this to $q \geq s$. Of particular interest is the case $q = s$, when the topologies induced by the inner products $G_c(\cdot, \cdot)$ coincide with the manifold topology.

Any positive Hermitian matrix $\mathbf{a}$ has a unique positive square root $\mathbf{b}$ which depends smoothly on $\mathbf{a}$. We have, moreover, the stronger result that an operator $A(\lambda) = \mathbf{a}_\lambda(D)$ in the class $\mathcal{E}_\lambda^{2s}$ has a square root $B(\lambda) = \mathbf{b}_\lambda(D)$ in the class $\mathcal{E}_\lambda^{s}$ (see Lemma B.2). With $B_c = R_{\psi_c} \circ B(\ell_c) \circ R_{\psi_c^{-1}}$ we have the identities

$$A(\ell_c) = B(\ell_c)^2 \,, \qquad\qquad A_c = B_c^2 \,; \tag{7.2}$$

furthermore, because $\mathbf{b}(\lambda, m)$ is a Hermitian matrix, the operator $B(\ell_c)$ is $L^2(d\theta)$-symmetric and for each curve c the operator B_c is $L^2(ds)$-symmetric. Therefore we can rewrite the metric G in the symmetric form

$$G_c(h, k) = \int_{\mathrm{S}^1} \langle A_c h, k \rangle \, \mathrm{d}s = \int_{\mathrm{S}^1} \langle B_c h, B_c k \rangle \, \mathrm{d}s \,. \tag{7.3}$$

We obtain therefore the following expression for the operator $\check{G}_c$ on $T\mathcal{I}^s$:

$$\check{G}_c = B_c^t \circ M_{|c'|} \circ B_c \,,$$

where

$$B_c : H^s(\mathrm{S}^1, \mathbb{R}^d) \to L^2(\mathrm{S}^1, \mathbb{R}^d), \qquad B_c^t : L^2(\mathrm{S}^1, \mathbb{R}^d) \to H^{-s}(\mathrm{S}^1, \mathbb{R}^d) \,,$$

and B_c^t is the transpose of B_c. The latter formula can now be used to obtain the following result concerning the smooth extension of this family of inner products into a strong Riemannian metric on $\mathcal{I}^s$, provided $s > 3/2$.

Theorem 7.1. *Let $s > 3/2$ and $B_\lambda \in \mathcal{E}_\lambda^s(\mathbb{Z})$. Then the expression*

$$G_c(h,k) = \int_{\mathrm{S}^1} \langle B_c h, B_c k \rangle \, \mathrm{d}s \,,$$

defines a smooth and strong *Riemannian metric on $\mathcal{I}^s(\mathrm{S}^1, \mathbb{R}^d)$.*

Remark 7.2. For strong metrics on a Lie group the invariance of the metric implies the geodesic and metric completeness of the space, see [20, Lemma 5.2]. This has been used in [5] to show completeness of the H^s-metric on $\mathrm{Diff}(\mathbb{R}^d)$, see also [10] for integer orders on the diffeomorphism group of a general manifold. Unfortunately, there is no automatic analogue of this result in our situation. To prove the global well-posedness on the space of regular curves additional assumptions on the dependence of the operator A on the length will be necessary. In future work, we plan to follow this line of research and use a similar strategy as in [7] for integer orders to obtain this result.

Proof. Since $B_\lambda \in \mathcal{E}_\lambda^s(\mathbb{Z})$ and $s > 3/2$, the mapping

$$\check{G}_c = B_c^t \circ M_{|c'|} \circ B_c$$

defines, for each $c \in \mathcal{I}^s$, a bounded isomorphism between $H^s(\mathrm{S}^1, \mathbb{R}^d)$ and its dual $H^{-s}(\mathrm{S}^1, \mathbb{R}^d)$. Thus we need only to show that the mapping

$$c \mapsto \check{G}_c, \qquad \mathcal{I}^s \to \mathcal{L}(H^s(\mathrm{S}^1, \mathbb{R}^d), H^{-s}(\mathrm{S}^1, \mathbb{R}^d))$$

is smooth. Since transposition between Banach spaces is itself a bounded operator it follows that the transpose

$$c \mapsto B_c^t, \quad \mathcal{I}^s \to \mathcal{L}(L^2(\mathrm{S}^1, \mathbb{R}^d), H^{-s}(\mathrm{S}^1, \mathbb{R}^d))$$

is smooth iff

$$c \mapsto B_c, \quad \mathcal{I}^s \to \mathcal{L}(H^s(\mathrm{S}^1, \mathbb{R}^d), L^2(\mathrm{S}^1, \mathbb{R}^d))$$

is smooth, which is the case by Proposition 5.6. Now, the mapping

$$c \mapsto M_{|c'|}, \qquad \mathcal{I}^s \to \mathcal{L}(L^2(\mathrm{S}^1, \mathbb{R}^d), L^2(\mathrm{S}^1, \mathbb{R}^d)),$$

