PT-symmetry, indefinite metric, and nonlinear quantum mechanics

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If a Hamiltonian of a quantum system is symmetric under space-time reflection, then the associated eigenvalues can be real. A conjugation operation for quantum states can then be defined in terms of space-time reflection, but the resulting Hilbert space inner product is not positive definite and gives rise to an interpretational difficulty. One way of resolving this difficulty is to introduce a *superselection rule* that excludes quantum states having negative norms. It is shown here that a quantum theory arising in this way gives an example of Kibble's nonlinear quantum mechanics, with the property that the state space has a constant negative curvature. It then follows from the positive curvature theorem that the resulting quantum theory is not physically viable. This conclusion also has implications to other quantum theories obtained from the imposition of analogous superselection rules.

I. INTRODUCTION

Over the past decade there has been a revival of interests in discrete symmetries of quantum mechanics arising in a variety of contexts [11, 14, 20, 24, 27]. One of the initiatives for the interests in discrete symmetries emerged out of the observation that complex Hamiltonians that possess space-time reflection (parity-time reversal) symmetry may have entirely real eigenvalues [3]. When the relation between the reality of the spectrum and PT symmetry was first observed, it was not immediately apparent whether a non-Hermitian Hamiltonian having such a symmetry describes a consistent quantum theory, because the associated Hilbert space (the metric space defined in terms of the PT conjugation) possesses indefinite inner product of the Pontryagin or the Kreĭn type [18, 23]. In other words, the probabilistic interpretation of the associated quantum theory was not apparent.

This issue was subsequently resolved by the introduction of a new symmetry [4, 21], denoted $\mathcal{C}$, that has an interpretation of a charge operator in the sense that the eigenvalues of this operator is $+1$ or -1, depending on the parity type of the corresponding eigenstate. Since $\mathcal{C}$ commutes with the Hamiltonian, one can define the Hilbert space inner product in terms of the $\mathcal{CPT}$ conjugation. In the Hilbert space thus constructed, the Hamiltonian operator enjoys self-adjointness. As a consequence, a consistent unitary theory of quantum mechanics is restored.

All of the above is by now well known, however, what is less known is the answer to the question "What would have happened had we used the indefinite PT-inner product and tried to formulate a quantum theory that way?" In this paper we address the possibility of formulating an alternative approach to define a quantum theory out of a PT-symmetric Hamiltonian. Specifically, we consider what happens if we introduce a superselection rule that truncates part of the state space associated with quantum states that do not have positive norm. We find that quantum theory obtained in this manner forms an example of Kibble's nonlinear quantum mechanics, that is, quantum mechanics defined on a nonlinear state-space manifold. This manifold, however, possesses negative curvature, leading to what one might call a hyperbolic quantum theory.

The paper is organised as follows. In Section II we consider an example of a PT-symmetric two-state system and show how indefinite metric arises from conjugation operation defined by space-time reflection. We work out this example explicitly because it suffices to consider just the two-state system in order to illustrate the main result of the paper. In Section III we review the idea of quantum state space geometry and explain how it can be generalised into the nonlinear domain. In Section IV we derive the metric of the quantum state space obtained by imposing the superselection rule, and show that the associated state-space curvature is negative. In Section V we offer one possible characterisation of measurements in hyperbolic quantum mechanics, which nonetheless leaves us with some interpretational issues. We conclude in Section VI with the observation, by evoking the positive curvature theorem of Brody and Hughston for the state-space manifold [8], that such an alternative formulation, although interesting mathematically, must be ruled out.

II. PT-SYMMETRIC SPIN-$\frac{1}{2}$ PARTICLE

Consider a quantum system described by a two-dimensional complex Hamiltonian

$$H = \begin{pmatrix} r\mathrm{e}^{\mathrm{i}\theta} & s \\ t & r\mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}. \tag{1}$$

We have in mind here a Hamiltonian for a PT-symmetric spin-$\frac{1}{2}$ particle system. It is now well understood that although the Hamiltonian (1) is not Hermitian in the conventional sense, it is nevertheless symmetric under space-time reflection, if we define the unitary parity reflection $\mathcal{P}$ by

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2}$$

and the antiunitary time reversal $\mathcal{T}$ by a Hermitian conjugation. Thus the energy eigenvalues

$$E_{\pm} = r\cos\theta \pm \sqrt{st - r^2\sin^2\theta} \tag{3}$$

are real and nondegenerate, provided that the parameters in the Hamiltonian belong to the region determined by $st > r^2\sin^2\theta$. We demand that this inequality be satisfied so that PT symmetry is respected [4].

