Risk Minimisation Using Options and Risky Assets

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This thesis is dedicated to
the memories of my father Maasar Maasin (1947-2012)
"standard-lah-si-kit.."

Declaration

I hereby declare that except where a specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

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Abstract

We consider mean-risk portfolio optimisation models, with risk quantified by symmetric measures (variance) as well as downside or tail measures (Lower Partial Moments, Conditional Value at Risk). A framework for including index options in the universe of assets, in addition to stocks, is provided. The exercise of index options is settled in cash, making this implementable with a variety of strike prices and maturities. We use a dataset with stocks from the FTSE 100 and index options on the FTSE100. Numerical results show that, for low to medium risk portfolios, the addition of an index put further reduces the risk to a considerable extent, particularly in the case of mean-CVaR efficient portfolios, where the left tail is dramatically improved. For higher risk portfolios, the inclusion of an index call improves the right tail of the portfolio distribution, creating thus the opportunity for considerably higher returns.

Contents

List of Tables

Chapter 1

Introduction

1.1 Decision Making Under Uncertainty and Risk

Decision-making involves the process of making choices among possible actions which offer different outcomes. Consequently, a preference relation on the set of outcomes is needed for one to make a rational decision. It is mathematically fundamental to establish this preference relation. If the possible outcomes are real numbers, the preference relation is straightforward to set. In this case, a decision has one single possible outcome. The standard "$\geq$" (larger than or equal) for real numbers is used if a high outcome (e.g. profit) is desirable. Likewise, if a lower result (e.g. cost) is desirable, the standard "$\leq$" (less than or equal) relation for real numbers is chosen as the preference relation.

Things are much more complicated if the outcomes of decisions are not known with certainty, as before. In this case, the outcome of a decision is represented by a random variable rather than a number. The distributions of the random variables representing decisions may be known or unknown, may be discrete or continuous. Usually, such a distribution is considered as discrete, represented by a finite set of possible realizations. If the probabilities of these realizations are known (and hence the distributions of the random variables are known), we have a case of decision making under risk; otherwise, it is a case of decision making under uncertainty (Whitmore & Findlay, 1978). Consider a simple example of a case when higher outcomes are preferred. Here we prefer to choose an action that offers a maximal result.

Example: Let n be the number of the available fixed-term deposit account, each with a known fixed interest rate over one investment period. Express A_j as the action of choosing a deposit account $j, j \in \{1, \ldots, n\}$ and let r_j be the interest rate offered by a deposit account j. If the initial amount of money to invest is W, then the outcome of deciding to invest in deposit j is Wr_j. Altogether we obtain a finite set of outcomes $\{Wr_1, \ldots, Wr_n\}$ with the preference relation of "$\geq$". The first thing to do is to identify the most preferred consequence, $Wr_j = \max\{Wr_1, \ldots, Wr_n\}$, and then to choose the action that leads to this consequence, A_j.

In this example, the outcomes of any actions are known, due to the certainty of return of a fixed-term account. However, if we consider n assets with variable returns, the outcome of choosing a specific asset is now unknown. One way to represent this uncertainty is to consider a set of m possible "scenarios". Denote by r_{ij} the return of asset $j \in \{1, \ldots, n\}$ under scenario $s_i, i \in \{1, \ldots, m\}$. If A_j is the action of choosing the $j-$th asset, A_j has m possible outcomes $r_{1j}, \ldots, r_{mj}$. If each scenario s_i has a probability of happening p_i, then the return of the asset $j \in \{1, \ldots, n\}$ is a well-defined random variable R_j with possible outcomes $r_{1j}, \ldots, r_{mj}$ occuring with probability $p_1, \ldots, p_m$ (see Whitmore and Findlay (1978) for a detailed framework).

Under this framework, the action A_j can be identified with a discrete random variable R_j. Choosing between actions means choosing between random variables, where a preference relation is not anymore straightforward, as some random variables offer good outcomes under some scenarios and poor outcomes under other scenarios. Moreover, the set of actions may not be finite anymore, as instead of choosing one of $\{A_1, \ldots, A_n\}$ (that is investing all money into one of these actions) one could fraction/share the money into available assets. This leads to the problem of portfolio selection, which is described in the next section 1.2.

1.2 Portfolio Selection as a Problem of Decision Making Under Risk

The portfolio selection problem is about how to divide an investor's wealth amongst a set of available securities. One basic principle in finance is that, due to the lack of perfect information about the future asset returns, financial decisions are made in the face of trade-offs. Markowitz (1952) identified the trade-off faced by the investors as risk versus expected return and proposed variance as a measure of risk He introduced the concepts of efficient portfolio and efficient frontier and proposed a computational method for finding efficient portfolios.

Following notations given in Roman and Mitra (2009), we consider an example of portfolio selection with one investment period. A rational investor is interested in investing their capital such that, at the end of the investment period, the return is maximised. The return of an asset between time t_0 and t_1 is defined as $\frac{P_1 - P_0}{P_0}$, where P_a =price of the asset at time t_a. If P_1 is unknown at time t_0 (which is the case of stocks), return is unknown.

Consider a set of n assets, with asset $j \in \{1 \ldots n\}$ having a return R_j at the end of the investment period. Since the future price of the asset is unknown, R_j is a random variable.

A portfolio is defined by the percentage of money invested in each asset j. Let x_j be the percentage of capital invested in asset j ($x_j = \frac{w_j}{w}$ where w_j is the amount of money invested in asset j and w is the total amount of capital to be invested), and let $x = (x_1, \ldots, x_n)$ denote the portfolio choice. This portfolio return is given as

$$R_x = x_1 R_1 + \ldots + x_n R_n$$

with distribution function $F(r) = P(R_x \leq r)$ depending on the choice of $x = (x_1, \ldots, x_n)$ (Roman & Mitra, 2009).

The weights $(x_1, \ldots, x_n)$ belong to a set of decision vectors given as

$$X = \{(x_1, \ldots, x_n) | \sum_{j=1}^{n} x_j = 1, x_j \geq 0, \forall j \in \{1, \ldots, n\}\} \tag{1.1}$$

This is the simplest way to represent a feasible set: by the requirement that the weights must sum to 1 and no short selling[1] is allowed.

[1]Short selling is the sale of a security that is not owned by the seller. In most of cases, the seller borrowed the securities to short sell. Short selling is motivated by the belief that a security's price will decline, enabling it to be bought back at a lower price to make a profit.

To interpret the portfolio selection problem, let us consider another portfolio R_y that is determined by the decision vector $y = (y_1, \ldots, y_n) \in X$, where y_j is the proportion of capital invested in asset j. Hence, the random variable $R_y = y_1 R_1 + \ldots + y_n R_n$ represent this portfolio return. Now , the problem of choosing between portfolios $x = (x_1, \ldots, x_n) \in X$ and $y = (y_1, \ldots, y_n) \in X$ becomes the problem of choosing between the two random variables R_x and R_y. This means that any portfolio is associated with a random variable describing its return. Thus, a model for choosing which random variable is considered "better" than another random variable is required. The first purpose of such model is to define a preference relation among random variables and the second purpose is to identify the non-dominated random variables with respect to that preference relation.

As per section 1.1, it is usual to represent a random variable associated to return as a discrete random variable with possible outcomes $\{r_{1j}, \ldots, r_{mj}\}$ occuring with probability $p_1, \ldots, p_m$.