is smooth for $s > 3/2$. Finally, since composition of bounded operators between Banach spaces is itself a bounded operator, it follows that the composition

$$c \mapsto B_c^t \circ M_{|c'|} \circ B_c, \qquad \mathcal{I}^s \to \mathcal{L}(H^s(\mathrm{S}^1, \mathbb{R}^d), H^{-s}(\mathrm{S}^1, \mathbb{R}^d)),$$

is smooth. □

Remark 7.3. A smooth, strong Riemannian metric on a Hilbert manifold has a smooth spray (see [25] for instance). However, formula (4.2) is no longer useful in that case because it is not clear that this expression extends to a smooth map from $\mathcal{I}^s$ to $H^s(\mathrm{S}^1, \mathbb{R}^d)$. Following Lang [25, Proposition 7.2], we introduce the H^s inner product on $H^s(\mathrm{S}^1, \mathbb{R}^d)$:

$$\langle\langle h, k \rangle\rangle_{H^s} := \int_{\mathrm{S}^1} \langle \Lambda^s h, \Lambda^s k \rangle \, \mathrm{d}\theta$$

so that

$$G_c(h,k) = \langle\!\langle P_c h, k\rangle\!\rangle_{H^s}, \quad \text{where} \quad P_c := \left(\Lambda^{-s}\circ B_c\right)^t \circ \Lambda^{-s} \circ M_{|c'|} \circ B_c.$$

Here, Q^t is the transpose of a bounded operator

$$Q : H^s(\mathrm{S}^1, \mathbb{R}^d) \to H^s(\mathrm{S}^1, \mathbb{R}^d).$$

In that case, there is an alternative expression for the spray, which is nevertheless equivalent to (4.2). It is given implicitly by the formula

$$\langle\!\langle S_c(h), P_c k\rangle\!\rangle_{H^s} = \frac{1}{2}\langle\!\langle (D_{c,k}P_c)\, h, h\rangle\!\rangle_{H^s} - \langle\!\langle (D_{c,h}P_c)\, h, k\rangle\!\rangle_{H^s}.$$

Appendix A. Smooth curves in $\mathcal{S}^r(\mathbb{Z})$

The right framework to work with smooth maps on infinite dimensional (other than Banach spaces) are *convenient vector spaces* (see [19, 24]). Note that every Fréchet space is a convenient vector space. We will need the following lemma about recognising smooth curves in convenient vector spaces.

Lemma A.1. [19, 4.1.19] *Let $c : \mathbb{R} \to E$ be a curve in a convenient vector space. Let $\mathcal{V} \subseteq E'$ be a point-separating subset of bounded linear functionals such that the bornology of E has a basis of $\sigma(E, \mathcal{V})$-closed sets. Then the following are equivalent:*

1. *c is smooth;*
2. *There exist locally bounded curves $c^k : \mathbb{R} \to E$ such that $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is smooth with $(\ell \circ c)^{(k)} = \ell \circ c^k$, for each $\ell \in \mathcal{V}$.*

To apply this lemma to the space $\mathcal{S}^r(\mathbb{Z})$ of Fourier multipliers we need to choose a suitable set $\mathcal{V}$ of linear functionals. This is accomplished in the following lemma.

Lemma A.2. *Let $r \in \mathbb{R}$ and $\mathcal{V} = \{\lambda \circ \mathrm{ev}_m : m \in \mathbb{Z},\ \lambda \in \mathcal{L}(\mathbb{C}^d)'\} \subset \mathcal{S}^r(\mathbb{Z})'$. Then the bornology of $\mathcal{S}^r(\mathbb{Z})$ has a basis consisting of $\sigma(\mathcal{S}^r(\mathbb{Z}), \mathcal{V})$-closed sets.*

Proof. We can regard $\mathcal{V}$ as a subset of both $\mathcal{S}^r(\mathbb{Z})'$ as well as $\ell^\infty(\mathbb{Z}, \mathcal{L}(\mathbb{C}^d))'$. It is shown in [19, 4.1.21] that the bornology of $\ell^\infty(\mathbb{Z}, \mathcal{L}(\mathbb{C}^d))$ has a basis of $\sigma(\ell^\infty(\mathbb{Z}, \mathcal{L}(\mathbb{C}^d)), \mathcal{V})$-closed sets. We embed $\mathcal{S}^r(\mathbb{Z})$ as a closed subspace into

$$\iota : \mathcal{S}^r(\mathbb{Z}) \to \prod_{\alpha\in\mathbb{N}} \ell^\infty(\mathbb{Z}, \mathcal{L}(\mathbb{C}^d)),\ (\mathbf{a}(m))_{m\in\mathbb{Z}} \mapsto \left(\langle m\rangle^{-(r-\alpha)}\Delta^\alpha \mathbf{a}(m)\right)_{m\in\mathbb{Z},\,\alpha\in\mathbb{N}}.$$