When the energy eigenvalues are real, the eigenvectors $|E_{\pm}\rangle$ of the Hamiltonian H are simultaneously eigenstates of the $\mathcal{PT}$ operator. We choose the overall phases of the eigenvectors so that their eigenvalues under $\mathcal{PT}$ are all unity. With respect to this choice of phases the eigenvectors are given by

$$|E_+\rangle = \frac{1}{\sqrt{2\sqrt{st}\cos\alpha}} \begin{pmatrix} \sqrt{s}\,\mathrm{e}^{\mathrm{i}\alpha/2} \\ \sqrt{t}\,\mathrm{e}^{-\mathrm{i}\alpha/2} \end{pmatrix}, \quad |E_-\rangle = \frac{\mathrm{i}}{\sqrt{2\sqrt{st}\cos\alpha}} \begin{pmatrix} \sqrt{s}\,\mathrm{e}^{-\mathrm{i}\alpha/2} \\ -\sqrt{t}\,\mathrm{e}^{\mathrm{i}\alpha/2} \end{pmatrix}. \tag{4}$$

Here we have set $\sin\alpha = (r/\sqrt{st})\sin\theta$, and the inequality $st > r^2\sin^2\theta$ for the reality of $E_{\pm}$ ensures that α is real and that both st and $\cos\alpha$ are positive.

We now examine properties of the PT-inner product. To this end we recall that in conventional Hermitian quantum mechanics the norm of a vector in a finite-dimensional Hilbert space is defined in terms of a Hermitian inner product, which has the form $\langle u|v\rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2$ in two dimensions. Thus, the norm $\langle v|v\rangle$ of a vector is positive definite. On the other hand, the PT-inner product is determined by the PT conjugation operation so that we have $\langle u\|v\rangle = \bar{u}_2 v_1 + \bar{u}_1 v_2$ in two dimensions. Just as in the case of the Hermitian norm, the PT norm $\langle v\|v\rangle$ is also independent of overall phase. However, with respect to the PT-inner product we have an indefinite norm given by $\langle E_+\|E_+\rangle = +1$ and $\langle E_-\|E_-\rangle = -1$, as well as the orthogonality conditions $\langle E_-\|E_+\rangle = \langle E_+\|E_-\rangle = 0$. These identities can easily be verified by use of (4).

As in the case of a Hermitian norm, because the PT norm is invariant under a change of overall complex phase of the vector, we can consider a space of rays through the origin of the Hilbert space, having the inner product structure of $(+,-)$. The result is the projective Hilbert space—also known as a complex projective space. In the present case of a two-state system the projective Hilbert space is just the complex projective line $\mathbb{CP}^1$, which in real terms is the surface S^2 of a sphere in three-dimensional space. We identify the north and south poles of this sphere with the eigenstates $|E_+\rangle$ and $|E_-\rangle$, respectively. A generic state in this quantum system is expressed as a linear superposition of $|E_\pm\rangle$. However, because the Hilbert space has an indefinite metric some of these states have positive norm while others have negative norm. With respect to our choice of poles, the states having positive unit norm lie in the northern hemisphere and can be represented in the form

$$|n\rangle = \cosh x|E_+\rangle + \mathrm{e}^{\mathrm{i}\phi}\sinh x|E_-\rangle. \tag{5}$$

Similarly, the states having negative norm lie in the southern hemisphere and are expressible in the form

$$|s\rangle = \sinh x|E_+\rangle + \mathrm{e}^{\mathrm{i}\phi}\cosh x|E_-\rangle. \tag{6}$$

Here, $0 \le x < \infty$ and $0 \le \phi < 2\pi$. The north and south poles correspond to $x = 0$ in (5) and (6), respectively. In the northern hemisphere of the state space $\mathbb{CP}^1 \sim S^2$ the states have positive unit norm:

$$\langle n\|n\rangle = \cosh^2 x - \sinh^2 x = 1. \tag{7}$$

On the other hand, in the southern hemisphere the states have negative unit norm:

$$\langle s\|s\rangle = \sinh^2 x - \cosh^2 x = -1. \tag{8}$$

The two hemispheres of S^2 are joined at the equator, where the states are expressible in the form $\frac{1}{\sqrt{2}}(|E_+\rangle + \mathrm{e}^{\mathrm{i}\phi}|E_-\rangle)$ and have vanishing norm. These null states on the equator are obtained by taking the limit $x \to \infty$ in $|n\rangle$ or $|s\rangle$.

III. STATE-SPACE GEOMETRY AND NONLINEAR QUANTUM MECHANICS

It is evident from the preceding discussion that if we endow the Hilbert space of states spanned by the eigenvectors of a PT-symmetric Hamiltonian with an inner product determined by a PT conjugation, then we obtain a metric with indefinite signature. As a consequence, this construction gives rise to a difficulty if we were to interpret the Hilbert space metric as probabilities in quantum mechanics. One possible way of circumventing this difficulty is to impose a superselection rule that in effect 'truncates' states that do not possess positive norms. One of the purposes of this paper to show that a quantum theory arising in this manner constitutes an example of Kibble's nonlinear theory. For the benefit of readers less acquainted with the ideas of nonlinear quantum mechanics we shall briefly review in this section the idea of state space geometry of quantum mechanics and how it may be extended into nonlinear domain.