1.3 The Modelling Paradigm

One paradigm for choosing among random variables is mean-risk. Here, a random variable R_x representing the return of a portfolio x is characterized using two statistics of its distribution: the expected value/mean (large value are desired) and a "risk" value (low values are desired). The preference relationship is defined based on these two statistics: one random variables is "preferred" to another if it has higher mean and lower risk. A non-dominated random variable under this relationship represents an "efficient" portfolio: one that has the lowest risk for a specified level of expected return. An efficient portfolio is found by solving an optimisation problem in which we minimise risk subject to a constraint on the expected return. Varying the level of expected return, we obtain different efficient portfolios[2].

[2]This corresponds to examples of low mean-low risk trade-offs up to high mean-high risk trade-offs.

1.4 Thesis Motivation

Markowitz (1952) proposed variance as a measure of risk. Criticism of variance, mainly due to its symmetric nature that penalizes upside potential as well as downside deviations, led to proposal of other risk measures, most notably below target risk measures such as Lower Partial Moments (see Fishburn (1977), Bawa and Lindenberg (1977)) and quantile based risk measures such as Value-at Risk (VaR) and Conditional VaR (see Rockafellar and Uryasev (2000), Tasche (2002)). A review of financial risk measures can be found in Roman and Mitra (2009) and Albrecht (2004). Mean risk models with various risk measures have been implemented (see for example Roman and Mitra (2009) and references within); this has been mainly in the context when the universe of assets consists of stocks and bonds.

A closely related area of research concerns financial scenario generation: this is about simulating future values for asset prices or returns with the purpose of serving as parameters in optimisation models. Commonly used methods of financial scenario generation include sampling or bootstrapping (see Efron and Tibshirani (1994)) from historical data, or methods based on econometric models (see Bollerslev (1986)). More recently, general purpose scenario generators were proposed and used in financial optimisation models, such as the moment matching method (see Høyland, Kaut, and Wallace (2003)) or Hidden Markov Models (see Messina and Toscani (2008), Roman and Mitra (2009), Erlwein, Mamon, and Davison (2011)). A review of desirable properties for scenario generators is given in Kaut and Wallace (2003).

Options are financial assets that give the right (not the obligation) to trade an asset at a specified price. They can be of great use as they put a limit on the losses that could be incurred. However, including options in portfolio optimisation is not an easy task due to several reasons.

Identifying the return distribution of an option is difficult if the option is traded before its maturity date. Options for each of the component stocks with the maturity equal to its investment period may not be available. Constructing a portfolio of stocks and adding options for each stock may seem one straightforward way to integrate options, however, even in the case when such options are available, it may be one very costly approach, leading to great decrease in portfolio return.

Research efforts towards portfolio optimisation in the presence of options include S. Alexander, Coleman, and Li (2006), Papahristodoulou and Dotzauer (2004), Horasanlı (2008), and Faias and Santa-Clara (2017). In S. Alexander et al. (2006), CVaR minimisation for portfolios of options only is considered. They demonstrated possible computation of optimal CVaR with low number of assets. In Faias and Santa-Clara(2017), an expected utility maximisation is used for a portfolio of options where the payoffs are simulated from distribution of the underlying asset. Since options put a limit on the maximum loss, options have been considered in the context of robust optimisation (see Zymler, Rustem, and Kuhn (2011)).

In this study, we propose to use index options, in addition to a set of stocks, in mean-risk scenario based optimisation models. Index options are settled in cash (an investor is not required to trade the underlying, i.e. the index) making it an implementable strategy with a variety of maturities and exercise prices and with transaction costs. The motivation and contribution of this work lies in finding answers to the following research questions:

1. Can improved (in terms of mean-risk trade-off) portfolios be obtained by adding index options?
2. Which risk measures are more sensitive to the introduction of index options?
3. What is the numerical framework to use when the universe of assets is composed of stocks and index options?

The rest of this thesis is organised as follows. Chapter 2 presents the literature review on the selected mean-risk models for this thesis. Risk measures and the algebraic formulations of the corresponding mean-risk optimisation models is presented in Chapter 3. Chapter 4 describes the background for incorporating an index option in the portfolio optimisation. Computational results are presented in Chapter 5. Conclusions are drawn in Chapter 6.

Chapter 2

Literature Review

2.1 Historical Background

In 1952, Markowitz identified the portfolio selection as a bi-criteria optimisation problem with a trade-off between maximum expected return and minimum risk. Such solving of the portfolio selection problem leads to the introduction of mean-risk models. Markowitz introduced variance as a risk measure (Markowitz, 1952). Mean-Variance analysis for optimal asset allocation is widely regarded as and remains to date a widely used model of portfolio allocation (Björk, Murgoci, & Zhou, 2014). However, because of its symmetrical property, practitioners and academics criticized variance as a risk measure (for example, see Lwin, Qu, and MacCarthy (2017)), as it also penalizes favourable outcomes. Since 1952, many alternative risk measures are introduced.

In the 1970's, lower partial moments (LPM) were introduced by Bawa and Lindenberg and Fishburn to generalize the "below target" risk measures (Fishburn, 1977; Bawa & Lindenberg, 1977). Fishburn is also the first to propose the (α, τ) model which is one of the mean-risk models in which the risk measure used is LPM of order α around τ (Fishburn, 1977). In 1992, LPM has been characterized as a risk measure by a general set of utility functions (Nawrocki, 1992).

The mean absolute deviation (MAD) has been introduced as a risk measure by Konno and Yamazaki (Konno & Yamazaki, 1991). They proposed a mean-risk model where the risk is measured by the absolute deviation instead of variance. The model was re-formulated as an LP (linear programming) model and designed to keep the advantages of the mean-variance model, while at the same time being computationally cheaper.

In 1993, the Value-at-Risk (VaR) was introduced to highlight the importance of measuring risk for regulatory purposes (see, for example Roman and Mitra (2009)). To estimate the VaR of a portfolio is, roughly speaking, to determine how much the value of a portfolio could decline over a given period of time with a given probability (Hendricks, 1997). Corresponding to a confidence level, VaR is a percentile of the profit and loss distribution.

The concept of "coherence" was introduced in (1999) as a set of desirable properties of risk measures concerning with the tail of the distribution (Artzner et al., 1999). Despite its benefits, VaR is not convex as a function of portfolio weights and thus difficult to optimise (Pang & Leyffer, 2004; Larsen, Mausser, & Uryasev, 2002).

In practice, VaR is only coherent when the underlying loss distribution is normal, because otherwise it lacks of sub-additivity (Artzner et al., 1999). Due to this problem, a new risk measure, the Conditional Value-at-Risk (CVaR), is introduced by Rockafellar and Uryasev (2000). VaR and CVaR are two important risk measures that have been used extensively in portfolio selection; VaR is closely related to a particular quantile of the profit and loss distribution and CVaR, formally defined by Rockafellar and Uryasev, quantifies the expected loss beyond VaR. They concluded that CVaR is an appropriate risk measure to use in insurance and has better theoretical and computational properties than VaR.

As presented by Alexander and Baptista (2004), a CVaR constraint is more effective than a VaR constraint as a tool to control risk-averse agents. Pflug (2000) independently showed that CVaR optimisation is much simpler and a mean-CVaR model can be reformulated as an LP. Mansini, Ogryczak, and Speranza (2007) studied the portfolio optimisation based on the use of multi-CVaR risk measures that is, using different confidence levels. The study allows for more detailed risk aversion modelling while preserving the original CVaR model (Mansini et al., 2007).

