

Newtonian and single layer potentials for the Stokes system with L^∞ coefficients and the exterior Dirichlet problem

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Dedicated to Professor H. Begehr on the occasion of his 80th birthday.

Abstract. A mixed variational formulation of some problems in L^2 -based Sobolev spaces is used to define the Newtonian and layer potentials for the Stokes system with L^∞ coefficients on Lipschitz domains in \mathbb{R}^3 . Then the solution of the exterior Dirichlet problem for the Stokes system with L^∞ coefficients is presented in terms of these potentials and the inverse of the corresponding single layer operator.

Mathematics Subject Classification (2010). Primary 35J25, 35Q35, 42B20, 46E35; Secondary 76D, 76M.

Keywords. Stokes system with L^∞ coefficients, Newtonian and layer potentials, variational approach, inf-sup condition, Sobolev spaces.

1. Introduction

Let \mathbf{u} be an unknown vector field, π be an unknown scalar field, and \mathbf{f} be a given vector field defined on an exterior Lipschitz domain $\Omega_- \subset \mathbb{R}^3$. Let also $\mathbb{E}(\mathbf{u})$ be the symmetric part of the gradient of \mathbf{u} , $\nabla \mathbf{u}$. Then the equations

$$\mathcal{L}_\mu(\mathbf{u}, \pi) := \operatorname{div}(2\mu\mathbb{E}(\mathbf{u})) - \nabla\pi = \mathbf{f}, \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega_- \quad (1.1)$$

determine the *Stokes system* with a known viscosity coefficient $\mu \in L^\infty(\Omega_-)$. This linear PDE system describes the flows of viscous incompressible fluids, when the inertia of such a fluid can be neglected. The coefficient μ is related to the physical properties of the fluid (for further details we refer the reader to the books [45] and [23] and the references therein).

The methods of layer potential theory have a main role in the analysis of boundary value problems for elliptic partial differential equations (see, e.g., [13, 17, 30, 36, 39, 43, 48]). Fabes, Kenig and Verchota [21] obtained mapping properties of layer potentials for the constant coefficient Stokes system in L^p

spaces. Mitrea and Wright [43] have used various methods of layer potentials in the analysis of the main boundary problems for the Stokes system with constant coefficients in arbitrary Lipschitz domains in \mathbb{R}^n . The authors in [32] have obtained mapping properties of the constant coefficient Stokes layer potential operators in standard and weighted Sobolev spaces by exploiting results of singular integral operators. Gatica and Wendland [24] used the coupling of mixed finite element and boundary integral methods for solving a class of linear and nonlinear elliptic boundary value problems. The authors in [33] used the Stokes and Brinkman integral layer potentials and a fixed point theorem to show an existence result for a nonlinear Neumann-transmission problem for the Stokes and Brinkman systems with data in L^p , Sobolev, and Besov spaces (see also [34, 35]). All above results are devoted to elliptic boundary value problems with constant coefficients.

Potential theory plays also a main role in the study of elliptic boundary value problems with variable coefficients. Dindoš and Mitrea [19] have obtained well-posedness results in Sobolev spaces for Poisson problems for the Stokes and Navier-Stokes systems with Dirichlet condition on C^1 and Lipschitz domains in compact Riemannian manifolds by using mapping properties of Stokes layer potentials in Sobolev and Besov spaces. Chkadua, Mikhailov and Natroshvili [14] obtained direct segregated systems of boundary-domain integral equations for a mixed boundary value problem of Dirichlet-Neumann type for a scalar second-order divergent elliptic partial differential equation with a variable coefficient in an exterior domain of \mathbb{R}^3 (see also [13]). Hofmann, Mitrea and Morris [29] considered layer potentials in L^p spaces for elliptic operators of the form $L = -\operatorname{div}(A\nabla u)$ acting in the upper half-space \mathbb{R}_+^n , $n \geq 3$, or in more general Lipschitz graph domains, with an L^∞ coefficient matrix A , which is t -independent, and solutions of the equation $Lu = 0$ satisfy interior De Giorgi-Nash-Moser estimates. They obtained a Calderón-Zygmund type theory associated to the layer potentials, and well-posedness results of boundary problems for the operator L in L^p and endpoint spaces.

Our variational approach is inspired by that developed by Sayas and Selgas in [46] for the constant coefficient Stokes layer potentials on Lipschitz boundaries, and is based on the technique of Nédélec [44]. Girault and Sequeira [26] obtained a well-posed result in weighted Sobolev spaces for the Dirichlet problem for the standard Stokes system in exterior Lipschitz domains of \mathbb{R}^n , $n = 2, 3$. Băcuță, Hassell and Hsiao [8] developed a variational approach for the standard Brinkman single layer potential and used it in the analysis of the time dependent exterior Stokes problem with Dirichlet boundary condition in \mathbb{R}^n , $n = 2, 3$. Barton [7] constructed layer potentials for strongly elliptic differential operators in general settings by using the Lax-Milgram theorem, and generalized various properties of layer potentials for harmonic and second order elliptic equations. Brewster et al. in [9] have used a variational approach and a deep analysis to obtain well-posedness results for boundary problems of Dirichlet, Neumann and mixed type for higher

order divergence-form elliptic equations with L^∞ coefficients in locally (ϵ, δ) -domains and in Besov and Bessel potential spaces. Choi and Lee [15] have studied the Dirichlet problem for the Stokes system with nonsmooth coefficients, and proved the unique solvability of the problem in Sobolev spaces on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) with a small Lipschitz constant when the coefficients have vanishing mean oscillations with respect to all variables. Choi and Yang [16] obtained the existence and pointwise bound of the fundamental solution for the Stokes system with measurable coefficients in \mathbb{R}^n , $n \geq 3$, whenever the weak solutions of the system are locally Hölder continuous. Alliot and Amrouche [3] have used a variational approach to obtain weak solutions for the exterior Stokes problem in weighted Sobolev spaces. Also, Amrouche and Nguyen [5] proved existence and uniqueness results in weighted Sobolev spaces for the Poisson problem with Dirichlet boundary condition for the Navier-Stokes system in exterior Lipschitz domains in \mathbb{R}^3 .

The purpose of this work is to show the well-posedness result of the Poisson problem of Dirichlet type for the Stokes system with L^∞ coefficients in L^2 -based Sobolev spaces on an exterior Lipschitz domain in \mathbb{R}^3 . We use a variational approach that reduces this boundary value problem to a mixed variational formulation. A similar variational approach is used to define the Newtonian and layer potentials for the Stokes system with L^∞ coefficients on Lipschitz surfaces in \mathbb{R}^3 , by using the weak solutions of some transmission problems in L^2 -based Sobolev spaces. Finally, the mapping properties of these layer potentials are used to construct explicitly the solution of the exterior Dirichlet problem for the Stokes system with L^∞ coefficients. The analysis developed in this paper confines to the case $n = 3$, due to its practical interest, but the extension to the case $n \geq 3$ can be done with similar arguments.

2. Functional setting and useful results

Let $\Omega_+ := \Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, i.e., an open connected set whose boundary $\partial\Omega$ is locally the graph of a Lipschitz function. Assume that $\partial\Omega$ is connected. Let $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}_+$ denote the exterior Lipschitz domain. Let \mathring{E}_\pm denote the operators of extension by zero outside Ω_\pm .

2.1. Standard L^2 -based Sobolev spaces and related results

Let \mathcal{F} and \mathcal{F}^{-1} denote the Fourier transform and its inverse defined on the the space of tempered distributions $\mathcal{S}^*(\mathbb{R}^3)$ (i.e., the topological dual of the space $\mathcal{S}(\mathbb{R}^3)$ of all rapidly decreasing infinitely differentiable functions on \mathbb{R}^3). The Lebesgue space of (equivalence classes of) measurable, square integrable functions on \mathbb{R}^3 is denoted by $L^2(\mathbb{R}^3)$, and by $L^\infty(\mathbb{R}^3)$ we denote the space of (equivalence classes of) essentially bounded measurable functions on \mathbb{R}^3 . Let $H^1(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)^3$ denote the L^2 -based Sobolev (Bessel potential) spaces

$$H^1(\mathbb{R}^3) := \{f \in \mathcal{S}^*(\mathbb{R}^3) : \|f\|_{H^1(\mathbb{R}^3)} = \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{1}{2}}\mathcal{F}f]\|_{L^2(\mathbb{R}^3)} < \infty\}, \quad (2.1)$$

$$H^1(\mathbb{R}^3)^3 := \{f = (f_1, f_2, f_3) : f_j \in H^1(\mathbb{R}^3), j = 1, 2, 3\}. \quad (2.2)$$

Now let Ω' be Ω_+ , Ω_- or \mathbb{R}^3 . We denote by $\mathcal{D}(\Omega') := C_0^\infty(\Omega')$ the space of infinitely differentiable functions with compact support in Ω' , equipped with the inductive limit topology. Let $\mathcal{D}^*(\Omega')$ denote the corresponding space of distributions on Ω' , i.e., the dual space of $\mathcal{D}(\Omega')$. Let us consider the space

$$H^1(\Omega') := \{f \in \mathcal{D}^*(\Omega') : \exists F \in H^1(\mathbb{R}^3) \text{ such that } F|_{\Omega'} = f\}, \quad (2.3)$$

where $\cdot|_{\Omega'}$ is the restriction operator to Ω' . The space $\tilde{H}^1(\Omega')$ is the closure of $\mathcal{D}(\Omega')$ in $H^1(\mathbb{R}^3)$. This space can be also characterized as

$$\tilde{H}^1(\Omega') := \left\{ \tilde{f} \in H^1(\mathbb{R}^3) : \text{supp } \tilde{f} \subseteq \overline{\Omega'} \right\}. \quad (2.4)$$

Similar to definition (2.2), $H^1(\Omega')^3$ and $\tilde{H}^1(\Omega')^3$ are the spaces of vector-valued functions whose components belong to the scalar spaces $H^1(\Omega')$ and $\tilde{H}^1(\Omega')$, respectively (see, e.g., [38]). The Sobolev space $\tilde{H}^1(\Omega')$ can be identified with the closure $\hat{H}^1(\Omega')$ of $\mathcal{D}(\Omega')$ in the norm of $H^1(\Omega')$ (see, e.g., [42, (3.11), (3.13)], [38, Theorem 3.33]). The space $\mathcal{D}(\overline{\Omega'})$ is dense in $H^1(\Omega')$, and the following spaces can be isomorphically identified (cf., e.g., [38, Theorem 3.14])

$$(H^1(\Omega'))^* = \tilde{H}^{-1}(\Omega'), \quad H^{-1}(\Omega') = (\tilde{H}^1(\Omega'))^*. \quad (2.5)$$

For $s \in [0, 1]$, the Sobolev space $H^s(\partial\Omega)$ on the boundary $\partial\Omega$ can be defined by using the space $H^s(\mathbb{R}^2)$, a partition of unity and the pull-backs of the local parametrization of $\partial\Omega$, and $H^{-s}(\partial\Omega) = (H^s(\partial\Omega))^*$. All the above spaces are Hilbert spaces. For further properties of Sobolev spaces on bounded Lipschitz domains and Lipschitz boundaries, we refer to [1, 31, 38, 43, 47].

