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A bending quasi-front generated by an instantaneous impulse loading at the edge of a semi-infinite pre-stressed incompressible elastic plate

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Abstract

A refined membrane-like theory is used to describe bending of a semi-infinite pre-stressed incompressible elastic plate subjected to an instantaneous impulse loading at the edge. A far-field solution for the quasi-front is obtained by using the method of matched asymptotic expansions. A leading-order hyperbolic membrane equation is used for an outer problem, whereas a refined (singularly perturbed) membrane equation of an inner problem describes a boundary layer, which smoothes a discontinuity predicted by the outer problem at the wave front. The inner problem is then reduced to one-dimensional by an appropriate choice of inner coordinates, motivated by the wave front geometry. Using the inherent symmetry of the outer problem, a solution for the quasi-front is derived that is valid in a vicinity of the tip of the wave front. Pre-stress is shown to affect geometry and type of the generated quasi-front; in addition to a classical *receding* quasi-front the pre-stressed plate can support propagation of an *advancing* quasi-front. Possible responses may even feature both types of quasi-front at the same time, which is illustrated by numerical examples. The case of a so-called *narrow* quasi-front, associated with a possible degeneration of contribution of singular perturbation terms to the governing equation, is studied qualitatively.

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1 Introduction

The problem of instantaneous impulse loading of a point at the edge of a two-dimensional elastic half-space was first formulated by Lamb (1904). On the one hand this problem featured possibly the simplest setup for observing transient motion in bounded elastic media, on the other hand the problem could be used as a basic model for studying earthquakes and developing electro-acoustic devices, which explains wide interest in Lamb's problem over the last hundred years. In his paper Lamb used the integral transforms technique to formulate qualitative solutions for the components of displacement. Subsequently, various analytical techniques were suggested for either direct derivation of the exact solutions (method of functionally-invariant solutions by Smirnov and Sobolev (1933)) or exact inversion of integrals in the transformed solutions (Cagniard's method, see Cagniard (1962); De Hoop (1960)). The paper by Petrashen' et al. (1950) established a connection between these two approaches as well as presented some new analytical solutions. Later Burridge (1971) extended the Cagniard's method for a general anisotropic medium and Willis (1973) developed a method for treating 'self-similar' boundary-initial value problems, in particular applicable to Lamb's problem for a generally anisotropic half-space.

It has long been established that motions of a thin elastic layer in a plane-stress state are asymptotically described by the equations, which at the leading order are formally equivalent to the equations for a plane-strain problem of elasticity. It would therefore be tempting to apply the techniques, obtained in the context of the two-dimensional Lamb's problem, directly to *generalised plane-stress* problems. However, the associated results would seriously contradict the exact three-dimensional theory, for example, solutions of the generalised plane-stress equations distort propagation speed of a dilatational disturbance. Moreover, quite similarly with transient waves in rods, see Skalak (1957), the dilatational front, predicted by the equations of generalised plane-stress, is actually a quasi-front, i.e. the dilatational wave features no discontinuity. The indication is that the generalised plane-stress is not a sufficient approximation for modelling a transient loading of thin plates. However, using the method of integral transforms Kaplunov and Nol'de (1992) demonstrated that there is no need to use the full three-dimensional theory to describe quasi-front phenomena in plates, as a refined long-wave low-frequency plate theory was shown to be sufficient. Higher-order derivatives, typical for such theories, may be considered as singular perturbations of hyperbolic equations describing boundary layers in a vicinity of quasi-fronts. This formulation of the problem, generally also applicable to thin shells, may be termed as *generalised Lamb's problem.* During the course of a further research Kaplunov and Nol'de (1995) and Emri et al. (2001) extended the results to thin shells of zero Gaussian curvature and to thin plates composed of a viscous material, respectively. In these papers the method of matched asymptotic expansions was used, with outer problem posed for non-refined long-wave low-frequency equations and inner problem posed for a refined theory in a boundary layer, which by an appropriate change of variables was reduced to a one-dimensional governing equation.

With a view to extend these ideas to more general media, this paper is devoted to the analysis of instantaneous impulse loading of the edge of a semi-infinite pre-stressed incompressible elastic plate. By *pre-stress* we mean a static homogeneous finite deformation applied to the plate prior to propagation of an infinitesimal disturbance. This concept of *incremental deformations* was found especially appropriate for modelling of rubbers and rubber-like materials, see Ogden (1984). The incompressibility constraint is then introduced both to account for a low compressibility of rubbers and to somewhat simplify our analysis. We envisage a thin rubber sheet stretched by the in-plane tensions as a basic physical setup for the problem analysed in this paper. The considered problem is therefore analogous to the plate bending problem in the classic linear elasticity. It deals with anti-symmetric deformations with respect to the mid-surface of the plate. An appropriate counterpart problem for the theory of generalised plane stress, dealing with symmetric deformations of the plate, is to be modelled by a vector equation and will become a subject of our separate forthcoming publication.

The influence of a pre-stress on the dynamic response of plates is profound. Generally, pre-stressed media behave as anisotropic in the sense that material properties depend on the considered direction. Another and more subtle difference with usual isotropic media is a long-wave behavior of the flexural fundamental mode of a pre-stressed plate. A long-wave low-frequency limit in a pre-stressed plate generally corresponds to a finite wave speed and therefore is associated with propagation of a quasi-front, see Kaplunov et al. (2000). This *bending quasi-front* has no analogue in an isotropic theory. Further difference lies in a possible structure of the quasi-front itself; while the quasi-front for an isotropic plate is always *receding,* it is possible to vary the pre-stress in such a way that the type of quasi-front changes to an *advancing* one, which was demonstrated by Kaplunov et al. (2000) for a pre-stressed incompressible strip. In fact, because of anisotropy of a pre-stressed plate it is possible to generate responses featuring both types of front at the same time, which was suggested by Pichugin and Rogerson (2002).

In Section 2 an asymptotic membrane-like governing equation for a pre-stressed incompressible elastic plate will be introduced. A boundary condition of an instantaneous impulse loading at the edge will also be introduced and its use in the context of the asymptotic plate theory is discussed. The solution will be performed by using the method of matched asymptotic expansions, see for example Cole (1968). The outer problem will be formulated and solved in Section 3 as a leading order (non-refined) governing equation subjected to an instantaneous edge impulse loading. A modified procedure of solving the outer problem will involve application of Fourier cosine transform with respect to the coordinate normal to the plate edge. This approach will make use of the symmetry of the problem with respect to the edge and enable matching with the inner solution at the tip of the wave front. The outer solution will have the form of an ellipse-shaped discontinuity propagating away from the impact point. Section 4 will be concerned with asymptotic solutions for the inner problem. By introducing a special system of inner coordinates, all of the considered inner problems will be reduced to one-dimensional equations. The asymptotic solution that we will derive first is generally valid for most part of the quasi-front, except in a vicinity of the edge. In contrast with previously published far-field solutions this asymptotic is valid in a vicinity of the tip of the quasi-front, in this sense we term it uniform. The situation that we termed *narrow* quasi-front (when the contribution of singular perturbation terms degenerates) will also be considered. In order to avoid complicated algebra we will only study the solution in this case qualitatively. Section 5 will be devoted to a more detailed analysis of the obtained solutions. Important analytical characteristics will be obtained for all considered types of quasi-fronts. Numerical examples will be used throughout Section 5 to demonstrate possible wave front profiles.