2.2 Comparisons of Mean-Variance and Mean-CVaR

We mainly consider in this thesis two mean-risk models, where the risk measures are Variance and Conditional Value-at-Risk. Mean-Variance and Mean-CVaR quantify risk from different perspectives: Variance is well-known as a measure of the spread around the expected value of a random variable and CVaR measures the expected loss corresponding to a number of worst cases, depending on the chosen confidence level (Roman, Darby-Dowman, & Mitra, 2004). Hence, the mean-variance and mean-CVaR approaches may lead to different solutions.

Benbachir, Gaboune, and Alaoui (2012) presented portfolio optimisation study by using mean-variance and mean-CVaR approaches. In their numerical work, the application of the mean-variance model to a portfolio of 30 stocks excludes 4 out of 30 stocks to achieve the target portfolio return. In contrast, the selection done by mean-CVaR optimisation excludes 13 out of 30 stocks to achieve the same portfolio return. They conclude that the portfolio selection under the mean-variance model reduces risk at a slower rate with the selection of assets made are less careful as compared to the selection made under the mean-CVaR model; that is, the mean-CVaR portfolios is constructed with less cardinality of assets.

We can interpret that in general, the mean-CVaR model has better properties in many aspects in comparison to mean-variance (and other mean-risk models). In recent studies, it is considered as the preferred approach for portfolio selection. However, the mean-variance is still considered as one of the main approaches in the studies of risk minimisation. This is because it can provide comparison when a new model is proposed and assists in validating the results of a new model. Choosing the suitable risk measures depending on the nature our problems is also crucial for a good investment choice.

2.3 Challenges ans Suggestions

According to Roman et al. (2004), the portfolio obtained in a solution of the mean-variance model may be considered as unacceptable by a regulator. This is because it may have an excessively large CVaR, leading to big losses under some unfavourable scenarios. At same time, the mean-CVaR model may also be unacceptable since it may have an excessively large variance, and thus results into a very small Sharpe ratio[1] (see Luenberger (1998)). Hence, many researchers revisit these risk measures and the resulting mean-risk models. They discuss about alternative models for portfolio selection, as well as the choice criteria based on stochastic dominance (Roman et al., 2004).

Due to the shortcomings of comparing the outcome from the two models, Roman, Darby-Dowman, and Mitra (2007) proposed a model for portfolio selection problems that uses both variance and CVaR in order to capture the pros of both mean-risk models. The problem is a quadratic program. The efficient solutions of this model may be found by solving a single objective optimisation problem in which the variance is minimised while constraints are imposed on the expected return and the CVaR level (Roman et al., 2007).

In conclusion, we can see the two models complement each other. Any new approach proposed could now be extended by combining other risk measures, which can lead to models with better results and performances in order to solve more complex problems (Dedu & Şerban, 2015).

[1]The Sharpe ratio is the average return earned in excess of the risk-free rate per unit of volatility or total risk. Subtracting the risk-free rate from the mean return, the performance associated with risk-taking activities can be isolated.

Chapter 3

Risk Measures

3.1 Risk measures

Risk measures are classified following Albrecht (2004), Rockafellar and Uryasev (2002), and Ogryczak and Śliwiński (2003) into two categories. The first category measures the deviation from a target and are concerned with the whole distribution of outcomes. The second category concerns only with the left tail of a return distribution (or the right tail of a loss distribution), that is, the unfavourable outcomes.

Adopting the terminology used in Albrecht (2004), *risk measures of the first kind* measure the magnitude of deviations from a specific point. These risk measures can be further divided into symmetric risk measures and asymmetric risk measures. Symmetric risk measures are calculated in terms of dispersion of results around a pre-specified target. Asymmetric risk measures quantify risk by taking into account only outcomes below target, that could be either fixed or distribution specific. Lower partial moments (LPM) and central semi-deviations are among the important asymmetric risk measures (see Fishburn (1977), Bawa and Lindenberg (1977), Ogryczak and Ruszczyński (1999)).

Risk measures of the second kind measure the overall significance of possible losses. These risk measures are concerned only with a certain number of worst outcomes (the left tail), of the return distribution. The commonly used risk measures in this category are Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) (see Jorion

(2001), Rockafellar and Uryasev (2000, 2002)). This section gives brief overview on risk measures (variance, LPM, VaR and CVaR) that will be used in this work.

- **Variance**

  Variance is a well-known indicator used in statistics for the spread around the mean of a random variable. The variance of a random variable R_x is defined as its second central moment, the expected value of the square of the deviations of R_x from its own mean;

  $$\sigma^2(R_x) = E[(R_x - E(R_x))^2]$$

  where $E(R_x)$ is the expected value of R_x.

  Variance of a portfolio return $R_x = x_1R_1 + \ldots + x_nR_n$ is a quadratic function of $x = (x_1, \ldots, x_n)$ (see Luenberger (1998), Markowitz (1952)):

  $$\sigma^2(R_x) = \sum_{j=1}^{n}\sum_{k=1}^{n} x_j x_k \sigma_{jk} \tag{3.1}$$

  Where σ_{jk} is the covariance between R_j and R_k.

- **Lower Partial Moments (LPM)**

  An asset pricing model using a mean-LPM was first developed by Bawa and Lindenberg (1977) and Fishburn (1977). LPM is a generic name for asymmetric measures that consider a fixed target below which an investor does not want the return to fall. Asymmetric measures provide a more intuitive representation of risk, since upside deviations are not penalised. LPM measures the expected value of deviation below a fixed target value τ.

  Let τ be a predefined target value for the portfolio return R_x, and let $\alpha \geq 0$.

  The LPM of order α around τ of the random variable R_x with distribution function F is defined as (see Fishburn (1977)):

  $$LPM_\alpha(\tau, R_x) = E\{[max(0, \tau - R_x)]^\alpha\} = \int_{-\infty}^{\tau} (\tau - r)^\alpha dF(r)$$

  While τ is a target fixed by decision maker (DM), α is a parameter describing the investor's risk aversion. The larger the α , the more risk-averse is the investor. A decision maker may be willing to take a risk in order to minimise the chance that the return falls below τ, provided that the main concern is the failure to meet the target return. For this case, choosing a small α is appropriate. Instead, if small deviations below the target are reasonably harmless when compare to large deviations, the DM may prefer a higher probability of falling below the target, as long as the shortfalls are not too large. In this case, a larger α is chosen (see Fishburn (1977)).