Because the bornology of the product is the product bornology it follows that a basis for the bornology on $\prod_{\alpha\in\mathbb{N}} \ell^\infty(\mathbb{Z}, \mathcal{L}(\mathbb{C}^d))$ is given by $\sigma(\prod_\alpha \ell^\infty, \mathcal{W})$-closed sets, where $\mathcal{W} = \bigcup_{\alpha\in\mathbb{N}} \mathcal{V} \circ \mathrm{pr}_\alpha$ and pr_α denotes the canonical projection onto the α-th factor. Here we note that on $\mathcal{S}^r(\mathbb{Z})'$ the set $\iota^*(\mathcal{W})$ is contained in the linear space of $\mathcal{V}$ and hence the bornology of $\mathcal{S}^r(\mathbb{Z})$ has a basis consisting of $\sigma(\mathcal{S}^r(\mathbb{Z}), \mathcal{V})$-closed sets. ☐

Appendix B. Square-root of an operator in $\mathcal{E}^r_\lambda(\mathbb{Z})$

Since a positive definite Hermitian matrix has a unique positive square root, which depends smoothly on its coefficients, we can define formally the square root $B_\lambda = \mathbf{op}\left(\mathbf{a}(\lambda, m)^{1/2}\right)$ of an element $A_\lambda = \mathbf{a}_\lambda(D)$ in the class $\mathcal{E}^r_\lambda(\mathbb{Z})$. In order to prove this we need the following lemma.

Lemma B.1. [5, Lemma 4.8] *Let $a, b, x \in \mathcal{L}(\mathbb{C}^d)$ be three matrices satisfying*

$$bx + xb = a\,,$$

with b Hermitian and positive definite. Then

$$\|x\| \le \sqrt{\frac{d}{2}}\,\|b^{-1}\|\,\|a\|\,,$$

where $\|\cdot\|$ denotes the Frobenius norm, i.e. $\|x\| = \sqrt{\operatorname{tr} xx^*}$.

The following lemma together with its proof is a generalisation of [5, Lemma 4.7] to our situation of one-parameter families of symbols.

Lemma B.2. *The positive square root of an operator in the class $\mathcal{E}^r_\lambda(\mathbb{Z})$ belongs to the class $\mathcal{E}^{r/2}_\lambda(\mathbb{Z})$. Conversely, the square of an operator in the class $\mathcal{E}^r_\lambda(\mathbb{Z})$ belongs to the class $\mathcal{E}^{2r}_\lambda(\mathbb{Z})$.*

Proof. We will prove the estimate

$$\left\|\partial^\beta_\lambda \Delta^\alpha \mathbf{b}(\lambda, m)\right\| \lesssim \langle m\rangle^{r/2-\alpha}\,,$$

which holds locally uniformly in λ, by induction over $\alpha+\beta$. If $\alpha+\beta = 0$ the statement is $\|\mathbf{b}(\lambda, m)\| \lesssim \langle m\rangle^{r/2}$. Assume that it has been proven for $\alpha+\beta \le k$. Then let $\alpha+\beta = k+1$ and, omitting the arguments (λ, m), we obtain using the product rule,

$$\partial^\beta_\lambda \Delta^\alpha \mathbf{a} = \mathbf{b}\left(\partial^\beta_\lambda \Delta^\alpha \mathbf{b}\right) + \left(\partial^\beta_\lambda \Delta^\alpha \mathbf{b}\right)\mathbf{b} + \sum_{(i,j)\in X} \binom{\alpha}{i}\binom{\beta}{j} \partial^j_\lambda \Delta^i \mathbf{b}\, \partial^{\beta-j}_\lambda \Delta^{\alpha-i}\mathbf{b}\,,$$

with

$$X = \{(i,j) \,:\, 0 \le i \le \alpha,\ 0 \le j \le \beta,\ i+j < \alpha+\beta\}\,.$$

Then by the induction assumption

$$\left\|\partial^j_\lambda \Delta^i \mathbf{b}\, \partial^{\beta-j}_\lambda \Delta^{\alpha-i}\mathbf{b}\right\| \lesssim \langle m\rangle^{r/2-i}\langle m\rangle^{r/2-\alpha+i} \lesssim \langle m\rangle^{r-\alpha}\,,$$

and hence we obtain via Lemma B.1,

$$\left\|\partial^\beta_\lambda \Delta^\alpha \mathbf{b}\right\| \lesssim \langle m\rangle^{-r/2}\langle m\rangle^{r-\alpha} \lesssim \langle m\rangle^{r/2-\alpha}\,.$$

This completes the induction. ∎

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