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Distributed Set-Membership Filtering for Multirate Systems Under the Round-Robin Scheduling Over Sensor Networks

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***Abstract*—In this paper, the distributed set-membership filtering problem is dealt with for a class of time-varying multirate systems in sensor networks with the communication protocol. For relieving the communication burden, the round-Robin (RR) protocol is exploited to orchestrate the transmission order, under which each sensor node only broadcasts partial information to both the corresponding local filter and its neighboring nodes. In order to meet the practical transmission requirements as well as reduce communication cost, the multirate strategy is proposed to govern the sampling/update rate of the plant, the sensors, and the filters. By means of the lifting technique, the augmented filtering error system is established with a unified sampling rate. The main purpose of the addressed filtering problem is to design a set of distributed filters such that, in the simultaneous presence of the RR transmission protocol, the multirate mechanism, and the bounded noises, there exists a certain ellipsoid that includes all possible error states at each time instant. Then, the desired distributed filter gains are obtained by minimizing such an ellipsoid in the sense of the minimum trace of the weighted matrix. The proposed resource-efficient filtering algorithm is of a recursive form, thereby facilitating the online implementation. A numerical simulation example is given to demonstrate the effectiveness of the proposed protocol-based distributed filter design method.**

***Index Terms*—Distributed filtering, multirate mechanism, round-Robin (RR) protocol, sensor networks, set-membership filtering.**

I. Introduction

OVER the last few years, the filtering or state estimation problem has gained increasing research attention due to its promising application prospects in areas of signal processing and control engineering, and a variety of filter design schemes have been reported in [3], [6], [14], [22], and [35]. Among others, the Kalman filtering algorithm and its variants (e.g., extended Kalman filtering and unscented Kalman filtering algorithms) have proven to be powerful tools for tackling the control/filtering problems where the noises are assumed to be Gaussian. Such a Gaussianity assumption is, unfortunately, a bit too restrictive owing to the fact that many kinds of noises are either deterministic or random but with non-Gaussian statistics. As such, the robust and/or H_∞ filtering approaches have been developed to cope with the non-Gaussian noises entering into the target plant and the observation model (see [9], [23], [30], [39]). It should be pointed out that, when considering the filtering issue with bounded noises confined to certain ellipsoids, the set-membership filtering serves as a well-appreciated robust filtering scheme ensuring the error states to be included in certain optimized ellipsoids with 100% confidence [10], [15], [17], [34], [45].

These days, the sensor network is undergoing a notable surge in popularity due primarily to its attractive characteristics, such as low cost and high scalability [13]. Typical applications of sensor networks include the health monitoring [33] and the forest fire detection for large-scale complex environment [37]. In this context, the so-called distributed filtering/estimation strategy, whose aim is to achieve the desired estimation performance in a collaborative manner, has drawn considerable research attention with a great deal of results available in the literature. Compared to the centralized scheme, a distinctive feature of the distributed strategy, as summarized in [7], is the dramatic improvement in the reliability (against the sensor failures) and efficiency (upon computation and communication capabilities). Up to now, some widely used distributed filtering schemes include the distributed H_∞ filtering, the distributed Kalman filtering, the distributed set-membership filtering, as well as the distributed moving-horizon filtering (see [2], [4], [16], [20], [26], [38]). Note that, in contrast with the fruitful results on Kalman-type filtering strategies, little attention has been paid to the distributed set-membership filtering issue despite its compelling advantage in dealing with bounded noises residing within prescribed ellipsoids, and this motivates our current investigation.

Typically, a sensor network consists of plenty of spatially distributed sensor nodes that have the capability of collecting

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and processing data as well as communicating with neighboring nodes according to a given topology. A well-recognized weakness of the sensor networks is that each sensor node has limited sensing/communication range resulting from its limited power storage capacity, and it thus makes practical sense to develop certain *energy-efficient* distributed filtering schemes. For example, for the purpose of reducing the frequency of data transmission, the event-based distributed filtering scheme has already been preferred and adopted in industry, see [8], [20] for some representative works. Another well-known energy-saving way is to employ appropriate data scheduling strategies in order to reduce communication traffic. By now, several commonly employed communication protocols have been successfully applied to the networked control systems (NCSs) to schedule the data transmission. These protocols include, but are not limited to, the try-once-discard (TOD) protocol, the round-Robin (RR) protocol, and the stochastic communication (SC) protocol [5], [19], [24], [25], [36], [44], [45].

Compared with the TOD protocol and the SC protocol where the transmission node is dynamically selected, the RR protocol is considered to be a kind of static scheduling protocols, under which the transmission order is given in advance following a circular manner. On account of its periodic property, a common method for analyzing/synthesizing the NCSs scheduled by the RR protocol is to transform the original system into a periodic switching system. Using this transformation method, the RR-protocol-based state estimation problem has been discussed in [29] for a group of discrete-time genetic regulatory networks and in [28] for a class of discrete-time nonlinear singularly perturbed complex networks. Nevertheless, when it comes to the time-varying systems, the method for periodic switching systems is no longer effective due mainly to the time-varying behaviors. Consequently, it is of both theoretical importance and practical significance to seek an efficient approach that is capable of characterizing the impact from the periodic property of the RR protocol on the control/filtering performance for the time-varying systems, and this gives rise to another motivation for this paper.

In order to facilitate the data transmission in a networked environment, the physical signals of interest are always subjected to A/D and D/A conversions. Generally speaking, frequent conversions might improve the system performance to an extent at the expense of the resource/cost. However, from a practical point of view, it is sometimes unnecessary or even impossible to sample/update all the different kinds of signals at the same rate. As such, instead of using the traditional synchronous sampling/update mechanism, it seems natural to develop the so-called multirate strategy, whose underlying idea is to adopt different sampling/update rates for different components (e.g., sensor, filter, controller, and actuator) while ensuring the desired system performance. Such a multirate scheme has been widely recognized in research communities and successfully deployed in industrial practice (see [11], [12], [18], [27], [41]–[43]). In particular, the lifting technique proposed in [21] has played a vital role in promoting the multirate schemes by transforming the multirate system into an equivalent linear time-invariant system with a unified sampling period. However, to the best of our knowledge, the multirate strategy has not gained adequate research attention for the set-membership filtering problem, and such a gap arouses us to look into the influence from the multiple rates onto the filter performance over a sensor network.

Motivated by the above discussion, in this paper, we proceed to launch an investigation on the distributed set-membership filtering problem for a class of time-varying systems in the simultaneous presence of the RR transmission protocol and the multirate mechanism. The main challenges stem from the following four aspects.

1) For the RR transmission protocol, traditional methods for periodic switching systems are not applicable to the time-varying systems and, therefore, the first challenge is to seek an effective methodology to reveal the impact from the RR protocol on the filtering performance over a finite horizon.
2) In sensor networks, the communication topology is preset according to the specific task and, consequently, the second challenge is to ensure that the introduced RR protocol will not change the fixed topology structure.
3) It is quite general for the plant, the sensors, and the filters to be governed by different sampling/update rates. Hence, the third challenge is how to transform such a multirate system into a single-rate one.
4) Due to the existence of the RR protocol and the multirate strategy, how to construct an effective filter to subtly coordinate these two kinds of transmission rules constitutes the fourth challenge.

In response to the aforementioned four challenges, the primary contributions of this paper can be highlighted as follows.

1) The multirate strategy is, for the first time, considered for the distributed set-membership filtering problem.
2) With the help of the lifting technique combined with the zero-order holder (ZOH) strategy, the impact from both the multirate scheme and the RR protocol is explicitly reflected in the distributed filter design.
3) A series of optimized ellipsoids in the sense of the minimum trace of the weighted matrix is obtained by solving the optimization problems with certain inequality constraints.