- **Value-at-Risk (VaR)**

  One of the most popular quantile-based risk measures is the Value-at-Risk (VaR) (see Jorion (2001)). The VaR at parameter $\alpha \in (0, 1)$, or confidence level $(1-\alpha)$, is defined as the negative of an $\alpha-$percentile of the portfolio return distribution, or as a $(1-\alpha)$-percentile of the portfolio loss distribution, where α is typically chosen as 0.01 or 0.05. Thus, with probability of at least $(1-\alpha)$, the loss[1] will not exceed VaR. Following definitions presented in Roman and Mitra (2009), the VaR at level α of R_x is defined using the notion of $\alpha-$quantiles, for the common case when the loss distribution is considered to be the negative of the return distribution:

  Definition 2.1: An $\alpha -$ *quantile* of R_x is a real number r such that

  $$P(R_x < r) \leq \alpha \leq P(R_x \leq r).$$

  Definition 2.2: The *lower* $\alpha -$ *quantile* of R_x, denoted by $q_\alpha(R_x)$ is defined as

  $$q_\alpha(R_x) = \inf\{r \in \mathbb{R} : F(r) = P(R_x \leq r) \geq \alpha\}.$$

  Definition 2.3: The *upper* $\alpha -$ *quantile* of R_x, denoted by $q^\alpha(R_x)$ is defined as

  $$q^\alpha(R_x) = \inf\{r \in \mathbb{R} : F(r) = P(R_x \leq r) > \alpha\}.$$

  Definition 2.4: The *Value-at-Risk at level* α *of* R_x is defined as the negative of the upper $\alpha-$quantile of $R_x : VaR_\alpha(R_x) = -q^\alpha(R_x)$.

  The minus sign in the definition of VaR is because $q^\alpha(R_x)$ is likely to be negative. Absolute values are considered in reporting this value in term of "loss".

  Although widely used in practice, VaR has been criticized for not being a coherent risk measure of risk (see Artzner et al. (1999)) and not being convex with

[1] In our context we refer negative returns as positive losses. Therefeore, any loss related to random variable R_x is represented by a random variable $-R_x$

respect to $x_1 \dots x_n$; this makes it difficult to optimise (see Pflug (2000)). This is also explained in Larsen et al. (2002), Pang and Leyffer (2004) and references therein.

- **Conditional Value-at-Risk (CVaR)**

  Conditional Value-at-Risk (see Rockafellar and Uryasev (2000, 2002)) was proposed as an alternative quantile-based risk measure. It has gained interest from practitioners and academics due to its desirable computational and theoretical properties. Informally speaking, given a percentage of worst-case scenarios, CVaR is the average of losses under these scenarios. A more formal definition involves the concept of "α - tail distribution". This is the distribution obtained by considering the worst A% outcomes of the return distribution (where $\alpha =$ A%) and scaling probabilities such that they sum up to 1 (hence a"proper" distribution is obtained). The CVaR at parameter α (confidence level $(1-\alpha)$)is the negative of the expected value of the alpha tail distribution.

  As pointed out in Roman and Mitra (2009), CVaR is approximately equal to the average of losses greater than or equal to VaR at the same confident level, α. If $P(R_x \leq q^\alpha(R_x)) = \alpha$, this approximation is exact:

$$CVaR_\alpha(R_x) = -\frac{1}{\alpha}E(R_x 1_{\{R_x \leq q^\alpha(R_x)\}})$$

  Definition 2.5: In general cases, the CVaR at level α of R_x is defined as:

$$CVaR_\alpha(R_x) = -\frac{1}{\alpha}\{E(R_x 1_{\{R_x \leq q^\alpha(R_x)\}}) - q^\alpha(R_x)[P(R_x \leq q^\alpha(R_x)) - \alpha]\}$$

  where

$$1_{Relation} = \begin{cases} 1, & \textit{if } \text{Relation } \textit{is true}; \\ 0, & \textit{if } \text{Relation } \textit{is false}. \end{cases}$$

3.2 CVaR calculation and optimisation

It has been shown by Rockafellar and Uryasev (2002) that CVaR can be optimised using an auxiliary function $F : X \times \mathbf{R} \mapsto \mathbf{R}$:

$$F_\alpha(x, v) = \frac{1}{\alpha}E[\max(-R_x + v, 0] - v$$

In practice, a portfolio return R_x is considered as a discrete random variable because the random returns are usually described by their realisations under various scenarios. This simplifies the calculation and optimisation of CVaR as it makes the optimisation problems above a linear programming problem. Suppose that R_x has m possible outcomes $r_{1x}, \ldots, r_{mx}$ with probabilities $p_1, \ldots, p_m$ with $r_{ix} = \sum_{j=1}^{n} x_j r_{ij}, \forall i \in \{1 \ldots m\}$, then:

$$F_\alpha(x, v) = \frac{1}{\alpha} \sum_{i=1}^{m} p_i[\max(v - r_{ix}, 0)] - v = \frac{1}{\alpha} \sum_{i=1}^{m} p_i[\max(v - \sum_{j=1}^{n} x_j r_{ij}, 0)] - v$$

This formulation will be used for the mean-CVaR optimisation model in the next section.

The following is proven in Rockafellar and Uryasev (2000, 2002). The optimal value of the objective function when minimizing F_α over v, for a fixed $x \in X$ is the CVaR of the portfolio. Also, the minimal CVaR is obtained by minimizing F_α over (x, v).

In contrast to VaR, CVaR is a convex function of portfolio weights $x_1, \ldots, x_n$. It is obvious that $CVaR_\alpha(x) \geq VaR_\alpha(x)$ for any portfolio $x \in X$. Thus, minimising CVaR can be used to limit the VaR of a portfolio. Furthermore, CVaR is known to be a coherent risk measure (see Artzner et al. (1999)).

We present below the algebraic formulation of the three mean-risk models used in our computational analysis. We use the following notation:

The input data are:

m	$=$	the number of (equally probable) scenarios;
n	$=$	the number of assets;
r_{ij}	$=$	the return of asset j under scenario i; $j = 1 \ldots n$, $i = 1 \ldots m$;
μ_j	$=$	the expected rate of return of asset j; $j = 1 \ldots n$;
σ_{kj}	$=$	the covariance between returns of asset k and asset j;$k, j = 1 \ldots n$;
d	$=$	target expected rate of return for the portfolio.

The decision variables are:

x_j $=$ the fraction of the portfolio value invested in asset j, $j = 1 \ldots n$.

3.2.1 The Mean-Variance Model (MV)

Mean-variance optimisation model is used to minimise variance of the portfolio return. Equation 3.1 is used to measure this risk, and Markowitz (1952) formulated the portfolio optimisation as a quadratic programming problem:

$$\begin{aligned} \min_{x} \quad & \sum_{j=1}^{n}\sum_{k=1}^{n}\sigma_{kj}x_j x_k \\ \text{subject to:} \quad & \\ & \sum_{j=1}^{n}\mu_j x_j \ \geq \ d \\ & x \ \in \ X \end{aligned}$$

3.2.2 The Mean-Expected Downside Risk model (M-LPM0)

For this model, we consider here $\tau = 0$ and $\alpha = 1$. In addition to the decision variables x_j, there are m decision variables, representing the magnitude of negative deviations of the portfolio return from the zero value, for every scenario $i \in \{1 \dots m\}$:

$$y_i = \begin{cases} -\sum_{j=1}^{n} r_{ij}x_j, & \text{if } \sum_{j=1}^{n} r_{ij}x_j \leq 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \min \quad & \frac{1}{m}\sum_{i=1}^{m} y_i \\ \text{subject to:} \quad & \\ & -\sum_{j=1}^{n} r_{ij}x_j \ \leq \ y_i \quad ; \ \forall i \in \{1 \dots m\} \\ & y_i \ \geq \ 0 \quad ; \ \forall i \in \{1 \dots m\} \\ & \sum_{j=1}^{n}\mu_j x_j \ \geq \ d \quad ; \\ & x \ \in \ X \end{aligned}$$

