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Primary pre-service teachers: reasoning and generalisation

Tim Rowland[1], Gwen Ineson[2], Julie Alderton[1], Gina Donaldson[3], Chronoula Voutsina[4] & Kirsty Wilson[5]

[1]University of Cambridge, UK [2]Brunel University London, UK [3]Canterbury Christchurch University, UK [4]University of Southampton, UK [5]University of Birmingham, UK

Generalising tasks, in the context of mathematical reasoning, have featured in our work with primary pre-service teachers (PSTs). We used two particular problems - 'matchstick squares' and 'flower beds' - to explore the generalisation approaches taken by PSTs. In this paper, we analyse the ways in which one of them, Terry, uses recursive or functional approaches to generalisation, and how he attends to looking for a relationship and seeing sameness and difference between figures in a sequence. We consider what motivates shifts in attention, the significance of the PST's prior experience and of PST-collaboration in our teaching sessions. We conclude with a discussion about the significance of this activity in the PST's preparation for teaching, with reference to Mason's (2010) notions of pro-spection and retro-spection.

Keywords: generalisation; reasoning; pre-service primary mathematics teacher education.

Introduction

The current National Curriculum (Department for Education, DfE, 2013) for children at primary schools in England now includes reasoning as an explicit aim of its programme of study for primary mathematics. This has renewed the place of reasoning in the debate about teaching and learning of children in primary school. For example, national testing for children aged 7 and 11 now includes written papers on mathematical reasoning (DfE, 2017).

However, the term ‘mathematical reasoning’ covers many different thinking processes and strategies, and DfE exemplification focuses on reasoning associated with answering closed questions (DfE 2016). This sort of reasoning does not necessarily match the aim of the National Curriculum, which focuses on conjecturing and generalisation.

The authors of this paper are members of a larger group of primary mathematics educators, each with a commitment to research in mathematics education. The group has met about twice a year, for 10 years. As primary mathematics teacher educators in five universities, we have found that we promote mathematical reasoning in similar ways in our programmes. We have a shared belief in the value of reasoning associated with pattern, algebra and generalisation, and find that we use very similar activities in our sessions. In order to enrich our work as tutors on pre-service teacher education programmes, we wanted to investigate how student teachers respond to university-based training sessions which aim to prepare them to teach reasoning, and to explore the approaches to generalisation that student teachers adopt themselves when engaging with such activities.

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Generalisation

Within the broader context of mathematical reasoning, a common context for generalising, sometimes referred to as 'growing patterns,' is a sequence of geometric figures constructed from, for example, matchsticks, squares or dots. Learners' attempts to generalise such a pattern can involve "manipulating the figure itself to make counting easier; finding a local rule (recursion) which reflects one way to build the next term from previous ones; (and) spotting a pattern which leads to a direct formula" (Mason, 1996, pp. 75-76). One important theme of the research on pattern generalising is this distinction between finding a local, *recursive* relationship and a direct, *functional* relationship. Research points to learners' preferences towards finding a local rule of recursion between figures in a sequence, and the relative difficulty of finding a functional relationship (MacGregor & Stacey, 1993; Stacey & MacGregor, 2001).

For example, in the 'growing pattern' of matchstick-squares shown below (Figure 1):

Figure 1. Matchstick squares

a recursive response would observe that each one has 3 more matches than the previous one. So 4, 7, 10, …. The number of matches in, say, the 10th configuration could be found by extending the number sequence …13, 16, etc. A functional insight would observe that when there are *n* squares, the number of matches can be expressed as 3*n*+1. In this way I can find how many matchsticks there would be if there were 10 squares, without having to list the previous 9.

Ferrara and Sinclair (2016) argue that while noticing a recursive relationship requires an understanding of horizontal 'mobilities,' identifying a functional rule requires an understanding of vertical 'mobilities,' i.e. understanding the relationship between the independent and dependent variable.

Wider literature also identifies the significance of visualisation in pattern generalisation. Wilkie and Clarke (2016) explored the different ways in which individual students *see* a pattern, by inviting them to use colour to show how they saw elements of the geometric shape. They found that the subsequent generalisations reflected the ways in which students initially perceived the pattern. Seeing the structure of a figure as the result of 'growth' from previous figures led to a recursive rule, while other ways of seeing led to a functional rule. Different ways of seeing and counting elements in a pattern can lead to different, equivalent generalisations.