2 Governing equations

Consider a thin semi-infinite plate with free faces, composed of a homogeneously pre-stressed incompressible elastic material. We introduce a Cartesian system of coordinates $Ox_1x_2x_3$ with origin O at the mid-plane of the plate, such that axis Ox_2 is orthogonal to the plate faces and axes Ox_1 and Ox_3 are aligned with the principal directions of the (static) initial deformation that generated the pre-stress. For the sake of algebraic simplicity Ox_1 is assumed to be normal to the plate edge, see Figure 1. Long-wave low-frequency flexu-

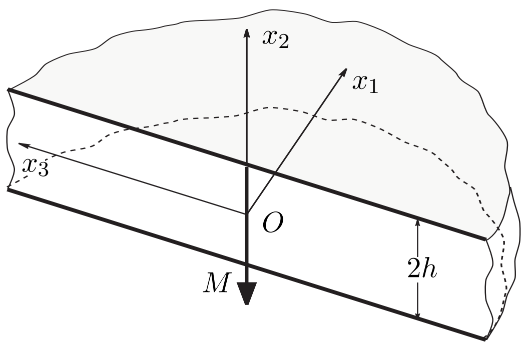

Fig. 1. Semi-infinite plate subjected to an instantaneous impulse loading at the edge.

ral motion of such a plate may be asymptotically described by the following two-dimensional governing equation valid to $O(h^4)$

$$\hat{c}_1^2\frac{\partial^2 v}{\partial x_1^2}+\hat{c}_3^2\frac{\partial^2 v}{\partial x_3^2}-\frac{\partial^2 v}{\partial t^2}+\frac{h^2}{3}\left(\hat{\mathcal{F}}_{cc}^{(2)}\frac{\partial^4 v}{\partial x_1^4}+\hat{\mathcal{F}}_{cs}^{(2)}\frac{\partial^4 v}{\partial x_1^2\partial x_3^2}+\hat{\mathcal{F}}_{ss}^{(2)}\frac{\partial^4 v}{\partial x_3^4}\right)=0\,, \quad (1)$$

in which $v = v(x_1, x_3, t)$ is a mid-surface deflection, h a half-thickness of the plate, ρ a material density and $\hat{c}_1$ and $\hat{c}_3$ are long-wave low-frequency phase speed limits when propagating along Ox_1 and Ox_3, respectively. The individual components of the displacement may be represented in terms of v using the following leading order relations

$$v_1=-\frac{g_1}{\gamma_{21}}\frac{\partial v}{\partial x_1}\,x_2\,,\quad v_2=v\,,\quad v_3=-\frac{g_3}{\gamma_{23}}\frac{\partial v}{\partial x_3}\,x_2\,. \quad (2)$$

A measure of the leading order incremental surface traction at the edge of a semi-infinite plate is given in this asymptotic regime by

$$\tau_2=\rho\hat{c}_1^2\frac{\partial v}{\partial x_1}\,. \quad (3)$$

All parameters occurring in (1)–(3) determine pre-stress and material properties and are defined in the Appendix A. It is worth noting that one of the consequences of the relations (2) is that to leading order the mid-surface deflection does not vary along the plate thickness. This asymptotic membrane-like theory for plates was derived by Pichugin and Rogerson (2002).

Higher-order derivative terms in the governing equation (1) may be considered as singular perturbation of the leading order hyperbolic membrane equation, see (1) when $h = 0$. These terms may be expected to smooth out the discontinuity at the front predicted by the hyperbolic equation, which is therefore called *a quasi-front.* Our main goal in this paper is to investigate the influence of plate boundaries and pre-stress on the quasi-front; in doing so we focus on studying the far-field response of the equation (1). This means that we consider a time instant when the original disturbance has travelled a distance L, which is much greater than the plate thickness. Physically L may be interpreted as the width of a thin and *finite* strip, which can be considered infinite before the first reflection occurs, or as the point where an observer places his device for recording the far-field response. The parameter $\eta \equiv h/L$ is then asymptotically small and may be used to reformulate the governing equation (1) in a non-dimensional form according to the following scalings

$$x_1=L\xi_1\,,\quad x_3=L\xi_3\,,\quad t=\frac{L}{c_0}\tau\,, \quad (4)$$

in which c_0 is a unit of long wave speed. Although generally the phase speed v of a plane harmonic wave in a pre-stressed medium depends on the direction of propagation, for long waves it may be assumed to be $O(1)$, which justifies the choice of scaling for time made in (4). In terms of non-dimensional variables (4), equation (1) takes the form

$$c_1^2\frac{\partial^2 u}{\partial \xi_1^2}+c_3^2\frac{\partial^2 u}{\partial \xi_3^2}-\frac{\partial^2 u}{\partial \tau^2}+\frac{\eta^2}{3}\left(\mathcal{F}_{cc}^{(2)}\frac{\partial^4 u}{\partial \xi_1^4}+\mathcal{F}_{cs}^{(2)}\frac{\partial^4 u}{\partial \xi_1^2\partial \xi_3^2}+\mathcal{F}_{ss}^{(2)}\frac{\partial^4 u}{\partial \xi_3^4}\right)=0\,, \quad (5)$$

which is valid to $O(\eta^4)$, where $u=u(\xi_1,\xi_3,\tau)\equiv v(x_1,x_3,t)/L$, $c_{\cdot}=\hat{c}_{\cdot}/c_0$ and $\mathcal{F}_{\cdot\cdot}^{(2)}=\hat{\mathcal{F}}_{\cdot\cdot}^{(2)}/c_0$. Please note that non-dimensional variables (4) will be used throughout this paper.

It is worth remarking that equation (5) is asymptotically consistent, which may be demonstrated by the fact that its dispersion relation exactly matches the second order long-wave low-frequency approximation of the exact dispersion equation for a three-dimensional homogeneously pre-stressed incompressible plate. More specifically, the range of applicability of (5) may be indicated by considering a plane harmonic wave of the form $u=\exp[i(kn_1\xi_1+kn_3\xi_3-\omega\tau)]$, $n_1^2+n_3^2=1$, see Kaplunov and Nol'de (1992). Upon substituting this solution into (5) and using appropriate simplifications it may be inferred that

$$\omega=k\left(c_1^2n_1^2+c_3^2n_3^2+\frac{1}{2}\left(\mathcal{F}_{cc}^{(2)}n_1^4+\mathcal{F}_{cs}^{(2)}n_1^2n_3^2+\mathcal{F}_{ss}^{(2)}n_3^4\right)k^2\eta^2+O(k^4\eta^4)\right)\,. \quad (6)$$

The phase of our harmonic wave will not be distorted at the observation distance L only provided $kO(k^4\eta^4)\to 0$ as $\eta\to 0$, therefore $k\ll\eta^{-4/5}$. It may also be remarked here that the range of applicability of non-refined membrane equation (i.e. $\eta^2=0$ in (5)) may be deduced from the requirement that $kO(k^2\eta^2)\to 0$ as $\eta\to 0$, thus the classical membrane equation may be used for wave numbers $k\ll\eta^{-2/3}$. These estimates coincide with the corresponding ranges of applicability for classical isotropic (both refined and non-refined) plate theories, see Goldenveizer et al. (1993) and Kaplunov et al. (1998).

It was shown by Pichugin and Rogerson (2002) that for long-wave low-frequency flexural motion, (3) is the dominant component of incremental traction at the plate edge. The instantaneous impulse loading boundary condition at the edge of the plate is therefore formulated as

$$c_1^2\left.\frac{\partial u}{\partial \xi_1}\right|_{\xi_1=0}=-M\delta(\tau)\delta(\xi_3)\,, \quad (7)$$

in which M is a load magnitude and $\delta(\cdot)$ is the Dirac's delta function. We remark that here as well as in the following discussion all of the derivatives are understood in a generalised sense. The two other components of surface traction may be shown to be of $O(\eta)$. Thus, a boundary condition that only prescribes τ_2 will introduce a discrepancy of $O(\eta)$ into the solution of (5), which may devalue the influence of $O(\eta^2)$ correction terms in the refined governing equation. Generally, boundary conditions for lower-dimensional asymptotic structural theories have to be specially derived, see Kaplunov et al. (1998) for a discussion on this topic. However, in this paper we will study the influence of the higher-order terms only in a vicinity of fronts predicted by the non-refined hyperbolic theory, where these terms become leading order. The boundary conditions will only be applied to a non-refined problem away from the wave front, so that the influence of the higher-order terms may be neglected. The error introduced by use of the non-refined boundary condition will therefore be asymptotically secondary.