3.2.3 The Mean-CVaR$_\alpha$ Model (M-CVaR$_\alpha$)

For this model, in addition to the decision variables x_j, there are $m+1$ decision variables. The variable v represents the negative of an α-quantile of the portfolio return distribution. Thus, when solving this model, the optimal value of the variable v may be used as an approximation for VaR$_\alpha$. If the $\alpha-$quantile is unique, the optimal value of v is the VaR$_\alpha$ of the return distribution of the solution portfolio. The other m decision variables represent the magnitude of negative deviations of the portfolio return from the α-quantile, for every scenario $i \in \{1 \ldots m\}$:

$$y_i = \begin{cases} -v - \sum_{j=1}^{n} r_{ij} x_j, & \text{if } \sum_{j=1}^{n} r_{ij} x_j \leq -v; \\ 0, & \text{otherwise.} \end{cases}$$

$$\min \quad v + \frac{1}{\alpha m} \sum_{i=1}^{m} y_i$$

subject to:

$$\begin{aligned} \sum_{j=1}^{n} -r_{ij} x_j - v &\leq y_i \quad ; \quad \forall i \in \{1 \ldots m\} \\ y_i &\geq 0 \quad ; \quad \forall i \in \{1 \ldots m\} \\ \sum_{j=1}^{n} \mu_j x_j &\geq d \quad ; \\ x &\in X \end{aligned}$$

Chapter 4

Portfolio Optimisation with Options

4.1 Basics of option pricing

An option is a financial derivative described as a contract when the holder of the contract is given a right to exercise a deal, but the holder is not obliged to exercise this right. Financial options are traded both on exchanges and in the over-the-counter market.

There are two basic types of options namely calls and puts. A call option gives the holder the right to buy the underlying asset (stock, real estate etc.) at a certain price at a specified period of time. A put option gives the holder the right to sell the underlying asset at a certain price at a specified period of time. The price of underlying stated in the contract is known as the exercise price or strike price while the date in the contract is known as the expiration date or maturity (see Hull and Basu (2016)).

Most common options that are being exercised today are either American options or European options; which the difference between them is as follows. American options can be exercised at any time from the date of writing up to the expiration date, while European options can only be exercised at maturity.

Mathematically, the value of an option is represented in terms of the option payoff function. An option payoff function, is a function of the underlying stock price S_T at maturity, T. Consider put and call options with strike price K, the payoff function for a put and a call option is given as:

$$V_{put}(S_T) = \max\{0, K - S_T\},$$

and

$$V_{call}(S_T) = \max\{0, S_T - K\},$$

respectively.

As an example, assume that an investor is holding a portfolio consisting of a stock (long) and a put option on the same stock (long) with strike price K. The payoff function of the portfolio,V_{pf}, is therefore given as

$$V_{pf}(S_T) = S_T + V_{put}(S_T) = \max\{K, S_T\}.$$

This payoff function shows that the put option with strike price K secure the portfolio value at maturity from dropping below K.

Determining the correct price of an option has been a widely researched subject. A price of European call and put options can be given in closed-form using Black-Scholes option pricing formula (see Bodie, Kane, and Marcus (2014)). lt is widely used, although often with adjustments and corrections, by options market participants (see Zvi, Alex, and Alan (2004)).

In this study, option prices are obtained directly from Datastream (see Reuters (2010)) for the at-the-money index options. Our dataset is explained further in section 5.

4.2 Incorporating index options into portfolio optimisation

We consider a one period investment problem with decisions made at time t and evaluation made at time $(t + 1)$. We consider an initial universe of assets consisting of the component stocks of the FTSE 100. To this, we add a call and a put option on the FTSE 100 index as two extra assets.

As in Faias and Santa-Clara (2017), we employ a scenario based approach. We simulate the price of the FTSE100 at the end of the investment period. We use historical rates of return for the component assets of FTSE100 as scenarios for the rates of return between t and $(t+1)$. These historical rates of return are computed using prices monitored between periods of time equal to the investment period.

In order to calculate the rates of return of the options (under the same scenarios), we simulate the scenario prices of FTSE 100. We use the historical returns of FTSE 100, calculated in the similar way as the historical returns of the component assets. Using the current (known) price of FTSE100, denoted by S_t, we simulate prices for FTSE100 at time $(t+1)$ by multiplying S_t with the simulated returns for FTSE100. We summarise these scenario generation as follows:

1. Using the historical prices for the FTSE 100 (monitorised over the same time periods as the stocks in the universe of assets - we do this using monthly rates of return) we compute the corresponding historical rates of return, r_{t+1}.
2. The returns from step 1 are used to simulate next period's underlying (FTSE100) value S_{t+1}, given its current value S_t:

$$S_{t+1} = S_t(1 + r_{t+1})$$

3. Denoting by the K_c the strike price of the call, and by the K_p the strike price of the put, and using one period simulated underlying asset value S_{t+1}, we simulate option payoffs at their maturity $t+1$. Thus the payoffs for call and put are given respectively as:

$$V_{t+1,C} = \max(S_{t+1} - K_c, 0),$$

and

$$V_{t+1,P} = \max(K_p - S_{t+1}, 0).$$

4. Using the simulated payoff above, the rates of return of the options are:

$$r_{t+1,C} = \frac{V_{t+1,C}}{C_t} - 1,$$

for index call options, and

$$r_{t+1,P} = \frac{V_{t+1,P}}{P_t} - 1$$

for index put options. Denoted here that C_t and P_t are the real prices of the call and put index option, respectively, at decision time t, as described in Section 4.1.

Chapter 5

Computational Results

5.1 Objectives, Dataset and Computational Setup

Our main objective is to investigate whether the inclusion of index options leads to a significant decrease in risk, and thus significantly better optimal portfolios in terms of mean-risk trade-off. A second objective is to investigate the composition of optimal portfolios in terms of: (a) the proportion of options in the optimal portfolios, and (b) whether the portfolios with options have similar stock composition compared to their 'stock-only' counterpart. A third objective is to investigate the effect of the risk measure employed on the return distribution of the optimal portfolios.

For the first two objectives, we implement mean-variance and mean-CVaR model with a universe of assets composed of (a) stocks only; (b) the same stocks and two index options, a call and a put, with maturity equal to the investment period.

For the third objective, we implement the mean-variance, mean-expected downside risk and M-CVaR$_{0.05}$ models with a universe of stocks consisting of the component stocks of FTSE 100. We choose efficient portfolios with the same expected return and investigate the properties of their return distributions.

The data used for this analysis is drawn from the FTSE100. The investment period is one month. Monthly returns of the 87 stock components of the index from January 2005 until May 2014 are considered. The dataset for the in-sample analysis has 100 time periods, initially from Jan 2005 until May 2013; we employ a one month rolling window approach in which we consider 12 in-sample data sets by adding a next month of data and removing the oldest data point (thus, always having 100 time periods in the in-sample data set). For backtesting analysis, the portfolio is examined over the twelve months period of June 2013 until May 2014.

We consider at-the-money (that is, strike price equals to current price) call and put index options, with maturity one month. The prices are taken from Datastream codes ESXC.SERIESC (for calls) and ESXC.SERIESP (for puts) for our analysis in section 5.3. All data are obtained from Datastream (see Reuters (2010)) and models were implemented in AMPL (see Fourer, Gay, and Kernighan (1993)) and solved using the CPLEX 12.5 (see ILOG (2012)) optimisation solver.