Bills and Rowland (1999) contrast two ways of arriving at a functional generalisation, which they call 'empirical' and 'structural'. The fundamental distinction is between knowing *that* and knowing *why*. In the case of the squares growing pattern (Figure 1), an *empirical* approach would reason: I have the numbers 4, 7, 10, 13, … and I observe that these can all be expressed as 3*n*+1. It's just a fact, and it works, though I don't know why. A *structural* insight might perceive some general structure in the situation – for example, that in every case, there is a row of C-shapes, each with 3 matchsticks, and one to complete the last square. So there are 3*n*+1 matchsticks in the *n*th configuration.

For a striking numerical example of the distinction, consider the sequence 1, 1+3, 1+3+5, etc. A functional generalisation – that the terms are all perfect squares, and the *n*th term equals n^2 – follows fairly readily. In the first instance this might well be an empirical generalisation – I don't (yet) know *why* these sums are squares. The generalisation becomes structural if, for example, we envisage a 3x3 square array of dots (Figure 2), with 1 dot bottom left, 3 dots adjacent to the first one, (building a 2x2 square array), then 5 dots above and to the right of that (2x2) square, completing the 3x3 square array. The first *n* odd numbers are then seen as a set of dots from which an *n* x *n* array is constructed.

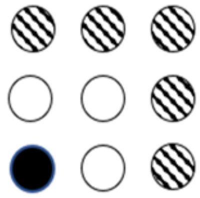

Figure 2. Square array

In summary, seeing the structure of a geometric figure supports what Bills and Rowland (1999) refer to as 'structural' generalisation. This is in contrast to 'empirical' generalisation which, in the context of a geometric sequence, describes a consistent relationship identified between quantifiable elements, such as the figure number and number of matchsticks. The resulting (empirical) generalisation is then "divorced from the structure of the pattern" (Küchemann, 2010, p.233). Küchemann (2010) makes a compelling case for focussing on structure *within* a single figure in a sequence rather than presenting learners with a systematic sequence of elements. Such analysis of the structure of a *generic example* fosters "seeing a generality through the particular" (Mason, 1996, p.65). (The above account of the 3x3 square of dots (with Figure 2) was intended to be generic in connection with 1+3+ … (2*n*-1)). Beyond working with a generic example, teachers have an array of pedagogic choices which may shape pattern perception and visualisation. These include the use of concrete materials, drawings, diagrams and technological environments (Wilkie & Clarke, 2016).

While, in the literature, relatively little attention has been paid to teacher knowledge in relation to generalising and functional thinking, there is evidence that this is an area of difficulty for primary teachers and primary pre-service teachers (Wilkie, 2016; Goulding *et al.*, 2002). Wilkie's research highlighted "the importance of teachers developing their own ability to generalise patterns and to learn to understand the process by which students develop functional thinking through recursive and explicit generalisation" (p.270). Our own study explores these important ideas, as pre-service teachers work on tasks which challenge them to reason, yet are 'sufficiently close' to primary mathematics.

The Study

This paper presents the approach that one student teacher - we call him Terry - took to tackling a problem involving reasoning and generalisation. Terry was on a one year graduate primary teacher education course, specialising in mathematics. The session that Terry reflects on below was designed to enable students to explore growing patterns, whilst working together with peers to explore possible alternative approaches. Students were presented with the Flowerbed pattern (original source unknown) where square slabs are placed around the border of a square flowerbed - see Figure 3 below. They were asked to generalise about the number of paving slabs required around each square bed. Students were given some time initially to consider the problem, then they worked together, sharing their approaches. Shortly after the taught session, Terry was interviewed about his approach to the problem. We used Wilkie's notions of 'recursive' and 'functional' thinking to analyse his responses.

Terry's response and our analysis

Terry had a degree in Theatre Studies and had studied mathematics at A level. In the interview, he said that he had been confident with the subject in the first year of his A level study but had found the second year "quite a lot more challenging". Terry was enjoying teaching mathematics and had found the experience of applying his mathematics knowledge in his teaching practice rewarding. The specialist course had changed his view of the subject by introducing him to mathematics pedagogy.

> Terry: I think my view of mathematics was quite narrow until coming onto the course and just seeing how everything can be broken down and made so much more accessible, even … even things like fractions which is like this feared term in primary schools.

Terry recounted his approach to the Flowerbeds problem with reference to his notes from the session. During the interview, and while he was explaining his train of thought, Terry made additional notes on a printed illustration of the pattern that was provided by the interviewer (one of the authors). He explained that his initial approach was to focus on the number of squares that formed the centre of the shape for each case. He wrote the corresponding numbers (1, 4, 9) under each case and then counted the number of white squares that surrounded the dark-shaded centre of the shape in each case (8, 12, 16) (Figure 1).