A useful result for the next arguments is given below (see, e.g., [17], [31, Proposition 3.3]).

Lemma 2.1. *Assume that $\Omega := \Omega_+ \subset \mathbb{R}^3$ is a bounded Lipschitz domain with connected boundary $\partial\Omega$ and denote by $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}$ the corresponding exterior domain. Then there exist linear and bounded trace operators $\gamma_\pm : H^1(\Omega_\pm) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$ such that $\gamma_\pm f = f|_{\partial\Omega}$ for any $f \in C^\infty(\overline{\Omega_\pm})$. These operators are surjective and have (non-unique) bounded linear right inverse operators $\gamma_\pm^{-1} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega_\pm)$.*

The jump of a function $u \in H^1(\mathbb{R}^3 \setminus \partial\Omega)$ across $\partial\Omega$ is denoted by $[\gamma(u)] := \gamma_+(u) - \gamma_-(u)$. For $u \in H_{\text{loc}}^1(\mathbb{R}^3)$, $[\gamma(u)] = 0$. The trace operator $\gamma : H^1(\mathbb{R}^3) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$ can be also considered and is linear and bounded¹.

If X is either an open subset or a surface in \mathbb{R}^3 , then we use the notation $\langle \cdot, \cdot \rangle_X$ for the duality pairing of two dual Sobolev spaces defined on X .

2.2. Some weighted Sobolev spaces and related results

For a point $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, its distance to the origin is denoted by $|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$. Let ρ denote the weight function

$$\rho(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{\frac{1}{2}}. \quad (2.6)$$

¹The trace operators defined on Sobolev spaces of vector fields on Ω_\pm or \mathbb{R}^3 are also denoted by γ_\pm and γ , respectively.

For $\lambda \in \mathbb{R}$, we consider the weighted space $L^2(\rho^\lambda; \mathbb{R}^3)$ given by

$$f \in L^2(\rho^\lambda; \mathbb{R}^3) \iff \rho^\lambda f \in L^2(\mathbb{R}^3), \quad (2.7)$$

which is a Hilbert space when it is endowed with the inner product and the associated norm,

$$(f, g)_{L^2(\rho^\lambda; \mathbb{R}^3)} := \int_{\mathbb{R}^3} fg\rho^{2\lambda} dx, \quad \|f\|_{L^2(\rho^\lambda; \mathbb{R}^3)}^2 := (f, f)_{L^2(\rho^\lambda; \mathbb{R}^3)}. \quad (2.8)$$

We also consider the weighted Sobolev space

$$\begin{aligned} \mathcal{H}^1(\mathbb{R}^3) &:= \{f \in \mathcal{D}'(\mathbb{R}^3) : \rho^{-1}f \in L^2(\mathbb{R}^3), \nabla f \in L^2(\mathbb{R}^3)^3\} \\ &= \{f \in L^2(\rho^{-1}; \mathbb{R}^3) : \nabla f \in L^2(\mathbb{R}^3)^3\}, \end{aligned} \quad (2.9)$$

which is a Hilbert space with respect to the inner product

$$(f, g)_{\mathcal{H}^1(\mathbb{R}^3)} := (f, g)_{L^2(\rho^{-1}; \mathbb{R}^3)} + (\nabla f, \nabla g)_{L^2(\mathbb{R}^3)^3} \quad (2.10)$$

and the associated norm

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^3)}^2 := \|\rho^{-1}f\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla f\|_{L^2(\mathbb{R}^3)^3}^2 \quad (2.11)$$

(cf. [28]; see also [5]). The spaces $L^2(\rho^\lambda; \Omega_-)$ and $\mathcal{H}^1(\Omega_-)$ can be similarly defined, and $\mathcal{D}(\overline{\Omega_-})$ is dense in $\mathcal{H}^1(\Omega_-)$ (see, e.g., [28, Theorem I.1], [27, Ch.1, Theorem 2.1]). The seminorm

$$|f|_{\mathcal{H}^1(\Omega_-)} := \|\nabla f\|_{L^2(\Omega_-)^3} \quad (2.12)$$

is equivalent to the norm of $\mathcal{H}^1(\Omega_-)$ defined as in (2.11), with Ω_- in place of \mathbb{R}^3 (see, e.g., [18, Chapter XI, Part B, §1, Theorem 1]). The weighted spaces $L^2(\rho^{-1}; \Omega_+)$ and $\mathcal{H}^1(\Omega_+)$ coincide with the standard spaces $L^2(\Omega_+)$ and $H^1(\Omega_+)$, respectively (with equivalent norms).

Note that the result in Lemma 2.1 extends also to the weighted Sobolev space $\mathcal{H}^1(\Omega_-)$. Therefore, there exists a linear bounded *exterior trace operator*

$$\gamma_- : \mathcal{H}^1(\Omega_-) \rightarrow H^{\frac{1}{2}}(\partial\Omega), \quad (2.13)$$

which is also surjective (see [46, p. 69]). Moreover, the embedding of the space $H^1(\Omega_-)$ into $\mathcal{H}^1(\Omega_-)$ and Lemma 2.1 show the existence of a (non-unique) linear and bounded right inverse $\gamma_-^{-1} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow \mathcal{H}^1(\Omega_-)$ of operator (2.13) (see [32, Lemma 2.2], [40, Theorem 2.3, Lemma 2.6]).

Let $\mathring{\mathcal{H}}^1(\Omega_-) \subset \mathcal{H}^1(\Omega_-)$ denote the closure of $\mathcal{D}(\Omega_-)$ in $\mathcal{H}^1(\Omega_-)$. This space can be characterized as

$$\mathring{\mathcal{H}}^1(\Omega_-) = \{v \in \mathcal{H}^1(\Omega_-) : \gamma_- v = 0 \text{ on } \partial\Omega\} \quad (2.14)$$

(cf., e.g., [38, Theorem 3.33]). Also let $\widetilde{\mathcal{H}}^1(\Omega_-) \subset \mathcal{H}^1(\mathbb{R}^3)$ denote the closure of $\mathcal{D}(\Omega_-)$ in $\mathcal{H}^1(\mathbb{R}^3)$. This space can be also characterized as

$$\widetilde{\mathcal{H}}^1(\Omega_-) = \{u \in \mathcal{H}^1(\mathbb{R}^3) : \text{supp } u \subseteq \overline{\Omega_-}\}, \quad (2.15)$$

and can be isomorphically identified with the space $\mathring{\mathcal{H}}^1(\Omega_-)$ via the extension by zero operator \mathring{E}_- , i.e., $\mathring{\mathcal{H}}^1(\Omega_-) = \mathring{E}_- \mathring{\mathcal{H}}^1(\Omega_-)$ (cf., e.g., [38, Theorem 3.29 (ii)]). In addition, consider the spaces (see, e.g., [5, p. 44], [37, Theorem 2.4])

$$\mathcal{H}^{-1}(\mathbb{R}^3) := (\mathcal{H}^1(\mathbb{R}^3))^*, \mathcal{H}^{-1}(\Omega_-) := (\mathring{\mathcal{H}}^1(\Omega_-))^*, \mathring{\mathcal{H}}^{-1}(\Omega_-) := (\mathcal{H}^1(\Omega_-))^*.$$

3. The conormal derivative operators for the Stokes system with L^∞ coefficients

In the sequel we assume that the viscosity coefficient μ of the Stokes system (1.1) belongs to $L^\infty(\mathbb{R}^3)$ and there exists a constant $c_\mu > 0$, such that

$$c_\mu^{-1} \leq \mu \leq c_\mu \text{ a.e. in } \mathbb{R}^3. \quad (3.1)$$

Let $\mathbb{E}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)$ be the strain rate tensor. If $(\mathbf{u}, \pi) \in C^1(\overline{\Omega_\pm})^3 \times C^0(\overline{\Omega_\pm})$, we can define the *classical* interior and exterior conormal derivatives (i.e., *the boundary traction fields*) for the Stokes system (1.1) with continuously differentiable viscosity coefficient μ by the well-known formula

$$\mathbf{t}_\mu^{\pm}(\mathbf{u}, \pi) := \gamma_\pm (-\pi \mathbb{I} + 2\mu \mathbb{E}(\mathbf{u})) \boldsymbol{\nu}, \quad (3.2)$$

$\boldsymbol{\nu}$ being the outward unit normal to Ω_+ , defined a.e. on $\partial\Omega$, and the symbol \pm refers to the limit and conormal derivative from Ω_\pm . Then for any function $\varphi \in \mathcal{D}(\mathbb{R}^3)^3$, we obtain

$$\pm \langle \mathbf{t}_\mu^{\pm}(\mathbf{u}, \pi), \varphi \rangle_{\partial\Omega} = 2\langle \mu \mathbb{E}(\mathbf{u}), \mathbb{E}(\varphi) \rangle_{\Omega_\pm} - \langle \pi, \operatorname{div} \varphi \rangle_{\Omega_\pm} + \langle \mathcal{L}_\mu(\mathbf{u}, \pi), \varphi \rangle_{\Omega_\pm}.$$

This formula suggests the following weak definition of the generalized conormal derivative for the Stokes system with L^∞ coefficients in the setting of L^2 -weighted Sobolev spaces (cf., e.g., [17, Lemma 3.2], [32, Lemma 2.9], [34, Lemma 2.2], [40, Definition 3.1, Theorem 3.2], [43, Theorem 10.4.1]).