The outline of this paper is as follows. Section II formulates the multirate systems under consideration, gives a novel description of the RR protocol and develops a time-varying distributed filter. In Section III, by means of the recursive linear matrix inequality (RLMI) technique, sufficient conditions are derived to guarantee the existence of the desired distributed filters for different filtering error constraints. Then, a set of optimized ellipsoids is derived by solving some optimization problems with matrix inequality constraints. Section IV provides a numerical simulation to show the validity of the proposed filter design algorithm. Section V concludes this paper.

Notations: The notation used here is fairly standard except where otherwise stated. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote, respectively, the n-dimensional Euclidean space and the set of all $m \times n$ real matrices. For a matrix M, M^{T}, and M^{-1} represent the transpose and the inverse of matrix M, respectively. The notation $X \geq Y$ ($X > Y$) with X and Y being the symmetric matrices means that $X - Y$ is non-negative definite (positive definite). trace(X) denotes the trace of square matrix X. diag$\{A_1, A_2, \ldots, A_n\}$ represents a block diagonal matrix whose elements are matrices $A_1, A_2, \ldots, A_n$. I stands for the identity matrix of compatible dimension. $\mathbf{I}_n$ denotes an n-dimensional column vector with all entries 1. The notation "mod(a, b)" represents the remainder of diving an integer a by a positive integer b. col$\{x_1, \ldots, x_n\}$ denotes the column vector $[x_1^{\mathrm{T}}, \ldots, x_n^{\mathrm{T}}]^{\mathrm{T}}$. $\otimes$ denotes the Kronecker product.

II. Problem Formulation

Consider a sensor network whose topology is represented by a directed graph $\mathbf{G} = \{\mathbf{V}, \mathbf{E}, \mathbf{A}\}$ with the set of nodes $\mathbf{V} = \{1, \ldots, N\}$, the set of edges $\mathbf{E} \subseteq \mathbf{V} \times \mathbf{V}$, and a weighted adjacency matrix $\mathbf{A} = [a_{ij}]$ with non-negative adjacency elements a_{ij}. An edge of $\mathbf{A}$ is denoted by (i, j). The adjacency elements associated with the edges of the graph are positive, that is, $a_{ij} > 0 \Leftrightarrow (i, j) \in \mathbf{E}$. Moreover, we assume $a_{ii} = 1$ for all $i \in \mathbf{V}$ and, therefore, (i, i) can be regarded as an additional edge. The set of all neighbors of node i is defined as $\mathcal{N}_i \triangleq \{j \in \mathbf{V} : (i, j) \in \mathbf{E}\}$.

Consider the system described by the following dynamics:

$$x(k_{i+1}) = A(k_i)x(k_i) + B(k_i)w(k_i) \tag{1}$$

where $x(k_i) \in \mathbb{R}^{n_x}$ is the state vector; $A(k_i)$ and $B(k_i)$ are the known time-varying matrices with appropriate dimensions; $h_p \triangleq k_{i+1} - k_i$ ($\forall i = 0, 1, \ldots$) denotes the update period of the plant (1) with $k_0 = 0$; and $w(k_i)$ is the process noise belonging to the following ellipsoidal set:

$$\mathcal{W}(k_i) \triangleq \left\{ w(k_i) \in \mathbb{R}^{n_w} | w^{\mathrm{T}}(k_i)Q^{-1}(k_i)w(k_i) \leq 1 \right\} \tag{2}$$

with $Q(k_i)$ being a given positive definite matrix sequence.

The measurement model of the jth sensor node with the timescale t_i is described by

$$y_j(t_i) = C_j(t_i)x(t_i) + D_j(t_i)v(t_i) \tag{3}$$

where $y_j(t_i) \in \mathbb{R}^{n_y}$ is the measurement output; $C_j(t_i)$ and $D_j(t_i)$ are the known time-varying matrices with appropriate dimensions; $h_m \triangleq t_{i+1} - t_i$ ($\forall i = 0, 1, \ldots$) is the sampling period of sensors with $t_0 = 0$; and $v(t_i)$ is the measurement noise confined to the following ellipsoidal set:

$$\mathcal{V}(t_i) \triangleq \left\{ v(t_i) \in \mathbb{R}^{n_v} | v^{\mathrm{T}}(t_i)R^{-1}(t_i)v(t_i) \leq 1 \right\} \tag{4}$$

with $R(t_i)$ being a given positive definite matrix sequence.

A. Round-Robin Communication Protocol

In this paper, we focus on the distributed filtering problem over sensor networks with the RR protocol as depicted in Fig. 1. For the underlying sensor network, each sensor node is capable of computing and communicating with its neighboring nodes. However, as mentioned in [40], frequent communications among sensors occupy most of the energy of sensors. Therefore, from the perspective of reducing unnecessary communication traffic, the RR protocol is adopted during data

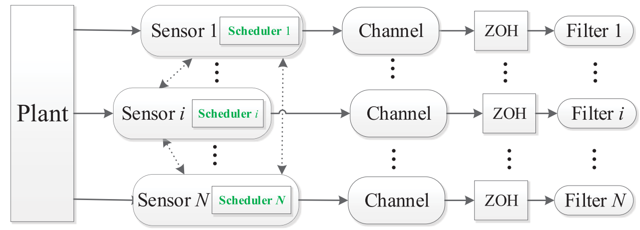

Fig. 1. Block diagram of sensor networks with the limited communication capacity.

interactions from each sensor node to both the corresponding local filter and its neighbors. In this case, at each time instant, only partial information of each node is transmitted according to the RR protocol.

Assume that the measurement information obtained from each sensor node consists of n_y data packets. Denote by $\theta^r(t_i) \in \mathcal{R} \triangleq \{1, \ldots, n_y\}$ the selected data packet from sensor node r at time t_i. Then, recalling the periodic property of the RR protocol, $\theta^r(t_i)$ is iteratively determined by

$$\theta^r(t_i) = \begin{cases} n_y, & i = 0 \\ \mathrm{mod}\big(i - 1, n_y\big) + 1, & \text{otherwise.} \end{cases} \tag{5}$$

Next, denote by $\bar{y}_j(t_i) \in \mathbb{R}^{n_y}$ the measurement signal after being transmitted and define the matrix $\Phi_i = \mathrm{diag}\{\delta(i - 1), \delta(i - 2), \ldots, \delta(i - n_y)\} \in \mathbb{R}^{n_y \times n_y}$, $\forall i \in \mathcal{R}$. By recurring to the ZOHs, $\bar{y}_j(t_i)$ is characterized by a sequence of delayed measurements with the following form:

$$\bar{y}_j(t_i) = \sum_{l=0}^{n_y - 1} \Phi_{\theta^j(t_{i-l})} y_j(t_{i-l}). \tag{6}$$

Here, $\theta^j(t_{i-l}) = l$ when $i - l < 0$ and $\bar{y}_j(t_i) = y_j(0)$ when $i \leq 0$, where $y_j(0)$ is the initial measurement output. Especially, let $x(t_i) \triangleq x(0)$ for $i \leq 0$ with $x(0)$ being the initial state.

Remark 1: From (6), the actual measurement signal after scheduling by the RR protocol is formulated by a sequence of time-delayed measurement outputs. Different from the periodic switching system method adopted in [29] and [44], the model proposed, in this paper, reflects the periodic feature of the RR protocol in a more concise way, where the main advantage lies in its applicability for the time-varying systems. Moreover, based on such a model, the existing theoretical tools related to the time-delay systems can be directly exploited to deal with the analysis/design issues of the filtering algorithm.

B. Multirate Scheme

For the sake of respecting physical restrictions on different devices (e.g., sensor, filter, controller, and actuator), the multirate sampling/update scheme is discussed here. To be more specific, it is assumed that the update period of the filters (to be designed in the sequel) is defined as $h_f \triangleq T_{i+1} - T_i$ ($\forall i = 0, 1, \ldots$) with $T_0 = 0$. The relationship among h_p, h_m, and h_f is $h_f = ah_p$ and $h_m = bh_f$, where a and b are positive integers. Fig. 2 provides an illustration of the multirate scheme among different devices.