The characteristics of efficient portfolios may vary depending on the target return, d. Based on our data set, the maximum level of asset expected return is 0.0349 and the minimum is at -0.007323. We chose three different level of d as $d_1 = 0.01$, $d_2 = 0.02$, and $d_3 = 0.03$. We solve the three mean-risk models considered above for every level of expected return d_1, d_2, and d_3.

5.2 In-sample analysis: stocks only

The return distributions of the efficient portfolios are discrete with 100 equally probable outcomes. We analyse these distributions using in sample parameters of standard deviation, skewness, minimum, maximum, and range. We compare sets of three distributions, each having the expected values of d_1, d_2, and d_3.

For a portfolio distribution, it is desirable to have smaller standard deviation and range, and to have larger median, skewness, minimum, and maximum. In all three cases (refer Table 5.1, 5.2 and 5.3) the LPM0 efficient portfolio has the highest median, while obviously the M-V portfolios has the lowest standard deviation. The other three statistics are consistently better for M-CVaR$_{0.05}$ portfolios. Furthermore, it is observed that M-CVaR$_{0.05}$ are the only ones with return distributions that are positively skewed.

Table 5.1: Statistics for the mean-risk efficient distributions with expected value $d_1 = 0.01$.

	M-V	M-LPM0	M-CVaR$_{0.05}$
Median	0.013642802	**0.015415625**	0.011657133
Standard Deviation	**0.033783896**	0.037980983	0.03830069313
Skewness	-0.572809564	-0.699597028	**0.361368572**
Minimum	-0.091575414	-0.13878571	**-0.075281385**
Maximum	0.096340696	0.115140014	**0.1340699**

Table 5.2: Statistics for the mean-risk efficient distributions with expected value $d_2 = 0.02$.

	M-V	M-LPM0	M-CVaR$_{0.05}$
Median	0.026989273	**0.019547821**	0.027136642
Standard Deviation	**0.045539089**	0.048306959	0.053504299
Skewness	-1.068379489	-0.541590574	**0.106153237**
Minimum	-0.169911236	-0.169067343	**-0.147248853**
Maximum	0.113089218	0.132563297	**0.189886327**

Table 5.3: Statistics for the mean-risk efficient distributions with expected value $d_3 = 0.03$.

	M-V	M-LPM0	M-CVaR$_{0.05}$
Median	0.034052445	**0.025471156**	0.033057945
Standard Deviation	**0.077244906**	0.080794207	0.084842553
Skewness	-0.577646499	-0.307744989	**0.199328725**
Minimum	-0.254267371	-0.269452992	**-0.213015378**
Maximum	0.186538415	0.200218938	**0.274484094**

5.3 Introducing index options in the universe of assets

We add the two index options as described in 5.1, and test the performance on two mean-risk models, the M-V and the M-CVaR$_{0.05}$, for d_1, d_2, and d_3. We perform optimisation on 12 data sets, obtained as described in 5.1 by using a rolling window of one month.

Table 5.4 displays the optimal weights of the put and call index options, together with the number of assets in the optimal portfolios. We can observe the following. Firstly, for low to medium expected rates of return (1% and 2%), the M-V efficient portfolios contain more assets than the M-CVaR$_{0.05}$ efficient portfolios. Secondly, the index put is in the composition of these portfolios (at 1% and 2% expected rate of return), for both M-V and M-CVaR efficient portfolios. The put is however in a higher proportion in M-CVaR efficient portfolios. Finally, for high expected returns (3%), it is the index call that is in the composition of the optimal portfolios, and in considerably higher amount in the M-CVaR$_{0.05}$ efficient portfolios.

The contribution of index call and put options into the expected rate of return of the M-CVaR efficient portfolio is provided in Table 5.5. It is observed here that the inclusion of put option contributes a negative return to the expected rate of return of the portfolio. This is the case because the put does work as an insurance for the portfolio to secure against its overall risk. Thus, this negative contribution is allowable as an 'insurance cost' to obtain an efficient portfolio. Whereas, the inclusion of more call option (in the high in-sample returns of 3%) contributes a positive return to the total expected return. This is happening because call options are used to achieve higher returns. Further explanation about the effect of this contribution to the level of risks will be discussed in Table 5.7.

These results are further explained by the correlation of the return distributions of the optimal portfolios with the return distribution of the index (see Table 5.6). While all efficient portfolios composed of stocks only are positively correlated to the index, by adding put options we obtain portfolios that are uncorrelated with the index. This is particularly true in the case of M-CVaR$_{0.05}$ efficient portfolios with low in-sample expected return (1%) and to somewhat a lesser extent, in the case of

Table 5.4: The number of assets in the composition of mean-risk portfolios with weight of index options.

		M-V			M-CVaR$_{0.05}$		
Optim. runs	In sample returns	Number of assets	Weight of put(%)	Weight of call(%)	Number of assets	Weight of put(%)	Weight of call(%)
1	0.01	24	1.91	0	19	2.31	0
	0.02	12	1.67	0	10	2.02	0
	0.03	7	0	0.93	7	1.10	3.37
2	0.01	25	1.98	0	19	2.32	0
	0.02	13	2.00	0	10	2.83	0
	0.03	7	0	0.74	6	1.07	5.19
3	0.01	25	1.98	0	20	2.33	0
	0.02	14	2.00	0	10	2.98	0
	0.03	7	0	1.77	6	1.81	4.71
4	0.01	25	1.98	0	21	2.27	0
	0.02	14	1.96	0	10	2.98	0
	0.03	7	0	1.72	6	2.02	4.74
5	0.01	23	1.99	0	21	2.27	0
	0.02	15	2.01	0	10	2.90	0
	0.03	7	0	1.30	8	1.84	5.93
6	0.01	23	1.98	0	21	2.27	0
	0.02	12	1.90	0	9	2.11	0
	0.03	7	0	2.32	8	1.73	7.09
7	0.01	24	1.96	0	21	2.27	0
	0.02	12	1.84	0	8	2.88	0
	0.03	6	0	2.77	6	1.39	9.56
8	0.01	23	1.97	0	19	2.26	0
	0.02	13	1.93	0	9	2.78	0
	0.03	6	0	1.46	6	1.35	10.75
9	0.01	23	2.01	0	19	2.27	0
	0.02	12	1.93	0	8	3.27	0
	0.03	4	0	1.52	6	**0**	**12.12**
10	0.01	23	1.99	0	19	2.24	0
	0.02	12	1.74	0	8	1.92	0
	0.03	5	0	1.86	8	0.66	8.92
11	0.01	23	2.00	0	18	2.23	0
	0.02	12	1.72	0	8	1.34	0
	0.03	3	**0**	**0**	2	**0**	**0**
12	0.01	21	1.98	0	19	2.22	0
	0.02	10	1.62	0	8	1.69	0
	0.03	4	0	2.11	4	**0**	7.52

Table 5.5: Contribution of Index Call and Put Options to the Expected Rate of Return of the In-sample M-CVaR efficient portfolios

	In-sample returns						
	$d_1 = 1\%$		$d_2 = 2\%$		$d_3 = 3\%$		
Optimisation runs	call(%)	put(%)	call(%)	put(%)	call(%)	put(%)	total(%)
1	0	-0.49	0	-0.60	0.43	-0.23	0.20
2	0	-0.34	0	-0.44	0.50	-0.16	0.34
3	0	-0.36	0	-0.41	0.68	-0.28	0.39
4	0	-0.37	0	-0.48	0.68	-0.33	0.35
5	0	-0.37	0	-0.35	0.75	-0.30	0.45
6	0	-0.37	0	-0.47	0.88	-0.28	0.60
7	0	-0.36	0	-0.44	0.90	-0.22	0.68
8	0	-0.36	0	-0.52	0.92	-0.21	0.71
9	0	-0.34	0	-0.29	0.72	0.00	0.72
10	0	-0.43	0	-0.26	0.75	-0.13	0.62
11	0	-0.35	0	-0.26	0.00	0.00	0.00
12	0	-0.35	0	0.00	0.38	0.00	0.38

medium expected return (2%). The M-V efficient portfolios at 1% and 2% expected return are still positively correlated to the index but to a lesser extent than their stocks only counterparts. For high risk - high return portfolios, index calls are in the composition of the efficient portfolios. This makes the resulting portfolio to be even more correlated with the index, in both M-V and M-$CVaR_{0.05}$ models.

