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Origin Preserving Path Formulation for Multiparameter $\mathbb{Z}_2$-Equivariant Corank 2 Bifurcation Problems

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A singularity theory, in the form of path formulation, is developed to analyze and organize the qualitative behavior of multiparameter $\mathbb{Z}_2$-equivariant bifurcation problems of corank 2 and their deformations when the trivial solution is preserved as parameters vary. Path formulation allows for an efficient discussion of different parameter structures with a minimal modification of the algebra between cases. We give a partial classification of one-parameter problems. With a couple of parameter hierarchies, we show that the generic bifurcation problems are 2-determined and of topological codimension-0. We also show that the preservation of the trivial solutions is an important hypotheses for multiparameter bifurcation problems. We apply our results to the bifurcation of a cylindrical panel under axial compression.

Keywords: Equivariant bifurcation problem; singularity theory; path formulation.

1. Introduction

The use of singularity theory to analyze bifurcation problems has some history. Based on the approach of [Golubitsky & Schaeffer, 1979], a comprehensive classification of $\mathbb{Z}_2$-equivariant one-parameter bifurcation problems of corank 2 up to codimension-5 was established in [Dangelmayer & Armbruster, 1983]. Here, we develop a singularity theory, in the form of the *path formulation*, to analyze and organize the qualitative behavior of multiparameter $\mathbb{Z}_2$-equivariant bifurcation problems of corank 2, and their perturbations, when the trivial solution is preserved as parameters vary. Path formulation allows for an efficient discussion of different parameter structures with a minimal modification of the algebra between cases. In the remaining of Sec. 1 we discuss further the main points of our work, namely: multiparameter bifurcation, origin preservation and $\mathbb{Z}_2$-equivariance, singularity theory and the path formulation point of view. In Sec. 2, we present our main results and compare them with existing results. Sections 3–5 are concerned with the singularity theory necessary for our analysis. Classifications are in Sec. 6 and we discuss examples in Sec. 7.

We first fix some terminology. We consider *bifurcation equations* $f(z, \lambda) = 0$ where $z \in \mathbb{R}^2$ is the state space, $\lambda \in \mathbb{R}^l$, $l = 1, 2$, are the *bifurcation parameter*(s) and f is the *bifurcation function* (*or map*). The zero-set $f^{-1}(0)$ of f is the *bifurcation diagram* of f. The perturbations of the bifurcation diagram of f are described from the zero-sets $F(z, \lambda, \alpha) = 0$ of any *unfolding* F (or "*deformation*") with a parameter $\alpha \in \mathbb{R}^a$ of f, satisfying

$F(z, \lambda, 0) = f(z, \lambda)$ for all z, λ. The "*qualitative*" description of the bifurcation diagram of f, and of its deformations, in any arbitrarily small neighborhood of the bifurcation points, is obtained modulo changes of coordinates respecting the fibers along the bifurcation parameters. To fix the ideas, two bifurcation functions f and g are *bifurcation equivalent* if there exist changes of coordinates (T, X, L) such that

$$g(z, \lambda) = T(z, \lambda) f(X(z, \lambda), L(\lambda)), \qquad (1)$$

where T is a (z, λ)-family of invertible matrices and $(z, \lambda) \mapsto (X, L)$ is a local diffeomorphism preserving the trivial branch and symmetry of the problems [see thereafter (2), and (10) for unfoldings, for the details]. It is thus sensible to consider the bifurcation maps and their unfoldings as germs of map to avoid making systematic references to neighborhoods. In general, the *germ of a map* $g : \mathbb{R}^n \to \mathbb{R}^p$ *around* $x_0 \in \mathbb{R}^n$, denoted by $g : (\mathbb{R}^n, x_0) \to \mathbb{R}^p$, is the equivalence class of maps identified if they agree on some neighborhood $\mathcal{U} \subset \mathbb{R}^n$ of x_0, $\mathcal{U}$ depending on the maps. There are similar definitions for *germs of varieties*, or other objects. The notation $x \in (\mathbb{R}^n, 0)$ means that we consider the variable x in any neighborhood of the origin. The concept of germ is useful to focus on the features of the bifurcation diagrams persisting in any neighborhood of the origin. Germs are also useful because they form sets with nice algebraic structures, making singularity theory an efficient tool for their classification. We do not always refer to germs but, unless otherwise stated, by function, map, set, etc., we mean their germ, often around the origin. Given a germ g, we denote by g^o its value at the origin, by g_x^o the value of its derivative with respect to x at the origin, etc. As a consequence of the Implicit Function Theorem [Chow & Hale, 1982], necessary conditions to have a local bifurcation at the origin for $f(z, \lambda) = 0$ are $f^o = 0$ and f_z^o is singular. When $x \in (\mathbb{R}^n, 0)$, we denote by $\mathcal{E}_x$ the ring of smooth germs $g : (\mathbb{R}^n, 0) \to \mathbb{R}$ of maximal ideal $\mathcal{M}_x = \{g \in \mathcal{E}_x : g^o = 0\}$. We denote by $\boldsymbol{\mathcal{E}}_{x,p}$ the $\mathcal{E}_x$-module of smooth germs $g : (\mathbb{R}^n, 0) \to \mathbb{R}^p$ and by $\boldsymbol{\mathcal{M}}_{x,p} = \{g \in \boldsymbol{\mathcal{E}}_{x,p} : g^o = 0\}$. When p is clear from the context, we denote $\boldsymbol{\mathcal{E}}_{x,p}$, resp., $\boldsymbol{\mathcal{M}}_{x,p}$, by $\boldsymbol{\mathcal{E}}_x$, resp., $\boldsymbol{\mathcal{M}}_x$.

1.1. *Multiparameter bifurcation*

Let f be a bifurcation map, singular at the origin. The generators of the kernel of f_z^o are called *linear modes*. For multidimensional kernels, linear modes are not uniquely defined but there is often in practice a sensible choice for them. Some bifurcation problems have naturally multiple parameters, for instance, when several linear modes interact. The following cases are possible.

(1) Each bifurcation parameter controls the critical eigenvalue of each interacting mode. Those parameters are thus of "*equal status*".
(2) With two linear modes, one parameter λ_2 could represent the detuning and the other λ_1 the overall load on the system. This leads to a "*hierarchy of parameters*". For instance, λ_2 could then be considered fixed when λ_1 varies.

To be concrete, in corank 2, a typical Jacobian matrix $f_z(0, \lambda)$ of the bifurcation function could be $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ in Case (1) and $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 + \lambda_2 \end{pmatrix}$ in Case (2). With parameters of equal status, the qualitative analysis of the bifurcation diagram via diffeomorphic changes of coordinates (1) only preserves the layout of the regions in the λ-plane separating bifurcations. We have a finer information with a hierarchy like in Case (2). We can preserve the qualitative information on the λ_2-sequences of the λ_1-slices of the bifurcation diagrams.

As an explicit example, consider the following one-dimensional model problem:

$$f(x, \lambda) = x(x^2 + \lambda_1^2 - \lambda_2) = 0.$$

The main bifurcation sheet of $f^{-1}(0)$ has the shape of a bowl. When both parameters are of equal status, the local diffeomorphism $L(\lambda) = (\lambda_1, \lambda_2 + \lambda_1^2)$ transforms f into $g(x, \lambda) = x(x^2 - \lambda_2)$ whose bifurcation sheet has the shape of a cylinder. This occurs because L does not control specifically the scale in the λ_1-direction, so the bowl can be stretched without bound in the λ_1-direction. On the other hand, any diffeomorphism $L(\lambda) = (L_1(\lambda), L_2(\lambda_2))$ in Case (2), respecting the hierarchy between λ_1 and λ_2, keeps the "*bowl shape*" of the main bifurcation sheets. Those are two types of situations we shall be able to distinguish using our approach.

1.2. *Origin preservation and* $\mathbb{Z}_2$*-symmetry*

Another element we take into account is the persistence of the origin as a solution of the bifurcation equation and of all its deformations. This has links with the $\mathbb{Z}_2$-symmetry of the equations. Suppose $\mathbb{Z}_2$ acts on the state space via a reflection κ (κ^2 is the identity) acting trivially on the parameters. Moreover, suppose that the bifurcation map and all its deformations have a sheet $\bar{z}(\lambda, \alpha)$ of symmetric solutions: $\kappa\bar{z} = \bar{z}$. Translating this sheet to the origin, the bifurcation problem and its perturbations become origin preserving. Explicitly, if f is a bifurcation map, by *preserving the origin* we mean that we have $f(0, \lambda) = 0$, for all $\lambda \in (\mathbb{R}^l, 0)$, and, similarly, for any deformation $F(z, \lambda, \alpha) = 0$, with additional parameters $\alpha \in (\mathbb{R}^a, 0)$, we have $F(0, \lambda, \alpha) = 0$ for all $(\lambda, \alpha) \in (\mathbb{R}^{l+a}, 0)$. It then makes sense to also impose this structure on the singularity theory used to systematically classify and analyze such problems. The example (6) in Sec. 2.2 shows that such structure can be actually important for the finite determinacy of multiparameter bifurcation maps.