Definition 3.1. Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy assumption (3.1). Let

$$\begin{aligned} \mathcal{H}^1(\Omega_\pm, \mathcal{L}_\mu) &:= \left\{ (\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \in \mathcal{H}^1(\Omega_\pm)^3 \times L^2(\Omega_\pm) \times \mathring{\mathcal{H}}^{-1}(\Omega_\pm)^3 : \right. \\ &\quad \left. \mathcal{L}_\mu(\mathbf{u}_\pm, \pi_\pm) = \tilde{\mathbf{f}}_\pm|_{\Omega_\pm} \text{ and } \operatorname{div} \mathbf{u}_\pm = 0 \text{ in } \Omega_\pm \right\}. \end{aligned} \quad (3.3)$$

Then define the conormal derivative operator $\mathbf{t}_\mu^\pm : \mathcal{H}^1(\Omega_\pm, \mathcal{L}_\mu) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)^3$,

$$\mathcal{H}^1(\Omega_\pm, \mathcal{L}_\mu) \ni (\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \longmapsto \mathbf{t}_\mu^\pm(\mathbf{u}_\pm, \pi_\pm; \tilde{\mathbf{f}}_\pm) \in H^{-\frac{1}{2}}(\partial\Omega)^3, \quad (3.4)$$

$$\begin{aligned} \pm \left\langle \mathbf{t}_\mu^\pm(\mathbf{u}_\pm, \pi_\pm; \tilde{\mathbf{f}}_\pm), \Phi \right\rangle_{\partial\Omega} &:= 2\langle \mu \mathbb{E}(\mathbf{u}_\pm), \mathbb{E}(\gamma_\pm^{-1} \Phi) \rangle_{\Omega_\pm} \\ &\quad - \langle \pi_\pm, \operatorname{div}(\gamma_\pm^{-1} \Phi) \rangle_{\Omega_\pm} + \langle \tilde{\mathbf{f}}_\pm, \gamma_\pm^{-1} \Phi \rangle_{\Omega_\pm}, \quad \forall \Phi \in H^{\frac{1}{2}}(\partial\Omega)^3, \end{aligned} \quad (3.5)$$

where $\gamma_\pm^{-1} : H^{\frac{1}{2}}(\partial\Omega)^3 \rightarrow \mathcal{H}^1(\Omega_\pm)^3$ is a (non-unique) bounded right inverse of the trace operator $\gamma_\pm : \mathcal{H}^1(\Omega_\pm)^3 \rightarrow H^{\frac{1}{2}}(\partial\Omega)^3$.

We use the simplified notation $\mathbf{t}_\mu^\pm(\mathbf{u}_\pm, \pi_\pm)$ for $\mathbf{t}_\mu^\pm(\mathbf{u}_\pm, \pi_\pm; \mathbf{0})$. The following assertion can be proved similar to [41, Theorem 5.3], [32, Lemma 2.9].

Lemma 3.2. *Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy assumption (3.1). Then for all $\mathbf{w}_\pm \in \mathcal{H}^1(\Omega_\pm)^3$ and $(\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \in \mathcal{H}^1(\Omega_\pm, \mathcal{L}_\mu)$ the following identity holds*

$$\begin{aligned} \pm \langle \mathbf{t}_\mu^\pm(\mathbf{u}_\pm, \pi_\pm; \tilde{\mathbf{f}}_\pm), \gamma_\pm \mathbf{w}_\pm \rangle_{\partial\Omega} &= 2\langle \mu \mathbb{E}(\mathbf{u}_\pm), \mathbb{E}(\mathbf{w}_\pm) \rangle_{\Omega_\pm} - \langle \pi_\pm, \operatorname{div} \mathbf{w}_\pm \rangle_{\Omega_\pm} \\ &\quad + \langle \tilde{\mathbf{f}}_\pm, \mathbf{w}_\pm \rangle_{\Omega_\pm}. \end{aligned} \quad (3.6)$$

Let γ denote the trace operator from $\mathcal{H}^1(\mathbb{R}^3)^3$ to $H^{\frac{1}{2}}(\partial\Omega)^3$ (cf., e.g., [40, Theorem 2.3, Lemma 2.6], [8, (2.2)]). For $(\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \in \mathcal{H}^1(\Omega_\pm, \mathcal{L}_\mu)$, let

$$\mathbf{u} := \mathring{E}_+ \mathbf{u}_+ + \mathring{E}_- \mathbf{u}_-, \quad \pi := \mathring{E}_+ \pi_+ + \mathring{E}_- \pi_-, \quad \mathbf{f} := \tilde{\mathbf{f}}_+ + \tilde{\mathbf{f}}_- \quad (3.7)$$

$$[\mathbf{t}_\mu(\mathbf{u}, \pi; \mathbf{f})] := \mathbf{t}_\mu^+(\mathbf{u}_+, \pi_+; \tilde{\mathbf{f}}_+) - \mathbf{t}_\mu^-(\mathbf{u}_-, \pi_-; \tilde{\mathbf{f}}_-). \quad (3.8)$$

Moreover, if $\mathbf{f} = \mathbf{0}$, we define

$$[\mathbf{t}_\mu(\mathbf{u}, \pi)] := [\mathbf{t}_\mu(\mathbf{u}, \pi; \mathbf{0})] = \mathbf{t}_\mu^+(\mathbf{u}_+, \pi_+) - \mathbf{t}_\mu^-(\mathbf{u}_-, \pi_-). \quad (3.9)$$

Then Lemma 3.2 leads to the following result.

Lemma 3.3. *Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy assumption (3.1). Also let $(\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \in \mathcal{H}^1(\Omega_\pm, \mathcal{L}_\mu)$ and let $(\mathbf{u}, \pi, \mathbf{f})$ be defined as in (3.7). Then for all $\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^3)^3$, the following formula holds*

$$\begin{aligned} \langle [\mathbf{t}_\mu(\mathbf{u}, \pi; \mathbf{f})], \gamma \mathbf{w} \rangle_{\partial\Omega} &= 2\langle \mu \mathbb{E}(\mathbf{u}_+), \mathbb{E}(\mathbf{w}) \rangle_{\Omega_+} + 2\langle \mu \mathbb{E}(\mathbf{u}_-), \mathbb{E}(\mathbf{w}) \rangle_{\Omega_-} \\ &\quad - \langle \pi, \operatorname{div} \mathbf{w} \rangle_{\mathbb{R}^3} + \langle \mathbf{f}, \mathbf{w} \rangle_{\mathbb{R}^3}. \end{aligned} \quad (3.10)$$

We also need the following particular case of Lemma 3.3 when $\mathbf{f} = \mathbf{0}$.

Lemma 3.4. *Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy assumption (3.1). Also let $(\mathbf{u}_\pm, \pi_\pm, \mathbf{0}) \in \mathcal{H}^1(\Omega_\pm, \mathcal{L}_\mu)$. Let \mathbf{u} and π be defined as in formula (3.7). Then for all $\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^3)^3$,*

$$\begin{aligned} \langle [\mathbf{t}_\mu(\mathbf{u}, \pi)], \gamma \mathbf{w} \rangle_{\partial\Omega} &= 2\langle \mu \mathbb{E}(\mathbf{u}_+), \mathbb{E}(\mathbf{w}) \rangle_{\Omega_+} + 2\langle \mu \mathbb{E}(\mathbf{u}_-), \mathbb{E}(\mathbf{w}) \rangle_{\Omega_-} \\ &\quad - \langle \pi, \operatorname{div} \mathbf{w} \rangle_{\mathbb{R}^3}. \end{aligned} \quad (3.11)$$

4. Newtonian and single layer potentials for the Stokes system with L^∞ coefficients

Recall that the function $\mu \in L^\infty(\mathbb{R}^3)$ satisfies conditions (3.1). Next, we define the Newtonian and single layer potentials for the L^∞ coefficient Stokes system (1.1).

4.1. Variational solution of the variable-coefficient Stokes system in \mathbb{R}^3 .

First we show the following useful well-posedness result.

Lemma 4.1. *Let $a_\mu(\cdot, \cdot) : \mathcal{H}^1(\mathbb{R}^3)^3 \times \mathcal{H}^1(\mathbb{R}^3)^3 \rightarrow \mathbb{R}$ and $b(\cdot, \cdot) : \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3) \rightarrow \mathbb{R}$ be the bilinear forms given by*

$$a_\mu(\mathbf{u}, \mathbf{v}) := 2\langle \mu \mathbb{E}(\mathbf{u}), \mathbb{E}(\mathbf{v}) \rangle_{\mathbb{R}^3}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^3)^3, \quad (4.1)$$

$$b(\mathbf{v}, q) := -\langle \operatorname{div} \mathbf{v}, q \rangle_{\mathbb{R}^3}, \quad \forall \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^3)^3, \quad \forall q \in L^2(\mathbb{R}^3). \quad (4.2)$$

Also let $\ell : \mathcal{H}^1(\mathbb{R}^3)^3 \rightarrow \mathbb{R}$ be a linear and bounded map. Then the mixed variational formulation

$$\begin{cases} a_\mu(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \ell(\mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^3)^3, \\ b(\mathbf{u}, q) = 0, \quad \forall q \in L^2(\mathbb{R}^3) \end{cases} \quad (4.3)$$

is well-posed. Hence, (4.3) has a unique solution $(\mathbf{u}, p) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ and there exists a constant $C = C(c_\mu) > 0$ such that

$$\|\mathbf{u}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} + \|p\|_{L^2(\mathbb{R}^3)} \leq C \|\ell\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3}. \quad (4.4)$$

Proof. By using conditions (3.1) and definition (2.11) of the norm of the weighted Sobolev space $\mathcal{H}^1(\mathbb{R}^3)$ we obtain that

$$\begin{aligned} |a_\mu(\mathbf{u}, \mathbf{v})| &\leq 2c_\mu \|\mathbb{E}(\mathbf{u})\|_{L^2(\mathbb{R}^3)^{3 \times 3}} \|\mathbb{E}(\mathbf{v})\|_{L^2(\mathbb{R}^3)^{3 \times 3}} \\ &\leq 2c_\mu \|\mathbf{u}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} \|\mathbf{v}\|_{\mathcal{H}^1(\mathbb{R}^3)^3}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^3)^3. \end{aligned} \quad (4.5)$$