3 Solution for a membrane

The solution will be performed by using the method of matched asymptotic expansions, see for example the book by Cole (1968). Since the higher-order correction terms featured in the governing equation (5) only contribute significantly in a vicinity of the quasi-front, for the outer problem we only need to consider the membrane equation [2]

$$c_1^2\frac{\partial^2 u_{\mathrm{out}}}{\partial \xi_1^2} + c_3^2\frac{\partial^2 u_{\mathrm{out}}}{\partial \xi_3^2} - \frac{\partial^2 u_{\mathrm{out}}}{\partial \tau^2} = 0\,, \tag{8}$$

where $u_{\mathrm{out}} = u_{\mathrm{out}}(\xi_1, \xi_3, \tau)$ is the outer solution for the mid-surface (membrane) deflection. This equation is to be solved subject to the appropriately modified boundary condition (7)

$$c_1^2\left.\frac{\partial u_{\mathrm{out}}}{\partial \xi_1}\right|_{\xi_1=0} = -M\delta(\tau)\delta(\xi_3)\,, \tag{9}$$

and zero initial conditions.

The common procedure of solving the boundary value problem (8), (9) by use of integral transforms involves Fourier and Laplace transforming with respect to ξ_3 and τ, respectively, see e.g. Petrashen' et al. (1950). So transformed solution is essentially built of monochromatic waves along $O\xi_3$, which prevents successful matching with the inner solution at the tip of the wave front; we will discuss this in greater detail at the end of this section. In order to avoid these difficulties we adopt another approach here. First, we apply Fourier cosine transform with respect to ξ_1 to the outer governing equation (8), yielding

$$\frac{\partial^2 u_{\mathrm{out}}^{\mathrm{F}}}{\partial \tau^2} - c_3^2\frac{\partial^2 u_{\mathrm{out}}^{\mathrm{F}}}{\partial \xi_3^2} + c_1^2 k_1^2 u_{\mathrm{out}}^{\mathrm{F}} = 2M\delta(\tau)\delta(\xi_3)\,, \tag{10}$$

[2] The classical membrane equation is usually derived with the assumption that in-plane tensions supporting the membrane are equal to each other, which in our case corresponds to $c_1 = c_3$. Equation (8) is best interpreted as equation of a membrane stretched by two *distinct* mutually orthogonal tensions.

in which $u_{\text{out}}^{\text{F}} = u_{\text{out}}^{\text{F}}(k_1, \xi_3, \tau)$ is a Fourier cosine transform of $u_{\text{out}}(\xi_1, \xi_3, \tau)$. The fact that we were able to represent our governing equation in terms of single Fourier cosine transform indicates that response of our problem is essentially symmetric with respect to the plate's edge. Thus u_{out} may be extended to negative values of ξ_1 as an even function, which allows us to assume that $u_{\text{out}}^{\text{F}}$ is a (complete) Fourier transform with respect to ξ_1. A further Fourier transform with respect to ξ_3 recasts (10) into

$$\frac{\partial^2 u_{\text{out}}^{\text{FF}}}{\partial \tau^2} + \left(c_1^2 k_1^2 + c_3^2 k_3^2\right) u_{\text{out}}^{\text{FF}} = 2M\delta(\tau)\,, \tag{11}$$

where $u_{\text{out}}^{\text{FF}} = u_{\text{out}}^{\text{FF}}(k_1, k_3, \tau)$ is a Fourier transform of $u_{\text{out}}^{\text{F}}(k_1, \xi_3, \tau)$. The solution of the non-homogeneous ordinary differential equation (11) that satisfies zero initial conditions has the form

$$u_{\text{out}}^{\text{FF}}(k_1, k_3, \tau) = \frac{2M \sin\left(\tau\sqrt{c_1^2 k_1^2 + c_3^2 k_3^2}\right)}{\sqrt{c_1^2 k_1^2 + c_3^2 k_3^2}}\,, \quad \tau > 0\,. \tag{12}$$

We therefore obtained double Fourier transform of exact solution of the outer problem. Since we only used non-refined governing equation (8), the $O(\eta)$ error in boundary condition can not affect the behaviour of our (leading order) solution. And because this solution will be used as a boundary condition for the inner problem it may be concluded that the error involved in application of the boundary condition (7) will not be asymptotically significant.

Transformed solution (12) may be inverted exactly using only the table integrals. Specifically, since function $u_{\text{out}}^{\text{FF}}$ is even with respect to k_3 the inverse Fourier transform may be performed by inspection as cosine transform, which after referring to Gradshteyn and Ryzhik (1980, p. 737) is given by

$$u_{\text{out}}^{\text{F}}(k_1, \xi_3, \tau) = \frac{M}{c_3} J_0\left(\frac{c_1}{c_3}\sqrt{c_3^2\tau^2 - \xi_3^2}\, k_1\right) H(c_3\tau - |\xi_3|)\,, \quad \tau > 0\,, \tag{13}$$

in which $J_0(\cdot)$ is a Bessel's function of the first kind and $H(\cdot)$ is a Heaviside unit step function. This result is quite reminiscent of the analysis of transient SH waves in an isotropic plate performed by Graff (1991). We mentioned in our previous discussion that $u_{\text{out}}^{\text{F}}$ may be treated as either Fourier or Fourier cosine transform. It is easier to invert solution (13) by cosine transform. The resulting integral may again be evaluated analytically, see e.g. Gradshteyn and Ryzhik (1980, p. 730), yielding

$$u_{\text{out}}(\xi_1, \xi_3, \tau) = \frac{MH\left(c_1\sqrt{c_3^2\tau^2 - \xi_3^2} - c_3\xi_1\right) H(c_3\tau - |\xi_3|)}{\pi\sqrt{c_1^2 c_3^2 \tau^2 - c_3^2\xi_1^2 - c_1^2\xi_3^2}}\,, \quad \tau > 0\,. \tag{14}$$

It is easy to check manually that this satisfies the boundary condition (9). Outer solution (14) predicts ellipse-shaped discontinuity propagating along

$O\xi_1$ and $O\xi_3$ with speeds c_1 and c_3, respectively. There are no disturbances of mid-surface before the arrival of this discontinuity.

In order to demonstrate the difficulties occurring when using a more traditional procedure we may first invert Fourier transform with respect to ξ_1 in (12). A singly transformed solution will then have the form

$$u_{\text{out}}^{\text{F}}(\xi_1, k_3, \tau) = \frac{M}{c_1} J_0 \left(\frac{c_3}{c_1} \sqrt{c_1^2 \tau^2 - \xi_1^2}\, k_3 \right) H(c_1 \tau - \xi_1)\,, \quad \tau > 0\,. \tag{15}$$

Whereas the simplicity of the outer governing equation (8) allowed us to obtain this result exactly, a typical procedure generally provides only an approximation of (15) established by the method of steepest descent or any similar approximate integration technique, see e.g. Petrashen' et al. (1950). For example, if in our case we use Fourier and Laplace transforms with respect to ξ_3 and τ, respectively, the steepest descent approximation of appropriate Mellin integral delivers

$$u_{\text{out}}^{\text{F}}(\xi_1, k_3, \tau) \sim \frac{\sqrt{2} M \sin\left(\dfrac{c_3 k_3}{c_1} \sqrt{c_1^2 \tau^2 - \xi_1^2} + \dfrac{\pi}{4} \right)}{\sqrt{\pi c_1 c_3 k_3} (c_1^2 \tau^2 - \xi_1^2)^{1/4}}\,, \tag{16}$$

which exactly matches the approximation of (15) for large values of argument of Bessel's function. However, this approximation fails in a vicinity of the tip of the wave front, because saddle points tend to infinity as $c_1^2\tau^2 - \xi_1^2 \to 0$, for instance this happens in paper by Kaplunov and Nol'de (1995) dealing with quasi-fronts in conical and cylindrical shells. Thus, when in a vicinity of the tip of the wave front, matching with inner solution is not possible because approximation of the outer solution is not valid. It is worth noting that the proposed solution (13) or (14) also has a matching difficulty, but only in a vicinity of the plate's edge, where the solution is dominated by a surface wave and a contribution of the quasi-front is probably of less importance.