1.3. *Singularity theory*

Singularity theory is a powerful tool to classify systematically and analyze qualitatively local bifurcation diagrams $f^{-1}(0)$. We already discussed two modes interaction without any symmetries in [Furter & Sitta, 2004]. Here we use singularity theory to discuss the bifurcations of steady states near the interaction of two linear modes, one symmetric, the other anti-symmetric with respect to an action of $\mathbb{Z}_2$, in nonlinear problems with a persistent trivial branch. Work has been done on corank 2, $\mathbb{Z}_2$-equivariant bifurcation maps. Good references are the classical analysis in [Chow & Hale, 1982] and [Golubitsky *et al.*, 1988] for a singularity theory approach for one-parameter maps. Our techniques can be readily adapted to any number of parameters, even with some special structure on them (like hierarchy, symmetry etc.), and to problems with other symmetries (see [Furter, 1997]). We apply our results to the work of Wu [1999, 2000]. Our starting point was the remark in [Wu, 2000] that the singularity theory in [Golubitsky *et al.*, 1988] did not apply directly. Here, we show how it can be done with an approach that can be extended to many other cases.

We discuss Wu's analysis in Sec. 2 and compare with our results. In Sec. 3 we describe the theories of bifurcation and path equivalences applied to the classification of bifurcation maps and their deformations. Bifurcation equivalence (or "*parametrized contact-equivalence*") provides the best approach to classify the normal forms, solve their recognition problem (a set of equations and inequalities that any other germ g equivalent to f must satisfy) and study their deformations [Golubitsky & Schaeffer, 1979, 1985; Golubitsky *et al.*, 1988]. To compare two bifurcation maps f and g, we use changes of coordinates (T, X, L) (*bifurcation equivalences*) such that

$$g(z, \lambda) = T(z, \lambda) f(X(z, \lambda), L(\lambda)) \tag{2}$$

where T is a (z, λ)-family of invertible matrices and $(z, \lambda) \mapsto (X, L)$ is a local diffeomorphism preserving the trivial branch. As stated in Sec. 3, because f and g are $\mathbb{Z}_2$-equivariant germs, T and X will be chosen to be $\mathbb{Z}_2$-equivariant. Moreover, as necessary, L is compatible with the hierarchy of the bifurcation parameters. Comparison between unfoldings of f is achieved via the notion of mapping of unfolding [see (10)]. When f is of finite codimension, distinguished unfoldings, called *miniversal*, describe qualitatively all the allowed perturbations of the diagram $f^{-1}(0)$. Note that two bifurcation equations obtained by exercising different choices during a Lyapunov–Schmidt type reduction process are bifurcation equivalent [Jepson & Spence, 1989].

1.4. *Path formulation*

In Sec. 3.4, we describe an alternative approach when applying singularity theory: the *path formulation*. A bifurcation map f is considered as a deformation of their *core* f_0, $f_0(z) = f(z, 0)$, with parameters $\lambda \in (\mathbb{R}^l, 0)$. Using the unfolding theory for f_0, f is identified by a path $\lambda \mapsto \overline{\alpha}(\lambda)$ in the parameter space of a miniversal unfolding F_0 of f_0, miniversal in the correct category (see (11) or [Montaldi, 1994; Furter *et al.*, 1998]). More precisely, every $f(z, \lambda)$ is bifurcation equivalent to $F_0(z, \overline{\alpha}(\lambda))$ where $\overline{\alpha}$ is a path associated with f [see (11)]. Equivalences between paths $\overline{\alpha}$, $\overline{\beta}$ with the same core are given by changes of co-ordinates

$$\overline{\alpha}(\lambda) = H(\lambda, \overline{\beta}(L(\lambda))), \tag{3}$$

with H either lifting to diffeomorphisms preserving $F_0^{-1}(0)$ [Furter, 1997] or, equivalently here, preserving the discriminant of F_0 [Damon, 1987] (see Sec. 3.4 for more details). Both points of view give the same result and, for finite codimension problems, both theories lead to the same classification as bifurcation equivalence. There are advantages in using path formulation for multiparameter bifurcations, because it organizes the classification of bifurcation maps, distinguishing between the singular behavior attributable to the core and to the paths. In Sec. 4, we get the $\mathbb{Z}_2$-equivariant cores of lowest codimension and represent the zero-set structures of their miniversal unfoldings in Figs. 1 and 2. To recover the bifurcation diagrams we look at sections of $F_0(z, \alpha) = 0$ over the paths. In Sec. 5, we give the explicit description of the module $\mathrm{Derlog}^*(F_0)$ of vector fields liftable from the parameter space of F_0 onto $F_0^{-1}(0)$. This module is the main algebraic ingredient we need for the path formulation. The algebraic techniques for path equivalence are easier to use because they involve modules over shorter systems of rings (see [Damon, 1984]) than for bifurcation equivalence. Moreover, $\mathrm{Derlog}^*(F_0)$ depends only on F_0, not on the parameter structure of the bifurcation germs. We discuss two-parameter problems with a hierarchical structure in parameter space. Finally, Sec. 6 is devoted to the new classifications of origin preserving bifurcation germs with the generic core and one or two parameters and in Sec. 7 we discuss some examples.

2. Abstract Reduced Bifurcation Equations

Wu [1999, 2000] discussed the bifurcations of steady states in $\mathbb{Z}_2$-equivariant nonlinear problems of corank 2 with two parameters arising from the interaction of two linear modes. The $\mathbb{Z}_2$-action corresponds to the equivariance of the original equation with respect to a $\mathbb{Z}_2$-action of some linear operator S with $S^2 = I$, such that the first linear mode v_1 is S-invariant, $Sv_1 = v_1$, and the second linear mode is S-anti-symmetric, $Sv_2 = -v_2$. Explicitly, S acts on the element of the kernel $xv_1 + yv_2$ as $S(xv_1 + yv_2) = xv_1 - yv_2$. After a Lyapunov–Schmidt reduction, Wu's reduced bifurcation equations are

$$Ax^2 + Cy^2 + (a_1\lambda_1 + b_1\lambda_2)x + h_1(z, \lambda) = 0, \quad (4)$$

$$2Cxy + (a_2\lambda_1 + b_2\lambda_2)y + h_2(z, \lambda) = 0, \quad (5)$$

where $z = (x, y) \in \mathbb{R}^2$ represents the coordinates on the kernel of the linearization, $\lambda \in \mathbb{R}^2$ are the two bifurcation parameters and h_1, h_2 represent terms of order three or more in (z, λ). There are two important assumptions for (4) and (5).

(1) (H0) The origin $x = y = 0$ is a solution for all small λ, so h_1 and h_2 vanish when $z = 0$.
(2) The systems (4) and (5) is equivariant with respect to the $\mathbb{Z}_2$-action $(x, y) \mapsto (x, -y)$.

In [Wu, 1999, 2000], there is a third assumption, that (4) and (5) is the gradient of some functional, imposing the presence of the same coefficient C in (4) and (5), but we shall see that this has no fundamental importance in our context.

2.1. *Analysis of (4) and (5)*

Under the nondegeneracy conditions

$$\mathrm{ND0} : A \cdot C \cdot (a_1b_2 - a_2b_1) \neq 0$$

and

$$\mathrm{ND1a} : a_1 \cdot a_2 \neq 0, \quad \mathrm{ND1b} : (2a_1C - a_2A) \neq 0,$$

Wu [1999, 2000] calculated secondary and tertiary bifurcation branches and their stability. His results follow from a careful nonlinear analysis of the branches of (4), and (5) and suggest that the quadratic terms are enough to determine fully the qualitative behavior of (4) and (5) and this behavior will not be altered by perturbation. In singularity theory terminology, (4) and (5) should be 2-determined and of zero (topological) codimension.