Moreover, by using the Korn type inequality for functions in $\mathcal{H}^1(\mathbb{R}^3)^3$,

$$\|\text{grad } \mathbf{v}\|_{L^2(\mathbb{R}^3)^{3 \times 3}} \leq 2^{\frac{1}{2}} \|\mathbb{E}(\mathbf{v})\|_{L^2(\mathbb{R}^3)^{3 \times 3}} \quad (4.6)$$

(cf., e.g., [46, (2.2)]) and since the seminorm

$$|g|_{\mathcal{H}^1(\mathbb{R}^3)} := \|\nabla g\|_{L^2(\mathbb{R}^3)^3} \quad (4.7)$$

is a norm in $\mathcal{H}^1(\mathbb{R}^3)^3$ equivalent to the norm defined by (2.11) (see, e.g., [18, Chapter XI, Part B, §1, Theorem 1]), there exists a constant $c_1 > 0$ such that

$$\begin{aligned} a_\mu(\mathbf{u}, \mathbf{u}) &\geq 2c_\mu^{-1} \|\mathbb{E}(\mathbf{u})\|_{L^2(\mathbb{R}^3)^{3 \times 3}}^2 \geq c_\mu^{-1} \|\nabla \mathbf{u}\|_{L^2(\mathbb{R}^3)^{3 \times 3}}^2 \\ &\geq c_\mu^{-1} c_1 \|\mathbf{u}\|_{\mathcal{H}^1(\mathbb{R}^3)^3}^2, \quad \forall \mathbf{u} \in \mathcal{H}^1(\mathbb{R}^3)^3. \end{aligned} \quad (4.8)$$

Inequalities (4.5) and (4.8) show that $a_\mu(\cdot, \cdot) : \mathcal{H}^1(\mathbb{R}^3)^3 \times \mathcal{H}^1(\mathbb{R}^3)^3 \rightarrow \mathbb{R}$ is a bounded and coercive bilinear form. Moreover, since the divergence operator

$$\text{div} : \mathcal{H}^1(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3) \quad (4.9)$$

is bounded, then the bilinear form $b(\cdot, \cdot) : \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3) \rightarrow \mathbb{R}$ is bounded as well. In addition, the operator in (4.9) is surjective (cf. [2, Proposition 2.1], [46, Proposition 2.4]) and also

$$\begin{aligned} \mathcal{H}_{\text{div}}^1(\mathbb{R}^3)^3 &:= \{\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^3)^3 : \text{div } \mathbf{w} = 0\} \\ &= \{\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^3)^3 : b(\mathbf{w}, q) = 0, \quad \forall q \in L^2(\mathbb{R}^3)\}. \end{aligned}$$

In addition, the operator in (4.9) is surjective (cf. [2, Proposition 2.1], [46, Proposition 2.4]), and hence the operator

$$-\text{div} : \mathcal{H}^1(\mathbb{R}^3)^3 / \mathcal{H}_{\text{div}}^1(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3)$$

is an isomorphism. Then by Lemma A.2(ii) the bounded bilinear form $b(\cdot, \cdot) : \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3) \rightarrow \mathbb{R}$ satisfies the inf-sup condition (A.7). Hence, there exists $\beta_0 \in (0, \infty)$ such that

$$\inf_{q \in L^2(\mathbb{R}^3) \setminus \{0\}} \sup_{\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^3)^3 \setminus \{\mathbf{0}\}} \frac{b(\mathbf{w}, q)}{\|\mathbf{w}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} \|q\|_{L^2(\mathbb{R}^3)}} \geq \beta_0. \quad (4.10)$$

By applying Theorem A.4, with $X = \mathcal{H}^1(\mathbb{R}^3)^3$, $M = L^2(\mathbb{R}^3)$, $V = \mathcal{H}_{\text{div}}^1(\mathbb{R}^3)^3$, we conclude that the mixed variational formulation (4.3) has a unique solution $(\mathbf{u}, p) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ and there exists a constant $C = C(c_\mu) > 0$ such that (\mathbf{u}, p) satisfies inequality (4.4). \square

Next we use the result of Lemma 4.1 in order to show the well-posedness of the L^∞ coefficient Stokes system in the space $\mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ (see also [2, Theorem 3] for the constant-coefficient case).

Theorem 4.2. *Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy conditions (3.1). Then the L^∞ coefficient Stokes system*

$$\begin{cases} \nabla \pi - \operatorname{div}(2\mu \mathbb{E}(\mathbf{u})) = \boldsymbol{\ell}, & \boldsymbol{\ell} \in \mathcal{H}^{-1}(\mathbb{R}^3)^3, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } \mathbb{R}^3, \end{cases} \quad (4.11)$$

has a unique solution $(\mathbf{u}, p) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$, and there exists a constant $C_0 = C_0(c_\mu) > 0$ such that

$$\|\mathbf{u}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} + \|p\|_{L^2(\mathbb{R}^3)} \leq C_0 \|\boldsymbol{\ell}\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3}. \quad (4.12)$$

Proof. Note that the Stokes system (4.11) is equivalent to the variational problem (4.3) as follows from the density of $\mathcal{D}(\mathbb{R}^3)^3$ in the space $\mathcal{H}^1(\mathbb{R}^3)^3$ (cf., e.g., [28], [46, Proposition 2.1]). Then the well-posedness result of the Stokes system with L^∞ coefficients (4.11) follows from Lemma 4.1. \square

4.2. Newtonian potential for the Stokes system with L^∞ coefficients

The well-posedness of problem (4.11) allows us to define the *Newtonian potential for the Stokes system with L^∞ coefficients* as follows.

Definition 4.3. For any $\boldsymbol{\ell} \in \mathcal{H}^{-1}(\mathbb{R}^3)^3$, we define the *Newtonian velocity and pressure potentials* for the Stokes system with L^∞ coefficients as

$$\mathcal{N}_{\mu; \mathbb{R}^3} \boldsymbol{\ell} := -\mathbf{u}, \quad \mathcal{Q}_{\mu; \mathbb{R}^3} \boldsymbol{\ell} := -\pi, \quad (4.13)$$

where $(\mathbf{u}, \pi) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ is the unique solution of problem (4.11) with the given datum $\boldsymbol{\ell}$.

Moreover, the well-posedness of problem (4.11) yields the continuity of the above operators as stated in the following assertion (cf. also [32, Lemma A.3] for $\mu = 1$).

Lemma 4.4. *The Newtonian velocity and pressure potential operators*

$$\mathcal{N}_{\mu; \mathbb{R}^3} : \mathcal{H}^{-1}(\mathbb{R}^3)^3 \rightarrow \mathcal{H}^1(\mathbb{R}^3)^3, \quad \mathcal{Q}_{\mu; \mathbb{R}^3} : \mathcal{H}^{-1}(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3) \quad (4.14)$$

are linear and continuous.

4.3. Single layer potential for the Stokes system with L^∞ coefficients

For a given $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\partial\Omega)^3$, we now consider the following transmission problem for the Stokes system with L^∞ coefficients

$$\begin{cases} \operatorname{div}(2\mu \mathbb{E}(\mathbf{u}_\varphi)) - \nabla \pi_\varphi = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ \operatorname{div} \mathbf{u}_\varphi = 0 & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ [\mathbf{t}_\mu(\mathbf{u}_\varphi, \pi_\varphi)] = \boldsymbol{\varphi} & \text{on } \partial\Omega, \end{cases} \quad (4.15)$$

and show that this problem has a unique solution $(\mathbf{u}_\varphi, \pi_\varphi) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ (cf. also [46, Proposition 5.1] for $\mu = 1$). Note that the membership of \mathbf{u}_φ in $\mathcal{H}^1(\mathbb{R}^3)^3$ implies the transmission condition

$$[\gamma(\mathbf{u}_\varphi)] = \mathbf{0} \text{ on } \partial\Omega, \quad (4.16)$$

and the first equation in (4.15) implies also that the jump $[\mathbf{t}_\mu(\mathbf{u}_\varphi, \pi_\varphi)]$ is well defined.

Theorem 4.5. *Let $\varphi \in H^{-\frac{1}{2}}(\partial\Omega)^3$ be given. Then the transmission problem (4.15) has the following equivalent mixed variational formulation: Find $(\mathbf{u}_\varphi, \pi_\varphi) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ such that*

$$\begin{cases} 2\langle \mu \mathbb{E}(\mathbf{u}_\varphi), \mathbb{E}(\mathbf{v}) \rangle_{\mathbb{R}^3} - \langle \pi_\varphi, \operatorname{div} \mathbf{v} \rangle_{\mathbb{R}^3} = \langle \varphi, \gamma \mathbf{v} \rangle_{\partial\Omega}, \quad \forall \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^3)^3, \\ \langle \operatorname{div} \mathbf{u}_\varphi, q \rangle_{\mathbb{R}^3} = 0, \quad \forall q \in L^2(\mathbb{R}^3). \end{cases} \quad (4.17)$$

Moreover, problem (4.17) is well-posed. Hence (4.17) has a unique solution $(\mathbf{u}_\varphi, \pi_\varphi) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$, and there exists a constant $C = C(c_\mu)$ such that

$$\|\mathbf{u}_\varphi\|_{\mathcal{H}^1(\mathbb{R}^3)^3} + \|\pi_\varphi\|_{L^2(\mathbb{R}^3)} \leq C \|\varphi\|_{H^{-\frac{1}{2}}(\partial\Omega)^3}. \quad (4.18)$$

Proof. The equivalence between the transmission problem (4.15) and the variational problem (4.17) follows from the density of the space $\mathcal{D}(\mathbb{R}^3)^3$ in $\mathcal{H}^1(\mathbb{R}^3)^3$ and formula (3.11), while the well-posedness of the variational problem (4.17) is an immediate consequence of Lemma 4.1 with the linear and continuous form $\ell : \mathcal{H}^1(\mathbb{R}^3)^3 \rightarrow \mathbb{R}$ given by

$$\ell(\mathbf{v}) := \langle \varphi, \gamma \mathbf{v} \rangle_{\partial\Omega} = \langle \gamma^* \varphi, \mathbf{v} \rangle_{\mathbb{R}^3}, \quad \forall \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^3)^3,$$

and hence $\ell = \gamma^* \varphi$, where $\gamma^* : H^{-\frac{1}{2}}(\partial\Omega)^3 \rightarrow \mathcal{H}^{-1}(\mathbb{R}^3)^3$ is the adjoint of the trace operator $\gamma : \mathcal{H}^1(\mathbb{R}^3)^3 \rightarrow H^{\frac{1}{2}}(\partial\Omega)^3$. \square

Theorem 4.5 leads to the following definition (cf. [46, p. 75] for $\mu = 1$).

