Domination analysis of greedy heuristics for the frequency assignment problem

A.E. Koller*† and S.D. Noble†

Abstract

We introduce the greedy expectation algorithm for the fixed spectrum version of the frequency assignment problem. This algorithm was previously studied for the travelling salesman problem. We show that the domination number of this algorithm is at least $\sigma^{n-\lceil \log_2 n\rceil -1}$ where σ is the available span and n the number of vertices in the constraint graph. In contrast to this we show that the standard greedy algorithm has domination number strictly less than $\sigma^n e^{-\frac{5(n-1)}{144}}$ for large n and fixed σ.

Keywords: Frequency Assignment Problem, Greedy Heuristic, Domination Number.

1 Introduction and Definitions

The frequency assignment problem has been well studied in many publications [2, 3, 10, 11, 12]. See [1] for a recent comprehensive survey.

Wireless communication plays an important role both in civil and military applications. In order to establish connection a transmitter and a corresponding receiver have to be tuned (assigned) to the same frequency. The frequency assignment problem therefore deals with the tuning of several wireless connections. Naturally, depending on the location of the sender and receiver

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†Department of Mathematical Sciences, Brunel University, Uxbridge, Middlesex, UB8 3PH, United Kingdom. Email: angela.koller@brunel.ac.uk, steven.derek.noble@brunel.ac.uk

and the frequency they are tuned to, interference is likely to occur. Since the spectrum of frequencies is a limited resource, it has become important to assign the frequencies in an optimum or near-optimum manner in such a way that the interference is kept as small as possible.

In practise, there are two main types of frequency assignment problems that can occur: the minimum span frequency assignment and the fixed spectrum frequency assignment (also sometimes called the minimum interference problem).

In this paper we are looking at the second model, the fixed spectrum problem. Here the transmitters are assigned with frequencies out of a given range of σ channels, 0, 1, ..., σ-1. The value σ is referred to as the span. This version of the problem is arguably the more important because in practice regulators assign blocks of channels to particular operators or companies and later the actual assignments are made using frequencies from the given blocks.

Due to the given fixed span of frequencies (which is usually not big enough) it is almost always the case that some interference does occur. In order to minimize interference we apply constraints to the frequencies assigned to sets of transmitters. We will only consider constraints applied to pairs of transmitters. All these constraints are represented in a constraint matrix $C = (c_{ij})$ where for all i, j the frequencies f_i and f_j assigned to transmitters i, j respectively must satisfy $|f_i - f_j| \geq c_{ij}$.

The most commonly used constraints are based solely on the geographical distance between pairs of transmitters, that is there exist constants $d_1, \ldots, d_k$ such that $|f_i - f_j| \geq k$ if $d_{ij} < d_k$ where d_{ij} is the distance between transmitters i and j.

For various reasons there can be the situation where a particular constraint between say transmitters i and j must not be broken. In order to take this matter into account, weight w_{ij} is put on constraint c_{ij}. The weights are intended to reflect the importance of the constraints. Often all constraints are equally important and all weights are equal to one.

Depending on whether weights are applied to constraints or not, the objective is to minimize the number of constraints broken or to minimize the sum of weights of constraints broken, respectively. Given an assignment f we denote its cost that is the sum of weights of constraints broken by $c(f)$.

The problem is modelled using a graph with a vertex for each transmitter and an edge between pairs of transmitters that are constrained. From now on we will use a mixture of graph theoretic and radio frequency terminology depending on which seems more appropriate at the time.

The initial stage of many algorithms to assign frequencies to transmitters consists of a greedy algorithm, consequently it is useful to have a theoretical method of differentiating between the performance of various greedy methods. In this paper we compare a standard greedy algorithm with the greedy expectation algorithm which we define later. In order to do this, we need to define the domination number. The *domination number*, $\mathrm{dom}(\mathcal{A}, n)$, for a heuristic $\mathcal{A}$ is the maximum integer $d(n)$, such that for every graph G on n vertices, $\mathcal{A}$ produces an assignment f which is not worse than at least $d(n)$ assignments in G including f itself. A heuristic with higher domination number may be considered better than a heuristic with lower domination number [7].