Our Theorem 8 shows that this is indeed the case when $\lambda \in (\mathbb{R}^2, 0)$, *provided we work with the set of germs satisfying (H0)*. More precisely, when the two bifurcation parameters are of equal importance, that is, we mix them freely in the change of coordinates (2)

$$L(\lambda) = (L_1(\lambda), L_2(\lambda)),$$

the bifurcation maps (4) and (5) satisfying (ND0) are 2-determined in the space of maps preserving the origin. If we want to privilege one bifurcation parameter over the other, say λ_2, using in (2)

$$L(\lambda) = (L_1(\lambda), L_2(\lambda_2)),$$

then we need to add (ND1a) to (ND0). The condition (ND1b) is not necessary for 2-determinacy. The result is false without (H0) because we show in Proposition 1 that (4) and (5) is of infinite codimension as a two-parameter bifurcation problem. That is, if the higher order terms depend on λ so that $f(0, \lambda)$ is not always 0 for all λ, there is an infinite number of possibilities of nonequivalent bifurcation diagrams with the same terms of order two (**jet** of order two) (Propositions 1 and 2).

Our approach gives also information on the stability under perturbation of the bifurcation diagrams. If the perturbations also satisfy (H0), (4) and (5) is stable under (ND0): every unfolding of

f maps into f itself, the codimension of f is zero. If perturbations are allowed to destroy the trivial solution, it depends on the number of parameters:

(1) as a one-parameter problem, it is of codimension 2: we need two perturbation parameters to describe qualitatively every possible perturbation (Proposition 3).
(2) with two, or more, bifurcation parameters, there is an infinite number of possible non-equivalent perturbations (Proposition 1 and Corollary 2.1).

In [Furter & Sitta, 2004] we also show that those results apply more widely to nondegenerate quadratic bifurcation problems. There are two types of normal forms depending on the sign $\epsilon_1\epsilon_2$. Positive, it corresponds to the gradient form (4) and (5). If the sign is negative, the problem cannot be a gradient but the same consequences still apply. The only effect is that one group of diagrams (those supporting secondary Hopf bifurcation) are now possible (see Fig. 2). Without symmetry, the second original assumption, the perturbations of variational problems of corank 2 are much more constrained that those of the nonvariational problem. They have less codimension [Furter & Sitta, 2009].

2.2. *Two-parameter origin preserving problem and finite determinacy*

The following result illustrates that some bifurcation germs, whose 2-jet has a zero-set containing the trivial branch $(0, \lambda)$, cannot be of finite codimension.

Proposition 1. *Let* $\epsilon_1^2 = \epsilon_2^2 = \delta_1^2 = \delta_2^2 = 1$, $m \in \mathbb{R}$. *When* $a_k \neq 0$, *the following bifurcation germs with two bifurcation parameters* $\lambda \in (\mathbb{R}^2, 0)$ *are not bifurcation equivalent for different integer* k:

$$f(z, \lambda) = \begin{pmatrix} x^2 + \epsilon_1 y^2 + \delta_1\lambda_1 x + a_k\lambda_2^k \\ 2\epsilon_2 xy + (m\lambda_1 + \delta_2\lambda_2)y \end{pmatrix}. \qquad (6)$$

Proof. The zero-set of the 2-jet of (6) has the trivial branch $(0, \lambda)$. From some simple algebra, the branches of solutions of $f^{-1}(0)$ with $y = 0$ are given by

$$\left(x^2 + \frac{1}{2}\delta_1\lambda_1\right)^2 = \frac{1}{4}\lambda_1^2 - a_k\lambda_2^k.$$

Hence, there are only solutions for $\lambda = (\lambda_1, \lambda_2)$ in the cuspidal wedge

$$\lambda_1^2 > 4a_k\lambda_2^k. \qquad (7)$$

These calculations show that the zero-set of (6) depends on the term $a_k\lambda_2^k$ that cannot indeed be ignored because the wedges (7) are not diffeomorphic for different k. ∎

Corollary 2.1. *As a corollary, there are perturbations of the 2-jet of* (6) *of arbitrary order that are not bifurcation equivalent. Hence, the codimension of the 2-jet of* f *is infinite.*

Using a more powerful theory, the geometrical characterization of finite codimension of path in $\mathbb{C}$, inspired from [Damon, 1984], we show that the result is general.

Proposition 2 [Furter & Sitta, 2004]. *With two* (*or more*) *bifurcation parameters the quadratic normal forms such that* $f(0, \lambda) = 0$ *are of infinite codimension.*

This is a two-parameter phenomenon because of the following result, that is proved using the usual techniques of [Golubitsky *et al.*, 1988] or [Furter & Sitta, 2004].

Proposition 3. *Let* $\epsilon_1^2 = \epsilon_2^2 = \delta^2 = 1$ *and* $m \in \mathbb{R}$. *With one bifurcation parameter* $\lambda \in (\mathbb{R}, 0)$, *the following miniversal unfolding is of* (*topological*) *codimension-2*:

$$\begin{pmatrix} x^2 + \epsilon_1 y^2 + \delta\lambda x + \beta_1 + \beta_2\lambda \\ 2\epsilon_2 xy + m\lambda y \end{pmatrix}, \qquad (8)$$

when $m(\epsilon_2 m - 2\delta_1) \neq 0$, *with unfolding parameters* $(\beta_1, \beta_2) \in (\mathbb{R}^2, 0)$. *The coefficient* m *is a* moduli, *an invariant of the equivalence classes of* (8).

3. Singularity Theory for Bifurcation Germs

We present here the main definitions and abstract results we need from singularity theory.

3.1. *Structure of* $\mathbb{Z}_2$*-equivariant problems*

The group $\mathbb{Z}_2$ acts on $\mathbb{R}^2$ by $\gamma(x, y) = (x, -y)$. The action of $\mathbb{Z}_2$ on any additional parameter is trivial. We denote $z = (x, y)$. A germ $f : (\mathbb{R}^{2+a}, 0) \to \mathbb{R}$ with a-parameters $\alpha \in (\mathbb{R}^a, 0)$ is $\mathbb{Z}_2$-*invariant* if $f(\gamma z, \alpha) = f(z, \alpha)$. A germ $f : (\mathbb{R}^{2+a}, 0) \to (\mathbb{R}^2, 0)$

with a-parameters $\alpha \in (\mathbb{R}^a, 0)$ is $\mathbb{Z}_2$-*equivariant* if $f(\gamma z, \alpha) = \gamma f(z, \alpha)$. The ring $\mathcal{E}_z^{\mathbb{Z}_2}$ of $\mathbb{Z}_2$-invariant germs is generated by x and $v = y^2$, that is, any element of $\mathcal{E}_z^{\mathbb{Z}_2}$ can be written as $h(x, y^2)$ for $h : (\mathbb{R}^2, 0) \to \mathbb{R}$, and the module $\boldsymbol{\mathcal{E}}_z^{\mathbb{Z}_2}$ of $\mathbb{Z}_2$-equivariant germs is freely generated over $\mathcal{E}_z^{\mathbb{Z}_2}$ by $X_1 = (1, 0)$ and $Y_1 = (0, y)$, that is, every f of $\boldsymbol{\mathcal{E}}_z^{\mathbb{Z}_2}$ is of the form $pX_1 + qY_1$ with unique $p, q \in \mathcal{E}_z^{\mathbb{Z}_2}$. We identify f with $[p, q]$. For parametrized germs, $\mathcal{E}_{(z,\alpha)}^{\mathbb{Z}_2}$ denotes the ring of $\mathbb{Z}_2$-invariant germs with parameters $\alpha \in (\mathbb{R}^a, 0)$ and $\boldsymbol{\mathcal{E}}_{(z,\alpha)}^{\mathbb{Z}_2}$ the $\mathcal{E}_{(z,\alpha)}^{\mathbb{Z}_2}$-module of $\mathbb{Z}_2$-equivariant germs with parameters α. A map $[P(x, v, \alpha), Q(x, v, \alpha)]$ is a gradient if and only if $2P_v \equiv Q_x$. When the ring is clear in the context, we denote by $\mathcal{I}_{a_1 \ldots a_n}$ the ideal generated by the elements $a_1, \ldots, a_n$. Let R be a ring, we denote by $\langle m_1, \ldots, m_r \rangle_R$ the R-module generated by the m_i's.

Origin preserving $\mathbb{Z}_2$-equivariant bifurcation germs with $\lambda \in (\mathbb{R}^l, 0)$, $l = 1, 2$, form the $\mathcal{E}_{(z,\lambda)}^{\mathbb{Z}_2}$-submodule $\boldsymbol{\mathcal{F}}_{(z,\lambda)}^{\mathbb{Z}_2}$ of $\boldsymbol{\mathcal{E}}_{(z,\lambda)}^{\mathbb{Z}_2}$ generated by $[x, 0], [v, 0]$ and $Y_1 = [0, 1] = (0, y)$, that is, $f \in \boldsymbol{\mathcal{F}}_{(z,\lambda)}^{\mathbb{Z}_2}$ can be written as

$$\begin{pmatrix} xp_1(x, v, \lambda) + vp_2(x, v, \lambda) \\ yq(x, v, \lambda) \end{pmatrix}.$$

Note that the generators $[x, 0]$ and $[v, 0]$ are not free therefore we continue to denote elements of $\boldsymbol{\mathcal{F}}_{(z,\lambda)}^{\mathbb{Z}_2}$ by $[xp_1 + vp_2, q]$. Their origin preserving unfoldings with a-parameter, say $\alpha \in (\mathbb{R}^a, 0)$, form the $\mathcal{E}_{(z,\lambda,\alpha)}^{\mathbb{Z}_2}$-module $\boldsymbol{\mathcal{F}}_{(z,\lambda,\alpha)}^{\mathbb{Z}_2}$ also generated by $[x, 0], [v, 0]$ and Y_1.