> Terry: I started off by noting down, we had case 1, case 2, case 3, and I noted down how many squares were in the centre of the flowerbed … Yeah, so I was drawn to that, so we had 1, 4 and 9. And then I calculated …

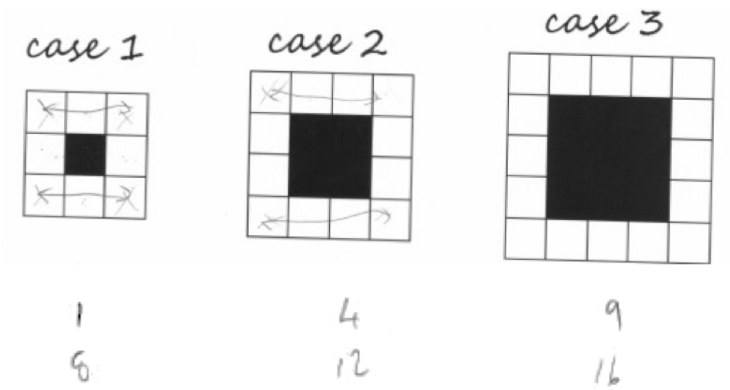

Figure 3. Terry's jottings while explaining his initial approach to the pattern.

Terry continued his explanation referring to his own notes from the session.

> Terry: And then I started off by trying to figure out some kind of pattern or link or connection between those numbers, and I wasn't really getting anywhere to be honest. And then I … I thought back to a previous university session, when we did something similar to generalising, where we found something that stays the same each time.
>
> Interviewer: OK.
>
> Terry: So this is obviously where I'd gone to in the middle, originally that is different each time …so I thought what is the same each time.…And it ended up being the four corners. …Were the same each time, there's always going to be four corners, so that's where I ended up going down this route.
>
> Interviewer: OK, and following that, after you saw the four corners as staying the same, what did you do next? Where did you go after that?

> Terry: So for this one I would have, N would be 1 (referring to case 1), so I'd have … four lots I think of N, and then I would be adding on … oh no hang on … this is 4 here, that's always … I've just confused myself.

Terry's initial approach was to count the squares of each case in the sequence with the view of identifying a functional relationship (determined empirically) between two quantifiable aspects of each case; the number of squares that constitute the central part of each case and the number of white squares that surround the central part (Figure 3).

The difficulty that he encountered in identifying a link between these numbers prompted a move to a recursive approach whereby he looked for what remained the same and what changed in each item of the sequence. This was supported by his recollection of a similar activity and strategy that he had learned in a previous university session. Terry found it difficult to conclude his explanation. The interviewer prompted a bit more.

> Interviewer: Right, so you have the four corners as a constant feature.
>
> Terry: Yes.
>
> Interviewer: And then what happens? Are you looking at the squares between the corners now?
>
> Terry: Yeah, so then there's, we've got … four here and then obviously one, two, so it's two lots …
>
> Interviewer: So you're still looking at the middle part or not anymore?
>
> Terry: I, yes, to base off this one.
>
> Interviewer: OK.
>
> Terry: So you've got the, I guess we call that, maybe that can be called N, so it's 4N….Plus 4 …
>
> Interviewer: … N is the centre one with four around it?
>
> Terry: Yes, so there's four lots of N around it.

In the above extract, Terry goes back to focusing his attention on a single case of the sequence (case 1) seeking to identify a general rule with attention to the *structure* of the shape. He associates N with the central black square. He refers to 4N as representing the four adjacent white squares and to "Plus 4" as representing the four constant corners. When moving his attention to case 2, he becomes confused and returns to recursive reasoning.

> Terry: And then … plus four, this one, but then I'm, I've not accounted for this one, have I? Or have I? No, I haven't.

Here, "plus four, this one" refers to Terry's observation that the sides of the square in case 2 (excluding the four corners) are formed out of eight, in total, white squares that are adjacent to the centre (i.e. four more than the squares that constitute the sides in case 1). However, at that point Terry realises that he has not accounted for how the central square has grown moving from case 1 to case 2 and remains puzzled.

At this point, Terry recalls his collaboration with one of his peers during the session, and describes an alternative approach that they took when seeking the general (functional) rule for the sequence.

> Terry: Yeah, well we had ways of looking at it, I mean I think, that was one way of seeing it. The other way I saw was I'd looked at this as like a 1, 2, 3 (draws a line across the three white squares in the first and third row of case 1).