The concept of domination number has until now been applied almost exclusively to the travelling salesman problem.

In 1997 the question was asked in [4] whether there exists a polynomial time algorithm $\mathcal{A}$ for the travelling salesman problem with $\mathrm{dom}(\mathcal{A}, n) \geq n!/p(n)$ for some polynomial $p(n)$. It was conjectured then, that unless $P = NP$ there is no such algorithm [4] . However, Gutin and Yeo [6] proved only a year later that for the greedy expectation algorithm (GEA) the domination number of the travelling salesman problem $\mathrm{dom}(\mathrm{GEA}, n) \geq (n-2)!$ for every $n \neq 6$. Their result had previously been established in the 1970s in [15] and [16]. Gutin, Yeo and Zverovich [7] showed in 2001, that if $n \geq 2$ the domination number of the greedy algorithm for the travelling salesman problem is 1. Further results on the domination number of TSP heuristics have been obtained in [13] and [14]. We have adapted these algorithms to the frequency assignment problem as follows.

Standard greedy algorithm We assume that we are given a fixed ordering $(v_1, \ldots, v_n)$ of the vertices. Initially all vertices are unassigned. At each stage of the algorithm we assign a frequency to one vertex. Once an assignment is made, it is not changed. We begin by assigning v_1 with frequency 0. Recall that w_{ij} is the cost associated with breaking the constraint involving the frequencies assigned to vertices v_i and v_j. In the ith stage of the algorithm, a frequency is assigned to v_i by finding the smallest possible frequency such that $\sum_{j=1}^{i-1} w_{ij} x_{ij}$ is minimized, where $x_{ij} = 1$ if constraint c_{ij} between vertex v_i and vertex v_j is broken, otherwise $x_{ij} = 0$. Clearly in the case when all weights are equal to one, this corresponds to choosing a frequency minimising the number of violated constraints with previously assigned transmitters.

Greedy expectation algorithm This algorithm is the same as the standard greedy algorithm in that vertices are assigned one after another in the specified order $(v_1, \ldots, v_n)$ and that assignments, once selected, remain fixed. The algorithm differs in the way in which the frequency is selected.

In the ith stage v_i is assigned with the smallest possible frequency such that

$$\left(\sum_{j=1}^{i-1} w_{ij}x_{ij} + \sum_{j=i+1}^{n} w_{ij}p_{ij}\right)$$

is minimised, where p_{ij} is the probability that the constraint between vertex v_i and vertex v_j is broken if vertex v_j is assigned a frequency chosen uniformly at random from $\{0, 1, \ldots, \sigma-1\}$. Again in the case when all weights are equal to one, this corresponds to minimising the sum of the number of violated constraints with previously assigned transmitters and the expected number of constraints broken when the remaining transmitters are assigned uniformly at random.

In the following sections we will give bounds on the domination number for the frequency assignment problem of both, the greedy expectation algorithm and the standard greedy algorithm.

2 Domination number of the GEA for the FAP

After having defined the greedy expectation algorithm for the FAP we now give a lower bound for its domination number:

Theorem 2.1 *The domination number of the greedy expectation algorithm for the fixed spectrum version of the frequency assignment problem is at least* $\sigma^{n-\lceil\log_2 n\rceil-1}$.

Before proving the theorem we first state and prove the following lemma relating the cost of the solution produced by the GEA to the expected cost of a solution generated uniformly at random.