3.2. *Bifurcation equivalence*

Qualitative description means modulo changes of coordinates that preserve the zero-sets and the special role of the bifurcation parameters. We use a form of parametrized contact-equivalence that preserves the λ-slices of the diagram [Golubitsky & Schaeffer, 1979, 1985; Golubitsky *et al.*, 1988]. Two bifurcation germs f, g are *bifurcation equivalent* if there exist changes of coordinates (T, X, L) such that

$$g(z, \lambda) = T(z, \lambda) f(X(z, \lambda), L(\lambda)), \tag{9}$$

where T is a (z, λ)-family of $\mathbb{Z}_2$-equivariant invertible matrices and (X, L) is a local $\mathbb{Z}_2$-equivariant diffeomorphism of $(\mathbb{R}^{2+l}, 0)$, fixing the origin, $X(0, \lambda) = 0$ for $\lambda \in (\mathbb{R}^l, 0)$. We require that the sets of X and L are path-connected to the identity to preserve the orientation of the diagrams. Note that (9) means that $f^{-1}(0)$ and $g^{-1}(0)$ are formed of diffeomorphic λ-slices. We consider different structures on L. When $\lambda \in (\mathbb{R}, 0)$, $L : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$. When $\lambda \in (\mathbb{R}^2, 0)$ we distinguish between $L(\lambda) = (L_1(\lambda), L_2(\lambda))$ and $L(\lambda) = (L_1(\lambda), L_2(\lambda_2))$. In the first case, we cannot guarantee that one-dimensional slices are preserved, *only two-dimensional regions in the λ-plane with diffeomorphic zero-set structure are preserved*. In the second case, bifurcation equivalence *preserves the λ_2-sequences of diffeomorphic λ_1-slices of the zero-set*. In that case it makes sense to draw *sequences in λ_2 of slices in (z, λ_1)-space* as in [Wu, 1999]. The bifurcation equivalences form groups by composition denoted by $\mathcal{K}_{o,\lambda}^{\mathbb{Z}_2}$, $\lambda \in (\mathbb{R}^l, 0)$, $l = 1, 2$, and $\mathcal{K}_{o,\lambda_1/\lambda_2}^{\mathbb{Z}_2}$, respectively, acting on $\boldsymbol{\mathcal{F}}_{(z,\lambda)}^{\mathbb{Z}_2}$. Perturbations of f are germs $F : (\mathbb{R}^{2+l+a}, 0) \to (\mathbb{R}^{2+l}, 0)$ such that $F(z, \lambda, 0) = f(z, \lambda)$, called *unfoldings* of f with a-parameters. To compare two unfoldings F_1, F_2 of f with a_1 (resp., a_2), parameters, we say that F_2 *maps into* F_1 if

$$F_2(z, \lambda, \alpha_2) = T(z, \lambda, \alpha_2) F_1(X(z, \lambda, \alpha_2), L(\lambda, \alpha_2), A(\alpha_2)), \tag{10}$$

where T, X, L are unfoldings of the identity in their category. Here X will preserve the origin, and $A : (\mathbb{R}^{a_2}, 0) \to (\mathbb{R}^{a_1}, 0)$ is in general not invertible. We say that an unfolding F of f is *versal* if any other unfolding G of f maps into F which means that one gets all possible perturbations of $f^{-1}(0)$ via $F^{-1}(0)$. A versal unfoldings of f with the minimum number of parameters is called *miniversal*. They are actually all equivalent.

The main goal of singularity theory is to classify normal forms and their miniversal unfoldings modulo the allowed changes of coordinates. A *normal form* is a particularly chosen "simple" representative of an equivalence class of germs. This allows for an *a priori* discussion independent of the particular bifurcation problem we look at, only its structure is important. To achieve this goal we could adapt the theory and algebraic calculations of [Golubitsky *et al.*, 1988; Lari-Lavassami, 1990]. Because we approach the question from a different angle that organizes better the results and illustrates how adaptable the calculations are, we are only going to give the fundamental abstract results applicable to the groups $\mathcal{K}^{\mathbb{Z}_2}_{o,\lambda}$ and $\mathcal{K}^{\mathbb{Z}_2}_{o,\lambda_1/\lambda_2}$, without the details we would need to apply the theory.

3.3. *Fundamental theorems for bifurcation equivalence*

Both $\mathcal{K}^{\mathbb{Z}_2}_{o,\lambda}$ and $\mathcal{K}^{\mathbb{Z}_2}_{o,\lambda_1/\lambda_2}$ are geometric subgroups of contact equivalences (in the sense of [Damon, 1984]) acting on $\boldsymbol{\mathcal{F}}^{\mathbb{Z}_2}_{(z,\lambda)}$, so the general theory about unfoldings and determinacy applies. Let $\mathcal{G} = \mathcal{K}^{\mathbb{Z}_2}_{o,\lambda}$ or $\mathcal{K}^{\mathbb{Z}_2}_{o,\lambda_1/\lambda_2}$. For $f \in \boldsymbol{\mathcal{F}}^{\mathbb{Z}_2}_{(z,\lambda)}$, we associate an *extended tangent space* $\mathcal{T}_e\mathcal{G}(f)$, derived in the usual way from the derivative of one-parameter unfoldings (see [Damon, 1984], Proposition 4.1, and Sec. 4.1.1 here) and its *extended normal space* $\mathcal{N}_e\mathcal{G}(f) = \boldsymbol{\mathcal{F}}^{\mathbb{Z}_2}_{(z,\lambda)}/\mathcal{T}_e\mathcal{G}(f)$. Assume that $c = \dim_{\mathbb{R}} \mathcal{N}_e\mathcal{G}(f) < \infty$ (as a $\mathbb{R}$-vector space), c is the *$\mathcal{G}$-codimension* of f. We summarize the results in the next theorems.

Theorem 1 [Unfolding Theory]. *Let $\mathcal{G} = \mathcal{K}^{\mathbb{Z}_2}_{o,\lambda}$ or $\mathcal{K}^{\mathbb{Z}_2}_{o,\lambda_1/\lambda_2}$ and $f \in \boldsymbol{\mathcal{F}}^{\mathbb{Z}_2}_{(z,\lambda)}$ of finite $\mathcal{G}$-codimension. A miniversal unfolding F of f is obtained from $f + \sum_{i=1}^{c} \alpha_i f_i$ where $\{f_i\}_{i=1}^{c}$ projects down onto a basis of the extended normal space $\mathcal{N}_e\mathcal{G}(f)$.*

The second theorem deals with how to determine a normal form. We need another tangent space. A *unipotent subgroup* of $\mathcal{UG}$ is formed of equivalences whose linearization form a unipotent subgroup of matrices. We define then the *unipotent tangent space* $\mathcal{TUG}(f)$ (see [Bruce *et al.*, 1987]) of f in the usual way.

Theorem 2 [Determinacy Theory]. *Let $\mathcal{G} = \mathcal{K}^{\mathbb{Z}_2}_{o,\lambda}$ or $\mathcal{K}^{\mathbb{Z}_2}_{o,\lambda_1/\lambda_2}$ and $f \in \boldsymbol{\mathcal{F}}^{\mathbb{Z}_2}_{(z,\lambda)}$ of finite $\mathcal{G}$-codimension.*

(1) *f is finitely determined, that is, f is $\mathcal{G}$-equivalent to a polynomial normal form.*
(2) *The* set of higher order terms, *terms that can be removed in any element of the $\mathcal{G}$-class of f, is contained in the intrinsic part of $\mathcal{TUG}(f)$ [Bruce et al., 1987].*

Such theorems indicate how singularity theory uses algebraic calculations of the tangent and normal spaces to calculate the codimension of f, its miniversal unfoldings and solve its recognition problem. The main issue in the explicit calculations is the structure of the tangent spaces as modules over systems of rings $\{\mathcal{E}^{\mathbb{Z}_2}_{(z,\lambda)}, \mathcal{E}_\lambda\}$ for $\mathcal{K}^{\mathbb{Z}_2}_{o,\lambda}$, and $\{\mathcal{E}^{\mathbb{Z}_2}_{(z,\lambda)}, \mathcal{E}_\lambda, \mathcal{E}_{\lambda_2}\}$ for $\mathcal{K}^{\mathbb{Z}_2}_{o,\lambda_1/\lambda_2}$. We are going to use a different approach with simpler systems of rings, so we do not need more details at present.