Definition 4.6. For any $\varphi \in H^{-\frac{1}{2}}(\partial\Omega)^3$, we define the *single layer velocity and pressure potentials* for the Stokes system with L^∞ coefficients (1.1) as

$$\mathbf{V}_{\mu; \partial\Omega} \varphi := \mathbf{u}_\varphi, \quad \mathcal{Q}_{\mu; \partial\Omega}^s \varphi := \pi_\varphi, \quad (4.19)$$

and the *potential operators* $\mathbf{V}_{\mu; \partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \rightarrow H^{\frac{1}{2}}(\partial\Omega)^3$ and $\mathbf{K}_{\mu; \partial\Omega}^* : H^{-\frac{1}{2}}(\partial\Omega)^3 \rightarrow H^{-\frac{1}{2}}(\partial\Omega)^3$ as

$$\mathbf{V}_{\mu; \partial\Omega} \varphi := \gamma \mathbf{u}_\varphi, \quad \mathbf{K}_{\mu; \partial\Omega}^* \varphi := \frac{1}{2} (\mathbf{t}_\mu^+(\mathbf{u}_\varphi, \pi_\varphi) + \mathbf{t}_\mu^-(\mathbf{u}_\varphi, \pi_\varphi)), \quad (4.20)$$

where $(\mathbf{u}_\varphi, \pi_\varphi)$ is the unique solution of problem (4.15) in $\mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$.

The next result shows the continuity of single layer velocity and pressure potential operators for the variable coefficient Stokes system (cf. [46, Proposition 5.2], [32, Lemma A.4, (A.10), (A.12)] and [43, Theorem 10.5.3] in the case $\mu = 1$).

Lemma 4.7. *The following operators are linear and continuous*

$$\mathbf{V}_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \rightarrow \mathcal{H}^1(\mathbb{R}^3)^3, \quad \mathcal{Q}_{\mu;\partial\Omega}^s : H^{-\frac{1}{2}}(\partial\Omega)^3 \rightarrow L^2(\mathbb{R}^3), \quad (4.21)$$

$$\mathcal{V}_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \rightarrow H^{\frac{1}{2}}(\partial\Omega)^3, \quad \mathbf{K}_{\mu;\partial\Omega}^* : H^{-\frac{1}{2}}(\partial\Omega)^3 \rightarrow H^{-\frac{1}{2}}(\partial\Omega)^3. \quad (4.22)$$

Proof. The continuity of operators (4.21) and (4.22) follows from the well-posedness of the transmission problem (4.15) and Definition 4.6. \square

The next result yields the jump relations of the single layer potential and its conormal derivative across $\partial\Omega$ (see also [46, Proposition 5.3] for $\mu = 1$).

Lemma 4.8. *Let $\varphi \in H^{-\frac{1}{2}}(\partial\Omega)^3$. Then almost everywhere on $\partial\Omega$,*

$$[\gamma \mathbf{V}_{\mu;\partial\Omega} \varphi] = \mathbf{0}, \quad (4.23)$$

$$[\mathbf{t}_\mu(\mathbf{V}_{\mu;\partial\Omega} \varphi, \mathcal{Q}_{\mu;\partial\Omega}^s \varphi)] = \varphi, \quad \mathbf{t}_\mu^\pm(\mathbf{V}_{\mu;\partial\Omega} \varphi, \mathcal{Q}_{\mu;\partial\Omega}^s \varphi) = \pm \frac{1}{2} \varphi + \mathbf{K}_{\mu;\partial\Omega}^* \varphi. \quad (4.24)$$

Proof. Formulas (4.23) and (4.24) follow from Definition 4.6 and the transmission condition in (4.16), as well as the transmission condition in the third line of (4.15). \square

Let $\mathbb{R}\boldsymbol{\nu} = \{c\boldsymbol{\nu} : c \in \mathbb{R}\}$. Let $\text{Ker}\{T : X \rightarrow Y\} := \{x \in X : T(x) = 0\}$ denote the null space of the map $T : X \rightarrow Y$.

We next obtain the main properties of the single layer potential operator (cf., e.g., [43, Theorem 10.5.3], and [8, Proposition 3.3(c)] and [46, Proposition 5.4] for $\mu = 1$ and $s \in [0, \infty)$).

Lemma 4.9. *The following properties hold*

$$\mathbf{V}_{\mu;\partial\Omega} \boldsymbol{\nu} = \mathbf{0} \text{ in } \mathbb{R}^3, \quad \mathcal{Q}_{\mu;\partial\Omega}^s \boldsymbol{\nu} = -\chi_{\Omega_+} \quad (4.25)$$

$$\text{Ker} \left\{ \mathcal{V}_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \rightarrow H^{\frac{1}{2}}(\partial\Omega)^3 \right\} = \mathbb{R}\boldsymbol{\nu}, \quad (4.26)$$

$$\mathcal{V}_{\mu;\partial\Omega} \varphi \in H_V^{\frac{1}{2}}(\partial\Omega)^3, \quad \forall \varphi \in H^{-\frac{1}{2}}(\partial\Omega)^3, \quad (4.27)$$

where $\chi_{\Omega_+} = 1$ in Ω_+ , $\chi_{\Omega_+} = 0$ in Ω_- , and

$$H_V^{\frac{1}{2}}(\partial\Omega)^3 := \left\{ \phi \in H^{\frac{1}{2}}(\partial\Omega)^3 : \langle \boldsymbol{\nu}, \phi \rangle_{\partial\Omega} = 0 \right\}. \quad (4.28)$$

Proof. First, we consider the transmission problem (4.15) with the datum $\varphi = \boldsymbol{\nu} \in H^{-\frac{1}{2}}(\partial\Omega)^3$. Then the solution of this problem is given by

$$(\mathbf{u}_\boldsymbol{\nu}, \pi_\boldsymbol{\nu}) = (\mathbf{0}, -\chi_{\Omega_+}) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3). \quad (4.29)$$

Indeed, the pair $(\mathbf{u}_\boldsymbol{\nu}, \pi_\boldsymbol{\nu})$ satisfies the equations and the transmission condition in (4.15), as well as the transmission condition (4.16), and, in view of formula (3.11) and the divergence theorem,

$$\langle [\mathbf{t}_\mu(\mathbf{u}_\boldsymbol{\nu}, \pi_\boldsymbol{\nu})], \gamma \mathbf{v} \rangle_{\partial\Omega} = -\langle \pi_\boldsymbol{\nu}, \text{div } \mathbf{v} \rangle_{\mathbb{R}^3} = \langle \boldsymbol{\nu}, \gamma \mathbf{v} \rangle_{\partial\Omega}, \quad \forall \mathbf{v} \in \mathcal{D}(\mathbb{R}^3)^3. \quad (4.30)$$

Then by formula (2.3), Lemma 2.1, the dense embedding of the space $\mathcal{D}(\mathbb{R}^3)^3$ in $\mathcal{H}^1(\mathbb{R}^3)^3$, and the above equality, we obtain that $\langle [\mathbf{t}_\mu(\mathbf{u}_\boldsymbol{\nu}, \pi_\boldsymbol{\nu})], \Phi \rangle_{\partial\Omega} = \langle \boldsymbol{\nu}, \Phi \rangle_{\partial\Omega}$ for any $\Phi \in H^{\frac{1}{2}}(\partial\Omega)^3$. Hence, $[\mathbf{t}_\mu(\mathbf{u}_\boldsymbol{\nu}, \pi_\boldsymbol{\nu})] = \boldsymbol{\nu}$, as asserted. Then

Definition 4.6 implies relations (4.25). Moreover, $\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\nu} = \mathbf{0}$, i.e., $\mathbb{R}\boldsymbol{\nu} \subseteq \text{Ker}\{\mathbf{V}_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \rightarrow H^{\frac{1}{2}}(\partial\Omega)^3\}$.

Now let $\boldsymbol{\varphi}_0 \in \text{Ker}\{\mathbf{V}_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \rightarrow H^{\frac{1}{2}}(\partial\Omega)^3\}$. Let $(\mathbf{u}_{\boldsymbol{\varphi}_0}, \pi_{\boldsymbol{\varphi}_0}) = (\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}_0, \mathcal{Q}_{\mu;\partial\Omega}^s\boldsymbol{\varphi}_0) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ be the unique solution of problem (4.15) with datum $\boldsymbol{\varphi}_0$. Since $\gamma\mathbf{u}_{\boldsymbol{\varphi}_0} = \mathbf{0}$ on $\partial\Omega$, formula (3.11) yields that

$$0 = \langle [\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\varphi}_0}, \pi_{\boldsymbol{\varphi}_0})], \gamma\mathbf{u}_{\boldsymbol{\varphi}_0} \rangle_{\partial\Omega} = a_\mu(\mathbf{u}_{\boldsymbol{\varphi}_0}, \mathbf{u}_{\boldsymbol{\varphi}_0}), \quad (4.31)$$

and hence $\mathbf{u}_{\boldsymbol{\varphi}_0} = \mathbf{0}$, $\pi_{\boldsymbol{\varphi}_0} = c\chi_{\Omega_+}$ in \mathbb{R}^3 , where $c \in \mathbb{R}$. In view of formula (3.11),

$$\langle [\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\varphi}_0}, \pi_{\boldsymbol{\varphi}_0})], \gamma\mathbf{w} \rangle_{\partial\Omega} = -\langle \pi_{\boldsymbol{\varphi}_0}, \text{div } \mathbf{w} \rangle_{\mathbb{R}^3} = -c\langle \boldsymbol{\nu}, \gamma\mathbf{w} \rangle_{\partial\Omega}, \quad \forall \mathbf{w} \in \mathcal{D}(\mathbb{R}^3)^3,$$

and, thus, $\boldsymbol{\varphi}_0 = [\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\varphi}_0}, \pi_{\boldsymbol{\varphi}_0})] = -c\boldsymbol{\nu}$. Hence, formula (4.26) follows.