Lemma 2.1 *Let $\tilde{f}$ be an assignment generated uniformly at random using frequencies from $\{0, ..., \sigma-1\}$ and A be the event that $\tilde{f}$ assigns frequencies $x_1, ..., x_{k-1}$ to transmitters $v_1, ..., v_{k-1}$ respectively. Then there exists a frequency j_0 such that*

$$E\left(c(\tilde{f})|A, \tilde{f}(v_k) = j_0\right) \leq E\left(c(\tilde{f})|A\right). \tag{1}$$

Proof of Lemma: Using the formula of total expectation [5] we obtain

$$E\left(c(\tilde{f})|A\right) = \sum_j E\left(c(\tilde{f})|A, \tilde{f}(v_k) = j\right) P\left(\tilde{f}(v_k) = j|A\right).$$

Let j_0 be the frequency j minimizing $E\left(c(\tilde{f})|A, \tilde{f}(v_k)=j\right)$ then

$$\sum_j E\left(c(\tilde{f})|A, \tilde{f}(v_k)=j\right) P\left(\tilde{f}(v_k)=j|A\right) \geq E\left(c(\tilde{f})|A, \tilde{f}(v_k)=j_0\right).$$

$\square$

Proof of Theorem: Let $\tilde{f}$ be an assignment generated uniformly at random. At each step the greedy expectation algorithm finds a frequency j_0 satisfying (1). Hence using induction and Lemma 2.1 we can show that the algorithm produces a solution with cost at most $E(c(\tilde{f}))$. Thus the domination number of the GEA is at least the number of solutions with cost at least $E(c(\tilde{f}))$. We now compute the number of these solutions.

We regard an assignment as an n-tuple of elements from $\{0,\ldots,\sigma-1\}$. Next we define the addition of two assignments by adding the components modulo σ. This gives the set of assignments the structure of the group $\mathbb{Z}_\sigma^n$.

Now we consider a collection of bipartitions of the vertex set V into (X_i, Y_i) for $i = 1,\ldots,\lceil\log_2 n\rceil$, such that for every edge e there is some i such that e joins a vertex in X_i to a vertex in Y_i. In addition we require that $\bigcap_{i=1}^n Y_i \neq \emptyset$. Given vertex set $V = \{v_1,\ldots,v_n\}$, one way to do this is as follows. Define X_i by letting $v_k \in X_i$ if and only if the i-th least significant bit in the binary representation of $k-1$, is equal to 1. Let $Y_i = V \setminus X_i$.

We consider the multiset $\hat{H}$ of all assignments f such that $f(v_k) = \left(\sum_{i:v_k\in X_i} a_i + b\right) \bmod \sigma$ as $a_1,\ldots,a_{\lceil\log_2 n\rceil}$ and b run through all possible combinations of values from $\{0,\ldots,\sigma-1\}$. Thus $|\hat{H}| = \sigma^{\lceil\log_2 n\rceil+1}$. We claim that the mean cost of an assignment in $\hat{H}$ equals $E(c(\tilde{f}))$. To see this first observe that choosing an assignment uniformly at random from $\hat{H}$ is the same procedure as selecting an assignment f by setting $f(v_k) = \left(\sum_{i:v_k\in X_i} A_i + B\right) \bmod \sigma$ where $A_i,\ldots,A_{\lceil\log_2 n\rceil}$ and B are independent random variables taking the values $0,\ldots,\sigma-1$ uniformly at random.

We now consider the joint distribution of the channels assigned to the endpoints u, v of an edge. Suppose without loss of generality that $u \in X_1$, $v \in Y_1$. Then

$$\begin{aligned}
&Pr\left((f(u), f(v)) = (i,j)\right) \\
&= \sum_{a_2,\ldots,a_{\lceil\log_2 n\rceil}} Pr\left((f(u), f(v)) = (i,j), A_2 = a_2,\ldots,A_{\lceil\log_2 n\rceil} = a_{\lceil\log_2 n\rceil}\right) \\
&= \sum_{a_2,\ldots,a_{\lceil\log_2 n\rceil}} Pr\left((f(u), f(v)) = (i,j) | A_2 = a_2,\ldots,A_{\lceil\log_2 n\rceil} = a_{\lceil\log_2 n\rceil}\right) \\
&\qquad \cdot Pr\left(A_2 = a_2,\ldots,A_{\lceil\log_2 n\rceil} = a_{\lceil\log_2 n\rceil}\right).
\end{aligned}$$

Given the values of $A_2,\ldots,A_{\lceil\log_2 n\rceil}$ there is one possible choice for A_1 and B, giving $f(u) = i$, $f(v) = j$. Hence the conditional probability in the sum is $1/\sigma^2$ and consequently when we carry out the summation we also obtain $1/\sigma^2$.