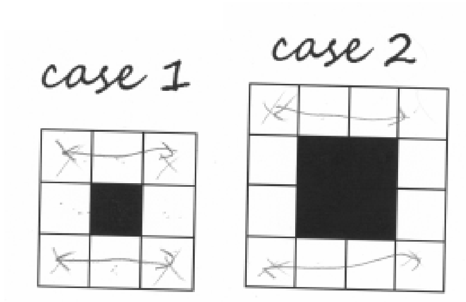

Figure 4. Terry's jottings on the printed pattern

> Terry: And then there was the middle ones and these, (referring to the central square of case 1 and the squares on either side of it) and then the same with this one (case 2), the top … (draws a line across the top and bottom rows of case 2, Figure 4).
>
> Interviewer: And you are still looking at the middle part, the dark part, yeah?
>
> Terry: Yes, so this one (goes back to case 1) I guess would be N and then there's, so there's two lots of N isn't there, and then on the top there's plus two, so two lots of N plus 2.
>
> Interviewer: Where are the two lots of N? What is the two lots of N? The four squares in the middle of case 2?
>
> Terry: Ehm … so 2, it's case 2 and then we've got on the top 1 and 2, 3, 4, so N plus 2 …Two lots of N plus 2.

With the assistance of one of his peers, the 'structure' perceived by Terry has now changed. Focusing on case 1, Terry associates N with 1 and explains that the number of squares in the top and bottom row is represented by N+2 so the top and bottom row are "two lots of N plus 2". He provides the same explanation for case 2 (Figure 4) noting the relationship between N and the number of squares that form the top and bottom row but without accounting the central, dark square and the adjacent white squares. Although he did not complete the formula here, he had generalised about all sections of the pattern separately by that point.

Towards the end of the session, the interviewer asked Terry to indicate one thing that he had learned from this session and would apply when he teaches mathematics.

> Terry: Giving children plenty of opportunity to discuss, I think that's quite important, and just to encourage people to discuss in the classroom because I know …
>
> Interviewer: Why do you think it's important?
>
> Terry: Because that's what helped me in terms of when I heard …
>
> Terry: … anything like that, that often was like a hook into allowing me to access the problem in which, without that I wouldn't have been able to. If it was just silent, I would have been sat there in my own space, staring at the one way I could identify it, trying to see it in some other way, but probably struggling and failing miserably. But being able to hear other people discuss it, allowed me like access into the problem a little bit more.

In his response, Terry highlights, on the basis of this experience, the value of opportunities for classroom discussion that encourage learners to see patterns in different ways, and to allow all learners to access tasks that might have been too challenging for them to tackle on their own.

Conclusion

Terry's account of different approaches to the exploration for a general rule indicated shifts of reasoning and attention to recursive as well as functional relationships (Ferrara & Sinclair, 2016). In this case, shifts of reasoning appeared to be prompted by difficulty in completing a particular line of exploration, which steered Terry to draw from his prior experience with similar activities, and also, by his observation of alternative approaches that others had adopted, in a setting that encouraged peer collaboration. Through the reported shifts between functional and recursive thinking, Terry appeared to maintain, largely, his focus and attention to the structural elements of the sequence (Küchemann, 2010).

Although Terry explicitly referred to "other ways of looking at it [the pattern]", we cannot know whether he was aware of his move between different kinds of mathematical reasoning. A question that is raised for us, as primary mathematics teacher educators, is whether this matters, and whether it would require greater and explicit emphasis as part of our sessions. Terry considered the opportunity to see the structure in different ways, in discussion with his peers, to be the key learning from this experience, and that that would influence his own teaching in the classroom. This highlights the value of including such activities in mathematics teacher-preparation sessions, offering pre-service teachers the opportunity to experience generalisation explorations for themselves, and to identify aspects of practice that would be important in their own classrooms.

Next Steps

In the next phase of this research, we are investigating how best to prepare our pre-service primary teachers to introduce and support children in school to work with generalising activities. The mathematics specialist PSTs in one of our universities have already worked on pattern generalisation tasks with a group of children in school, and discussed that experience at a follow-up session with their university tutor (one of the authors). As a theoretical framework for analysing their feedback, we are working with Mason's (2010) dictum that "in order to learn from experience it is necessary to do more than engage in activity" (p. 23). Mason (2010) suggests that teachers can do the following – for themselves and each other – to engage with pro-spection (anticipation) and retro-spection (reflection) on teaching: (i) work on mathematics for themselves to "sensitise themselves to the struggles that pupils experience" (p. 43), and (ii) collaborate in their enquiries – "to direct each other's attention to salient features so that finer distinctions can be made" (p. 43). This pro-spective and retro-spective activity relate both to their own learning (about generalisation) and their own teaching. Analysis of data from the university-based follow-up session is ongoing, against a framework derived from these insights from Mason (2010).

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