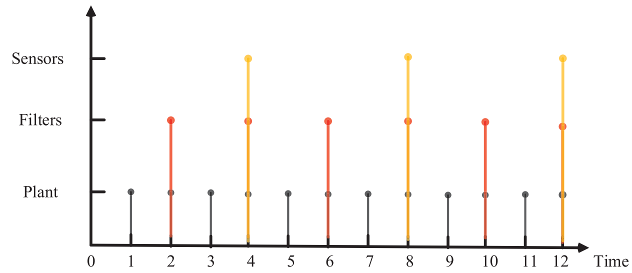

Fig. 2. Multirate mechanism for the plant, the filters, and the sensors with $a = b = 2$.

By means of the lifting technique, we derive the following system state equation with the timescale T_i:

$$x(T_{i+1}) = A_a(T_i)x(T_i) + B_a(T_i)w_a(T_i) \tag{7}$$

where, for $j = 1, \ldots, a-1$

$$\begin{aligned}
A_a(T_i) &= \prod_{j=1}^{a} A(T_{i+1} - jh_p) \\
w_a(T_i) &= \mathrm{col}\{w(T_i), w(T_i + h_p), \ldots, w(T_i + (a-1)h_p)\} \\
B_a(T_i) &= [\vec{B}_1(T_i) \quad \ldots \quad \vec{B}_{a-1}(T_i) \quad B(T_i + (a-1)h_p)] \\
\vec{B}_j(T_i) &= \prod_{s=1}^{a-j} A(T_{i+1} - sh_p)B(T_i + (j-1)h_p).
\end{aligned}$$

Then, the system state equation can be further reformulated with the timescale t_i

$$\begin{cases}
x(t_{i+1}) = \bar{A}_b(t_i)x(t_i) + \bar{B}_b^{(1)}(t_i)\bar{w}_b(t_i) \\
x(t_{i+1} - h_f) = \bar{A}_{b-1}(t_i)x(t_i) + \bar{B}_{b-1}^{(2)}(t_i)\bar{w}_{b-1}(t_i) \\
\quad\vdots \\
x(t_i + 2h_f) = \bar{A}_2(t_i)x(t_i) + \bar{B}_2^{(b-1)}(t_i)\bar{w}_2(t_i) \\
x(t_i + h_f) = \bar{A}_1(t_i)x(t_i) + \bar{B}_1^{(b)}(t_i)\bar{w}_1(t_i)
\end{cases} \tag{8}$$

where, for $m = 1, \ldots, b$ and $j = 1, \ldots, m-1$

$$\begin{aligned}
\bar{A}_{b-m+1}(t_i) &= \prod_{s=m}^{b} A_a\big(t_{i+1} - sh_f\big) \\
\bar{w}_m(t_i) &= \mathrm{col}\big\{w_a(t_i), \ldots, w_a\big(t_i + (m-1)h_f\big)\big\} \\
\bar{B}_m^{(l)}(t_i) &= \begin{cases} B_a(t_i), & m = 1,\ l = b \\ \breve{B}_m^{(l)}(t_i), & m = 2, \ldots, b,\ l = b - m + 1 \end{cases} \\
\vec{B}_j^{(l)}(t_i) &= \prod_{s=l}^{b-j} A_a\big(t_{i+1} - sh_f\big)B_a\big(t_i + (j-1)h_f\big) \\
\breve{B}_m^{(l)}(t_i) &= \Big[\vec{B}_1^{(l)}(t_i) \ldots \vec{B}_{m-1}^{(l)}(t_i)\ B_a\big(t_i + (m-1)h_f\big)\Big].
\end{aligned}$$

For notational simplicity, we define

$$\begin{aligned}
\breve{x}(t_i) &= \mathrm{col}\big\{x(t_i), x\big(t_i - h_f\big), \ldots, x\big(t_i - (b-1)h_f\big)\big\} \\
\bar{w}(t_i) &= \mathrm{col}\{\bar{w}_b(t_i), \bar{w}_{b-1}(t_i), \ldots, \bar{w}_1(t_i)\} \\
\breve{A}(t_i) &= \mathrm{col}\Big\{\vec{A}_b(t_i), \vec{A}_{b-1}(t_i), \ldots, \vec{A}_1(t_i)\Big\} \\
\breve{B}(t_i) &= \mathrm{diag}\Big\{\bar{B}_b^{(1)}(t_i), \bar{B}_{b-1}^{(2)}(t_i), \ldots, \bar{B}_1^{(b)}(t_i)\Big\}
\end{aligned}$$

$$\vec{A}_j(t_i) = \mathrm{col}^{\mathrm{T}}\Bigg\{\bar{A}_j^{\mathrm{T}}(t_i), \underbrace{0, \ldots, 0}_{b-1}\Bigg\},\ j = 1, \ldots, b$$

$$\bar{C}_r(t_i) = \Bigg[C_r(t_i), \underbrace{0, \ldots, 0}_{b-1}\Bigg],\ r = 1, \ldots, N.$$

Consequently, we have the following compact form:

$$\breve{x}(t_{i+1}) = \breve{A}(t_i)\breve{x}(t_i) + \breve{B}(t_i)\bar{w}(t_i). \tag{9}$$

In this paper, we construct the following RR-protocol-based time-varying distributed filter:

$$\begin{aligned}
\hat{x}_j(t_{i+1}) = \breve{A}(t_i)\hat{x}_j(t_i) &+ \sum_{r\in\mathcal{N}_j} a_{jr}L_{jr}(t_i) \\
&\times \left(\bar{y}_r(t_i) - \sum_{l=0}^{n_y-1} \Phi_{\theta^r(t_{i-l})}\bar{C}_r(t_{i-l})\hat{x}_r(t_{i-l})\right)
\end{aligned} \tag{10}$$

where $\hat{x}_j(t_i) \in \mathbb{R}^{bn_x}$ is the estimate of $\breve{x}(t_i)$ by the jth sensor node, and $\hat{x}_j(t_i) \triangleq \hat{x}_j(0)$ for $i \le 0$, $j = 1, \ldots, N$ with $\hat{x}_j(0)$ being the known initial estimation, as well as $L_{jr}(t_i) \in \mathbb{R}^{bn_x \times n_y}$ is the filter gain to be determined.

Remark 2: In this paper, the multirate scheme is considered for the distributed set-membership filtering problem. To be more specific, we assume that the update period of the filter is the integer multiples of that of the plant, that is, $h_f = ah_p$, and the sampling period of the measurement output (also called the transmission period) is the integer multiples of the update period of the filter, that is, $h_m = bh_f$. As we know, the lifting technique is an effective method to transform a multirate system into a corresponding single-rate system. To this end, by successively using such a technique, the update period of the plant is first transformed into that of the filter, i.e., (7), and then transformed into the sampling period of the sensor, i.e., (8). Subsequently, by resorting to the augmentation method, the transformed system (9) is obtained with the largest period, that is, the sensor sampling period. On the other hand, it can be observed from (10) that, at those non-sampling instants, the filter input will hold the last value by the ZOH strategy and, therefore, the difficulty caused by the asynchronous sampling/update is successfully eliminated. In addition, an implicit fact that should be stressed is that, due primarily to the decrease of the communication frequency, the energy of the sensor nodes is significantly saved.

In what follows, denoting $e_j(t_i) \triangleq \breve{x}(t_i) - \hat{x}_j(t_i)$, we have:

$$\begin{aligned}
e_j(t_{i+1}) = \breve{A}(t_i)e_j(t_i) &+ \breve{B}(t_i)\bar{w}(t_i) \\
&- \sum_{r\in\mathcal{N}_j}\sum_{l=0}^{n_y-1} a_{jr}L_{jr}(t_i)\Phi_{\theta^r(t_{i-l})}\bar{C}_r(t_{i-l})e_r(t_{i-l}) \\
&- \sum_{r\in\mathcal{N}_j}\sum_{l=0}^{n_y-1} a_{jr}L_{jr}(t_i)\Phi_{\theta^r(t_{i-l})}D_r(t_{i-l})v(t_{i-l}).
\end{aligned} \tag{11}$$

Assumption 1: The initial filtering error satisfies the following condition:

$$\varphi_j(0) \triangleq e_j(0)e_j^{\mathrm{T}}(0) \le P(0),\ j = 1, \ldots, N \tag{12}$$

with $P(0)$ being a given positive definite matrix.