Therefore the probability that the constraint corresponding to edge e is broken is the same as in an assignment where all the channels are chosen uniformly at random. Thus by linearity of expectation the mean cost of an assignment in $\hat{H}$ equals $E(c(\tilde{f}))$.

Let H be obtained by removing duplicates from $\hat{H}$. Now H is a subgroup of $\mathbb{Z}_\sigma^n$. Let $C_1, ..., C_k$ be the cosets of H. Since $|H| \leq \sigma^{\lceil \log_2 n \rceil + 1}$, we must have $k \geq \sigma^{n - \lceil \log_2 n \rceil - 1}$. Let g_i be an assignment such that $C_i = g_i + H$. Now let $\hat{C}_i$ be the multiset $g_i + \hat{H}$. Given an edge e, if we choose an element g of $\hat{C}_i$ uniformly at random, the probability that e is broken in g is the same as the probability that e is broken in $\tilde{f}$. Therefore the mean cost of an assignment in $\hat{C}_i$ is $E(c(\tilde{f}))$, which implies that at least one assignment has cost greater than or equal to $E(c(\tilde{f}))$. Since there are at least $\sigma^{n - \lceil \log_2 n \rceil - 1}$ cosets, at least $\sigma^{n - \lceil \log_2 n \rceil - 1}$ assignments have cost greater than or equal to $E(c(\tilde{f}))$. $\square$

3 Domination number of the SGA for the FAP

In this section we give an upper bound on the domination number of the standard greedy algorithm for the frequency assignment problem. The basic idea is to construct a graph on which the standard greedy algorithm works very badly and then use Chernoff type bounds to show that the probability that an assignment chosen uniformly at random, performs as badly as the assignment produced by the standard greedy algorithm, decreases exponentially with the size of the graph.

We will use the following theorem, originally due to Hoeffding [8]

Theorem 3.1 *Let the random variables $X_1, \ldots, X_N$ be independent, with $a_k \leq X_k \leq b_k$ for each k, for suitable constants a_k, b_k. Let $S = \sum_{k=1}^N X_k$ and let $\mu = E[S]$. Then for any $t \geq 0$,*

$$P\left(S - \mu \geq t\right) \leq e^{-2t^2 / \sum\limits_{k=1}^{N} (b_k - a_k)^2}.$$

Using this result we are able to show

Theorem 3.2 *The domination number of the standard greedy algorithm for the frequency assignment problem is at most $\sigma^n e^{-\frac{5(n-1)}{144}}$.*

Before proving the main theorem we establish the following lemma.

Lemma 3.1 *Let σ be fixed and e be an edge in a constraint graph having constraint*

$$c = \begin{cases} \frac{\sigma+1}{2} & \textit{if } \sigma \textit{ is odd,} \\ \frac{\sigma+2}{2} & \textit{if } \sigma \textit{ is even.} \end{cases}$$

Suppose the endpoints of e are labelled by choosing labels independently and uniformly at random from $\{0, 1, \ldots, \sigma - 1\}$. Then the probability that the constraint, corresponding to edge e, is broken is at least $\frac{3}{4}$.

Proof: Case 1: σ is odd.

Out of all possible assignments the endpoints of e can receive, the number of assignments when the constraint is not broken is

$$2\sum_{i=1}^{\frac{\sigma-1}{2}} i = \left(\frac{\sigma-1}{2}\right)\left(\frac{\sigma+1}{2}\right).$$

From this it follows that the probability that the constraint corresponding to edge e is broken is

$$1 - \frac{\sigma^2-1}{4\sigma^2} = 1 - \frac{1}{4} + \frac{1}{4\sigma^2} \geq \frac{3}{4}.$$

Case 2: σ is even.