3.4. *Path equivalence*

The *core* f_0 of $f \in \mathcal{F}$ at the origin is the germ obtained by setting $\lambda = 0$, $f_0(z) = f(z,0)$. It represents the singular behavior *independently* of the way the bifurcation parameter(s) enter. We assume that the core has a miniversal unfolding F_0 in the *relevant* category ($\mathbb{Z}_2$-equivariant, preserving the zero-branch and gradient if needed), with a-parameters, say (see Sec. 4.1). We consider f as a perturbation of f_0 with l-parameters. And so, from the theory of miniversal unfoldings ([Damon, 1984] or (10)), there exists $\overline{\alpha} : (\mathbb{R}^l, 0) \to (\mathbb{R}^a, 0)$ and local diffeomorphisms (T, X) such that

$$f(z,\lambda) = T(z,\lambda)F_0(X(z,\lambda), \overline{\alpha}(\lambda)). \tag{11}$$

The germ f and the pull-back $\overline{\alpha}^*F_0$ are bifurcation equivalent with equivalence (T, X, I). We call $\overline{\alpha}$ a *path* associated with f. The study of $f^{-1}(0)$ is transformed into the study of the section of $F_0^{-1}(0)$ over $\overline{\alpha}$. More precisely, let $\pi_{F_0} : (F_0^{-1}(0), 0) \to (\mathbb{R}^a, 0)$ be the restriction of the natural projection $\pi : (\mathbb{R}^{2+a}, 0) \to (\mathbb{R}^a, 0)$. Let Σ_{F_0} be the local bifurcation set of F_0, then $\Delta^{F_0} = \pi_{F_0}(\Sigma_{F_0})$ is the *discriminant* variety of F_0. The position of $\overline{\alpha}$ with respect to Δ^{F_0} monitors when, and "how", a path $\overline{\alpha}$ induces a crossing of Σ_{F_0}, that is, when there is a local change in behavior of the zero-set of $\overline{\alpha}^*F_0$. We say that the paths $\overline{\alpha}, \overline{\beta} : (\mathbb{R}^l, 0) \to (\mathbb{R}^a, 0)$ are *path equivalent* if

$$\overline{\alpha}(\lambda) = H(\lambda, \overline{\beta}(L(\lambda))) \tag{12}$$

where $L : (\mathbb{R}^l, 0) \to (\mathbb{R}^l, 0)$ is an orientation preserving diffeomorphism, path connected to the identity, and $H : (\mathbb{R}^{l+a}, 0) \to (\mathbb{R}^a, 0)$ is a λ-parametrized family of local diffeomorphism on $(\mathbb{R}^a, 0)$, path connected to the identity, that lifts to a λ-family of diffeomorphism $\Phi : (\mathbb{R}^{l+2+a}, 0) \to (\mathbb{R}^{2+a}, 0)$ preserving $F_0^{-1}(0)$ such that $\pi_{F_0} \circ \Phi = H \circ \pi_{F_0}$. Note that we cannot in general simplify H in (12) to a λ-parametrized matrix like with the usual contact-equivalence. An explicit description of the diffeomorphisms H is in general impossible, but the tangent space of paths can be calculated using vector fields liftable over π_{F_0}.

The idea of path formulation goes back at least to Martinet and was the original starting point of the work in [Golubitsky & Schaeffer, 1979]. Eventually the fruitful approach using (9) has been developed because the technicalities of path formulation could not be easily overcome at the time. Those ideas have been resurrected in [Mond & Montaldi, 1994; Montaldi, 1994] for the usual contact-equivalence and in [Bridges & Furter, 1993] for (symmetric) gradient problems. It follows recent progresses in singularity theory to handle variety preserving contact-equivalence [Damon, 1987]. Since then, the algebraic formulation derived in [Furter *et al.*, 1998] showed that the main features of the path formulation occur naturally in the algebra of Golubitsky–Schaeffer theory.

3.5. *Abstract theory for path equivalence*

For a fixed core f_0, the path equivalences (12) form groups acting on the *space of paths*, the $\mathcal{E}_\lambda$-module $\boldsymbol{\mathcal{M}}_{\lambda,a} = \{\overline{\alpha} : (\mathbb{R}^l, 0) \to (\mathbb{R}^a, 0)\} \subset \boldsymbol{\mathcal{E}}_{\lambda,a}$. Depending on if there is, or not, a hierarchy of parameters, we have groups $\mathcal{K}^*_\lambda$ and $\mathcal{K}^*_{\lambda_1/\lambda_2}$, respectively depending on F_0, that are also geometric subgroups of contact equivalences (in the sense of [Damon, 1984]), so his general theory about unfoldings and determinacy applies.

Theorem 3. *Let* $\mathcal{G} = \mathcal{K}^*_\lambda$ *or* $\mathcal{K}^*_{\lambda_1/\lambda_2}$. *For* $\overline{\alpha} \in \boldsymbol{\mathcal{M}}_{\lambda,a}$ *we associate an* extended tangent space $\mathcal{T}_e\mathcal{G}(\overline{\alpha})$ *and its* extended normal space $\mathcal{N}_e\mathcal{G}(\overline{\alpha}) = \boldsymbol{\mathcal{E}}_{\lambda,a}/\mathcal{T}_e\mathcal{G}(\overline{\alpha})$. *Assume that* $c = \dim_{\mathbb{R}} \mathcal{N}_e\mathcal{G}(\overline{\alpha}) < \infty$, *c is the* $\mathcal{G}$-*codimension of* $\overline{\alpha}$.

(1) *The path* $\overline{\alpha}$ *has a universal unfolding that can be obtained from* $\overline{\alpha} + \sum_{i=1}^c \alpha_i\overline{\alpha}_i$ *where* $\{\overline{\alpha}_i\}_{i=1}^c$ *projects down a basis of the extended normal space* $\mathcal{N}_e\mathcal{G}(\overline{\alpha})$.
(2) *The path* $\overline{\alpha}$ *is finitely determined*, *that is*, $\mathcal{G}$-*equivalent to a polynomial normal form.*
(3) (*Recognition problem*) *If* $\mathcal{UG}$ *is a unipotent subgroup of* $\mathcal{G}$, *the set of higher order terms that can be removed in any element of the* $\mathcal{G}$-*class of* $\overline{\alpha}$ *is contained in the intrinsic part of* $\mathcal{TUG}(\overline{\alpha})$ (*see* [*Bruce et al., 1987*]).

To get explicitly $\mathcal{T}_e\mathcal{G}(\overline{\alpha})$ we consider unfoldings with one-parameter $t \in (\mathbb{R}, 0)$ of the correct path equivalence (12) $(H(t, \lambda, \alpha), L(t, \lambda))$ such that (H, L) is the identity when $t = 0$. The elements of $\mathcal{T}_e\mathcal{G}(\overline{\alpha})$ are the tangent vectors at $t = 0$ of $H(t, \lambda, \overline{\alpha}(L(t, \lambda)))$ for all possible (H, L). Explicit examples are in [Damon, 1987] and Proposition 4 in Sec. 4.1.1. We find that the extended tangent spaces of path $\overline{\alpha}$ are as follows:

(1) $\mathcal{T}_e\mathcal{K}^*_\lambda(\overline{\alpha}) = \langle\overline{\alpha}_\lambda\rangle_{\mathcal{E}_\lambda} + \overline{\alpha}^*(\mathrm{Derlog}^*(F_0))_{\mathcal{E}_\lambda}$,
(2) $\mathcal{T}_e\mathcal{K}^*_{\lambda_1/\lambda_2}(\overline{\alpha}) = \langle\overline{\alpha}_\lambda\rangle_{\mathcal{E}_{\lambda_1/\lambda_2}} + \overline{\alpha}^*(\mathrm{Derlog}^*(F_0))_{\mathcal{E}_\lambda}$,

where $\mathrm{Derlog}^*(F_0)$ is the $\mathcal{E}_\alpha$-module of liftable vector fields, liftable over π_{F_0} to $F^{-1}(0)$ (see Sec. 5). The ring $\mathcal{E}_{\lambda_1/\lambda_2}$ is the system of ring $\{\mathcal{E}_\lambda, \mathcal{E}_{\lambda_2}\}$. For the $\mathbb{Z}_2$-equivariant corank 2 generic core (17), $\mathrm{Derlog}^*(F_0)$ is a free module generated by two vector fields ξ_1, ξ_2 in (22). Therefore $\mathcal{T}_e\mathcal{K}^*_\lambda(\overline{\alpha}) = \langle\overline{\alpha}_\lambda, \xi_1(\overline{\alpha}), \xi_2(\overline{\alpha})\rangle_{\mathcal{E}_\lambda}$, $\lambda \in (\mathbb{R}^l, 0)$, $l = 1, 2$, and, for $\lambda \in (\mathbb{R}^2, 0)$,

$$\mathcal{T}_e\mathcal{K}^*_{\lambda_1/\lambda_2}(\overline{\alpha}) = \langle\overline{\alpha}_{\lambda_1}, \xi_1(\overline{\alpha}), \xi_2(\overline{\alpha})\rangle_{\mathcal{E}_\lambda} + \langle\overline{\alpha}_{\lambda_2}\rangle_{\mathcal{E}_{\lambda_2}}. \tag{13}$$