The main objectives of this paper are threefold.

1) Establish sufficient conditions for the existence of the filter gains $\{L_{jr}(t_i)\}_{i\geq 0}$ in (10) such that the filtering error system (11) satisfies the following $P(t_i)$-dependent constraint over the finite-horizon $[t_0\ t_M]$:
$$\varphi_j(t_i) \triangleq e_j(t_i)e_j^{\mathrm{T}}(t_i) \leq P(t_i),\ j = 1, \ldots, N \tag{13}$$
where $\{P(t_i)\}_{i>0}$ is a positive definite matrix sequence to be determined.

2) Under the constraint condition (13), at each time instant t_i, an optimized ellipsoid is obtained by minimizing the following objective function:
$$f(P(t_i)) = \omega_1 P_{11}(t_i) + \cdots + \omega_c P_{cc}(t_i) \tag{14}$$
where $c \triangleq bn_x$, $P_{jj}(t_i)$ is the jth diagonal element of $P(t_i)$ and ω_j stands for the weighted coefficient satisfying $\omega_j > 0$ and $\sum_{j=1}^{c} \omega_j = 1$.

3) The results obtained from the objectives 1) and 2) are further generalized to the case with respect to the weighted average filtering error.

Remark 3: Along the similar line of [15] and [23], the objective function (14) lays stress on the importance of the state-vector entry of interest. This means that, if the jth state component plays a crucial role in a specific filtering task, a larger weighted coefficient ω_j should be selected. In particular, letting $w_j = (1/c)$ $(j = 1, \ldots, c)$, the objective function (14) can degenerate into trace(P), which illustrates that our results are more general.

III. Protocol-Based Distributed Set-Membership Filter Design

In this section, by applying the RLMI technique, sufficient conditions are first derived such that the filtering error satisfies the $P(t_i)$-dependent constraint and the weighted average filtering error constraint, respectively. Then, by solving optimization problems, a set of optimized ellipsoids containing all possible system states is recursively obtained and the desired distributed filter gains are parameterized.

A. Distributed Filter Design With the $P(t_i)$-Dependent Constraint

To simplify the notations, we define
$$\begin{aligned}
e(t_i) &= \mathrm{col}\{e_1(t_i), \ldots, e_N(t_i)\}\\
\mathcal{A}(t_i) &= I_N \otimes \breve{A}(t_i),\ \ \mathcal{L}(t_i) = [a_{jr}L_{jr}(t_i)]_{N\times N}\\
\mathcal{C}(t_i) &= \mathrm{diag}\{\bar{C}_1(t_i), \ldots, \bar{C}_N(t_i)\}\\
\bar{\Phi}_{\theta(t_i)} &= \mathrm{diag}\{\Phi_{\theta^1(t_i)}, \ldots, \Phi_{\theta^N(t_i)}\}\\
\mathcal{B}(t_i) &= \mathrm{col}\{\breve{B}(t_i), \ldots, \breve{B}(t_i)\}\\
\mathcal{D}(t_i) &= \mathrm{col}\{D_1(t_i), \ldots, D_N(t_i)\}.
\end{aligned}$$

With the defined notations, the filtering error dynamics can be expressed in the following compact form:
$$\begin{aligned}
e(t_{i+1}) = \mathcal{A}(t_i)e(t_i) &- \mathcal{L}(t_i)\sum_{l=0}^{n_y-1}\bar{\Phi}_{\theta(t_{i-l})}\mathcal{C}(t_{i-l})e(t_{i-l})\\
&- \mathcal{L}(t_i)\sum_{l=0}^{n_y-1}\bar{\Phi}_{\theta(t_{i-l})}\mathcal{D}(t_{i-l})v(t_{i-l}) + \mathcal{B}(t_i)\bar{w}(t_i).
\end{aligned} \tag{15}$$

Bearing in mind the fact that $a_{jr} = 0$ (if $r \notin \mathcal{N}_j$), it is obvious that $\mathcal{L}(t_i)$ is a sparse matrix with the following expression:
$$\mathcal{L}(t_i) \in \mathcal{H}_{bn_x\times n_y} \tag{16}$$
where $\mathcal{H}_{bn_x\times n_y} \triangleq \{\bar{H} = [H_{jr}] \in \mathbb{R}^{bn_xN\times n_yN} | H_{jr} \in \mathbb{R}^{bn_x\times n_y},\ H_{jr} = 0 \text{ if } r \notin \mathcal{N}_j\}$.

The following theorem is provided to establish a sufficient condition to satisfy the $P(t_i)$-dependent index (13).

Theorem 1: Consider the system (1), the RR protocol (6), and the distributed filter (10). Let the sequence of the positive definite matrices $P(t_i)$ be given. The filtering error constraint condition (13) is achieved if there exist a sequence of real matrices $\{\mathcal{L}(t_i)\}_{i\geq 0} \in \mathcal{H}_{bn_x\times n_y}$ and the sequences of positive scalars $\{\epsilon_{r,j}(t_i)\}_{\{j=1,\ldots,N,\ r\in\mathcal{R},\ i\geq 0\}}$, $\{\varepsilon_s(t_i)\}_{\{s=1,\ldots,ab,\ i\geq 0\}}$ and $\{\lambda_r(t_i)\}_{\{r\in\mathcal{R},\ i\geq 0\}}$ satisfying the following N RLMIs for $j = 1, \ldots, N$:
$$\begin{bmatrix} -\Theta(t_i) & * \\ \mathcal{I}_j\Omega(t_i) & -P(t_{i+1}) \end{bmatrix} < 0 \tag{17}$$
where, for $q = 2, \ldots, n_y$
$$\begin{aligned}
\Theta(t_i) = \mathrm{diag}\Bigg\{&\Upsilon_1(t_i),\ \sum_{j=1}^{N}\epsilon_{1,j}(t_i)\mathcal{I}_j^{\mathrm{T}}\mathcal{I}_j, \ldots, \sum_{j=1}^{N}\epsilon_{n_y,j}(t_i)\mathcal{I}_j^{\mathrm{T}}\\
&\times\mathcal{I}_j, \varepsilon_1(t_i)Q^{-1}(t_i), \ldots, \varepsilon_a(t_i)Q^{-1}(t_i+(a-1)h_p)\\
&\lambda_1(t_i)R^{-1}(t_i), \ldots, \lambda_{n_y}(t_i)R^{-1}(t_{i-n_y+1})\Bigg\}
\end{aligned}$$
$$\Upsilon_1(t_i) = 1 - \sum_{j=1}^{N}\sum_{q=1}^{n_y}\epsilon_{q,j}(t_i) - \sum_{r=0}^{b-1}\sum_{s=1}^{a(b-r)}\varepsilon_s(t_i) - \sum_{l=1}^{n_y}\lambda_l(t_i)$$
$$\mathcal{I}_j = \Big[\underbrace{0_{bn_x}\ \ldots\ 0_{bn_x}}_{j-1}\ I_{bn_x}\ \underbrace{0_{bn_x}\ \ldots\ 0_{bn_x}}_{N-j}\Big]$$
$$\begin{aligned}
\Omega(t_i) &= \big[0\quad \Omega_1(t_i)\quad \Omega_2(t_i)\quad \ldots\quad \Omega_{n_y}(t_i)\quad \mathcal{B}(t_i)\quad \Xi_1(t_i)\\
&\qquad \Xi_2(t_i)\quad \ldots\quad \Xi_{n_y}(t_i)\big]\\
\Omega_1(t_i) &= \big(\mathcal{A}(t_i) - \mathcal{L}(t_i)\bar{\Phi}_{\theta(t_i)}\mathcal{C}(t_i)\big)(I_N\otimes E(t_i))\\
\Omega_q(t_i) &= -\mathcal{L}(t_i)\bar{\Phi}_{\theta(t_{i-q+1})}\mathcal{C}(t_{i-q+1})\big(I_N\otimes E(t_{i-q+1})\big)\\
\Xi_1(t_i) &= -\mathcal{L}(t_i)\bar{\Phi}_{\theta(t_i)}\mathcal{D}(t_i)\\
\Xi_q(t_i) &= -\mathcal{L}(t_i)\bar{\Phi}_{\theta(t_{i-q+1})}\mathcal{D}(t_{i-q+1}).
\end{aligned}$$