Here, the number of assignments leading to an unbroken constraint on edge e is

$$2\sum_{i=1}^{\frac{\sigma-2}{2}} i = \left(\frac{\sigma-2}{2}\right)\left(\frac{\sigma}{2}\right).$$

The probability that edge e is broken is

$$1 - \frac{\sigma^2-2\sigma}{4\sigma^2} = 1 - \frac{1}{4} + \frac{1}{2\sigma} \geq \frac{3}{4}.$$

Proof of Theorem 3.2: Let σ be fixed and for each n consider the graph, $G_n = (V, E)$, where
$V = \{v_1, \ldots, v_n\}$. Now let $N = \lfloor \frac{n-1}{4} \rfloor$, and let

$$E = \{v_1 v_{1+k}, v_{1+k} v_{1+N+k}, v_{1+k} v_{1+2N+k}, v_{1+k} v_{1+3N+k} : k = 1, \ldots, N\}.$$

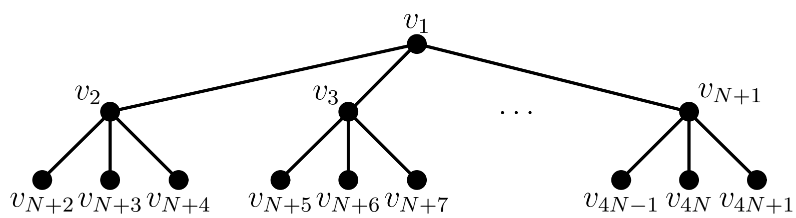
The constraint on any edge incident with v_1 is

$$c_1 = \begin{cases} \frac{\sigma-1}{2} & \text{if } \sigma \text{ is odd,} \\ \frac{\sigma}{2} & \text{if } \sigma \text{ is even.} \end{cases}$$

The constraint on all the other edges is

$$c_2 = \begin{cases} \frac{\sigma+1}{2} & \text{if } \sigma \text{ is odd,} \\ \frac{\sigma+2}{2} & \text{if } \sigma \text{ is even.} \end{cases}$$

Figure 1: The graph G_n



G_n is shown in Figure 1.

If we apply the standard greedy algorithm to G_n, labelling vertices in the order $v_1, \ldots, v_n$ then v_1 receives label 0, and $v_2, \ldots, v_{N+1}$ receive label c_1. This means that the other constraints cannot be satisfied and so $3N$ constraints are broken.

Now consider an assignment $\tilde{f}$ where all the labels are chosen independently and uniformly at random from $\{0, \ldots, \sigma - 1\}$. For $i = 1, \ldots, N$ and $j = 1, 2, 3$ let $X_{i,j} = 1$ if the constraint corresponding to $v_{1+i}v_{1+i+jN}$ is broken and 0 otherwise.

For $i = 1, \ldots, N$, let $X_i = \sum_{j=1}^{3} X_{i,j}$ and let $Y_i = 1$ if the constraint corresponding to $v_1 v_{1+i}$ is broken and 0 otherwise. Since there are no edges between vertices in X_r, X_s for $r \neq s$ it follows that $\{X_1, \ldots, X_N\}$ forms a collection of independent random variables. Let $S = \sum_{i=1}^{N} X_i$ and $T = \sum_{i=1}^{N} Y_i$.

The probability that $\tilde{f}$ breaks at least as many constraints as the standard greedy algorithm is at most $P\left(S + T \geq 3N\right) \leq P\left(S \geq 3N - N\right) = P\left(S \geq 2N\right)$. The X_{ij} are all identically distributed, so let $\alpha = P\left[X_{ij} = 1\right]$.