It will be useful first to state how the structures of metric geometry arise in standard quantum theory. The simplest way to determine the metric structure of quantum state space $\mathbb{CP}^n$ of an $(n+1)$-level system is to note that the transition probability

$$\cos^2\left(\tfrac{1}{2}\theta\right) = \frac{\langle\psi|\eta\rangle\langle\eta|\psi\rangle}{\langle\psi|\psi\rangle\langle\eta|\eta\rangle} \tag{9}$$

between a pair of states $|\psi\rangle$ and $|\eta\rangle$ gives rise to the notion of distance in projective spaces [16]. This can be seen as follows. First we set $|\psi\rangle = Z^\alpha$, $|\eta\rangle = Z^\alpha + \mathrm{d}Z^\alpha$, and $\theta = \mathrm{d}s$ in (9), where $\{Z^\alpha\}_{\alpha=1,2,\ldots,n+1}$ is viewed as the homogeneous coordinates for a point in $\mathbb{CP}^n$. With these substitutions, Taylor expand each side of (9) and retain terms of quadratic order, we recover the standard Fubini-Study metric on $\mathbb{CP}^n$:

$$\mathrm{d}s^2 = 4\frac{\bar{Z}_\alpha Z^\alpha \mathrm{d}\bar{Z}_\beta \mathrm{d}Z^\beta - \bar{Z}_\alpha Z^\beta \mathrm{d}\bar{Z}_\beta \mathrm{d}Z^\alpha}{(\bar{Z}_\gamma Z^\gamma)^2}. \tag{10}$$

For further detail of the relevance of the metric (10) in quantum mechanics, see, for example, Refs [1, 2, 7, 22].

In the case of a two-level system we have (Z^1, Z^2) for the homogeneous coordinate on $\mathbb{CP}^1$. Since the coordinates cannot all vanish simultaneously, we assume without loss of generality that $Z^1 \neq 0$ and introduce an inhomogeneous coordinate by setting $z = Z^2/Z^1$. Then a short calculation shows that the Fubini-Study metric simplifies to

$$\mathrm{d}s^2 = 4\frac{\mathrm{d}\bar{z}\mathrm{d}z}{(1+\bar{z}z)^2}. \tag{11}$$

In the usual stereographic representation this is just the metric on a sphere S^2 of radius one-half. This sphere, of course, is the Bloch sphere for a spin-$\frac{1}{2}$ particle system.

We now provide an alternative derivation for the metric structure of quantum state space by use of the method of kernel functions, because this construction applies to the more general context of nonlinear quantum theory. To put the matter differently, the derivation of the metric from the transition probability (9) is a special feature of linear quantum mechanics; in the nonlinear context, one *defines* what one means by a transition probability from the metric geometry of the state space.

For the kernel function method we consider complex analytic functions $f(z)$ defined in a bounded domain of $\mathbb{C}$, that is, a connected open set D in the complex plane. Since D is bounded we have the Hilbert space $\mathcal{L}^2(D)$ of square-integrable functions satisfying

$$\int\int_D \mathrm{d}x\,\mathrm{d}y\,|f(z)|^2 < \infty, \tag{12}$$

where $z = x + \mathrm{i}y$. Let $\{\phi_n(z)\}$ be a set of complex orthonormal functions on the domain D. Then the kernel K of this set is defined by the expression

$$K(z, \bar{w}) = \sum_{n=1}^{\infty} \phi_n(z)\overline{\phi_n(w)}. \tag{13}$$

Note that for a real orthonormal basis the kernel function generally diverges, whereas in the complex case it always converges uniformly [5, 15]. The function $K(z, \bar{w})$ is called the kernel, or sometimes Bergman kernel, because of the identity

$$g(z) = \int_D \mathrm{d}x\, \mathrm{d}y\, K(z, \bar{w}) g(w), \tag{14}$$

where $w = x + \mathrm{i}y$, that holds for any smooth function $g \in \mathcal{L}^2(D)$.

Given the kernel function K associated with a domain D in the complex plane, one can determine a natural Riemannian metric, called the Bergman metric, on the domain according to the prescription

$$\mathrm{d}s^2 = K(z, \bar{z})\mathrm{d}\bar{z}\mathrm{d}z. \tag{15}$$

Some of the key properties of this metric are that it is invariant under the conformal transformations, and that it is monotonic in the sense that the associated line element satisfies $\mathrm{d}s' > \mathrm{d}s$ if $D' \subset D$. More generally, for a bounded domain in $\mathbb{C}^n$ the Bergman metric is given by

$$\mathrm{d}s^2 = \partial_a \bar{\partial}_b \log K(z, \bar{z}) \mathrm{d}z^a \mathrm{d}\bar{z}^b, \tag{16}$$

where the coordinates are $\{z^a\}$, $\partial_a = \partial/\partial z^a = \frac{1}{2}(\partial/\partial x - \mathrm{i}\partial/\partial y)$, and $\bar{\partial}_b = \partial/\partial \bar{z}^b$. In the complex plane this definition reduces to (15). For standard quantum mechanics we define the kernel function by taking the product of unnormalised state vectors, that is, we set $K(z, \bar{z}) = \bar{Z}_\alpha(\bar{z}) Z^\alpha(z)$, where we regard z as the inhomogeneous coordinates. Then a straightforward exercise shows that the Fubini-Study metric on $\mathbb{CP}^n$ can be recovered from the expression (16). Further details of the derivation of the metric in this manner for conventional quantum mechanics are outlined in Ref. [9]. The expression (15) will be used below to derive the metric of the PT-symmetric quantum theory along with a superselection rule.