Now let $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\partial\Omega)^3$. By using the first formula in (4.20), we obtain for any $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\partial\Omega)^3$ that $\langle \mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}, \boldsymbol{\nu} \rangle_{\partial\Omega} = \langle \gamma\mathbf{u}_\boldsymbol{\varphi}, \boldsymbol{\nu} \rangle_{\partial\Omega} = \langle \text{div } \mathbf{u}_\boldsymbol{\varphi}, 1 \rangle_\Omega = 0$, where $\mathbf{u}_\boldsymbol{\varphi} = \mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}$. Thus, we get relation (4.27). \square

Next we use the notation $[\cdot]$ for the equivalence classes of the quotient space $H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}$. Thus, any $[\boldsymbol{\varphi}] \in H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}$ can be written as $[\boldsymbol{\varphi}] = \boldsymbol{\varphi} + \mathbb{R}\boldsymbol{\nu}$, where $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\partial\Omega)^3$.

Exploiting properties (4.26) and (4.27), we now show the following invertibility result (cf. [43, Theorem 10.5.3], [8, Proposition 3.3(d)], [46, Proposition 5.5] for $\mu = 1$ and $\alpha \geq 0$ constant).

Lemma 4.10. *The following operator is an isomorphism*

$$\mathbf{V}_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu} \rightarrow H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^3. \quad (4.32)$$

Proof. We use arguments similar to those for Proposition 5.5 in [46]. First, Lemma 4.7 and the membership relation (4.27) imply that the linear operator in (4.32) is continuous. We show that this operator is also $H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}$ -elliptic, i.e., that there exists a constant $c = c(\partial\Omega) > 0$ such that

$$\langle \mathbf{V}_{\mu;\partial\Omega} [\boldsymbol{\varphi}], [\boldsymbol{\varphi}] \rangle_{\partial\Omega} \geq c \|[\boldsymbol{\varphi}]\|_{H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}}^2, \quad \forall [\boldsymbol{\varphi}] \in H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}. \quad (4.33)$$

Let $[\boldsymbol{\varphi}] \in H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}$. Thus, $[\boldsymbol{\varphi}] = \boldsymbol{\varphi} + \mathbb{R}\boldsymbol{\nu}$, where $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\partial\Omega)^3$. In view of formula (3.11), Definition 4.6, relations (4.26), (4.27), and inequality (4.8),

$$\begin{aligned} \langle \mathbf{V}_{\mu;\partial\Omega}([\boldsymbol{\varphi}]), [\boldsymbol{\varphi}] \rangle_{\partial\Omega} &= \langle \mathbf{V}_{\mu;\partial\Omega}(\boldsymbol{\varphi}), \boldsymbol{\varphi} \rangle_{\partial\Omega} = \langle \gamma\mathbf{u}_\boldsymbol{\varphi}, [\mathbf{t}_\mu(\mathbf{u}_\boldsymbol{\varphi}, \pi_\boldsymbol{\varphi})] \rangle_{\partial\Omega} \\ &= a_\mu(\mathbf{u}_\boldsymbol{\varphi}, \mathbf{u}_\boldsymbol{\varphi}) \geq c_\mu^{-1} \|\mathbf{u}_\boldsymbol{\varphi}\|_{H^1(\mathbb{R}^3)^3}^2, \end{aligned} \quad (4.34)$$

where $\mathbf{u}_\boldsymbol{\varphi} = \mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}$ and $\pi_\boldsymbol{\varphi} = \mathcal{Q}_{\mu;\partial\Omega}^s\boldsymbol{\varphi}$. Now we use the property that the trace operator

$$\gamma : \mathcal{H}_{\text{div}}^1(\mathbb{R}^3)^3 \rightarrow H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^3 \quad (4.35)$$

is surjective having a bounded right inverse $\gamma^{-1} : H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^3 \rightarrow \mathcal{H}_{\text{div}}^1(\mathbb{R}^3)^3$ (cf., e.g., [46, Proposition 4.4]). Hence, for any $\boldsymbol{\Phi} \in H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^3$, we have that $\mathbf{w} = \gamma^{-1}\boldsymbol{\Phi} \in \mathcal{H}_{\text{div}}^1(\mathbb{R}^3)^3$. Then there exists $c' \equiv c'(\partial\Omega) \in (0, \infty)$ such that

$$|\langle [\boldsymbol{\varphi}], \boldsymbol{\Phi} \rangle_{\partial\Omega}| = |\langle \boldsymbol{\varphi}, \boldsymbol{\Phi} \rangle_{\partial\Omega}| = |\langle [\mathbf{t}_\mu(\mathbf{u}_\boldsymbol{\varphi}, \pi_\boldsymbol{\varphi})], \gamma\mathbf{w} \rangle_{\partial\Omega}| = |a_\mu(\mathbf{u}_\boldsymbol{\varphi}, \mathbf{w})| \quad (4.36)$$

$$\leq 2c_\mu \|\mathbf{u}_\varphi\|_{\mathcal{H}^1(\mathbb{R}^3)^3} \|\gamma^{-1}\Phi\|_{\mathcal{H}^1(\mathbb{R}^3)^3} \leq 2c_\mu c' \|\mathbf{u}_\varphi\|_{\mathcal{H}^1(\mathbb{R}^3)^3} \|\Phi\|_{H^{\frac{1}{2}}(\partial\Omega)^3},$$

where the first equality in (4.36) follows from the relation $\llbracket \varphi \rrbracket = \varphi + \mathbb{R}\nu$ and the membership of Φ in $H^{\frac{1}{2}}(\partial\Omega)^3$, the second equality follows from Definition 4.6, and the third equality is a consequence of formula (3.11). Since the space $H^{\frac{1}{2}}(\partial\Omega)^3$ is the dual of the space $H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\nu$, formula (4.36) yields that

$$\|\llbracket \varphi \rrbracket\|_{H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\nu} \leq 2c_\mu c' \|\mathbf{u}_\varphi\|_{\mathcal{H}^1(\mathbb{R}^3)^3}. \quad (4.37)$$

Then by (4.34) and (4.37) we obtain inequality (4.33), and the Lax-Milgram lemma yields that operator (4.32) is an isomorphism. \square

Remark 4.11. The fundamental solution of the constant-coefficient Stokes system in \mathbb{R}^3 is well known and leads to the construction of Newtonian and boundary layer potentials via the integral approach (see, e.g., [17, 36, 43, 48]). In view of Theorems 4.2 and 4.5, the Newtonian and single layer potentials provided by the variational approach (in the case $\mu = 1$) coincide with classical ones expressed in terms of the fundamental solution, since they satisfy the same boundary value problems (4.11) and (4.15), respectively (see also [46, Proposition 5.1] for $\mu = 1$). The assumption $\mu = 1$ is a particular case of a more general case of L^∞ coefficients analyzed in this paper. We also note that an alternative approach, reducing various boundary value problems for variable-coefficient elliptic partial differential equations to *boundary-domain integral equations*, by employing the explicit parametrix-based integral potentials, was explored in, e.g., [12, 13, 14].

5. Exterior Dirichlet problem for the Stokes system with L^∞ coefficients

In this section we analyze the exterior Dirichlet problem for the Stokes system with L^∞ coefficients

$$\begin{cases} \operatorname{div}(2\mu\mathbb{E}(\mathbf{u})) - \nabla\pi = \mathbf{f} & \text{in } \Omega_-, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega_-, \\ \gamma_- \mathbf{u} = \phi & \text{on } \partial\Omega, \end{cases} \quad (5.1)$$

with given data $(\mathbf{f}, \phi) \in \mathcal{H}^{-1}(\Omega_-)^3 \times H^{\frac{1}{2}}(\partial\Omega)^3$.

5.1. Variational approach

First, we use a variational approach and show that problem (5.1) has a unique solution $(\mathbf{u}, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$ (see also [26, Theorem 3.4] and [3, Theorem 3.16] for the constant-coefficient Stokes system).

Theorem 5.1. *Assume that $\mu \in L^\infty(\Omega_-)$ satisfies conditions (3.1). Then for all given data $(\mathbf{f}, \phi) \in \mathcal{H}^{-1}(\Omega_-)^3 \times H^{\frac{1}{2}}(\partial\Omega)^3$ the exterior Dirichlet problem for the L^∞ coefficient Stokes system (5.1) is well posed. Hence problem (5.1)*

has a unique solution $(\mathbf{u}, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$ and there exists a constant $C \equiv C(\partial\Omega; c_\mu) > 0$ such that

$$\|\mathbf{u}\|_{\mathcal{H}^1(\Omega_-)^3} + \|\pi\|_{L^2(\Omega_-)} \leq C \left(\|\mathbf{f}\|_{\mathcal{H}^{-1}(\Omega_-)^3} + \|\phi\|_{H^{\frac{1}{2}}(\partial\Omega)^3} \right). \quad (5.2)$$

Proof. First, we note that the density of the space $\mathcal{D}(\Omega_-)^3$ in $\tilde{\mathcal{H}}^1(\Omega_-)^3$ implies that the exterior Dirichlet problem (5.1) has the following equivalent variational formulation: Find $(\mathbf{u}, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$ such that

$$\begin{cases} 2\langle \mu \mathbb{E}(\mathbf{u}), \mathbb{E}(\tilde{\mathbf{v}}) \rangle_{\Omega_-} - \langle \pi, \operatorname{div} \tilde{\mathbf{v}} \rangle_{\Omega_-} = -\langle \mathbf{f}, \tilde{\mathbf{v}} \rangle_{\Omega_-}, \quad \forall \tilde{\mathbf{v}} \in \tilde{\mathcal{H}}^1(\Omega_-)^3, \\ \langle \operatorname{div} \mathbf{u}, q \rangle_{\Omega_-} = 0, \quad \forall q \in L^2(\Omega_-), \\ \gamma_-(\mathbf{u}) = \phi \text{ on } \partial\Omega. \end{cases} \quad (5.3)$$

Next, we consider $\mathbf{u}_0 \in \mathcal{H}^1(\Omega_-)^3$ such that

$$\begin{cases} \operatorname{div} \mathbf{u}_0 = 0 & \text{in } \Omega_-, \\ \gamma_- \mathbf{u}_0 = \phi & \text{on } \partial\Omega. \end{cases} \quad (5.4)$$