Proof: It is easy to see that, if $\varphi_j(t_i) \leq P(t_i)$, there exists a vector $z_j(t_i) \in \mathbb{R}^{bn_x}$ satisfying
$$z_j^{\mathrm{T}}(t_i)z_j(t_i) \leq 1 \tag{18}$$
such that
$$e_j(t_i) = E(t_i)z_j(t_i) \tag{19}$$
with $E(t_i)$ being a factorization of $P(t_i)$, that is, $P(t_i) = E(t_i)E^{\mathrm{T}}(t_i)$.

By denoting $z(t_i) \triangleq \mathrm{col}\{z_1(t_i), \dots, z_N(t_i)\}$, (19) can be readily rewritten as

$$e(t_i) = (I_N \otimes E(t_i))z(t_i). \tag{20}$$

Defining $\xi(t_i) \triangleq [1 \; z^{\mathrm{T}}(t_i) \; z^{\mathrm{T}}(t_{i-1}) \; \dots \; z^{\mathrm{T}}(t_{i-n_y+1}) \; \bar{w}^{\mathrm{T}}(t_i) \; v^{\mathrm{T}}(t_i) \; v^{\mathrm{T}}(t_{i-1}) \; \dots \; v^{\mathrm{T}}(t_{i-n_y+1})]^{\mathrm{T}}$, we have

$$e(t_{i+1}) = \Omega(t_i)\xi(t_i). \tag{21}$$

Furthermore, according to (2), (4), and (18), the following constraint conditions should be satisfied:

$$\begin{cases} z^{\mathrm{T}}(t_{i-r})\mathcal{I}_j^{\mathrm{T}}\mathcal{I}_j z(t_{i-r}) \le 1 \\ w^{\mathrm{T}}(t_i + sh_p)Q^{-1}(t_i + sh_p)w(t_i + sh_p) \le 1 \\ v^{\mathrm{T}}(t_{i-r})R^{-1}(t_{i-r})v(t_{i-r}) \le 1 \end{cases} \tag{22}$$

for $j = 1, \dots, N$, $s = 0, \dots, ab - 1$, and $r = 0, 1, \dots, n_y - 1$.

By some simple operations, (22) can be further written as

$$\begin{cases} \xi^{\mathrm{T}}(t_i)\Lambda_{1,j}\xi(t_i) & \le 0 \\ & \vdots \\ \xi^{\mathrm{T}}(t_i)\Lambda_{n_y,j}\xi(t_i) & \le 0 \\ \xi^{\mathrm{T}}(t_i)\tilde{\Lambda}_1(t_i)\xi(t_i) & \le 0 \\ & \vdots \\ \xi^{\mathrm{T}}(t_i)\tilde{\Lambda}_{\bar{a}}(t_i)\xi(t_i) & \le 0 \\ \xi^{\mathrm{T}}(t_i)\bar{\Lambda}_1(t_i)\xi(t_i) & \le 0 \\ & \vdots \\ \xi^{\mathrm{T}}(t_i)\bar{\Lambda}_{n_y}(t_i)\xi(t_i) & \le 0 \end{cases} \tag{23}$$

where $\bar{a}(ab(b+1)/2)$, for $q = 2, \dots, n_y$ and $s = 1, \dots, \bar{a}$

$$\Lambda_{1,j} = \mathrm{diag}\left\{-1, \mathcal{I}_j^{\mathrm{T}}\mathcal{I}_j, \underbrace{0, \dots, 0}_{n_y-1}, \underbrace{0, \dots, 0}_{\bar{a}+n_y}\right\}$$

$$\Lambda_{q,j} = \mathrm{diag}\left\{-1, \underbrace{0, \dots, 0}_{q-1}, \mathcal{I}_j^{\mathrm{T}}\mathcal{I}_j, \underbrace{0, \dots, 0}_{n_y-q}, \underbrace{0, \dots, 0}_{\bar{a}+n_y}\right\}$$

$$\tilde{\Lambda}_1(t_i) = \mathrm{diag}\left\{-1, \underbrace{0, \dots, 0}_{n_y}, Q^{-1}(t_i), \underbrace{0, \dots, 0}_{n_y+\bar{a}-1}\right\}$$

$$\tilde{\Lambda}_s(t_i) = \mathrm{diag}\left\{-1, \underbrace{0, \dots, 0}_{n_y+s-1}, Q^{-1}(t_i + \bar{t}h_p), \underbrace{0, \dots, 0}_{n_y+\bar{a}-s}\right\}$$

$$\bar{t} = \begin{cases} s-1, & 1 \le s \le ab \\ s - z(r) - 1, & z(r) < s \le z(r+1) \end{cases}$$

$$z(r) = a\left(rb - \sum_{i=0}^{r-1} i\right), \quad r = 1, \dots, b-1$$

$$\bar{\Lambda}_1(t_i) = \mathrm{diag}\left\{-1, \underbrace{0, \dots, 0}_{n_y+\bar{a}}, R^{-1}(t_i), \underbrace{0 \dots, 0}_{n_y-1}\right\}$$

$$\bar{\Lambda}_q(t_i) = \mathrm{diag}\left\{-1, \underbrace{0, \dots, 0}_{n_y+\bar{a}+q-1}, R^{-1}(t_{i-q+1}), \underbrace{0, \dots, 0}_{n_y-q}\right\}.$$

It follows readily from (21) that, at time instant t_{i+1}, the constraint condition (13) can be written as:

$$\begin{aligned} & e_j^{\mathrm{T}}(t_{i+1})P^{-1}(t_{i+1})e_j(t_{i+1}) \\ & = e^{\mathrm{T}}(t_{i+1})\mathcal{I}_j^{\mathrm{T}}P^{-1}(t_{i+1})\mathcal{I}_j e(t_{i+1}) \\ & = \xi^{\mathrm{T}}(t_i)\Omega^{\mathrm{T}}(t_i)\mathcal{I}_j^{\mathrm{T}}P^{-1}(t_{i+1})\mathcal{I}_j\Omega(t_i)\xi(t_i) \\ & \le 1 \end{aligned} \tag{24}$$

for $j = 1, \dots, N$. Based on the definition of the $\xi(t_i)$, the above inequality can be further expressed as

$$\begin{aligned} & \xi^{\mathrm{T}}(t_i)\Omega^{\mathrm{T}}(t_i)\mathcal{I}_j^{\mathrm{T}}P^{-1}(t_{i+1})\mathcal{I}_j\Omega(t_i)\xi(t_i) \\ & \quad - \xi^{\mathrm{T}}(t_i)\mathrm{diag}\left\{1, \underbrace{0, \dots, 0}_{2n_y+\bar{a}}\right\}\xi(t_i) \le 0. \end{aligned} \tag{25}$$

