A branching process with fitness.

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The model. The model under consideration is the reinforced branching process analysed in a recent series of papers (see [1] and references therein). Variants of this model include the Bianconi-Barabasi (BB) model [2] of preferential attachment with fitness. The model describes a growing population of individuals (e.g., bacteria), each characterised by a fitness $x \in [0,\infty)$ which controls its reproduction rate. Each individual can independently divide with rate x, and each division event can either produce a 'descendant' of the same fitness x (this happens with probability γ), or (with probability β) a 'mutant' with a different fitness y drawn independently from a distribution with density $\mu(y)$. Generically $\alpha+\beta>1$, so that a division event can produce *both* a descendant and a mutant, with probability $\gamma+\beta-1$. All individuals are immortal. Setting $\gamma=\beta=1$ reproduces (in continuous time) the dynamics of the degree distribution in the BB model (but not the full topology of the BB network): an 'individual' is now a half-link emanating from a node, with each 'mutant' corresponding to a new node.

A population grown according to these rules can be viewed as a collection of 'families', each with its own fitness (growth rate) x. Quantities of interest include the overall size of the population $N(t)$ at time t, the number of distinct families $M(t)$, the population profile $N(x,t)$, and the distribution of family sizes $P(n,t)=\mathbb{E}\left[\sum_{i=1}^{M(t)}\delta_{n_i(t),n}\right]$, where $n_i(t)$ is the population of the i-th family at time t. An issue that engendered a considerable amount of discussion in the literature is whether this model allows a "winner takes all" behaviour whereby a single family contains a finite fraction of the whole population, analogous to Bose-Einstein (B.-E.) condensation into the lowest available energy eigenstate [2]. This question has been answered in the negative in Ref. [1] in the case of a class of distributions $\mu(x)$ with finite support. The purpose of this note is to investigate the cases of finite and infinite support within a unified framework, explicitly studying the time evolution of the quantities of interest rather than just the limiting behaviour. Expectations of the relevant population sizes are considered, as well as the family size distribution. The distributions of the whole population size require a different set of techniques [3].

Methodology. The starting point is the generating function $G(x,t,z)=\mathbb{E}\left[\sum_{i=1}^{M(t)} z^{n_i(t)}\delta(x-x_i)\right]$, where x_i is the fitness of the i-th family. It follows immediately that $\int dx G(x,t,1)=\mathbb{E}[M(t)]$, similarly $z\partial_z G(x,t,1)|_{z=1}=\mathbb{E}[N(x,t)]$, and $\int dx \oint \frac{G(x,t,z)}{z^{n+1}}\frac{dz}{2\pi i}=P(n,t)$.

Applying the standard techniques one finds that $G(x,t,z)$ satisfies

$$\partial_t G(x,t,z) = \gamma x(z-1)z\partial_z G(x,t,z) + z\beta\mu(x)F(t), \tag{1}$$

with $F(t) = \int \partial_z G(z,x,t)\Big|_{z=1} x\,dx$. Solving Eq. (1) one finds the following self-consistency equation on $F(t)$:

$$F(t) = m(t) + \beta\int_0^t F(\varphi)m(t-\varphi)d\varphi, \tag{2}$$

with $m(t) = \int \mu(x)e^{\gamma xt}xdx$. In order for the population not to explode in finite time $\mu(x)$ must either have a finite support, or decay at large x faster than any exponential.

Case 1: Finite support. Without loss of generality one can restrict x to $[0,1]$. Consequently, $m(t)$ possesses a Laplace transform

$$\tilde{m}(p) = \int_0^1 \frac{x\mu(x)dx}{p-\gamma x}. \tag{3}$$

Equation (2) is then solved straightforwardly:

$$F(t) = \int_{c-i\infty}^{c+i\infty} \frac{dp}{2\pi i} e^{pt} \frac{\tilde{m}(p)}{1-\beta\tilde{m}(p)}, \tag{4}$$

where $c > \max\gamma, p^*$, and the Malthusian parameter p^* (if it exists!) is the root of $1 = \beta\tilde{m}(p)$. Existence of the root distinguishes the regime without “condensation”, in analogy with the formalism of genuine B.-E. condensation. It can be shown that p^* exists if $\mu(1)$ is finite (both cases are possible if $\mu(x\to 1)\to 0$), is unique, and that $p^* > \gamma$. This last property ensures that the integral (4) is dominated by the pole at p^* rather than the cut along $[0,\gamma]$, and so $F(t) \to e^{p^*t}/\beta^2\rho(p^*)$, with $\rho(p^*) = \int \frac{x\mu(x)dx}{(p^*-\gamma x)^2}$. Correspondingly, the asymptotically dominant part of the expected population profile evaluates to $\mathbb{E}[N(x,t)] \to \frac{e^{p^*t}}{\beta\rho(p^*)(p^*-\gamma x)}$, with the overall population growth and the total number of families both proportional to e^{p^*t}.