Returning to the discussion on standard quantum mechanics, we thus see from (9) and (10) that the metric geometry is intimately related to the probabilistic interpretation of quantum mechanics. As regards quantum observables, they are defined as functions on the space of pure states. That is, if H_α^β represents an observable acting on the Hilbert space, then we have the homogeneous function $H(\bar{Z}, Z) = \bar{Z}_\beta H_\alpha^\beta Z^\alpha / \bar{Z}_\gamma Z^\gamma$ on $\mathbb{CP}^n$. The unitary time evolution of the state is then given by a Hamiltonian symplectic flow on the state space manifold generated by the function $H(\bar{Z}, Z)$. In particular, the quadratic nature of the function $H(\bar{Z}, Z)$ ensures that the Hamiltonian flow is an isometry, that is, transition probabilities are preserved in time.

Based on these geometric properties of standard quantum mechanics (as well as earlier work of Mielnik [19] on nonlinear observables), Kibble proposed two alternative nonlinear generalisations of quantum mechanics [17]. The first generalisation is to retain the linear structure of the quantum state space but to modify the dynamics to a class of nonlinear wave equations that nonetheless can be expressed as Hamiltonian flows. In other words, the first alternative is to drop the assumption that the Hamiltonian be quadratic—this implies that the time evolution is no longer an isometry so that transition probabilities are not preserved. This particular nonlinear generalisation of quantum mechanics has been investigated extensively by many authors (see, e.g., [12, 26]).

The second alternative proposed by Kibble, which is of interest to us here, is to consider the possibility of taking a generic Kähler manifold having symmetries as the state space of quantum mechanics. In this case, the dynamics are associated with isometries of the manifold, generated by Hamiltonian symplectic flows. In other words, the second proposal is to consider a nonlinear state space, while retaining in some sense the linearity of the dynamics. We note here that a Kähler manifold is a complex manifold equipped with a Riemannian structure compatible with the complex structure.

The state space of standard quantum mechanics, the complex projective space $\mathbb{CP}^n$, is a special example of a Kähler manifold—special in that it admits the structure of Segré embedding, which allows us to talk about such notions as linear superposition of states or particle states [7, 13]. These linear features are a direct consequence of the corresponding Hilbert space picture.

A generic Kähler manifold, on the other hand, does not admit such linear structures, and this justifies the use of the term 'nonlinear' quantum mechanics, although this theory must be distinguished from theories associated with nonlinear wave equations, for which the nonlinearities emerge purely dynamically. Also in the general nonlinear state space context there is no Hilbert space of states or linear observables acting on these states. Nevertheless, many of the fundamental ideas required in quantum mechanics—if not all—survive; for example, observables are given by a class of functions on the state space manifold, the eigenstates of the observables are the fixed points of these functions, the eigenvalues are the values of these functions at fixed points, and so on. Also, the time evolution is given by an isometry so that relative separation of pairs of states are preserved (nonlinear analogue of unitarity).

IV. SUPERSELECTION AND HYPERBOLIC QUANTUM MECHANICS

In the context of a PT-symmetric quantum theory with a Hilbert space inner product defined by the space-time reflection (PT conjugation), if we insist that *all physical states be associated with positive norm* and exclude from the state space those states having null or negative norm, then the resulting state space turns out to have the structure of a Kähler manifold, with dynamics governed by an isometry. Therefore, in this case the PT-symmetric theory becomes an example of Kibble's second alternative for nonlinear generalisation of quantum mechanics. To see this, it suffices to consider the simplest two-level system considered in Section II.

For this system we have states on the Bloch sphere S^2 that is expressible in the form $|\psi\rangle = \rho|E_+\rangle + z|E_-\rangle$, where without loss of generality we assume $\rho \in \mathbb{R}$ and $z \in \mathbb{C}$. It is convenient for the moment not to set the normalisation. However, for physical states we require the norm $\langle\psi\|\psi\rangle$ to be positive. This is the superselection rule that we impose here. It then follows from $\langle E_+\|E_+\rangle = +1$ and $\langle E_-\|E_-\rangle = -1$ that physical states must lie on the northern hemisphere of S^2 correspond to the domain D that is determined by the relation