The terms that can be ignored in the Taylor series expansion of the path are in the intrinsic part of the unipotent tangent spaces equal to $\langle\overline{\alpha}_\lambda\rangle_{\mathcal{M}^2_\lambda} + \overline{\alpha}^*(\langle\lambda\xi_1, \xi_2\rangle)$, where ξ_1, ξ_2 are the generators of $\mathrm{Derlog}^*(F_0)$. As in [Furter *et al.*, 1998], we call $\mathrm{Derlog}^*(F_0)$ the *algebraic* or *liftable Derlog*. By projecting down onto $(\mathbb{R}^a, 0)$, H in (12) preserves Δ^{F_0}, $H(\lambda, \Delta^{F_0}) \subset \Delta^{F_0}$ for $\lambda \in (\mathbb{R}^l, 0)$. And so $\mathrm{Derlog}^*(F_0)$ is a submodule of $\mathrm{Derlog}(\Delta^{F_0})$, the module of vector fields tangent to the discriminant Δ^{F_0} (the *geometric Derlog*). As for the nonequivariant case, those modules are here the same (see Sec. 5, Theorem 5).

3.6. *Comparison of bifurcation and path equivalences*

For finite codimension problem, bifurcation and path equivalences lead to the same classification and same miniversal unfoldings. More precisely we have the following result.

Theorem 4 [Furter *et al.*, 1998]. *With finite codimensions*, $\overline{\alpha}$ *is path equivalent to* $\overline{\beta}$ *if and only if* $\overline{\alpha}^*F_0$ *is bifurcation equivalent to* $\overline{\beta}^*F_0$. *For the unfoldings*: *A is a miniversal unfolding of* $\overline{\alpha}$ *for path equivalence if and only if* A^*F_0 *is a mininiversal unfolding of* $\overline{\alpha}^*F_0$ *for bifurcation equivalence.*

The theory for multidimensional $\lambda \in (\mathbb{R}^l, 0)$ is easier because the contribution of $\mathrm{Derlog}^*(F_0)$ in the tangent spaces does not depend explicitly on l and tangent spaces are $\mathcal{E}_\lambda$- or $\mathcal{E}_{\lambda_1/\lambda_2}$-modules, not always modules over a system of rings involving z-dependent rings as for $\mathcal{K}_\lambda$ (see [Damon, 1984]).

Moreover, $\mathrm{Derlog}^*(F_0)$ can be calculated once for all, independently of the parameter structure. But, using the group action of bifurcation equivalence via (9) is usually easier for *explicit simplification* of germs to the normal forms (to solve the recognition problem) because it is very difficult to have explicit access to H.

4. Cores and Zero-Sets for $\mathbb{Z}_2$-Equivariant Problems

Now we describe the cores we need, their miniversal unfoldings and zero-sets. Origin preserving $\mathbb{Z}_2$-equivariant germs are the $\mathcal{E}_z^{\mathbb{Z}_2}$-submodule $\boldsymbol{\mathcal{F}}_z^{\mathbb{Z}_2}$ of $\boldsymbol{\mathcal{E}}_z^{\mathbb{Z}_2}$ generated by $[x,0], [v,0]$ and $Y_1 = [0,1] = (0,y)$, that is, $f_0 \in \boldsymbol{\mathcal{F}}_z^{\mathbb{Z}_2}$ can be written as

$$\begin{pmatrix} xp_1(x,v) + vp_2(x,v) \\ yq(x,v) \end{pmatrix}.$$

Their origin preserving unfoldings with a-parameters $\alpha \in (\mathbb{R}^a, 0)$, say, form the $\mathcal{E}_{(z,\alpha)}^{\mathbb{Z}_2}$-module $\boldsymbol{\mathcal{F}}_{(z,\alpha)}^{\mathbb{Z}_2}$ also generated by $[x,0], [v,0]$ and Y_1.

4.1. *Origin preserving contact equivalence*

Two germs $f, g \in \boldsymbol{\mathcal{E}}_z^{\mathbb{Z}_2}$ are *contact equivalent* if there exist $\mathbb{Z}_2$-equivariant changes of coordinates, $X \in \boldsymbol{\mathcal{E}}_z^{\mathbb{Z}_2}$ and a family of matrices $T : (\mathbb{R}^2, 0) \to \mathrm{GL}(2,\mathbb{R})$, such that

$$g(z) = T(z)f(X(z)), \tag{14}$$

with $X^o = 0$, X_z^o is a matrix with positive diagonal, $T(\gamma z)\gamma = \gamma T(z)$ and T^o has a positive diagonal. Details of this equivalence can be found in [Golubitsky *et al.*, 1988, Chapter XIX], ignoring the dependance of λ in their problem. It forms a group under composition we denote by $\mathcal{K}^{\mathbb{Z}_2}$. It acts similarly on unfoldings via families of contact equivalences.

To work with origin preserving germs in $\boldsymbol{\mathcal{F}}_z^{\mathbb{Z}_2}$, we consider the subgroup $\mathcal{K}_o^{\mathbb{Z}_2}$ of $\mathcal{K}^{\mathbb{Z}_2}$ where $X \in \boldsymbol{\mathcal{F}}_z^{\mathbb{Z}_2}$. Note that, on $\boldsymbol{\mathcal{F}}_z^{\mathbb{Z}_2}$, we do not see any difference between $\mathcal{K}_o^{\mathbb{Z}_2}$ and $\mathcal{K}^{\mathbb{Z}_2}$. Indeed, the origin is always preserved by the local diffeomorphisms in $\mathcal{K}^{\mathbb{Z}_2}$. The difference is on unfoldings. For the $\mathcal{K}^{\mathbb{Z}_2}$-theory, unfoldings vanish only at the origin $(0,0)$ in (z,α)-space. For the origin preserving equivalence, we require that unfoldings vanish at $(0,\alpha)$ for $\alpha \in (\mathbb{R}^a, 0)$. Therefore, the tangent spaces with respect to the subgroups of the equivalence acting on spaces of unfoldings will be different. This fits with the point made in [Damon, 1984] that a group of contact-equivalence is actually a *quadruple* consisting of a space of germs, spaces of unfoldings (indexed by the number of parameters) and groups acting on each space with special relations between them that make the general abstract theory work. Here, we already defined the space of germs $\boldsymbol{\mathcal{F}}_z^{\mathbb{Z}_2}$ and the spaces of its deformations, unfoldings $\boldsymbol{\mathcal{F}}_{(z,\alpha)}^{\mathbb{Z}_2}$. The element $(T, X) \in \mathcal{K}_o^{\mathbb{Z}_2}$ if $T : (\mathbb{R}^2, 0) \to \mathrm{GL}(2,\mathbb{R})$,

$$T(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

satisfies $a(x,-y) = a(z)$, $b(x,-y) = -b(z)$, $c(x,-y) = -c(z)$ and $d(x,-y) = d(z)$. This means that $a, b \in \mathcal{E}_z^{\mathbb{Z}_2}$ and $b = yb_1$, $c = yc_1$, with $b_1, c_1 \in \mathcal{E}_z^{\mathbb{Z}_2}$. The local diffeomorphism $X = [p,q] \in \boldsymbol{\mathcal{E}}_z^{\mathbb{Z}_2}$ with $p^o = 0$ and $p_x^o, q^o > 0$. Again it is the maps (T, X) between unfoldings that are different. Because T preserves anyway the origin, we assume that $X = [P, Q]$ satisfy $X(0,\alpha) = 0$, with the nondegeneracy values $P_x^o, Q^o > 0$.