Particularly, we can choose \mathbf{u}_0 as the solution of the Dirichlet problem for a constant-coefficient Brinkman system

$$\begin{cases} (\Delta - \alpha \mathbb{I})\mathbf{u}_0 - \nabla \pi_0 = 0, \quad \operatorname{div} \mathbf{u}_0 = 0 & \text{in } \Omega_-, \\ \gamma_- \mathbf{u}_0 = \phi & \text{on } \partial\Omega, \end{cases} \quad (5.5)$$

where $\alpha > 0$ is an arbitrary constant. The solution is given by the double layer potential

$$\mathbf{u}_0 = \mathbf{W}_{\alpha; \partial\Omega} \left(\frac{1}{2} \mathbb{I} + \mathbf{K}_{\alpha; \partial\Omega} \right)^{-1} \phi, \quad (5.6)$$

where $\mathbf{K}_{\alpha; \partial\Omega} : H^{\frac{1}{2}}(\partial\Omega)^3 \rightarrow H^{\frac{1}{2}}(\partial\Omega)^3$ is the corresponding Brinkman double-layer boundary potential operator. Note that

$$(\mathbf{W}_\alpha \mathbf{h})_j(\mathbf{x}) := \int_{\partial\Omega} S_{ij\ell}^\alpha(\mathbf{x}, \mathbf{y}) \nu_\ell(\mathbf{y}) h_i(\mathbf{y}) d\sigma_{\mathbf{y}}. \quad (5.7)$$

The explicit form of the kernel $S_{ij\ell}^\alpha(\mathbf{x}, \mathbf{y})$ can be found in [48, (2.14)-(2.18)] and [36, Section 3.2.1].

In addition, the operator $\frac{1}{2} \mathbb{I} + \mathbf{K}_{\alpha; \partial\Omega} : H^{\frac{1}{2}}(\partial\Omega)^3 \rightarrow H^{\frac{1}{2}}(\partial\Omega)^3$ is an isomorphism (see, e.g., [39, Proposition 7.1]), and \mathbf{u}_0 belongs to the space $H^1(\Omega_-)^3$ (cf., e.g., [32, Lemma A.8]) and satisfies (5.5), and hence (5.4). Moreover, the embedding $H^1(\Omega_-)^3 \subset \mathcal{H}^1(\Omega_-)^3$ shows that \mathbf{u}_0 belongs also to the space $\mathcal{H}^1(\Omega_-)^3$ (see also [26, Lemma 3.2, Remark 3.3]).

Then with the new variable $\hat{\mathbf{u}} := \mathbf{u} - \mathbf{u}_0 \in \mathring{\mathcal{H}}^1(\Omega_-)^3$, the variational problem (5.3) reduces to the following mixed variational formulation (c.f. Problem (Q) in p. 324 of [26] for the constant-coefficient Stokes system): Find $(\hat{\mathbf{u}}, \pi) \in \mathring{\mathcal{H}}^1(\Omega_-)^3 \times L^2(\Omega_-)$ such that

$$\begin{cases} a_{\mu; \Omega_-}(\hat{\mathbf{u}}, \mathbf{v}) + b_{\Omega_-}(\mathbf{v}, \pi) = \mathfrak{F}_{\mu; \mathbf{u}_0}(\mathbf{v}), \quad \forall \mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3, \\ b_{\Omega_-}(\hat{\mathbf{u}}, q) = 0, \quad \forall q \in L^2(\Omega_-), \end{cases} \quad (5.8)$$

where $a_{\mu;\Omega_-} : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times \mathring{\mathcal{H}}^1(\Omega_-)^3 \rightarrow \mathbb{R}$ and $b_{\Omega_-} : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times L^2(\Omega_-) \rightarrow \mathbb{R}$ are the bilinear forms given by

$$a_{\mu;\Omega_-}(\mathbf{w}, \mathbf{v}) := 2\langle \mu \mathbb{E}(\mathbf{w}), \mathbb{E}(\mathbf{v}) \rangle_{\Omega_-}, \quad \forall \mathbf{v}, \mathbf{w} \in \mathring{\mathcal{H}}^1(\Omega_-)^3, \quad (5.9)$$

$$b_{\Omega_-}(\mathbf{v}, q) := -\langle \operatorname{div} \mathbf{v}, q \rangle_{\Omega_-}, \quad \forall \mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3, q \in L^2(\Omega_-), \quad (5.10)$$

and $\mathfrak{F}_{\mu;\mathbf{u}_0} : \mathring{\mathcal{H}}^1(\Omega_-)^3 \rightarrow \mathbb{R}$ is the linear form given by

$$\mathfrak{F}_{\mu;\mathbf{u}_0}(\mathbf{v}) := -\left(\langle \mathbf{f}, \mathring{E}_- \mathbf{v} \rangle_{\Omega_-} + 2\langle \mu \mathbb{E}(\mathbf{u}_0), \mathbb{E}(\mathbf{v}) \rangle_{\Omega_-} \right), \quad \forall \mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3. \quad (5.11)$$

Here we took into account that the spaces $\mathring{\mathcal{H}}^1(\Omega_-)^3$ and $\tilde{\mathcal{H}}^1(\Omega_-)^3$ can be identified through the isomorphism $\mathring{E}_- : \mathring{\mathcal{H}}^1(\Omega_-)^3 \rightarrow \tilde{\mathcal{H}}^1(\Omega_-)^3$. Note that

$$\begin{aligned} \mathring{\mathcal{H}}_{\operatorname{div}}^1(\Omega_-)^3 &:= \left\{ \mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3 : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_- \right\} \\ &= \left\{ \mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3 : b_{\Omega_-}(\mathbf{v}, q) = 0, \quad \forall q \in L^2(\Omega_-) \right\}. \end{aligned} \quad (5.12)$$

Now, formula (2.11), inequality (3.1) and the Hölder inequality yield that

$$\begin{aligned} |a_{\mu;\Omega_-}(\mathbf{v}_1, \mathbf{v}_2)| &\leq 2c_\mu \|\mathbb{E}(\mathbf{v}_1)\|_{L^2(\Omega_-)^{3 \times 3}} \|\mathbb{E}(\mathbf{v}_2)\|_{L^2(\Omega_-)^{3 \times 3}} \\ &\leq 2c_\mu \|\mathbf{v}_1\|_{\mathring{\mathcal{H}}^1(\Omega_-)^3} \|\mathbf{v}_2\|_{\mathring{\mathcal{H}}^1(\Omega_-)^3}, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in \mathring{\mathcal{H}}^1(\Omega_-)^3. \end{aligned} \quad (5.13)$$

Moreover, the formula

$$2\|\mathbb{E}(\mathbf{v})\|_{L^2(\Omega_-)^{3 \times 3}}^2 = \|\operatorname{grad} \mathbf{v}\|_{L^2(\Omega_-)^{3 \times 3}}^2 + \|\operatorname{div} \mathbf{v}\|_{L^2(\Omega_-)}^2, \quad \forall \mathbf{v} \in \mathcal{D}(\Omega_-)^3 \quad (5.14)$$

(cf., e.g., the proof of Corollary 2.2 in [46]), and the density of the space $\mathcal{D}(\Omega_-)^3$ in $\mathring{\mathcal{H}}^1(\Omega_-)^3$ show that the same formula holds also for any function in $\mathring{\mathcal{H}}^1(\Omega_-)^3$. Therefore, we obtain the following Korn type inequality

$$\|\operatorname{grad} \mathbf{v}\|_{L^2(\Omega_-)^{3 \times 3}} \leq 2^{\frac{1}{2}} \|\mathbb{E}(\mathbf{v})\|_{L^2(\Omega_-)^{3 \times 3}}, \quad \forall \mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3. \quad (5.15)$$

Then by using inequality (5.15), the equivalence of seminorm (2.12) to the norm (2.11) in the space $\mathring{\mathcal{H}}^1(\Omega_-)^3$, and assumption (3.1) we deduce that there exists a constant $C = C(\Omega_-) > 0$ such that

$$\begin{aligned} \|\mathbf{u}\|_{\mathring{\mathcal{H}}^1(\Omega_-)^3}^2 &\leq C \|\operatorname{grad} \mathbf{u}\|_{L^2(\Omega_-)^{3 \times 3}}^2 \leq 2C \|\mathbb{E}(\mathbf{u})\|_{L^2(\Omega_-)^{3 \times 3}}^2 \\ &\leq 2Cc_\mu \|\mu \mathbb{E}(\mathbf{u})\|_{L^2(\Omega_-)^{3 \times 3}}^2 = 2Cc_\mu a_{\mu;\Omega_-}(\mathbf{u}, \mathbf{u}), \quad \forall \mathbf{u} \in \mathring{\mathcal{H}}^1(\Omega_-)^3, \end{aligned}$$

and accordingly that

$$a_{\mu;\Omega_-}(\mathbf{u}, \mathbf{u}) \geq \frac{1}{2Cc_\mu} \|\mathbf{u}\|_{\mathring{\mathcal{H}}^1(\Omega_-)^3}^2, \quad \forall \mathbf{u} \in \mathring{\mathcal{H}}^1(\Omega_-)^3. \quad (5.16)$$