$$x^2 + y^2 < \rho^2, \tag{17}$$

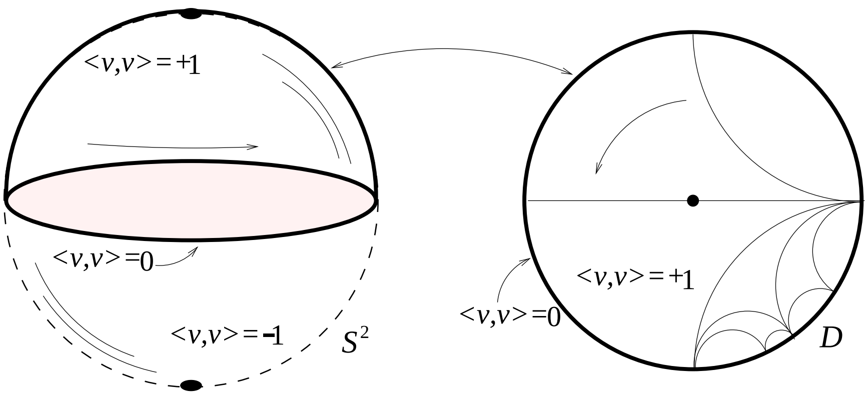

FIG. 1: (colour online) *Upper-half state space and the Poincaré disc. In the case of PT-symmetric two-state quantum mechanics equipped with a PT-norm, the physical states possessing positive norm lie on the northern hemisphere of* $\mathbb{CP}^1 \sim S^2$*. Such states are identified with points on the Poincaré unit disk D. Thus, the resulting state space is characterised by hyperbolic geometry. The evolution of the state then corresponds to the rotation of the disk, having a single fixed point at the centre of the disk D corresponding to the energy eigenstate* $|E_+\rangle$*. Geodesics on D are half-circles, as illustrated in the figure.*

where we set $z = x + \mathrm{i}y$. This is just a region inside a circle of radius ρ, that is, $z\bar{z} < \rho^2$. Now the standard set of orthonormal functions $\phi_n(z)$ in this region is given by

$$\phi_n(z) = \left(\frac{n}{\pi}\right)^{\frac{1}{2}} \frac{z^{n-1}}{\rho^n}. \tag{18}$$

It then follows from the identity

$$\sum_{n=1}^{\infty} \frac{n}{\pi} \frac{z^{n-1}\bar{w}^{n-1}}{\rho^{2n}} = \frac{\rho^2}{\pi(\rho^2 - z\bar{w})^2} \tag{19}$$

for the Bergman kernel $K(z, \bar{w})$ that the metric for the state space $D = S^2_+$ is given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}\bar{z}\mathrm{d}z}{\pi(1 - \bar{z}z)^2}, \tag{20}$$

where at this point we set the normalisation $\rho = 1$. This is just the classical metric of Poincaré on the unit disk, having constant negative curvature $\mathcal{R} = -4\pi$. In Figure 1 we illustrate this correspondence.

The result led by the foregoing discussion can be stated more formally as follows: *If we impose a superselection rule that demands that physical states must have positive PT norm, then the resulting state space for a two-state system is given by the Poincaré disk.* Therefore, we recover a hyperbolic geometry in the ambient state space, as opposed to the spherical geometry associated with the state space of standard quantum mechanics. A quantum theory defined on such a state space thus might appropriately be called 'hyperbolic quantum mechanics'. To our knowledge this is the first example of an explicit construction of Kibble's second alternative for nonlinear quantum mechanics, and it is interesting to see how such a theory emerges from imposing a superselection rule in PT-symmetric quantum theory, equipped with an indefinite PT-norm.

We remark in passing that the Poincaré disk can be obtained from the upper-half complex plane through a Möbius transformation. Indeed, any connected bounded domain in complex plane is analytically equivalent to the Poincaré disk, and these domains possess negative curvatures. Consequently, all quantum theories obtained by restricting the state space $\mathbb{CP}^1$ to any connected domain are analytically equivalent—all these theories possess hyperbolic state spaces. From this point of view it might be of interest to consider the possibility of formulating a quantum theory on nonconnected domain, such as a ring domain defined by $r < |z| < 1$ where z denotes the inhomogeneous coordinate of $\mathbb{CP}^1$. For such a domain the orthonormal system is given by

$$\phi_{2n-1}(z) = z^{n-1}[n/\pi(1-r^{2n})]^{1/2} \tag{21}$$

for the odd elements and

$$\phi_{2n}(z) = z^{-n}[(1-n)/\pi(1-r^{2(n-1)})]^{1/2} \tag{22}$$

for the even elements. A short calculation then shows (see Ref. [5]) that the Bergman kernel can be expressed as

$$K(z,\bar{z}) = \frac{1}{\pi\bar{z}z}\left[\wp(\log\bar{z}z) + \frac{\eta}{2\pi\mathrm{i}} - \frac{1}{2\log r}\right] \tag{23}$$

in terms of the Weierstrassian $\wp$-function, where η is the increment of the Weierstrassian ζ-function related to the period πi. Although unrelated to PT symmetry or hyperbolic quantum theory, a formulation of quantum mechanics on such a domain would be of considerable mathematical interest.