4 Solutions in a vicinity of quasi-front

We expect higher-order terms of the governing equation (5) to contribute significantly only when the variation of the mid-surface deflection u is large, which happens in a vicinity of wave front predicted by the outer solution (14). It is therefore natural to study inner problem using generalised system of coordinates, motivated by the form of solutions obtained by Kaplunov and Nol'de (1992) and similar to the one introduced by Kaplunov and Nol'de (1995), see Figure 2. This coordinate system magnifies the vicinity of the wave front, which is where the boundary layer associated with a higher-order correction to the governing equations generates quasi-front. In contrast with previous studies of quasi-fronts in plates the shape of the associated hyperbolic wave front is not circular but elliptic. In Figure 2 the period of time $\tau = 1$ passed since instantaneous impulse loading was applied at the point O. The wave front travelled the distances c_1 along $O\xi_1$ and c_3 along $O\xi_3$. We choose to represent a point A in a vicinity of the wave front by $\xi^*_{\rm in}$, the distance from the wave front along $O\xi_3$, and θ, the angle between $O\xi_1$ and point where $\xi^*_{\rm in} = 0$. The sign of $\xi^*_{\rm in}$ is chosen so that positive values of $\xi^*_{\rm in}$ correspond to the area before the arrival of the wave front. The inner variables may now be introduced according to

$$\xi_1 = c_1(t_{\rm in}\cos\theta + \eta^\alpha \xi_{\rm in})\,, \quad \xi_3 = c_3 t_{\rm in}\sin\theta\,, \quad \tau = t_{\rm in}\,, \tag{17}$$

within which we assumed $\xi_{\rm in} = \xi^*_{\rm in}/\eta^\alpha$, where α is a logarithmic measure of the width of the boundary layer. Upon the substitution of the inner variables defined by (17) into the governing equation (5) it may be seen that only the value of $\alpha = 2/3$ asymptotically balances the leading order terms of the resulting equation, given to $O(\eta^{2/3})$ by

$$\frac{\partial^2 u_{\rm in}}{\partial t_{\rm in}\partial \xi_{\rm in}} + \frac{1}{2t_{\rm in}}\frac{\partial u_{\rm in}}{\partial \xi_{\rm in}} + \frac{F_{\rm in}(\theta)}{\cos^3\theta}\frac{\partial^4 u_{\rm in}}{\partial \xi_{\rm in}^4} = 0\,, \tag{18}$$

where $u_{\rm in} \equiv u_{\rm in}(\xi_{\rm in}, \theta, t_{\rm in})$ and

$$F_{\rm in}(\theta) = \frac{1}{6}\left(\frac{\mathcal{F}^{(2)}_{cc}\cos^4\theta}{c_1^4} + \frac{\mathcal{F}^{(2)}_{cs}\cos^2\theta\sin^2\theta}{c_1^2c_3^2} + \frac{\mathcal{F}^{(2)}_{ss}\sin^4\theta}{c_3^4}\right)\,.$$

We note in passing that this equation is formally equivalent to the equation obtained by Kaplunov and Nol'de (1995) for the case of cylindrical and conical shells. However, in our case function $F_{\rm in}(\theta)$ may change its sign for various angles of propagation, which will be shown to affect the type of generated quasi-front. Asymptotic equation (18) may only be considered to be leading order governing equation provided its coefficients are all $O(1)$. This is not obviously always true for the last term on the left hand side; $F_{\rm in}(\theta)$ might

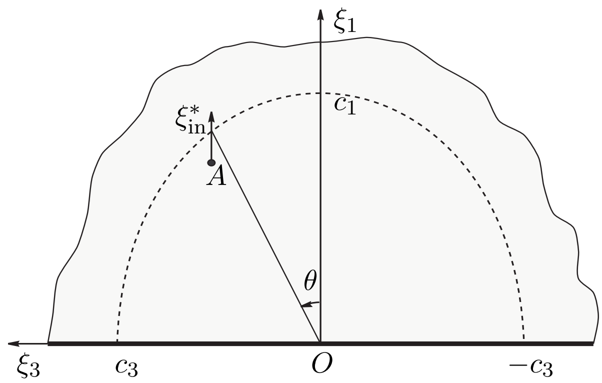

Fig. 2. Top up view of the wave front (dashed line) at the moment of time $\tau = 1$, superimposed with the system of inner coordinates.

generally be of $O(\eta^\gamma)$, $\gamma > 0$, which corresponds to the so-called case of *narrow* quasi-front briefly considered later in the paper. Conversely, $1/\cos^3\theta$ rapidly grows in a vicinity of the plate's edge that also requires addressing. Let us investigate these situations separately.

4.1 Ellipse-shaped quasi-front

Governing equation (18) is only valid provided the asymptotically secondary terms of the equation remain $o(1)$ as $\cos\theta \to 0$. The situation may be made clearer by using slightly modified system of inner variables, written as

$$\xi_1 = c_1 \left(t_{\mathrm{in}} \cos\theta + \frac{\eta^\alpha \xi_{\mathrm{in}}}{\cos\theta} \right), \quad \xi_3 = c_3 t_{\mathrm{in}} \sin\theta\,, \quad \tau = t_{\mathrm{in}}\,, \tag{19}$$

with $\alpha = 2/3$. In terms of these variables equation (18) recasts as

$$\frac{\partial^2 u_{\mathrm{in}}}{\partial t_{\mathrm{in}} \partial \xi_{\mathrm{in}}} + \frac{1}{2t_{\mathrm{in}}} \frac{\partial u_{\mathrm{in}}}{\partial \xi_{\mathrm{in}}} + F_{\mathrm{in}}(\theta) \frac{\partial^4 u_{\mathrm{in}}}{\partial \xi_{\mathrm{in}}^4} = 0\,, \tag{20}$$

in which we assume $|F_{\mathrm{in}}(\theta)| \sim 1$. The largest term that was not included into (20) is $O(\eta^{(2/3)}/\cos^2\theta)$ thus the governing equation (20) is only valid when $\cos\theta \gg \eta^{1/3}$, i.e. away from the edge of the plate. Another useful interpretation may be given by considering $(19)_1$, indeed $\xi_{\mathrm{in}}^* \equiv \eta^{2/3}\xi_{\mathrm{in}}/\cos\theta \ll t_{\mathrm{in}}\cos\theta$ when $\cos\theta \gg \eta^{1/3}$, thus for this range of angles θ the elliptic shape of the wave front is not distorted to the leading order. We therefore term this case as solution for an ellipse-shaped quasi-front.

Solution of (20) may be obtained by the method of integral transforms. Fourier transform with respect to ξ_{in} gives the governing equation in the form

$$\frac{\partial u_{\mathrm{in}}^{\mathrm{F}}}{\partial t_{\mathrm{in}}} + \frac{u_{\mathrm{in}}^{\mathrm{F}}}{2t_{\mathrm{in}}} - \mathrm{i}k_{\mathrm{in}}^3 F_{\mathrm{in}}(\theta)\, u_{\mathrm{in}}^{\mathrm{F}} = 0\,, \tag{21}$$

in which $u_{\mathrm{in}}^{\mathrm{F}} = u_{\mathrm{in}}^{\mathrm{F}}(k_{\mathrm{in}}, \theta, t_{\mathrm{in}})$. The general solution of this equation is

$$u_{\mathrm{in}}^{\mathrm{F}}(k_{\mathrm{in}}, \theta, t_{\mathrm{in}}) = \frac{C_{\mathrm{in}}(k_{\mathrm{in}}, \theta)}{\sqrt{t_{\mathrm{in}}}} \exp\left(\mathrm{i}F_{\mathrm{in}}(\theta) t_{\mathrm{in}} k_{\mathrm{in}}^3 \right)\,, \tag{22}$$

where unknown function $C_{\mathrm{in}}(k_{\mathrm{in}}, \theta)$ must be determined from the outer solution (13) by an asymptotic matching procedure. The standard matching procedure requires that the inner limit of the outer solution must be equal to the outer limit of the inner solution. The outer solution (13) may be represented in terms of the inner variables (19) by the following Fourier integral

$$u_{\text{out}}(\xi_{\text{in}},\theta,t_{\text{in}}) = \frac{M\cos\theta}{2\pi c_1 c_3 \eta^{2/3}} \int_{-\infty}^{\infty} J_0\left(\frac{k_{\text{in}} t_{\text{in}} \cos^2\theta}{\eta^{2/3}}\right) \exp\left(\frac{\mathrm{i} k_{\text{in}} t_{\text{in}} \cos^2\theta}{\eta^{2/3}}\right) e^{\,\mathrm{i}k_{\text{in}}\xi_{\text{in}}}\, dk_{\text{in}}\,. \quad (23)$$