By virtue of the S-procedure [1], [20] and taking account of the special structure of $\bar{w}(t_i)$, (25) is satisfied if the following inequality holds for $j = 1, \dots, N$:

$$\begin{aligned} & \xi^{\mathrm{T}}(t_i)\Omega^{\mathrm{T}}(t_i)\mathcal{I}_j^{\mathrm{T}}P^{-1}(t_{i+1})\mathcal{I}_j\Omega(t_i)\xi(t_i) \\ & \quad - \xi^{\mathrm{T}}(t_i)\mathrm{diag}\{1, 0, \dots, 0\}\xi(t_i) \\ & \quad - \sum_{j=1}^{N} \epsilon_{1,j}(t_i)\xi^{\mathrm{T}}(t_i)\Lambda_{1,j}\xi(t_i) \\ & \quad \qquad \vdots \\ & \quad - \sum_{j=1}^{N} \epsilon_{n_y,j}(t_i)\xi^{\mathrm{T}}(t_i)\Lambda_{n_y,j}\xi(t_i) \\ & \quad - \varepsilon_1(t_i)\xi^{\mathrm{T}}(t_i)\tilde{\Lambda}_1(t_i)\xi(t_i) \; \cdots \; - \varepsilon_a(t_i)\xi^{\mathrm{T}}(t_i)\tilde{\Lambda}_{\bar{a}}(t_i)\xi(t_i) \\ & \quad - \lambda_1(t_i)\xi^{\mathrm{T}}(t_i)\bar{\Lambda}_1(t_i)\xi(t_i) \; \cdots \; - \lambda_{n_y}(t_i)\xi^{\mathrm{T}}(t_i)\bar{\Lambda}_{n_y}(t_i)\xi(t_i) \\ & \le 0 \end{aligned} \tag{26}$$

where, for $\varepsilon_{\tilde{t}}(t_i)$ $(1 \le \tilde{t} \le ab)$, the subscript $\tilde{t}$ is denoted as $\tilde{t} = \bar{t} + 1$ with $\bar{t}$ being defined in (23).

After performing some simple matrix operations, (26) can be further ensured by the following matrix inequality:

$$\Omega^{\mathrm{T}}(t_i)\mathcal{I}_j^{\mathrm{T}}P^{-1}(t_{i+1})\mathcal{I}_j\Omega(t_i) - \Theta(t_i) \le 0. \tag{27}$$

In addition, by using Schur complement lemma [1], the inequality (27) holds if and only if (17) holds. Therefore, the proof is now complete. ■

B. Distributed Filter Design With the Weighted Average Filtering Error Constraint

In many application domains, in order to better achieve the filtering performance, we are more interested in the information from all local sensors rather than the individual sensor. Therefore, based on the results obtained in the previous section, a weighted average method is adopted to fuse the estimates from all local filters.

To begin with, we define

$$\bar{x}(t_i) \triangleq \sum_{j=1}^{N} \gamma_j \hat{x}_j(t_i) \tag{28}$$

where γ_j ($j = 1, \dots, N$) are the weighting coefficients with respect to the corresponding sensor nodes and satisfy $\sum_{j=1}^{N} \gamma_j = 1$.

Denoting $\Gamma \triangleq \mathrm{diag}\{\gamma_1, \dots, \gamma_N\}$, we obtain the following weighted average filtering error:

$$\begin{aligned}\eta(t_i) &\triangleq \check{x}(t_i) - \bar{x}(t_i)\\ &= \big((\mathbf{I}_N^{\mathrm{T}}\Gamma) \otimes I_{bn_x}\big)e(t_i). \end{aligned} \tag{29}$$

Assumption 2: The initial weighted average filtering error possesses the following constraint:

$$\pi(0) \triangleq \eta(0)\eta^{\mathrm{T}}(0) \le Y(0) \tag{30}$$

where $Y(0)$ is a given positive definite matrix.

Next, we shall design the distributed filter (10) such that, for the weighted average filtering error (29), the following $Y(t_i)$-dependent constraint is satisfied over the finite-horizon $[t_0 \ t_M]$:

$$\pi(t_i) \triangleq \eta(t_i)\eta^{\mathrm{T}}(t_i) \le Y(t_i) \tag{31}$$

where $\{Y(t_i)\}_{i>0}$ is a positive definite matrix sequence to be determined.

Theorem 2: Consider the discrete time-varying system (1) and the distributed filter (10). Let the sequence of the positive definite matrices $Y(t_i)$ be given. The filtering error constraint condition (31) is achieved if there exist a sequence of real matrices $\{\mathcal{L}(t_i)\}_{\{i\ge0\}} \in \mathcal{H}_{bn_x\times n_y}$ and the sequences of positive scalars $\{\epsilon_{r,j}(t_i)\}_{\{j=1,\dots,N,\ r\in\mathcal{R},\ i\ge0\}}$, $\{\varepsilon_s(t_i)\}_{\{s=1,\dots,ab,\ i\ge0\}}$, and $\{\lambda_r(t_i)\}_{\{r\in\mathcal{R},\ i\ge0\}}$ satisfying the following RLMI:

$$\begin{bmatrix} -\Theta(t_i) & * \\ \big((\mathbf{I}_N^{\mathrm{T}}\Gamma)\otimes I_{bn_x}\big)\Omega(t_i) & -Y(t_{i+1}) \end{bmatrix} < 0. \tag{32}$$

Proof: The proof of Theorem 2 is similar to that of Theorem 1 and thus is skipped here. ■

C. Two Optimization Problems

Having established sufficient conditions to ensure the desired filtering performance for different filtering error constraints, the corresponding optimization problems are then proposed to obtain the optimized ellipsoid by minimizing the objective function (14).

Corollary 1: Consider the system (1), the RR protocol (6), and the distributed filter (10). A sequence of minimized $\{P(t_i)\}_{i\ge0}$ is guaranteed if there exist a sequence of real matrices $\{\mathcal{L}(t_i)\}_{i\ge0} \in \mathcal{H}_{bn_x\times n_y}$ and the sequences of positive scalars $\{\epsilon_{r,j}(t_i)\}_{\{j=1,\dots,N,\ r\in\mathcal{R},\ i\ge0\}}$, $\{\varepsilon_s(t_i)\}_{\{s=1,\dots,ab,\ i\ge0\}}$ and $\{\lambda_r(t_i)\}_{\{r\in\mathcal{R},\ i\ge0\}}$ such that the following optimization problem:

$$\text{OP1:} \qquad \min f(P(t_i)) \tag{33}$$

is solvable subject to the constraint condition (17).

Corollary 2: Consider the system (1), the RR protocol (6), and the distributed filter (10). A sequence of minimized $\{Y(t_i)\}_{i\ge0}$ is guaranteed if there exist a sequence of real matrix $\{\mathcal{L}(t_i)\}_{i\ge0} \in \mathcal{H}_{bn_x\times n_y}$ and the sequences of positive scalars $\{\epsilon_{r,j}(t_i)\}_{\{j=1,\dots,N,\ r\in\mathcal{R},\ i\ge0\}}$, $\{\varepsilon_s(t_i)\}_{\{s=1,\dots,ab,\ i\ge0\}}$ and $\{\lambda_r(t_i)\}_{\{r\in\mathcal{R},\ i\ge0\}}$ such that the following optimization problem:

$$\text{OP2:} \qquad \min f(Y(t_i)) \tag{34}$$

is solvable subject to the constraint condition (32).

After obtaining the optimization problems OP1 and OP2, we summarize the following RR-protocol-based distributed set-membership filtering algorithm (Algorithm 1).

Algorithm 1 RR-Protocol-Based Distributed Set-Membership Filtering Algorithm

Step 0. Initialize the parameters $P(0)$, $x(0)$ and $\hat{x}_j(0)$ ($j = 1, \dots, N$) such that the initial constraint condition (12) (or (30)) is satisfied. Set $i = 0$ and the maximum step M.

Step 1. Based on $P(t_j) = E(t_j)E^{\mathrm{T}}(t_j)$ (or $Y(t_j) = E(t_j)E^{\mathrm{T}}(t_j)$), calculate matrix factorizations $E(t_j)$ for $i-(n_y-1) \le j \le i$.

Step 2. Solving optimization problem **OP1** or **OP2** according to the RR scheduling protocol, the filter parameters $L_{jr}(t_i)$ and positive definite matrices $P(t_{i+1})$ (or $(Y(t_{i+1}))$) can be obtained, respectively. Then, $\hat{x}_j(t_{i+1})$ ($j = 1, \dots, N$) can be computed by (10).