Returning to the PT-symmetric quantum theory, the situation in higher dimensions is somewhat more elaborate: there are natural generalisations of the upper-half plane in higher dimensions, and these are known as the Siegel domains [25]. In higher dimensions these domains become relevant to PT-symmetric theories if we once again demand the positivity condition on the PT norm. For example, if the Hamiltonian is a four-dimensional matrix and thus the state space is $\mathbb{CP}^3_+$, the corresponding positive-norm domain is given by $z\bar{z} + v\bar{v} - w\bar{w} < \rho^2$ instead of the condition $z\bar{z} < \rho^2$ in two dimensions. In general if the Hamiltonian is a $2n \times 2n$ matrix, then the associated state space $\mathfrak{M} = \mathbb{CP}^{2n-1}_+$ is determined by the bounded domain

$$\sum_{\alpha=1}^{n} \bar{z}^\alpha z^\alpha - \sum_{\beta=1}^{n-1} \bar{w}^\beta w^\beta < \rho^2 \tag{24}$$

of $\mathbb{C}^{2n}$, where $\{z^\alpha\}, \{w^\beta\} \in \mathbb{C}$. It follows that the hyperbolic nature of the state space persists in higher dimensions.

V. REDUCED HERMITIAN ENERGY MEASUREMENT

On a hyperbolic state space, various questions naturally arise, including that on measurement theory of physical observables. Here we describe one possible procedure for describing energy measurements. To begin we note the interpretational issue we encounter in PT-symmetric theory endowed with the superselection rule, when we consider only the upper-half of the state space $\mathfrak{M} = \mathbb{CP}^{2n-1}_{+}$: namely, a generic quantum state cannot be expressed as a superposition of the eigenstates contained in $\mathfrak{M}$. Therefore, instead of the full PT-symmetric Hamiltonian H, let us consider here a degenerate Hermitian Hamiltonian H_{+} defined by

$$H_{+} = \sum_{i=1}^{n} E^{i}_{+} |E^{i}_{+}\rangle\langle E^{i}_{+}|. \tag{25}$$

In the state space $\mathfrak{M}$ there is a hyperplane spanned by the positive eigenstates $|E^{i}_{+}\rangle$ of the original Hamiltonian H. As a consequence, the operator H_{+} represents, in this hyperplane, a standard quantum mechanical Hermitian Hamiltonian having n eigenstates. However, when we view the whole of $\mathfrak{M}$ as our quantum state space, H_{+} represents a degenerate operator. That is, while each point $|E^{i}_{+}\rangle \in \mathfrak{M}$ is a distinct eigenstate of H_{+}, the entire span of each such point with the negative states $|E^{i}_{-}\rangle$ also represents the eigenstates of H_{+}. This is because any state $|E^{i}_{-}\rangle$ is annihilated by H_{+} and hence is automatically an eigenstate of H_{+} with vanishing eigenvalue. Therefore, there is a continuum of degenerate eigenstates of H_{+} in $\mathfrak{M}$ associated with each nondegenerate eigenvalue E^{i}_{+} of H. An example of this for $n = 2$ is illustrated in Fig. 2.

A direct consequence of having such a degenerate Hamiltonian is that any state $|\psi\rangle \in \mathfrak{M}$ can now be represented as various superpositions of the eigenstates of H_{+}. For $n = 1$, the situation trivialises because any state in the state space $\mathfrak{M} = D$ is automatically an eigenstate of H_{+}. Therefore, if the Hermitian energy H_{+} is measured, the resulting expectation value is given by $\cosh^2(x)E_{+}$. The first nontrivial case is when $n = 2$, for which there is a pair of nonintersecting planes in $\mathfrak{M}$ representing degenerate eigenstates of H_{+}. Recall that, in standard quantum mechanics, when the outcome of an energy measurement is associated with degenerate eigenvalues, the resulting state of the system, after measurement, is given by the Lüders state. This is the state obtained by the minimum projection of the initial state into the subspace of the Hilbert space spanned by the degenerate eigenstates.

For a PT-symmetric theory with a PT norm and superselection rule, given a generic state $|\psi\rangle$ there are hyperplanes of complex dimension $n - 1$ through that point that are entirely contained in $\mathfrak{M} = \mathbb{CP}^{2n-1}_{+}$, and these hyperplanes intersect the hyperplanes of degenerate eigenstates of H_{+}, having complex dimension $2n - 2$, at distinct points. In this way, we see that a generic state can now be expressed as a linear superposition of the eigenstates of the Hermitian energy H_{+}. As a consequence, it may be possible that one of the planes through $|\psi\rangle$ determines a minimum projection of the state $|\psi\rangle$ onto the planes of degenerate eigenstates. If so, then the intersection points determine generalised Lüders states in the hyperbolic theory that we consider here. In particular, if this is the case, then we recover the standard quantum mechanical transition probabilities associated with measurement outcomes. However, corresponding expectation values will be different from the standard theory, as we have indicated above for the state space D.