In the opposite case when p^* does not exist ($\mu(x\to 1)\to 0$ is a necessary but not a sufficient condition for that), the integral (4) is controlled by the cut inherited from $\tilde{m}(p)$. The overall population growth is then proportional to $e^{\gamma t}I(t)$, where $I(t) = \int_0^1 \mu(1-\xi)e^{-\xi\gamma t}d\xi$. For example, if $\mu(x)\sim(1-x)^\alpha$ with some finite parameter α, the integral gives a power-law correction to the dominant exponential behaviour. The expected number profile evaluates to the sum of two qualitatively different contributions: a smooth term proportional to $\frac{\beta}{\gamma+\beta}\frac{\mu(x)}{1-x}$, which accounts for a finite fraction of the overall population, and an asymptotically singular term proportional to $\mu(x)e^{-\gamma t(1-x)}$. When t is large (and taking into account that $\mu(x\to 1)\to 0$) the second term has the form of a sharp peak ‘squeezed’ ever closer to 1. If normalised, this term would converge to a δ-function, consistent with [1], with the peak profile generalising Conjecture 8.1 of [1] obtained for a power-law $\mu(x\to 1)$ in a different setting. The expected number of families under the peak is macroscopic, generalising the conclusion of Ref. [1] that there are no ‘winner-take-all’ families to arbitary $\mu(x)$ consistent with fintie support and absence of maltusian parameter.

Case 2: Infinite support. The solution of Eq. (2) is now complicated by the fact that Laplace transform of $m(t)$ does not exist. This difficulty can be circumvented by performing an analytical continuation into the complex t plane, solving the equation along a direction where $m(t)$ is 'well-behaved', and analytically continuing the result back to the real axis. (see, e.g., [4]). One therefore obtains $F(t) = \int_{\mathscr{C}_H} \frac{dp}{2\pi i} e^{pt} \frac{\tilde{m}(p)}{1-\beta\tilde{m}(p)}$, where $\mathscr{C}_H$ is the Hankel contour. Subsequent evaluation of the integral gives the following asymptotic result: $\mathbb{E}[N(t)] \to (1+\beta/\gamma)\int_0^\infty \mu(x)e^{\gamma xt}dx$. The asymptotic behaviour of the number profile is subtle, due to the fact that the corresponding normalised density profile does not exhibit uniform convergence as $t\to\infty$. If x is fixed, one obtains $\mathbb{E}[N(x,t)] \to \mathbb{E}[N(t)]\frac{\beta\mu(x)}{\beta+\gamma}$, analogous to the smooth piece in the finite support case. However, the integral of this contribution contains only a finite fraction of the overall population. The second contribution is a smooth peak having the shape $\mu(x)e^{\gamma xt}$, centered at an increasing with time ('travelling') position $x_0(t)$ given by the solution of $\gamma t = -\mu'(x)/\mu(x)$. The area under the peak contributes the 'missing piece' of the overall expected number.

It is illuminating to specialise to $\ln\mu(x) \sim -x^{1+\alpha}$, with $\alpha > 0$ to ensure existence of $m(t)$. One then finds $x_0(t) = (\gamma t/(1+\alpha))^{1/\alpha}$, and, up to subleading corrections, the total number of individuals under the 'travelling' peak is $\exp\{\alpha(\gamma t/(1+\alpha))^{1+1/\alpha}\}$. However, the expected number of families there is proportional to $\exp\{(\alpha-1)(\gamma t/(1+\alpha))^{1+1/\alpha}\}$. Therefore in the regime $1<\alpha<2$ this number decays at large t, and eventually becomes less than 1. Thus a family with fitness in the peak region exists only occasionally, with a population much larger than the expected value.

Such a picture may be consistent with occasional existence of a 'winner-take-all' family. Further insight can be gained by analysing the family size distribution:

$$P(n,t) = \int_0^\infty \mu(x)dx \int_{\mathscr{C}_H} \frac{dp}{2\pi i} \frac{\beta F(p)e^{pt}}{\gamma x} \int_{e^{-\gamma xt}}^1 d\zeta\,\zeta^{p/\gamma x}(1-\zeta)^{n-1}. \qquad (5)$$

For a finite n this expression reduces to the Beta-function, reproducing the known results [5]. Assuming, however, that n scales with $\mathbb{E}[N(t)]$, one can find that the probability of family size greater than Const.$\cdot\mathbb{E}[N(t)]$ behaves as $\exp\{[1-(1+\alpha)^{1+1/\alpha}]\alpha(\gamma t/(1+\alpha)^2)^{1+1/\alpha}\}$, and therefore decays as t becomes large for any α. This decay, however, can be slower than the probability of finding a family under the travelling peak (estimated as the inverse expected number of such families) if $1/(1+\alpha)^{1+1/\alpha} > \alpha$, and therefore not inconsistent with the picture of occasional (with probability going to zero as $t\to\infty$) 'winner-take-all' families, large enough to ensure a finite contribution to the expected overall number. A more subtle analysis exploring the correlations between the numbers of families and their sizes is needed to bring full clarity to the issue.

References

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