Table 5.7 presents the optimal CVaR values in case of stocks only (S-portfolio) versus stocks + options (OS-portfolio). It is remarkable that risk is substantially decreased, especially at low in-sample expected returns. In the case of CVaR, risk is drastically reduced for each target returns of d_1 and d_2. This is because for these two target returns, the optimal portfolios includes a higher weight of put option as part of the portfolio. For high risk - high return portfolios ($d_3 = 3\%$) the decrease in risk obtained by adding index options is marginal. This is explained by the fact that index calls are mostly present in the composition of optimal portfolios (rather than puts) and these are used in order to achieve even higher return, rather than to reduce risk. A similar pattern, but to a lesser extent, is observed in case of M-V efficient portfolios (Table 5.8).

We investigate the composition of the efficient portfolios in the models considered. More precisely, we are interested to see whether by including an option we obtain a similar portfolio with the case of stocks only, scaled down to include the option

Table 5.6: Correlation coefficients of the return of FTSE 100 index with the return of efficient portfolios, composed of stocks only ("S") and composed of stocks + index options ("OS").

		M-V		$M\text{-}CVaR_{0.05}$	
Optim. runs	In sample mean return	OS (%)	S (%)	OS (%)	S (%)
1	0.01	23.40	63.36	-0.27	47.14
	0.02	41.80	70.00	24.84	53.52
	0.03	68.89	61.67	58.71	47.38
2	0.01	20.46	68.04	-0.56	59.16
	0.02	36.17	71.96	4.05	55.90
	0.03	64.66	59.52	62.70	55.99
3	0.01	21.97	67.78	1.76	60.85
	0.02	36.31	72.10	2.37	58.90
	0.03	73.82	60.97	57.26	57.05
4	0.01	23.14	68.77	4.41	58.69
	0.02	37.89	72.36	11.35	58.83
	0.03	73.90	59.25	55.93	52.64
5	0.01	22.76	68.86	4.50	59.49
	0.02	35.79	72.40	4.83	58.32
	0.03	73.29	63.49	66.23	52.72
6	0.01	24.14	69.40	4.21	59.29
	0.02	38.91	71.98	24.42	57.81
	0.03	76.74	60.69	69.36	55.68
7	0.01	23.92	69.35	4.59	59.62
	0.02	40.66	71.94	8.62	57.50
	0.03	74.84	57.84	68.75	55.83
8	0.01	24.36	69.16	4.04	63.51
	0.02	39.28	72.50	15.37	57.31
	0.03	70.80	61.69	69.92	57.23
9	0.01	22.59	68.93	3.27	59.56
	0.02	37.78	72.24	-0.45	56.90
	0.03	67.25	58.32	79.01	58.32
10	0.01	23.78	67.96	1.29	58.46
	0.02	41.66	71.32	32.42	55.83
	0.03	73.25	60.09	74.38	55.45
11	0.01	21.92	67.52	-1.08	57.54
	0.02	42.37	71.25	42.91	55.91
	0.03	54.66	54.66	52.44	52.44
12	0.01	22.70	67.10	2.66	56.00
	0.02	45.06	70.31	34.46	54.58
	0.03	66.41	54.60	79.30	54.60

Table 5.7: Optimal CVaR for M-CVaR$_{0.05}$ efficient portfolio; with (OS) and without (S) options.

	M-CVaR$_{0.05}$					
In sample returns	Optim. runs	S-portfolio(%)	OS-portfolio(%)	Optim. runs	S-portfolio(%)	OS-portfolio(%)
0.01		5.63	1.88		6.91	4.59
0.02	1	8.05	5.23	7	9.33	8.62
0.03		13.42	11.87		24.84	16.76
0.01		6.95	1.88		7.00	1.94
0.02	2	9.42	5.25	8	9.60	5.85
0.03		14.56	14.00		23.00	17.98
0.01		6.80	1.88		6.92	1.94
0.02	3	9.14	4.96	9	9.36	5.56
0.03		16.93	11.61		27.24	23.92
0.01		6.81	1.89		6.83	1.94
0.02	4	9.06	5.20	10	8.84	5.72
0.03		16.78	11.70		21.49	17.80
0.01		6.89	1.89		6.91	1.94
0.02	5	9.07	5.19	11	9.35	6.52
0.03		17.04	12.58		29.14	29.14
0.01		6.91	1.89		7.00	1.95
0.02	6	9.36	5.76	12	9.73	7.02
0.03		20.98	13.62		29.15	25.53

Table 5.8: Optimal standard deviations for M-V efficient portfolio; with (OS) and without (S) options.

	M-V					
In sample returns	Optim. runs	S-port-folio(%)	OS-port-folio(%)	Optim. runs	S-port-folio(%)	OS-port-folio(%)
0.01		3.36	2.25		3.64	2.31
0.02	1	4.53	3.81	7	4.97	4.02
0.03		7.68	7.53		11.98	10.89
0.01		3.61	2.26		3.66	2.34
0.02	2	4.92	3.89	8	5.07	4.06
0.03		9.02	8.96		10.60	10.38
0.01		3.60	2.24		3.66	2.27
0.02	3	4.77	3.71	9	4.96	3.89
0.03		8.99	8.20		12.39	12.20
0.01		3.64	2.28		3.61	2.28
0.02	4	4.85	3.84	10	4.78	3.88
0.03		9.07	8.39		10.46	9.99
0.01		3.64	2.27		3.63	2.26
0.02	5	4.85	3.79	11	4.97	4.05
0.03		8.77	8.48		13.20	13.20
0.01		3.65	1.89		3.65	2.33
0.02	6	4.96	2.31	12	5.12	4.34
0.03		10.38	9.37		13.13	12.75

weight. The difference of the composition of two portfolios $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ using the Euclidean distance for an n-dimensional space. This is indicated by $D_{x,y} = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$.