For the considered range of angles θ, $\cos\theta \gg \eta^{1/3}$ thus the argument of the Bessel's function in (23) grows large when $\eta \to 0$. We therefore use leading order approximation of Bessel's function $J_0(\cdot)$ for a large argument, see e.g. Gradshteyn and Ryzhik (1980, p. 961). Asymptotically negligible contributions of rapidly oscillating terms within the integral (23) may then be ignored, yielding the inner limit of the outer solution

$$u_{\text{out}}(\xi_{\text{in}},\theta,t_{\text{in}}) \sim \frac{M\eta^{-1/3}}{4\pi^{(3/2)} c_1 c_3 \sqrt{t_{\text{in}}}} \int_{-\infty}^{\infty} \frac{(1+\mathrm{i})}{\sqrt{k_{\text{in}}}}\, e^{\mathrm{i}k_{\text{in}}\xi_{\text{in}}}\, dk_{\text{in}}\,, \quad \eta \to 0\,. \quad (24)$$

By writing the general solution for the inner problem (22) in terms of inverse Fourier integral and transforming it to the outer variables we may obtain the outer limit of the inner solution in the form

$$u_{\text{in}}(\xi_1,\theta,\tau) \sim \frac{c_3\eta^{2/3}}{2\pi\sqrt{\tau}\cos\theta} \int_{-\infty}^{\infty} C_{\text{in}}(k_1,\theta)\, e^{-\,\mathrm{i}c_1\tau k_1\cos\theta}\, e^{\,\mathrm{i}k_1\xi_1}\, dk_1\,, \quad \eta \to 0\,, \quad (25)$$

where we kept in mind that θ is a *slow* variable that does not vary as $\eta \to 0$, i.e. as we approach the wave front. Limits (24) and (25) can be compared after the outer limit of the inner solution (25) is rewritten in terms of inner variables. The matching procedure yields then the unknown function

$$C_{\text{in}}(k_{\text{in}},\theta) = \frac{M(1+\mathrm{i})\eta^{-1/3}}{2\sqrt{\pi} c_1 c_3 \sqrt{k_{\text{in}}}}\,. \quad (26)$$

Therefore the inner solution (22) may now be formulated as

$$u_{\text{in}}(\xi_{\text{in}},\theta,t_{\text{in}}) = \frac{M\eta^{-1/3}}{4\sqrt{\pi^3 t_{\text{in}}} c_1 c_3} \int_{-\infty}^{\infty} \frac{(1+\mathrm{i})}{\sqrt{k_{\text{in}}}} \exp\left(\mathrm{i}F_{\text{in}}(\theta) t_{\text{in}} k_{\text{in}}^3\right) e^{\mathrm{i}k_{\text{in}}\xi_{\text{in}}}\, dk_{\text{in}}\,. \quad (27)$$

For the computational purposes it is convenient to represent this solution in terms of outer variables. Making use of the symmetry of $u_{\text{in}}^{\text{F}}(k_{\text{in}},\theta,t_{\text{in}})$ with respect to k_{in} we may also represent the inverse Fourier transform as an integral, which in terms of outer variables has the form

$$u_{\text{in}}(\xi_{\text{in}}^*,\theta,\tau) = \frac{M}{\sqrt{2\pi^3} c_1 c_3 \sqrt{\tau\cos\theta}}\, \Phi\left(\xi_{\text{in}}^*, \frac{F_{\text{in}}(\theta)\,\tau\eta^2}{6\cos^3\theta}\right)\,, \quad (28)$$

in which we introduced the following special function

$$\Phi(x,\varepsilon) = \int_0^{\infty} \cos\left(xk + \varepsilon k^3 + \frac{\pi}{4}\right) \frac{dk}{\sqrt{k}} \equiv 2\int_0^{\infty} \cos\left(xt^2 + \varepsilon t^6 + \frac{\pi}{4}\right) dt\,. \quad (29)$$

Function $\Phi(x, \varepsilon)$ was previously obtained in a slightly different form by Kaplunov and Nol'de (1995). We will investigate some of its properties in Section 5. The inner solution (28) is only valid provided $\cos\theta \gg \eta^{1/3}$. It is possible to use the same coordinate system as Kaplunov and Nol'de (1995) or Emri et al. (2001) to construct a solution valid for $\sin\theta \gg \eta^{1/3}$, however this is not of great use since any solution near the plate's egde is dominated by a surface wave and a contribution of the quasi-front is not as important. Thus, for most practical uses the approximation derived in this section may be considered uniform.

4.2 Narrow quasi-front

We may expect that for certain combinations of pre-stress, function of the material parameters and angle of propagation $F_{\text{in}}(\theta)$ vanishes. This phenomenon must not be associated with propagation of discontinuity, although it appears that our correction is not able to smooth the hyperbolic wave front in this case. Vanishing of $F_{\text{in}}(\theta)$ only indicates that our $O(\eta^4)$ plate theory is not valid for the associated angles and a yet more refined theory valid to $O(\eta^6)$ is necessary to describe the boundary layers in a vicinity of the wave front. The derivation of such a theory would involve very complicated algebra and is hardly worth the trouble. Fortunately, it is possible to determine qualitative properties of the solution without actually deriving a more accurate model. Indeed, an $O(\eta^4)$ correction to the governing equation (5) will consist of a linear combination of sixth-order space derivatives. Suppose now that $F_{\text{in}}(\theta) \sim O(\eta^\gamma)$ and let us consider the governing equation in terms of the inner coordinates (19). The appropriate leading order terms may only be balanced when $\alpha = 4/5$ and for any $\gamma \geqslant 2/5$ the governing equation may be written to $O(\eta^{4/5})$ as

$$\frac{\partial^2 u_{\text{in}}}{\partial t_{\text{in}} \partial \xi_{\text{in}}} + \frac{1}{2t_{\text{in}}} \frac{\partial u_{\text{in}}}{\partial \xi_{\text{in}}} + \eta^{-2/5} F_{\text{in}}(\theta) \frac{\partial^4 u_{\text{in}}}{\partial \xi_{\text{in}}^4} - G_{\text{in}}(\theta) \frac{\partial^6 u_{\text{in}}}{\partial \xi_{\text{in}}^6} = 0\,, \qquad (30)$$

where $G_{\text{in}}(\theta)$ is a function of coefficients of the refined governing equation valid to $O(\eta^6)$. Simulataneous vanishing of both $F_{\text{in}}(\theta)$ and $G_{\text{in}}(\theta)$ is quite unrealistic, thus we assume in the following that $G_{\text{in}}(\theta)$ is $O(1)$. The width of the quasi-front in this case is $\eta^{4/5} \ll \eta^{2/3}$, which justifies the notion of *a narrow quasi-front* that we introduced.