Step 3. Set $i = i + 1$. If $i > M$ stop, else jump to *Step 1.*

Remark 4: It is observed from the optimization problem OP1 (OP2) that the distributed filter design issue is solved under the performance requirement that the filtering error (the weighted average filtering error) is confined to an optimized upper bound. Moreover, from Algorithm 1, all important factors contributing to the complexities on the filter design have been covered, which include: 1) time-varying parameters of the system; 2) topology structure of the wireless sensor network; 3) the period of the RR protocol; and 4) the individual sampling/update periods of different devices.

Remark 5: Compared with the existing literature, there are three distinguishing features of the filtering problem addressed in this paper.

1) The RR protocol is introduced during the data transmission from each sensor node to both the corresponding local filter and its neighbors, thereby reducing the data collisions resulting from the limited network resources.
2) The multirate strategy is employed to govern different sampling/update rates of the plant, the sensors, and the filters in order to fulfill the physical limitation and engineering specifications.
3) The set-membership filtering scheme is carried out to ensure all possible states confined into the optimized ellipsoidal sets under the bounded noises.

IV. Illustrative Example

In this section, we shall display the effectiveness of the filter design scheme proposed in this paper via a numerical example.

Consider the sensor network (with three nodes) whose topology structure is denoted by a directed graph $\mathbf{G} = \{\mathbf{V}, \mathbf{E}, \mathbf{A}\}$ with the set of nodes $\mathbf{V} = \{1, 2, 3\}$, set of edges

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

TABLE I
FILTER PARAMETERS FOR THE SENSOR NODE 1

i	1	2	3	...
L_{11}	$10^{-3}\begin{bmatrix}0.2601 & 0\\ -0.0756 & 0\\ 0.1924 & 0\\ -0.0108 & 0\end{bmatrix}$	$\begin{bmatrix}0 & -3.3393\\ 0 & 0.4253\\ 0 & -2.2737\\ 0 & -0.1063\end{bmatrix}$	$\begin{bmatrix}0.1745 & 0\\ 0.0105 & 0\\ 0.1469 & 0\\ -0.0232 & 0\end{bmatrix}$	
L_{13}	$10^{-5}\begin{bmatrix}-0.5301 & 0\\ 0.1552 & 0\\ -0.3974 & 0\\ 0.0221 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0.0842\\ 0 & -0.0107\\ 0 & 0.0574\\ 0 & 0.0027\end{bmatrix}$	$10^{-5}\begin{bmatrix}-0.6923 & 0\\ -0.0419 & 0\\ -0.5830 & 0\\ 0.0922 & 0\end{bmatrix}$	

TABLE II
FILTER PARAMETERS FOR THE SENSOR NODE 2

i	1	2	3	...
L_{21}	$10^{-5}\begin{bmatrix}-0.4334 & 0\\ 0.1266 & 0\\ -0.3208 & 0\\ 0.0152 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0.0959\\ 0 & -0.0122\\ 0 & 0.0653\\ 0 & 0.0031\end{bmatrix}$	$10^{-5}\begin{bmatrix}-0.4878 & 0\\ -0.0295 & 0\\ -0.4107 & 0\\ 0.0650 & 0\end{bmatrix}$	
L_{22}	$10^{-3}\begin{bmatrix}0.2852 & 0\\ -0.0828 & 0\\ 0.2107 & 0\\ -0.0199 & 0\end{bmatrix}$	$\begin{bmatrix}0 & -3.0762\\ 0 & 0.3918\\ 0 & -2.0945\\ 0 & -0.0979\end{bmatrix}$	$\begin{bmatrix}0.1577 & 0\\ 0.0095 & 0\\ 0.1328 & 0\\ -0.0210 & 0\end{bmatrix}$	

TABLE III
FILTER PARAMETERS FOR THE SENSOR NODE 3

i	1	2	3	...
L_{31}	$10^{-5}\begin{bmatrix}-0.4795 & 0\\ 0.1372 & 0\\ -0.3465 & 0\\ 0.0188 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0.1238\\ 0 & -0.0158\\ 0 & 0.0843\\ 0 & 0.0039\end{bmatrix}$	$10^{-5}\begin{bmatrix}-0.6124 & 0\\ -0.0370 & 0\\ -0.5157 & 0\\ 0.0816 & 0\end{bmatrix}$	
L_{33}	$10^{-3}\begin{bmatrix}0.3169 & 0\\ -0.0919 & 0\\ 0.2335 & 0\\ -0.0131 & 0\end{bmatrix}$	$\begin{bmatrix}0 & -2.4517\\ 0 & 0.3123\\ 0 & -1.6693\\ 0 & -0.0780\end{bmatrix}$	$\begin{bmatrix}0.1980 & 0\\ 0.0120 & 0\\ 0.1667 & 0\\ -0.0264 & 0\end{bmatrix}$	

$\mathbf{E} = \{(1, 1), (1, 3), (2, 1), (2, 2), (3, 1), (3, 3)\}$, and the adjacency matrix $\mathbf{A} = [a_{ij}]_{3\times 3}$, where adjacency elements $a_{ij} = 1$ when $(i, j) \in \mathbf{E}$; otherwise, $a_{ij} = 0$.

The discrete time-varying system considered here is given by the following form over a prescribed finite horizon $i \in [0, 18]$:

$$x(k_{i+1}) = A(k_i)x(k_i) + B(k_i)w(k_i)$$

with $x(k_i) = \left[x_1(k_i)\ x_2(k_i)\right]^{\mathrm{T}}$ and

$$A(k_i) = \begin{bmatrix}1.19 + 0.2\sin(10k_i) & 0.25\\ -0.1 + \cos(2k_i) & 0.16\end{bmatrix}$$
$$B(k_i) = \begin{bmatrix}0.13 + 0.2\sin(10k_i)\\ 0.15 - 0.2\cos(2k_i)\end{bmatrix}.$$

Moreover, the sensor model (3) is governed by the following parameters:

$$C_1(t_i) = \begin{bmatrix}9 + 0.3\cos(1 - 2t_i) & 0\\ 0 & 1.8 - 0.1\cos(3t_i)\end{bmatrix}$$
$$C_2(t_i) = \begin{bmatrix}10 + 0.23\cos(1 - 2t_i) & 0\\ 0 & 2 - 0.05\cos(3t_i)\end{bmatrix}$$
$$C_3(t_i) = \begin{bmatrix}8 + 0.1\cos(1 - 2t_i) & 0\\ 0 & 2.5 - 0.1\cos(3t_i)\end{bmatrix}$$

and

$$D_1(t_i) = \begin{bmatrix}0.12\\ 0.17 + 0.27\cos(t_i)\end{bmatrix}$$

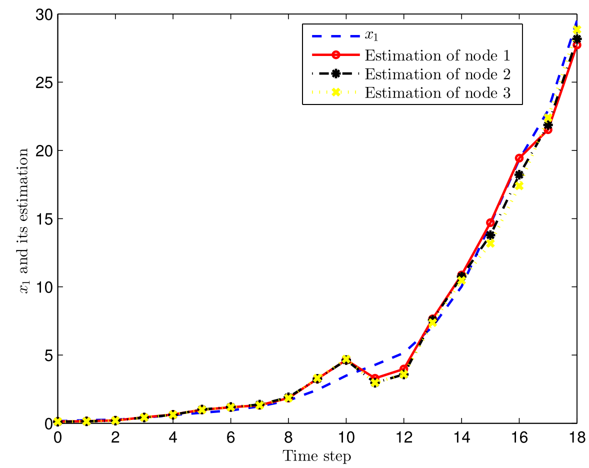

Fig. 3. State x_1 and its estimates with $a = b = 2$.