Then $E\left[S\right] = 3\alpha N$. Applying Theorem 3.1

$$P\left(S \geq 2N\right) = P\left(S - 3\alpha N \geq (2 - 3\alpha)N\right) \leq e^{-2N^2(2-3\alpha)^2 / \sum\limits_{1}^{N}(3-0)^2} = e^{-cN},$$

where $c = \left(2\alpha - 1\right)^2 + \frac{4}{3}\alpha - \frac{10}{9}$.

Hence the domination number of the standard greedy algorithm is at most $\sigma^n e^{-cN}$.

From Lemma 3.1 we deduce that $c \geq \frac{5}{36}$ and

$$\sigma^n e^{-cN} \leq \sigma^n e^{-\frac{5(n-1)}{144}}.$$

$\square$

Corollary: For fixed σ and large enough n the domination number of the SGA is strictly less than the domination number of the GEA.

4 Conclusion

We have shown, that for the greedy expectation algorithm, the domination number is greater than or equal to $\sigma^{n-\lceil\log_2 n\rceil-1}$, whereas the standard greedy algorithm has a domination number which is less than or equal to $\sigma^n e^{-\frac{5(n-1)}{144}}$. This shows that for fixed σ, asymptotically the worst case behaviour of the standard greedy algorithm is not as good as that of the greedy expectation algorithm.

References

[1] K.I. Aardal, S.P.M. van Hoesel, A.M.C.A. Koster, C. Mannino and A. Sassano, Models and Solution Techniques for Frequency Assignment Problems, Konrad-Zuse-Zentrum für Informationstechnik Berlin, ZIB-Report 01-40 (2001).

[2] R. Borndörfer, A. Eisenblätter, M. Grötschel and A. Martin, Frequency Assignment in Cellular Phone Networks, Ann. Oper. Res. 76 73-93 (1998).

[3] A. Eisenblätter, Frequency Assignment in GSM Networks: Models, Heuristics and Lower Bounds, Ph.D. Thesis, Technische Universität Berlin, 2001.

[4] F. Glover and A. Punnen, The traveling salesman problem: New solvable cases and linkages with the development of approximation algorithms, J. Oper. Res. Soc. 48 (1997) 502-510.

[5] G.R. Grimmett and D.R. Stirzaker, Probability and Random Processes (Clarendon Press, Oxford, 1992).

[6] G. Gutin and A. Yeo, Polynomial algorithms for the TSP and the QAP with a factorial domination number, to appear in Discrete Appl. Math.

[7] G. Gutin, A. Yeo and A. Zverovich, Traveling salesman should not be greedy: domination analysis of greedy-type heuristics for the TSP, to appear in Discrete Appl. Math.

[8] W.J. Hoeffding, Probability inequalities for sums of bounded random variables, J. Amer. Statist. Assoc. 58 (1963) 713-721.

[9] S. Hurley, D.H. Smith and S.U. Thiel, FASoft: A system for discrete channel frequency assignment, Radio Sci. 32 5 (1997) 1921-1939.

[10] A. W. J. Kolen, A genetic algorithm for frequency assignment, Tech. report, Universiteit Maastricht, 1999.

[11] A.M.C.A. Koster, Frequency Assignment - Models and Algorithms, Ph.D. Thesis, Maastricht University, 1999.

[12] R. Montemanni, D.H. Smith and S.M. Allen, Lower bounds for fixed spectrum frequency assignment, Ann. Oper. Res. 107 (2001) 237-250 (2002).

[13] A.P. Punnen, The traveling salesman problem: new polynomial approximation algorithms and domination analysis, J. Inform. Optim. 22 (2001) 191-206.

[14] A.P. Punnen, F. Margot and S.N. Kabadi, TSP heuristics: domination analysis and complexity, to appear in Algorithmica.

[15] V.I. Rublineckii, Estimates of the accuracy of procedures in the travelling salesman problem, Numer. Math. Comput. Technol. IV (1973) 18-23 (Russian).

[16] V.I. Sarvanov, The mean value of the functional of the assignment problem. Vestsi. Akad. Navuk Belarusi Ser. Fiz.-Mat. Navuk 143 2 (1976) 111-114.