Let us consider the simplest nontrivial case for which $n = 2$. In this dimensionality there are lines through a generic point $|\psi\rangle$ that are contained entirely in $\mathbb{CP}^{3}_{+}$, and these lines meet the pair of planes associated with the eigenstates of H_{+}. If one of these lines corresponds to a minimum projection of $|\psi\rangle$ onto these planes, then the associated intersection points determine generalised Lüders states, indicated by $|\alpha\rangle$ and $|\beta\rangle$ in Fig. 2. In this case the transition probabilities from the initial state $|\psi\rangle$ to $|\alpha\rangle$ and $|\beta\rangle$ are given by the cosine squares of the spherical angles between these states.

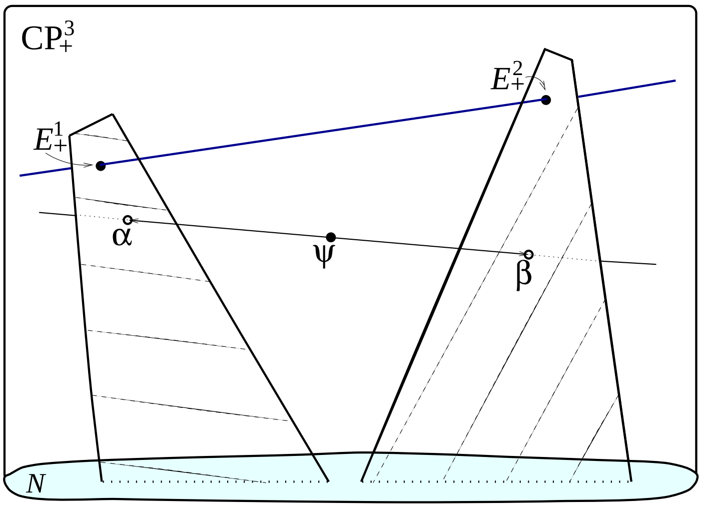

FIG. 2: (colour online) *Generalised Lüders states. For a generic nondegenerate PT-symmetric Hamiltonian in four dimensions there are four distinct eigenvalues* $\{E_\pm^{1,2}\}$*. If we construct a degenerate Hamiltonian* H_+ *from the positive-norm eigenstates* $|E_+^{1,2}\rangle$*, then the associated eigenstates of* H_+ *are represented in* $\mathbb{CP}^3_+$ *by a pair of nonintersecting planes. These are given by the plane spanned by three points* $\{|E_+^1\rangle, |E_-^1\rangle, |E_-^2\rangle\}$ *and the plane spanned by three points* $\{|E_+^2\rangle, |E_-^1\rangle, |E_-^2\rangle\}$*, respectively. These two planes intersect in* $\mathbb{CP}^3_-$ *at a line that joins two points* $|E_-^1\rangle$ *and* $|E_-^2\rangle$*, not shown in the figure. Through a generic point* $|\psi\rangle$ *of* $\mathbb{CP}^3_+$ *there are complex lines that do not meet the boundary* N *of null states, and hence contained entirely in* $\mathbb{CP}^3_+$*. If generalised Lüders states* $|\alpha\rangle$ *and* $|\beta\rangle$ *exist, these are given by minimum projection of* $|\psi\rangle$ *onto the planes of degenerate eigenstates of* H_+*, to which the measurement of* H_+ *results in.*

VI. DISCUSSION

The hyperbolic quantum theory emerging in this way from PT symmetry along with the selection rule is intriguing, but it leaves us with the question of whether such a quantum theory is physically viable. A candidate measurement theory proposed in Sec. V is interesting, but at the same time it is intricate, and furthermore, the expectation values resulting from the proposal differs from the conventional theory.

To this end we draw attention to the positive curvature theorem concerning the admissible nonlinear extension of quantum state space [8]. In short, this theorem states, in the context of Kibble's second alternative for the nonlinear extension, that for the resulting quantum theory to embody a fully consistent probabilistic interpretation, the holomorphic sectional curvature of the state space manifold has to be positive. Conversely, if the curvature is not positive, then measurement of an observable, for example, need not yield a definite outcome. Indeed, in the hyperbolic situation, one can envisage a situation whereby measurement of the energy takes the initial state further and further away from the eigenstates, towards the boundary, so that energy uncertainty increases as a result of an energy measurement.

In the case of standard quantum mechanics the state space manifold is just the complex projective space with constant positive curvature so that a probabilistically consistent measurement theory can be formulated. On the other hand, in the case of hyperbolic quantum theory we have formulated here, the state space possesses constant negative curvature. We thus conclude that, while mathematical properties of such a quantum theory is interesting, it has to be ruled out from physical point of view as a viable nonlinear extension. This conclusion also justifies the introduction by Bender *et al.* [4] and by Mostafazadeh [21] of an alternative symmetry to construct a positive definite inner product in order to circumvent the issues arising from indefinite metric. Then one can prove [6] that the resulting theory is in fact indistinguishable from a conventional quantum theory described by Hermitian Hamiltonians. We conclude by remarking that other quantum theories obtained by superselection rules analogous to the one considered here (for example, to eliminate 'ghost' states) evidently suffer from the problem of curvature negativity, and thus have to be rules out as an admissible physical theory.