This inner problem may be solved in a very similar manner as the previous one. After using Fourier transform with respect to ξ_{in} and subsequent matching with the outer solution, the resulting solution in a vicinity of a narrow quasi-front may be given in terms of outer variables by

$$u_{\text{in}}(\xi^*_{\text{in}}, \theta, \tau) = \frac{M}{\sqrt{2\pi^3 \tau \cos\theta}\, c_1 c_3}\, \Phi_n\!\left(\xi^*_{\text{in}}, \frac{F_{\text{in}}(\theta)\, \tau \eta^{8/5}}{\cos^3\theta}, \frac{G_{\text{in}}(\theta)\, \tau \eta^4}{\cos^5\theta}\right), \qquad (31)$$

in which

$$\Phi_n(x, \varepsilon_1, \varepsilon_2) = 2\int_0^\infty \cos\left(xt^2 + \varepsilon_1 t^6 + \varepsilon_2 t^8 + \frac{\pi}{4}\right) dt\,, \tag{32}$$

where we represented function $\Phi_n(x, \varepsilon_1, \varepsilon_2)$ in a form that highlights its relation to the special function $\Phi(x, \varepsilon)$ introduced in (29).

5 Analysis of the solutions

Since elastic membrane stretched by mutually orthogonal tensions T_1 and T_3 is the most obvious object modelled by the governing equation (1), for numerical examples we choose pre-stress in such way that the in-plane Cauchy stresses $\sigma_1 = T_1$, $\sigma_3 = T_3$ and the normal stress $\sigma_2 = 0$. It may easily be shown, see Pichugin and Rogerson (2002), that this corresponds to

$$c_1^2 = T_1\,, \quad c_3^2 = T_3\,. \tag{33}$$

Numerical examples in this section were generated in respect of a Varga material with the strain energy function $W_0 = \mu(\lambda_1 + \lambda_2 + \lambda_3 - 3)$, where μ is a shear modulus and λ_i, $i \in \{1, 2, 3\}$, are the principal stretches of static deformation, $\lambda_2 = 1/\lambda_1\lambda_3$. Non-zero components of the elasticity tensor $\boldsymbol{B}$ are given in this case by

$$B_{ijij} = \frac{\mu\lambda_i^2}{\lambda_i + \lambda_j}\,, \quad B_{ijji} = -\frac{\mu\lambda_i\lambda_j}{\lambda_i + \lambda_j}\,, \quad i \neq j\,, \qquad i, j \in \{1, 2, 3\}\,, \tag{34}$$

which allows us to determine all of the parameters introduced in Appendix A. We also have to ensure incremental stability [3] of the used pre-stress configurations, which for a Varga material necessarily means satisfying the inequality

$$1 - 2\frac{\lambda_1\cos^2\theta + \lambda_3\sin^2\theta}{\lambda_1\cos^2\theta + \lambda_3\sin^2\theta + \lambda_2} < \frac{\sigma_2}{\mu\lambda_2} < 1\,, \tag{35}$$

for all values of θ, see Pichugin (2002).

It is hardly possible to see much detail in three-dimensional plots of quasi-fronts. We opted for using diagrams, which show a magnified vicinity of the

[3] Typically, in addition to *local* stability conditions, such as strong ellipticity, extra *global* stability conditions have to be introduced to ensure dynamic stability of elastic structures. Inequality (35) is a *necessary* stability condition for a pre-stressed incompressible plate, obtained by the direct analysis of the bifurcation equation in the spirit of Ogden and Roxburgh (1993) (upper bound in (35) was also shown to be *sufficient* analytically, while numerical investigation strongly suggests that the lower bound must also be sufficient).

hyperbolic wave fronts. On these diagrams we depict quasi-fronts corresponding to various pre-stresses assuming $\mu = 8.0$, $\rho = 1.0$, $h = 1.0$, $\eta = 0.01$, $\tau = 5.0$ and $M = 100.0$.

5.1 *Quasi-fronts and the function* $\Phi(x, \varepsilon)$

Let us study behaviour of the special function $\Phi(x, \varepsilon)$ defined by (29) in greater detail. Firstly we note that it may be represented in form of the series

$$\Phi(x,\varepsilon) = \frac{1}{3}\sum_{m=0}^{\infty} \frac{(-x\,\mathrm{sgn}\,\varepsilon)^m}{m!\,|\varepsilon|^{(2m+1)/6}}\,\Gamma\Big(\frac{2m+1}{6}\Big)\,\sin\Big(\frac{\pi}{4}+\frac{\pi}{12}(4m-1)\,\mathrm{sgn}\,\varepsilon\Big)\,, \tag{36}$$

motivated by De Morgan (1842, p. 649). It may be shown that this series converges absolutely for all x and $\varepsilon \neq 0$. Secondly, we remark that this function fully describes outer solution in a vicinity of quasi-front and may be considered as a perturbation of it. Indeed,

$$\Phi(x,0) \equiv 2\int_0^{\infty} \cos\left(xt^2 + \frac{\pi}{4}\right) dt = \begin{cases} \sqrt{\dfrac{\pi}{|x|}} & x \leqslant 0\,, \\ 0 & x > 0\,, \end{cases} \tag{37}$$

see Gradshteyn and Ryzhik (1980, p. 399). In view of (37) it is clear that when $\eta \to 0$, (28) degenerates into a wave front approximation for the outer solution (14). Infinitely long waves dominate this solution, associated with the only stationary point $t = 0$ of the integral (37). The main effect of the small perturbation ε in the integral (29) is connected with introduction of the second real non-zero stationary point, whose contribution smoothes the discontinuity displayed by (37) to produce a quasi-front. The type of generated quasi-front depends on the sign of ε and we will study both possible situations separately.

5.1.1 $\varepsilon > 0$*: a receding quasi-front*

When $\varepsilon \neq 0$, a non-zero stationary point perturbes the value of the integral (37). The method of stationary phase may then be used to evaluate the special function $\Phi(x, \varepsilon)$. By extracting a large parameter

$$\lambda = \frac{2}{\sqrt{|\varepsilon|}}\left(\frac{|x|}{3}\right)^{3/2}, \tag{38}$$

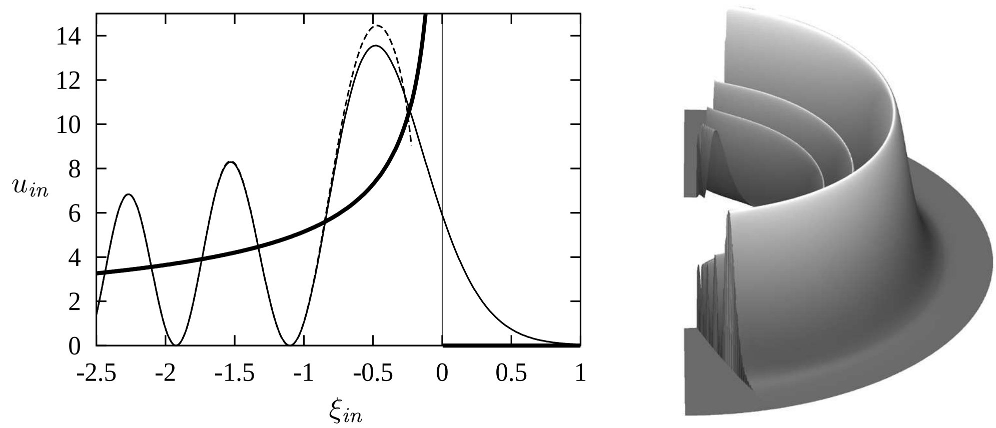

Fig. 3. Receding quasi-front in a membrane stretched by $\lambda_1 = 1.6$ and $\lambda_3 = 1.5$. The left hand plot illustrates the correspondence between outer (thick), exact (thin) and approximate (dashed) solutions for $\theta = 0$.

the stationary phase approximation for $\Phi(x, \varepsilon)$ may be written as

$$\Phi(x,\varepsilon) = \begin{cases} \sqrt{\dfrac{\pi}{|x|}}\left(1 + \sin\lambda - \dfrac{5}{18\lambda}\cos\lambda\right) + O(\lambda^{-2}) & x \ll -\sqrt[3]{|\varepsilon|}\,, \\ 0 & x \gg \sqrt[3]{|\varepsilon|}\,. \end{cases} \tag{39}$$

The first term in the expansion $(39)_1$ is associated with the outer solution, whereas other two account for the contribution of the non-zero stationary point. We compare (38) with (28) and (39) to conclude that approximations (39) may be used to describe the solutions for which $\xi^*_{\rm in} \ll -\eta^{2/3}$ or $\xi^*_{\rm in} \gg \eta^{2/3}$. Moreover, the absolute value of the non-zero stationary point may serve as a squared measure of the wave number dominating the solution in a vicinity of the wave front, so that the smoothing of the wave front is performed by waves with wave numbers $k \sim \sqrt{\xi^*_{\rm in}}/\eta$. Keeping in mind that our plate theory is only valid for $k \ll \eta^{-4/5}$ we infer that our solution (28) is only valid for $\xi^*_{\rm in} \ll -\eta^{2/5}$ (or $\xi_{\rm in} \ll -\eta^{-4/15}$, see (19)).