Table 5.9: Composition of efficient portfolios: stocks only (S) and stocks + options (OS) in M-CVaR$_{0.05}$ and M-V models (for d=2%)

	Model Implementation			
	M-CVaR$_{0.05}$		M-V	
Components	S (%)	OS (%)	S (%)	OS (%)
Admiral	0	5.97	2.96	4.19
Aggreko	24.22	21.40	5.62	11.77
Arm Holdings	0	18.80	11.47	18.00
Ashtead	0	3.70	0	1.63
Babcock	0	8.00	8.07	16.99
B.A.T	4.13	3.46	18.84	12.73
BT Group	4.18	0	0	0
Capita	0	0	2.15	0
Dixons	0	0	0.03	0
Easyjet	7.55	9.08	11.33	11.45
Glaxosmithkline	5.63	0	0	0
Intertek	0	0	4.87	0
National Grid	21.85	0	4.84	0
Rangold	22.31	12.38	9.99	7.46
Shire	10.13	0	4.88	5.45
Tullow Oil	0	0	0	2.72
Unilever	0	15.18	14.97	5.95
INDEX PUT	n/a	2.02	n/a	1.67
Total	100	100	100	100
D	34.58%		18.60%	

We provide an example of the composition of efficient portfolios in the case of $d = 0.02\%$ as shown in Table 5.9. We also display the Euclidean distance values at the bottom of every composition to show the overall difference between OS and S-portfolios for the two models. From this (value of Euclidean distance), we can see that the difference in portfolio construction is more obvious and significantly higher for M-CVaR$_{0.05}$ model implementation. Thus, we conclude that the 'addition' of options into the universe of assets is more sensitive to M-CVaR$_{0.05}$ model because the nature of minimising CVaR as a left-tail risk.

Apart from the value of Euclidean distance, we can emphasize this 'sensitivity' by looking at the proportion of wealth invested in options for both M-V and M-

$\text{CVaR}_{0.05}$ portfolios. It is clear that the proportion for index put options is lower for the M-V portfolios at 1.67% compared to 2.02% for portfolio under M-$\text{CVaR}_{0.05}$ implementation.

Based on this observation of portfolio compositions, we see that the stock only portfolios (S-portfolios) shows substantial reshuffling after we include index options. This is more obvious for the case of M-$\text{CVaR}_{0.05}$ optimal portfolio. Whereas for M-V optimal portfolio, the change from S-portfolios is somewhat close to scaling up (and down) of the proportion of investment in each stocks when options are included.The same reshuffling also happening for different optimisation runs, with higher Eucliden distance is found for M-$\text{CVaR}_{0.05}$ compared to M-V implementation.

5.4 Backtesting

We run backtesting on a monthly basis using as out of sample data the 12 months June 2013 - May 2014. We use as in-sample data the 100 months preceding the "backtested period"; for example we use data from Jan 2005 to May 2013 in optimisation and the optimal weights are used to compute an "actual" return on June 2013; we repeat this by removing the oldest data point and adding the next month of data. In general, this backtesting exercise is done to see how the 12 in-sample portfolios obtained in Section 5.3 would have perfomed in reality.

We compare the 12 realised returns of mean-CVaR efficient portfolios composed of stocks only (S-portfolios) and composed of stocks and index options (OS-portfolios)

We summarise the performance of S-portfolios and OS-portfolios by looking at mean, minimum, maximum and standard deviation of the realised returns. Table 5.10 shows the realised returns for the M-$\text{CVaR}_{0.05}$ model. The performance of portfolios is different based on the in-sample target portfolio expected return. Performance of OS-portfolios under target returns $d_1 = 1\%$ and $d_2 = 2\%$ shows better statistics in its standard deviation and its minimum. The mean of the realised returns is slightly lower in the case of OS-portfolios. However, if we take into account worst case realisations, OS-portfolios perform substantially better as they avoid extreme losses. This is explained by the fact that OS portfolios, at 1% and 2% in-sample

Table 5.10: Returns for S-portfolios and OS-portfolios under M-CVaR$_{0.05}$ optimisation for each target returns d

Backtest Periods	Realised Returns S(%)	OS(%)	S(%)	OS(%)	S(%)	OS(%)
	$d_1 = 1\%$		$d_2 = 2\%$		$d_3 = 3\%$	
1	-12.62	1.23	-10.89	-1.40	-11.63	-11.00
2	8.44	5.05	9.33	7.97	10.85	27.41
3	-2.76	-2.49	2.80	-4.29	-4.39	-6.03
4	-1.85	1.89	0.21	-0.22	-3.30	-2.86
5	0.42	-1.22	-0.97	-4.31	-2.75	2.55
6	0.44	-1.84	5.01	-0.34	-3.70	-7.22
7	-3.31	-2.53	-2.60	-0.22	12.08	-8.75
8	5.86	0.63	8.27	1.56	5.09	-7.66
9	4.43	-0.83	9.15	-2.36	2.65	19.80
10	-1.68	5.62	-3.97	2.76	5.35	-9.77
11	-3.45	-2.80	-1.48	-3.03	-1.87	-1.87
12	2.56	-2.72	4.62	-3.46	-6.49	8.41
Average	-0.29	**0.00**	**1.62**	-0.61	0.16	**0.25**
Std. deviation	5.45	**2.95**	6.08	**3.50**	**7.12**	12.33
Minimum	-12.62	**-2.80**	-10.89	**-4.31**	-11.63	**-11.00**
Maximum	**8.44**	5.62	**9.33**	7.97	12.08	**27.41**

expected return, have index put options in their composition, which make profit from the decrease in price of the index; thus they help in reducing the loss.

There is a different situation in the case of portfolios with in-sample expected return $d_3 = 3\%$. The OS-portfolios incur highest losses comparable to those of their stocks only counterparts. What remarkable is their “best case” realisations, similarly to a right tail. While the worst case realisations are somewhat similar for S and OS-portfolios, the best case realisations are much better in the case of OS-portfolios (as a consequence, there is more variability in the realised returns). The OS-portfolios, which include index call options, can generate much higher returns than their stocks only counterparts.

Chapter 6

Conclusion

We have presented a framework for introducing index options, in addition to stocks, in scenario based mean-risk models. Our numerical results indicate that index options can be used to substantially improve the risk-return trade-off, especially when risk is quantified by a tail measure such as CVaR; in this case, the proportion of index options in the portfolios is higher than in the case when risk is measured by variance. The way index options are selected and their effect on the portfolio return distribution depends on the (in-sample) expected portfolio return.

Portfolios in "low risk - low return" or "medium risk - medium return" areas of the efficient frontier have index *put* options in their composition. The addition of the put acts as a safety net, as it substantially reduces worst case scenario losses. The stocks-only portfolios have a return distribution that is positively correlated with the return distribution of the index; by introducing put options, we obtain portfolios very different in composition, whose return distributions are uncorrelated with index (in the case of mean-CVaR portfolios) or with low correlation with the index (in case of mean-variance portfolios). Hence, when the index falls in price - and when stocks only portfolios incur losses to some extent - put options help in curtailing this loss without reducing much the upside potential.

Portfolios in the "high risk - high return" area of the efficient frontier have index *call* options in their composition. In-sample, the risk (either measured by CVaR or by variance) is reduced in comparison to the risk of their stocks only counterparts, but only marginally; while, in contrast, the inclusion of put option in less aggressive portfolios can dramatically reduce the risk. The new portfolios have return distributions that are even more correlated with the index, as compared to the stocks only portfolios. In-sample summary of risk-return characteristics would indicate in a first instance that, with the addition of index call options, there is only a marginal improvement. However, a more detailed analysis of return distribution show that, while there might not be a substantial improvement in the left tail, there is a substantial improvement in the right tail: by considering the call options, in addition to stocks, much higher returns can be achieved, as compared to stocks only portfolios or the index itself. These observations are consistent with the backtesting results.

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