The group $\mathcal{K}_o^{\mathbb{Z}_2}$ with its extension to unfoldings, together with the spaces $\boldsymbol{\mathcal{F}}_z^{\mathbb{Z}_2}$ and $\boldsymbol{\mathcal{F}}_{(z,\alpha)}^{\mathbb{Z}_2}$, forms a geometric subgroup in the sense of [Damon, 1984] because it satisfies the four properties needed. To see it, remark that it is a subgroup of $\mathcal{K}^{\mathbb{Z}_2}$ that itself satisfies those four properties. The only additional result to check is that the exponential maps that map the vector fields to the diffeomorphisms in $\mathcal{K}_o^{\mathbb{Z}_2}$ give rise to origin preserving diffeomorphisms. Because the vector fields are in $\boldsymbol{\mathcal{F}}_{(z,\alpha)}^{\mathbb{Z}_2}$, they vanish at the origin for all $\alpha \in (\mathbb{R}^a, 0)$. The origin is thus a fixed point of the exponential map for all $\alpha \in (\mathbb{R}^a, 0)$.

4.1.1. *Tangent spaces*

The tangent spaces we need for the algebraic calculations for $\mathcal{K}_o^{\mathbb{Z}_2}$ are as follows.

Proposition 4. *Let* $f = [p,q] \in \boldsymbol{\mathcal{F}}_z^{\mathbb{Z}_2}$.

(1) *The* $\mathcal{K}_o^{\mathbb{Z}_2}$*-unipotent tangent space for* f *is given by*

$$\langle [vq,0],[0,p]\rangle_{\mathcal{E}_{x,v}} + \langle [p,0],[0,q],[vp_v,vq_v]\rangle_{\mathcal{M}_{(x,v)}} + \langle [p_x,q_x]\rangle_{\mathcal{M}^2_{(x,v)}}. \tag{15}$$

(2) *The* $\mathcal{K}_o^{\mathbb{Z}_2}$*-extended tangent space for* f *is generated by* $[p, 0]$, $[vq, 0]$, $[0, p]$, $[0, q]$, $[vp_v, vq_v]$ *over* $\mathcal{E}_{(x,v)}$ *and* $[p_x, q_x]$ *over* $\mathcal{M}_{(x,v)}$.

Proof. To calculate the tangent spaces we consider the one-parameter paths through the identity when $t = 0$ in the appropriate subgroup of contact-equivalence, and apply it to f and calculate their initial tangent vectors when $t = 0$. Explicitly, we get the following.

(1) The path $t \to (T(., t), X(., t))$ is taken in the subgroup of unipotent equivalence of $\mathcal{K}_o^{\mathbb{Z}_2}$ (see also [Golubitsky *et al.*, 1988, Chapter XIX(2.3), pp. 419–420]). It means that $T(z, 0) = I + tT_1(z, t)$, where $T_1(0, 0) = 0$, and $X(z, 0) = z + tX_1(z, t)$, where $X_1(0, 0)$ and $X_z(0, 0)$ vanish. The t-derivative at $t = 0$ of $t \to (T, X) \cdot f$ is

$$T_1(z, 0)f + f_z X_1(z, 0)$$

where T_1 and X_1 are arbitrary, of order one, respectively two. This means that the unipotent tangent space is thus given by (15).

(2) We have a similar calculation for the extended tangent space. It means that $T(z, 0) = I + tT_1(z, t)$ and $X(z, 0) = z + tX_1(z, t)$, where T_1 and X_1 are $\mathbb{Z}_2$-equivariant, but arbitrary. The t-derivatives at $t = 0$ of $t \to (T, X) \cdot f$ is

$$T_1(z, 0)f + f_z X_1(z, 0)$$

where T_1 and X_1 are arbitrary. We find that it is generated by $[p, 0]$, $[vq, 0]$, $[0, p]$, $[0, q]$, $[vp_v, vq_v]$ over $\mathcal{E}_{(x,v)}$ and $[p_x, q_x]$ over $\mathcal{M}_{(x,v)}$. ■

4.2. *Generic cores and their miniversal unfoldings*

We are only interested in cores with zero first jet so as to have a bifurcation of corank 2. The generic (i.e. of lowest codimension) core is as follows.

Proposition 5

(1) *Let* $f_0 \equiv [p, q]$ *such that* $p_x^o = q^o = 0$, $p_{xx}^o > 0$ *and* $p_v^o \cdot q_x^o \neq 0$. *Then* f_0 *can be cast into the following normal form*:

$$f_0(z) = (x^2 + \epsilon_1 y^2, 2\epsilon_2 xy). \tag{16}$$

When $p_{xx}^o < 0$, *by multiplying through by* -1 *we get the previous case but* with the stability assignments reversed. *Therefore*, $\epsilon_1 = \operatorname{sign}(p_{xx}^o \cdot p_v^o)$ *and* $\epsilon_2 = \operatorname{sign}(p_{xx}^o \cdot q_x^o)$. *If* f_0 *is a gradient we can use changes of coordinates preserving the gradient structure.*

(2) *For* $\mathcal{K}_o^{\mathbb{Z}_2}$, f_0 *has the following miniversal unfolding, that is also a gradient map,*

$$F_0(z, \alpha_1, \alpha_2) = (x^2 + \epsilon_1 y^2 + \alpha_1 x, 2\epsilon_2 xy + \alpha_2 y). \tag{17}$$

Proof. We use the usual techniques of [Golubitsky *et al.*, 1988].

(1) A simple rescaling casts the quadratic coefficients to (16). We can then remove the higher order terms using the intrinsic part of the unipotent tangent space. Here $p = x^2 + \epsilon_1 v$ and $q = 2\epsilon_2 x$. From (15), after straightforward simplifications, the unipotent tangent space is $\langle [x^3, 0], [vx, 0], [v^2, 0], [0, x^2], [0, v] \rangle_{\mathcal{E}_{(x,v)}}$, equal to

$$[\mathcal{I}^3_{(x,v)} + \mathcal{I}_v \mathcal{I}_{(x,v)}, \mathcal{I}^2_{(x,v)} + \mathcal{I}_v]. \tag{18}$$

From Lemma XIX 2.1 in [Golubitsky *et al.*, 1988, p. 419], $[\zeta_1, \zeta_2]$ is intrinsic if ζ_1, ζ_2 are intrinsic ideals such that $\langle v \rangle \zeta_2 \subset \zeta_1 \subset \zeta_2$. The ideals $\mathcal{I}_{(x,v)}$ and $\mathcal{I}_v$ are intrinsic and so (18) is intrinsic and so we can eliminate all the terms ignored in (16). In the gradient case the rescaling must preserve the gradient structure, so we choose $T = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and $X(z) = (\alpha x, \beta y)$. The higher order terms are eliminated as before (see [Bridges & Furter, 1993]).

(2) The extended tangent space of f_0 for $\mathcal{K}_o^{\mathbb{Z}_2}$ is generated by $[x^2 + \epsilon_1 v, 0]$, $[2\epsilon_2 xv, 0]$, $[0, x^2 + \epsilon_1 v]$, $[0, 2\epsilon_2 x]$, $[\epsilon_1 v, 0]$ over $\mathcal{E}_{(x,v)}$ and $[2x, 2\epsilon_2]$ over $\mathcal{M}_{(x,v)}$. Following similar calculations as before, the extended normal space is generated by $[x, 0]$ and $[0, 1]$. ■

4.3. *Zero-set structure of* $\mathbb{Z}_2$*-equivariant unfoldings*

Let $F = [P, Q] \in \boldsymbol{\mathcal{E}}^{\mathbb{Z}_2}_{(z,\alpha)}$, $\alpha \in (\mathbb{R}^a, 0)$. The zero-set of F is given by the roots of

$$P(x, v, \alpha) = yQ(x, v, \alpha) = 0.$$

The Jacobian matrix of F is

$$F_z(z, \alpha) = \begin{pmatrix} P_x(x, v, \alpha) & 2yP_v(x, v, \alpha) \\ yQ_x(x, v, \alpha) & (Q + 2vQ_v)(x, v, \alpha) \end{pmatrix}.$$

The zero-set of F consists of three pieces corresponding to solutions with different internal symmetries. We have the following.

(1) The trivial branch $Z_0^{\mathbb{Z}_2} = \{(0,0,\alpha)\}$ always belong to the zero-set of F because of the origin preserving property. Note that it is a subset of $\mathrm{Fix}(\mathbb{Z}_2)$. The linearization $F_z(0,0,\alpha)$ on $Z_0^{\mathbb{Z}_2}$ is a diagonal matrix with eigenvalues $P_x(0,0,\alpha)$ and $Q(0,0,\alpha)$.
(2) The set of $\mathbb{Z}_2$-symmetric solutions not on the trivial branch are
$$Z^{\mathbb{Z}_2} = \{(x,0,\alpha) : x \neq 0, P(x,0,\alpha) = 0\}.$$
The eigenvalues of $F_z(x,0,\alpha)$ on $Z^{\mathbb{Z}_2}$ are $P_x(x,0,\alpha)$ and $Q(x,0,\alpha)$.
(3) The set of solutions without symmetry
$$Z^{\mathbf{1}} = \{(x,v,\alpha) : v \neq 0, \quad P(x,v,\alpha) = Q(x,v,\alpha) = 0\}.$$
The eigenvalues of F_z on $Z^{\mathbf{1}}$ satisfy
$$\det(F_z) = 2v(P_xQ_v - Q_xP_v) \quad \text{and}$$
$$\mathrm{tr}(F_z) = P_x + 2vQ_v.$$

Explicitly, the zero-set of (17) is the following with $\alpha = (\alpha_1, \alpha_2)$.