Since for $\varepsilon > 0$ the non-zero stationary point is only real for negative values of $\xi^*_{\rm in}$ we may say that it perturbes the outer solution *behind* the wave front. The quasi-front obtained by such perturbation is well-known and is usually referred to as a *receding* quasi-front. The left-hand plot in Figure 3 depicts a typical configuration of the receding wave front together with its stationary phase approximation (39) and the outer solution (14). The three-dimensional diagramme in Figure 3 illustrates a (magnified) disturbance in a vicinity of the wave front at the specified time instant. The plots indicate that the disturbance is exponentially small before and strongly oscillating after the arrival of the wave front. It is important to remark that the absolute maximum of quasi-front's amplitude is delayed in comparison with arrival of the wave front predicted by the classical solution for a membrane.

5.1.2 $\varepsilon < 0$: an advancing quasi-front

Stationary phase approximations may also be obtained for $\Phi(x, \varepsilon)$ when $\varepsilon < 0$. In terms of the same large parameter λ, defined in (38), the resulting behaviour is given by

$$\Phi(x,\varepsilon) = \begin{cases} \sqrt{\dfrac{\pi}{|x|}} & x \ll -\sqrt[3]{|\varepsilon|}\,, \\ \sqrt{\dfrac{\pi}{|x|}}\left(\cos\lambda + \dfrac{5}{18\lambda}\sin\lambda\right) + O(\lambda^{-2}) & x \gg \sqrt[3]{|\varepsilon|}\,. \end{cases} \tag{40}$$

Using a similar reasoning as in the previous section we may conclude that asymptotics (40) are only valid when $\xi^*_{\text{in}} \ll -\eta^{2/3}$ or $\xi^*_{\text{in}} \gg \eta^{2/3}$ and the solution (28) may only be used when $\xi^*_{\text{in}} \gg \eta^{2/5}$ (or $\xi_{\text{in}} \gg \eta^{-4/15}$).

When $\varepsilon < 0$ the non-zero stationary point is only real for positive values of ξ^*_{in}, therefore it perturbes the outer solution *before* arrival of the wave front. This type of quasi-front is usually referred to as an *advancing* quasi-front. Figure 4 demonstrates a typical wave profile for this type of quasi-front. This

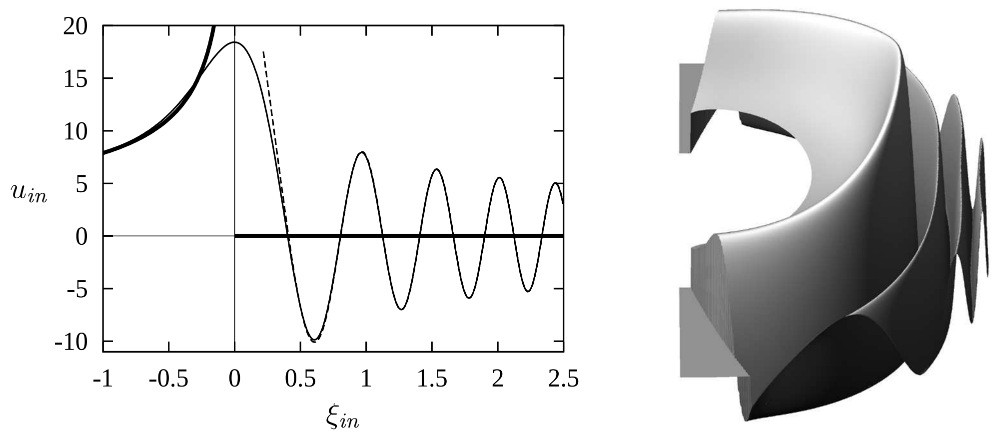

Fig. 4. Advancing quasi-front in a membrane stretched by $\lambda_1 = 1.6$ and $\lambda_3 = 1.1$. The left hand plot illustrates the correspondence between outer (thick), exact (thin) and approximate (dashed) solutions for $\theta = 0$.

quasi-front features oscillations before the arrival of the wave front, and after the arrival it rapidly converges to the outer solution. Such quasi-fronts were known to exist in non-isotropic media, see Kaplunov et al. (2000), where they were obtained for a case of the Heaviside unit step function loading. However, the associated response represented in terms of an integral of the Airy function Ai$(\cdot)$ did not indicate an important feature of an advancing quasi-front. It is apparent from the plot in Figure 4 that the maximum peak of an advancing quasi-front *is not* delayed in comparison with arrival of the front predicted by the membrane equation. In other words, the non-refined membrane theory does not distort the speed of the receding wave front. This may be explained by the fact that the non-zero stationary point only perturbes the disturbance

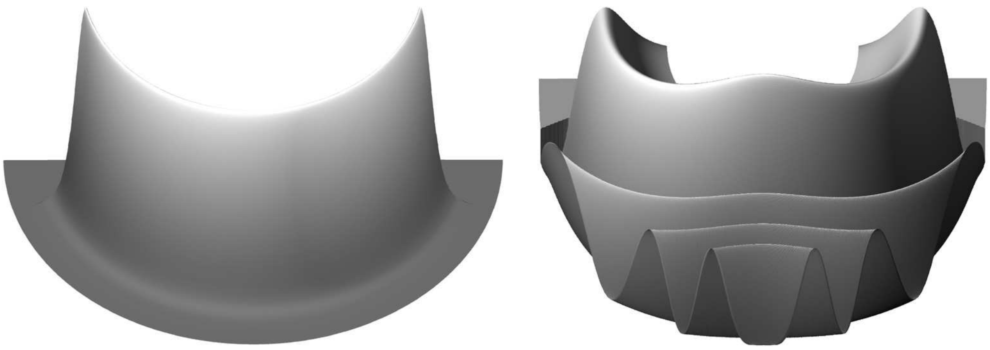

Fig. 5. Variation of amplitude along quasi-fronts. The membrane deformation is the same as in Figure 3 (left hand diagram) and Figure 4 (right hand diagram).

before the arrival of the wave front and therefore does not seriously affect the outer solution. It can easily be verified that $\Phi(x,\varepsilon)$ reaches a local maximum when $\xi^*_{\text{in}} = 0$ and $\varepsilon < 0$, and the amplitude of the disturbance is given by

$$\Phi(0,\varepsilon) \equiv 2\int_0^\infty \cos\left(\varepsilon t^6 + \frac{\pi}{4}\right) dt = \frac{\Gamma\left(\dfrac{1}{6}\right)}{2\sqrt{3}} \, |\varepsilon|^{-1/6} \;, \quad \varepsilon < 0 \,, \tag{41}$$

where $\Gamma(\cdot)$ is the gamma function. Thus, the peak amplitude of the receding wave front may be estimated to be $O(\eta^{-1})$, see (28).

Another point worth noticing is the apparent variation of the advancing quasi-front amplitude along the quasi-front, which was not observed for a receding quasi-front. The situation is further illustrated in Figure 5. By referring to the solution (28) and using (41) we conclude, that since the phase of the maximum peak of the advancing quasi-front does not depend on the angle θ, the variation of the amplitude along the quasi-front is solely determined by the factor $|F_{\text{in}}(\theta)|^{-1/6}$.