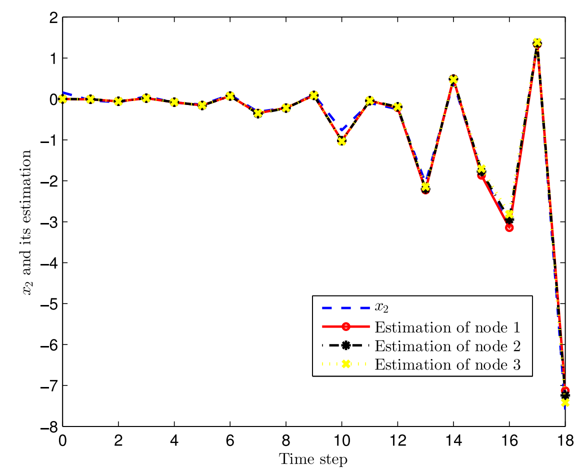

Fig. 4. State x_2 and its estimates with $a = b = 2$.

$$D_2(t_i) = \begin{bmatrix}0.1\\ 0.12 + 0.27\cos(t_i)\end{bmatrix}$$
$$D_3(t_i) = \begin{bmatrix}0.1\\ 0.17 + 0.27\cos(t_i)\end{bmatrix}.$$

The process noise and the measurement noise are selected as $w(k_i) = 0.13\sin(-k_i)$ and $v(t_i) = -0.12\sin(-t_i)$, whose corresponding weighted matrices are denoted as $Q(k_i) = 10$ and $R(t_i) = 10$, respectively. The initial states are $x(0) = [0.15\ 0.16]^{\mathrm{T}}$, $\hat{x}_1(0) = [0.125 \quad 0]^{\mathrm{T}}$, $\hat{x}_2(0) = [0.115 \quad 0]^{\mathrm{T}}$, and $\hat{x}_3(0) = [0.12 \quad 0]^{\mathrm{T}}$. The optimization problem (33) is recursively solved by means of the MATLAB LMI toolbox (YALMIP 3.0) with weighted coefficients $w_j = (1/c)$ $(j = 1, \ldots, c)$. The desired distributed filter parameters are shown in Tables I–III with respect to the directed graph $\mathbf{G}$ for the sampling/update rates $a = b = 2$.

Simulation results are given in Figs. 3–8. For the case of the sampling/update rates $a = b = 2$, Figs. 3 and 4 plot the true states and their estimates for sensor nodes 1–3, respectively. Figs. 5 and 6 depict the filtering errors of different sensor nodes and their common upper bounds where $e_j^{(i)}$ $(i = 1, 2; j = 1, 2, 3)$ denote the ith component of the filtering error associated with the sensor node j. Figs. 7 and 8 reveal the impact from the different sampling/update rates

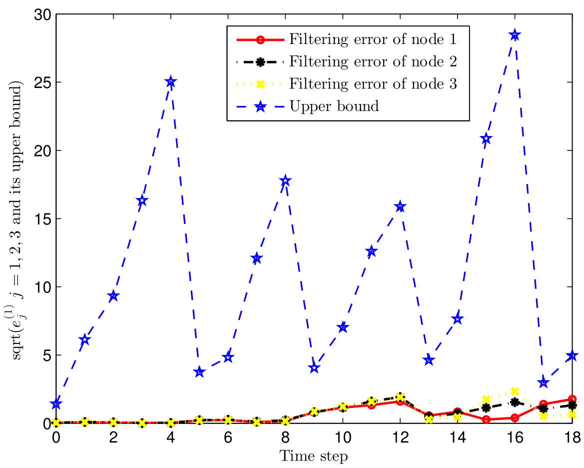

Fig. 5. Filtering errors $e_j^{(1)}$ ($j = 1, 2, 3$) and their common upper bound P_{11}.

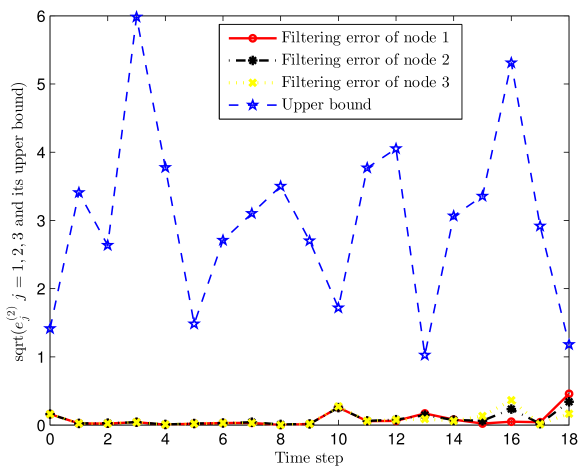

Fig. 6. Filtering errors $e_j^{(2)}$ ($j = 1, 2, 3$) and their common upper bound P_{22}.

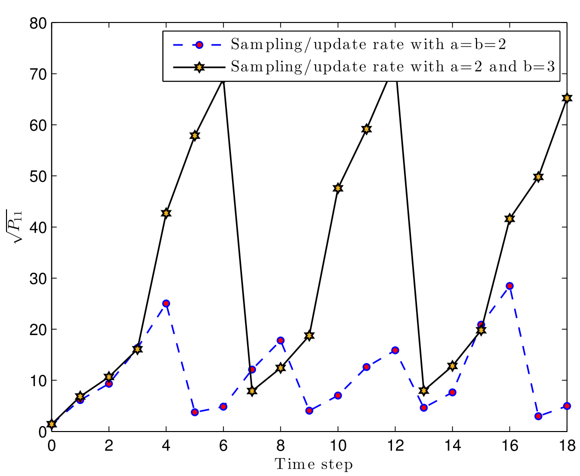

Fig. 7. Comparisons of the upper bound P_{11} with different sampling/update rates.

on the upper bounds of the filtering errors. It implies that, with the increase of the sampling/update period, the filtering performance degrades to a certain extent and this coincides with the reality. As a result, the simulation results illustrate the effectiveness of the proposed RR-protocol-based distributed filtering algorithm.

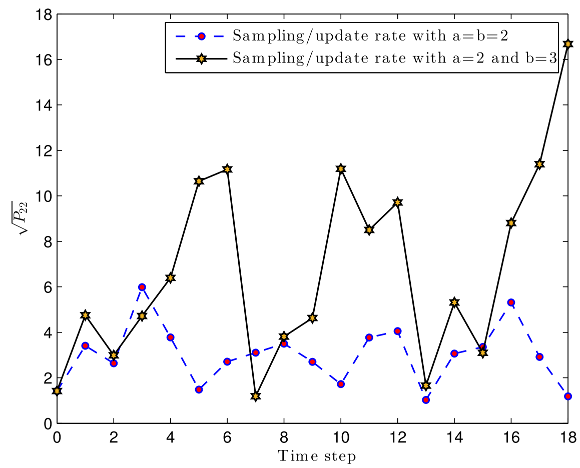

Fig. 8. Comparisons of the upper bound P_{22} with different sampling/update rates.

V. Conclusion

This paper has studied the distributed set-membership filtering problem for a class of time-varying multirate systems with the RR communication protocol. Under such a communication protocol, the actual signal received by the filter has been characterized by a sequence of delayed measurements. Moreover, the multirate strategy has been considered to govern the sampling/update rate of the plant, the sensors, and the filters. By virtue of the RLMI technique, sufficient conditions have been derived to guarantee the existence of the distributed set-membership filters in the simultaneous presence of the multirate sampling/update strategy and the RR protocol. By solving optimization problems with certain inequality constraints, a set of the optimized ellipsoids has been obtained in the sense of the minimum weighted matrix trace. A numerical simulation has been provided to demonstrate the feasibility of the proposed protocol-based distributed filter design algorithm. Further research topics include the extension of our results to more complex systems, such as a class of nonlinear systems [27], [31]; time-delay systems [29]; and state-saturated systems [32].

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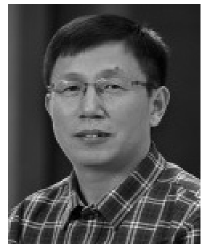

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