The author thanks C. M. Bender and L. P. Hughston for discussion, and the Russian Science Foundation for support (project 16-11-10218).

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[1] Anandan, J. & Aharonov, Y. 1990 "Geometry of quantum evolution" Phys. Rev. Lett. **65**, 1697–1700.

[2] Ashtekar, A. & Schilling, T. A. 1995 "Geometry of quantum mechanics" CAM-94 Physics Meeting, in AIP Conf. Proc. **342**, 471–478, ed. Zapeda, A. (AIP Press, Woodbury, New York).

[3] Bender, C. M. & Boettcher, S. 1998 "Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$-symmetry" *Phys. Rev. Lett.* **80**, 5243–5246.

[4] Bender, C. M., Brody, D. C. & Jones, H. F. 2002 "Complex extension of quantum mechanics" *Phys. Rev. Lett.* **89**, 27040-1∼4.

[5] Bergman, S. 1970 *The Kernel Function and Conformal Mapping* (Providence: AMS Press).

[6] Brody, D. C. 2016 "Consistency of PT-symmetric quantum mechanics" *J. Phys. A: Math. Theor.* **49**, 10LT03.

[7] Brody, D. C. & Hughston, L. P. 2001 "Geometric quantum mechanics" *J. Geom. Phys.* **38**, 19–53.

[8] Brody, D. C. & Hughston, L. P. 2002 "Stochastic reduction in nonlinear quantum mechanics" *Proc. P. Soc. London* A**458**, 1117–1127.

[9] Burbea, J. & Rao, C. R. 1984 "Differential metrics in probability spaces" *Prob. Math. Statist.* **3**, 241–258.

[10] Dorey, P., Dunning, C. & Tateo, R. 2001b "Spectral equivalences, Bethe ansatz equations, and reality properties in $\mathcal{PT}$-symmetric quantum mechanics" *J. Phys.* A**34**, 5679–5704.

[11] Duff, M. J. and Kalkkinen, J. 2006 "Signature reversal invariance" Nucl. Phys. B **757** (doi:10.1016/j.nuclphysb.2006.09.014, to appear).

[12] Faddeev, L. D. & Takhtajan, L. A. 1987 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer-Verlag).

[13] Gibbons, G. W. 1992 "Typical states and density matrices" J. Geom. Phys. **8**, 147-162.

[14] Gibbons, G. W. 2006 "Discrete symmetries and gravity", talk given at Andrew Chamblin Memorial Conference, 14 October 2006, Cambridge.

[15] Helgason, S. 1978 *Differential Geometry, Lie Groups, and Symmetric Spaces* (San Diego: Academic Press).

[16] Hughston, L. P. 1995 "Geometric aspects of quantum mechanics" In *Twistor Theory* ed S Huggett (New York: Marcel Dekker, Inc.).

[17] Kibble, T. W. B. 1979 "Geometrisation of quantum mechanics" *Commun. Math. Phys.* **65**, 189–201.

[18] Kreĭn, M. G. 1965 "An introduction to the geometry of indefinite J-spaces and the theory of operators in these spaces" in *Proc. Second Math. Summer School, Part I*, 15–92 (Kiev: Naukova Dumka).

[19] Mielnik, B. 1974 "Generalized quantum mechanics" *Commun. Math. Phys.* **37**, 221–256.

[20] Moffat, J. W. (2006) "Positive and negative energy symmetry and the cosmological constant problem" Preprint: hep-th/0610162.

[21] Mostafazadeh, A. 2002 *J. Math. Phys.* "Pseudo-Hermiticity versus PT-symmetry III" **43**, 3944–3951.

[22] Page, D. N. 1987 *Phys. Rev. A* "Geometrical description of Berry's phase" **36**, 3479–3481.

[23] Pontryagin, L. S. 1944 "Hermitian operators in spaces with indefinite metrics" *Bull. Acad. Sci. URSS. Ser. Math. [Izvestiya Akad. Nauk SSSR]*, **8**, 243–280.

[24] 't Hooft, G. and Nobbenhuis, S. 2006 "Invariance under complex transformations, and its relevance to the cosmological constant problem" Class. Quantum Grav. **23**, 3819–3832.

[25] Vinberg, É. B., Gindikin, S. G. and Pjateckiĭ-Šapiro, I. I. 1963 "Classification and canonical realisation of complex bounded homogeneous domains" *Trudy Moskov. Mat. Obšč* **12**, 359–388.

[26] Weinberg, S. 1989 "Testing quantum mechanics" Ann. Phys. **194**, 336–386.

[27] Znojil, M. (ed.) Proceedings of the First, Second, and Third International Workshops on Pseudo-Hermitian Hamiltonians in Quantum Mechanics, in *Czech. J. Phys.* **54**, issues #1 and #10 (2004) and **55**, issues #1 (2005).