5.2 A narrow quasi-front

It was previously remarked that the response may feature both types of quasi-front at the same time, Figure 6 illustrates this situation. The most interesting aspect of these diagrams is the transition area between the receding and advanicing quasi-fronts. Since the type of the quasi-front is specified by the sign of ε, (28) indicates that $F_{\text{in}}(\theta)$ is the function that determines the type of the quasi-front for a particular angle θ. Therefore, change of type of the quasi-front is always associated with vanishing of $F_{\text{in}}(\theta)$ and generation of a narrow quasi-front. Since we only derived a qualitative solution for narrow quasi-fronts, the relevant response in corresponding areas of Figure 6 is not computed, which is the reason for gaps in the diagrams. However, the processes in a vicinity of a narrow quasi-front are quite easy to observe. Both diagrams in Figure 6

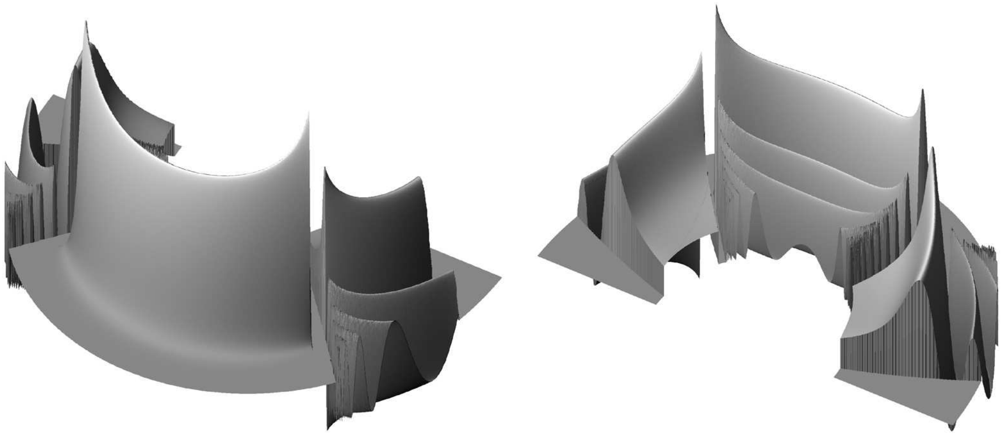

Fig. 6. A response of a stretched membrane featuring both receding and advancing quasi-fronts, $\lambda_1 = 1.6$ and $\lambda_3 = 1.3$.

demonstrate decrease in a relevant wave length in a vicinity of narrow quasi-fronts, which corresponds to the change of quasi-front's width mentioned earlier. Amplitude of the disturbance also seems to increase in a vicinity of a narrow quasi-front. The integral in function $\Phi_n(x, \varepsilon_1, \varepsilon_2)$, describing a narrow quasi-front, has three non-zero stationary points, which makes general analysis of it difficult. Nevertheless we still can determine the amplitude of $\Phi_n(0, 0, \varepsilon_2)$, which may serve as asymptotic indication of the actual amplitude of narrow quasi-front. Using (28) and (32) we observe that $\Phi_n(0, 0, \varepsilon_2) \sim \eta^{-1/2}$. This is certainly larger than a typical amplitude of a quasi-front described by $\Phi(x, \varepsilon)$, which may be estimated from the leading order term of the series (36) as $\Phi(0, \varepsilon) \sim \eta^{-1/3}$.

6 Conclusions

This investigation establishes the existence as well as essential characteristics of bending quasi-fronts in thin pre-stressed incompressible plates. It is shown that two types of generated responses are possible, namely receding (classical) and advancing quasi-fronts. The discontinuity predicted by the non-refined membrane-like theory is smoothed by both types of quasi-fronts, moreover the speed of a classical membrane wave front is shown to be distorted in comparison with the speed of a receding quasi-front. Some pre-stress configurations may feature both types of quasi-front propagating in different directions. This phenomenon is nessasarily accompanied by the so-called narrow quasi-front, situated in the areas of transition between receding and advancing quasi-fronts and characterised by a sharp increase of the disturbance's amplitude.

Our solution for the outer problem also indicated certain limitations of the traditional outer solutions obtained by the method of steepest descent, which are usually not valid in a vicinity of the tip of the wave front. The method of solving the outer problem implemented in this paper may be used to improve matching in previously solved problems that possess the necessary symmetry with respect to the plate edge. The success of our approach must be attributed to the non-traditional application of integral transforms with respect to the coordinate normal to the plate edge.

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A Modelling of pre-stressed incompressible media

The linearised equations of incremental motion in pre-stressed incompressible media may be written in the component form as

$$B_{milk} u_{k,lm} - p_{t,i} = \rho \ddot{u}_i \,, \tag{A.1}$$

in which we assume summation over repeated suffices, comma and overdot denote differentiation with respect to the indicated space and time variable, $\boldsymbol{u}$ is an infinitesimal displacement and p_t is a dynamic component of 'pressure' $p \equiv \bar{p} + p_t$, a reaction stress necessary to accomodate the incompressibility constraint. $\boldsymbol{B}$ is the fourth order elasticity tensor, components of which are evaluated in a statically deformed configuration $\mathcal{B}_e$ and defined by

$$B_{milk} = \bar{F}_{mA} \bar{F}_{lB} \left. \frac{\partial^2 W_0}{\partial F_{kB}\, \partial F_{iA}} \right|_{\mathcal{B}_e} ,$$

where W_0 is a strain energy function and $\bar{\boldsymbol{F}}$ is the deformation gradient tensor in $\mathcal{B}_e$. Equations (A.1) must be solved subject to linearised incompressibility condition $\operatorname{div} \boldsymbol{u} = 0$. The detailed derivation of these equations as well as representation for the components of $\boldsymbol{B}$ in terms of principal stretches λ_i, $i \in \{1, 2, 3\}$, of the initial static deformation may be found in Ogden (1984). It is sufficint for us to note that in the case of homogeneous finite deformation of initially isotropic body, the only non-zero components of $\boldsymbol{B}$ in a principal system of coordinates have the form B_{iijj}, B_{ijij} or B_{ijji}, $i, j \in \{1, 2, 3\}$. It is convenient to introduce the following parameters

$$\gamma_{ij} = B_{ijij} \,, \quad 2\beta_{ij} = 2\beta_{ji} = B_{iiii} + B_{jjjj} - 2B_{iijj} - 2B_{ijji} \,, \quad i \neq j \,,$$

$i, j \in \{1, 2, 3\}$. Coefficients of the governing equation (1) may then be represented in terms of these parameters as follows

$$\rho\hat{c}_1^2 = \gamma_{12} - g_1^2/\gamma_{21}\,, \qquad \rho\hat{c}_3^2 = \gamma_{32} - g_3^2/\gamma_{23}\,,$$
$$\rho\hat{\mathcal{F}}_{cc}^{(2)} = \frac{(\rho\hat{c}_1^2 - 2\beta_{12} - 2g_1)g_1^2}{\gamma_{21}^2}\,, \qquad \rho\hat{\mathcal{F}}_{ss}^{(2)} = \frac{(\rho\hat{c}_3^2 - 2\beta_{23} - 2g_3)g_3^2}{\gamma_{23}^2}\,,$$
$$\rho\hat{\mathcal{F}}_{cs}^{(2)} = \frac{(\rho\hat{c}_1^2 - \gamma_{13})g_3^2}{\gamma_{23}^2} - \frac{2g_1g_3(g_1 + g_3 - \beta_{13} + \beta_{12} + \beta_{23})}{\gamma_{21}\gamma_{23}} + \frac{(\rho\hat{c}_3^2 - \gamma_{31})g_1^2}{\gamma_{21}^2}\,,$$

within which $g_i = \gamma_{2i} - \sigma_2$, where σ_2 is a normal Cauchy stress in a statically deformed configuration.

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