In view of inequalities (5.13) and (5.16) it follows that the bilinear form $a_{\mu;\Omega_-}(\cdot, \cdot) : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times \mathring{\mathcal{H}}^1(\Omega_-)^3 \rightarrow \mathbb{R}$ is bounded and coercive. Moreover, arguments similar to those for inequality (5.13) imply that the bilinear form $b_{\Omega_-}(\cdot, \cdot) : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times L^2(\Omega_-) \rightarrow \mathbb{R}$ and the linear form $\mathfrak{F}_{\mu;\mathbf{u}_0} : \mathring{\mathcal{H}}^1(\mathbb{R}^3)^3 \rightarrow \mathbb{R}$ given by (5.10) and (5.11), are also bounded. Since the operator

$$\operatorname{div} : \mathring{\mathcal{H}}^1(\Omega_-)^3 \rightarrow L^2(\Omega_-) \quad (5.17)$$

is surjective (cf., e.g., [26, Theorem 3.2]), then by Lemma A.2, the bounded bilinear form $b_{\Omega_-}(\cdot, \cdot) : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times L^2(\Omega_-) \rightarrow \mathbb{R}$ satisfies the inf-sup condition

$$\inf_{q \in L^2(\Omega_-) \setminus \{0\}} \sup_{\mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3 \setminus \{0\}} \frac{b_{\Omega_-}(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathring{\mathcal{H}}^1(\Omega_-)^3} \|q\|_{L^2(\Omega_-)}} \geq \beta_D \quad (5.18)$$

with some constant $\beta_D > 0$ (cf. [26, Theorem 3.3]). Then Theorem A.4 (with $X = \mathring{\mathcal{H}}^1(\Omega_-)^3$ and $M = L^2(\Omega_-)$) implies that the variational problem (5.8) has a unique solution $(\mathring{\mathbf{u}}, \pi) \in \mathring{\mathcal{H}}^1(\Omega_-)^3 \times L^2(\Omega_-)$. Moreover, the pair $(\mathbf{u}, \pi) = (\mathring{\mathbf{u}} + \mathbf{u}_0, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$, where $\mathbf{u}_0 \in \mathcal{H}^1(\Omega_-)^3$ satisfies relations (5.4), is the unique solution of the mixed variational formulation (5.3) and depends continuously on the given data $(\mathbf{f}, \phi) \in \mathcal{H}^{-1}(\Omega_-)^3 \times H^{\frac{1}{2}}(\partial\Omega)^3$. The equivalence between the variational problem (5.3) and the exterior Dirichlet problem (5.1) shows that problem (5.1) is also well-posed, as asserted. \square

5.2. Potential approach

Theorem 5.1 asserts the well-posedness of the exterior Dirichlet problem for the Stokes system with L^∞ coefficients. However, if the given data (\mathbf{f}, ϕ) belong to the space $\mathcal{H}^{-1}(\Omega_-)^3 \times H_V^{\frac{1}{2}}(\partial\Omega)^3$, then the solution can be expressed in terms of the Newtonian and single layer potential and of the inverse of the single layer operator as follows (cf. [26, Theorem 3.4] for $\mu > 0$ constant, [22, Theorem 10.1] and [37, Theorem 5.1] for the Laplace operator).

Theorem 5.2. *If $\mathbf{f} \in \mathcal{H}^{-1}(\Omega_-)^3$ and $\phi \in H_V^{\frac{1}{2}}(\partial\Omega)^3$ then the exterior Dirichlet problem (5.1) has a unique solution $(\mathbf{u}, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$, given by*

$$\mathbf{u} = \mathcal{N}_{\mu; \mathbb{R}^3}(\tilde{\mathbf{f}})|_{\Omega_-} + \mathbf{V}_{\mu; \partial\Omega} \left(\mathcal{V}_{\mu; \partial\Omega}^{-1}(\phi - \gamma_-(\mathcal{N}_{\mu; \mathbb{R}^3}(\tilde{\mathbf{f}}))) \right), \quad (5.19)$$

$$\pi = \mathcal{Q}_{\mu; \mathbb{R}^3}(\tilde{\mathbf{f}})|_{\Omega_-} + \mathcal{Q}_{\mu; \partial\Omega}^s \left(\mathcal{V}_{\mu; \partial\Omega}^{-1}(\phi - \gamma_-(\mathcal{N}_{\mu; \mathbb{R}^3}(\tilde{\mathbf{f}}))) \right) \text{ in } \Omega_-, \quad (5.20)$$

where $\tilde{\mathbf{f}}$ is an extension of \mathbf{f} to an element of $\mathcal{H}^{-1}(\mathbb{R}^3)^3$.

Proof. The result follows from Definition 4.3 and Lemmas 4.7, 4.8, 4.10. \square

Appendix A. Mixed variational formulations and their well-posedness property

Here we make a brief review of well-posedness results due to Babuška [6] and Brezzi [10] for mixed variational formulations related to bounded bilinear forms in reflexive Banach spaces. We follow [20, Section 2.4], [11], [25, §4].

Let X and \mathcal{M} be reflexive Banach spaces, and let X^* and \mathcal{M}^* be their dual spaces. Let $a(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}$, $b(\cdot, \cdot) : X \times \mathcal{M} \rightarrow \mathbb{R}$ be bounded bilinear forms. Then we consider the following abstract mixed variational formulation.

For $f \in X^$, $g \in \mathcal{M}^*$ given, find a pair $(u, p) \in X \times \mathcal{M}$ such that*

$$\begin{cases} a(u, v) + b(v, p) &= f(v), \quad \forall v \in X, \\ b(u, q) &= g(q), \quad \forall q \in \mathcal{M}. \end{cases} \quad (\text{A.1})$$

Let $A : X \rightarrow X^*$ be the bounded linear operator defined by

$$\langle Av, w \rangle = a(v, w), \quad \forall v, w \in X, \quad (\text{A.2})$$

where $\langle \cdot, \cdot \rangle := X^* \langle \cdot, \cdot \rangle_X$ is the duality pairing of the dual spaces X^* and X . We also use the notation $\langle \cdot, \cdot \rangle$ for the duality pairing $\mathcal{M}^* \langle \cdot, \cdot \rangle_{\mathcal{M}}$. Let $B : X \rightarrow \mathcal{M}^*$ and $B^* : \mathcal{M} \rightarrow X^*$ be the bounded linear and transpose operators given by

$$\langle Bv, q \rangle = b(v, q), \quad \langle v, B^*q \rangle = \langle Bv, q \rangle, \quad \forall v \in X, \forall q \in \mathcal{M}. \quad (\text{A.3})$$

In addition, we consider the spaces

$$V := \text{Ker } B = \{v \in X : b(v, q) = 0, \quad \forall q \in \mathcal{M}\}, \quad (\text{A.4})$$

$$V^\perp := \{T \in X^* : \langle T, v \rangle = 0, \quad \forall v \in V\}. \quad (\text{A.5})$$

Then the following well-posedness result holds (cf., e.g., [20, Theorem 2.34]).

Theorem A.1. *Let X and \mathcal{M} be reflexive Banach spaces, $f \in X^*$ and $g \in \mathcal{M}^*$, and $a(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}$ and $b(\cdot, \cdot) : X \times \mathcal{M} \rightarrow \mathbb{R}$ be bounded bilinear forms. Let V be the subspace of X defined by (A.4). Then the variational problem (A.1) is well-posed if and only if $a(\cdot, \cdot)$ satisfies the conditions*

$$\left\{ \begin{array}{l} \exists \lambda > 0 \text{ such that } \inf_{u \in V \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{a(u, v)}{\|u\|_X \|v\|_X} \geq \lambda, \\ \{v \in V : a(u, v) = 0, \quad \forall u \in V\} = \{0\}, \end{array} \right. \quad (\text{A.6})$$

and $b(\cdot, \cdot)$ satisfies the inf-sup (Ladyzhenskaya-Babuška-Brezzi) condition,

$$\exists \beta > 0 \text{ such that } \inf_{q \in \mathcal{M} \setminus \{0\}} \sup_{v \in X \setminus \{0\}} \frac{b(v, q)}{\|v\|_X \|q\|_{\mathcal{M}}} \geq \beta. \quad (\text{A.7})$$

Moreover, there exists a constant C depending on β , λ and the norm of $a(\cdot, \cdot)$, such that the unique solution $(u, p) \in X \times \mathcal{M}$ of (A.1) satisfies the inequality

$$\|u\|_X + \|p\|_{\mathcal{M}} \leq C (\|f\|_{X^*} + \|g\|_{\mathcal{M}^*}). \quad (\text{A.8})$$

In addition, we have (see [20, Theorem A.56, Remark 2.7], [4, Theorem 2.7]).

Lemma A.2. *Let X, \mathcal{M} be reflexive Banach spaces. Let $b(\cdot, \cdot) : X \times \mathcal{M} \rightarrow \mathbb{R}$ be a bounded bilinear form. Let $B : X \rightarrow \mathcal{M}^*$ and $B^* : \mathcal{M} \rightarrow X^*$ be the operators defined by (A.3), and let $V = \text{Ker } B$. Then the following results are equivalent:*

- (i) *There exists a constant $\beta > 0$ such that $b(\cdot, \cdot)$ satisfies condition (A.7).*
- (ii) *$B : X/V \rightarrow \mathcal{M}^*$ is an isomorphism and $\|Bw\|_{\mathcal{M}^*} \geq \beta \|w\|_{X/V}$ for any $w \in X/V$.*
- (iii) *$B^* : \mathcal{M} \rightarrow V^\perp$ is an isomorphism and $\|B^*q\|_{X^*} \geq \beta \|q\|_{\mathcal{M}}$ for any $q \in \mathcal{M}$.*

Remark A.3. Let X be a reflexive Banach space and V be a closed subspace of X . If a bounded bilinear form $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ is *coercive* on V , i.e., there exists a constant $c_a > 0$ such that

$$a(w, w) \geq c_a \|w\|_X^2, \quad \forall w \in V, \quad (\text{A.9})$$

then the conditions (A.6) are satisfied as well (see, e.g., [20, Lemma 2.8]).

The next result known as the *Babuška-Brezzi theorem* is the version of Theorem A.1 for Hilbert spaces (see [6], [10, Theorems 0.1, 1.1, Corollary 1.2]).

Theorem A.4. *Let X and \mathcal{M} be two real Hilbert spaces. Let $a(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}$ and $b(\cdot, \cdot) : X \times \mathcal{M} \rightarrow \mathbb{R}$ be bounded bilinear forms. Let $f \in X^*$ and $g \in \mathcal{M}^*$. Let V be the subspace of X defined by (A.4). Assume that $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ is coercive and that $b(\cdot, \cdot) : X \times \mathcal{M} \rightarrow \mathbb{R}$ satisfies the inf-sup condition (A.7). Then the variational problem (A.1) is well-posed.*

Acknowledgements

The research has been supported by the grant EP/M013545/1: "Mathematical Analysis of Boundary-Domain Integral Equations for Nonlinear PDEs" from the EPSRC, UK. Part of this work was done in April/May 2018, when M. Kohr visited the Department of Mathematics of the University of Toronto. She is grateful to the members of this department for their hospitality.

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