(1) The trivial branch $Z_0^{\mathbb{Z}_2}$ with eigenvalues $P_x(0, 0,\alpha) = \alpha_1$ and $Q(0,0,\alpha) = \alpha_2$.
(2) The set $Z^{\mathbb{Z}_2} = \{(x,0,\alpha) : x = -\alpha_1\}$ with eigenvalues $P_x(x,0,\alpha) = -\alpha_1$ and $Q(x,0,\alpha) = \alpha_2 - 2\epsilon_2\alpha_1$.
(3) The set
$$Z^{\mathbf{1}} = \left\{(x,v,\alpha) : x = -\frac{1}{2}\epsilon_2\alpha_2, \quad v = \frac{1}{4}\epsilon_1\alpha_2(2\epsilon_2\alpha_1 - \alpha_2)\right\}$$
with $\det((F_0)_z) = 2v(P_xQ_v - Q_xP_v) = -2\epsilon_1\epsilon_2 v$ and $\mathrm{tr}((F_0)_z) = -\epsilon_2\alpha_2$ on $Z^{\mathbf{1}}$.

4.4. *Discriminants*

From the Implicit Function Theorem (IFT), as long as F_z remains nonsingular on the zero-set of F, the zero-set remains qualitatively similar. To understand and describe how the zero-set of F varies as α changes we need to determine the values of α where the IFT does not apply. They belong to the *discriminant* Δ of F, of equation
$$\{\alpha \in (\mathbb{R}^2,0) : \exists\, z \in (\mathbb{R}^2,0), \quad F(z,\alpha) = \det(F_z)(z,\alpha) = 0\}. \tag{19}$$

We have seen that the zero-set of F has three pieces. This means that there will be three pieces of the discriminant coming from the bifurcation between each pair of the three zero sets. The bifurcations from $Z_0^{\mathbb{Z}_2}$ or $Z^{\mathbb{Z}_2}$ (subsets of $\mathrm{Fix}(\mathbb{Z}_2)$) to $Z^{\mathbf{1}}$ (subset of $\mathrm{Fix}(\mathbf{1})$) are of "spontaneous symmetry breaking" type, and so they are typically of pitchfork type (see [Golubitsky *et al.*, 1988]). The third case from $Z_0^{\mathbb{Z}_2}$ to $Z^{\mathbb{Z}_2}$ does not involve any change of symmetry, both sets are in $\mathrm{Fix}(\mathbb{Z}_2)$, but the existence of the trivial branch implies that the typical bifurcation is transcritical (see [Golubitsky & Schaeffer, 1985]). Moreover, the zero-sets $Z_0^{\mathbb{Z}_2}$, $Z^{\mathbf{1}}$ can have internal bifurcation points without symmetry breaking (obviously, there is no bifurcation inside the trivial branch). From [Golubitsky *et al.*, 1988] we know that the typical bifurcation without symmetry breaking are fold points.

Hence, for general $F \in \boldsymbol{\mathcal{E}}_{(z,\alpha)}^{\mathbb{Z}_2}$, the discriminant is *a priori* formed of five pieces, we call (local) *bifurcation varieties*. Explicitly, we have the following sets,

(1) $\mathcal{B}_x^o = \{\alpha : P_x(0,0,\alpha) = 0\}$ is formed of the bifurcation points from the trivial branch $Z_0^{\mathbb{Z}_2}$ to $Z^{\mathbb{Z}_2}$,
(2) $\mathcal{P}_\kappa^o = \{\alpha : Q(0,0,\alpha) = 0\}$ for the symmetry breaking bifurcation points from $Z_0^{\mathbb{Z}_2}$ to $Z^{\mathbf{1}}$,
(3) $\mathcal{P}_\kappa = \{\alpha : \exists x \neq 0, P(x,0,\alpha) = Q(x,0,\alpha) = 0\}$ for the symmetry breaking bifurcation points from $Z^{\mathbb{Z}_2}$ to $Z^{\mathbf{1}}$,
(4) $\mathcal{B}_x = \{\alpha : \exists x \neq 0, P(x,0,\alpha) = P_x(x,0,\alpha) = 0\}$ for the bifurcation points inside $Z^{\mathbb{Z}_2}$ and
(5) $\mathcal{B}^{\mathbf{1}} = \{(x,v,\alpha) : \exists v \neq 0, P(x,v,\alpha) = Q(\ldots) = (P_xQ_v - Q_xP_v)(\ldots) = 0\}$ for bifurcation points inside $Z^{\mathbf{1}}$.

The discriminant of (17) is formed of only three local bifurcation varieties, namely $\mathcal{P}_\kappa^o$, $\mathcal{P}_\kappa$ and $\mathcal{B}_x^o$, because the bifurcation varieties $\mathcal{B}_x$ and $\mathcal{B}^{\mathbf{1}}$ are empty. Their equations are $\alpha_1 = 0$ for $\mathcal{B}_x^o$, $\alpha_2 = 0$ for $\mathcal{P}_\kappa^o$ and $2\alpha_1 - \epsilon_2\alpha_2 = 0$ for $\mathcal{P}_\kappa$. Together, Δ^{F_0} is given by
$$h_0^{\mathbb{Z}_2}(\alpha) = \alpha_1\alpha_2(2\alpha_1 - \epsilon_2\alpha_2) = 0. \tag{20}$$
Hopf bifurcation points are possible on $Z^{\mathbf{1}}$, although they are not invariant of the $\mathcal{K}_o^{\mathbb{Z}_2}$-equivalence. They occur near the set $\mathcal{H}$ of points $(x,v,\alpha) \in Z^{\mathbf{1}}$ with zero trace and positive determinant: $(P_x + 2vQ_v)(x,v,\alpha) = 0$ and $(P_xQ_v - Q_xP_v)(x,v,\alpha) > 0$. We can ascertain their existence using continuity arguments along branches of solutions where the

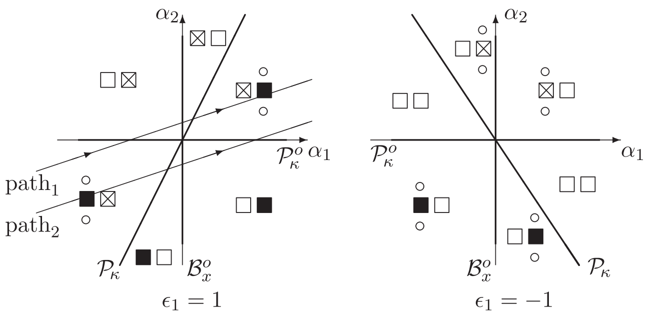

Fig. 1. Solution set of $F_0(z, \alpha) = 0$ as a function of α when $\epsilon_1\epsilon_2 = 1$. The two paths, path$_1$ and path$_2$, correspond to our example in Sec. 7

determinant does not change sign but the trace does [Golubitsky & Schaeffer, 1985].

4.4.1. *Zero-set diagrams of the cores*

The zero-set of F_0 is in $(\mathbb{R}^4, 0)$. We represent the solution set of (17) as its projections on the (α_1, α_2)-plane. The pattern of solutions is the same in each connected component of $\mathbb{R}^2 - \Delta$ where Δ is the relevant discriminant, namely (20). It is only composed of the local bifurcation varieties $\mathcal{B}_x^o$, $\mathcal{P}_\kappa^o$ and $\mathcal{P}_\kappa$. The *squares* represent $\mathbb{Z}_2$-symmetric solutions ($y = 0$). There are always two of them. We take the convention that the one on the left represents the trivial solution in $Z_0^{\mathbb{Z}_2}$ and the second on the right is in $Z^{\mathbb{Z}_2}$. The other solutions in $Z^{\mathbf{1}}$ (without any symmetry) are represented by *circles*. When they exist, they appear as $\mathbb{Z}_2$-symmetric pairs because the symmetry forces them to do so. This explains why they appear across the transition varieties $\mathcal{P}_\kappa^o$ and $\mathcal{P}_\kappa$ as pitchfork bifurcations. A black square ■ or circle • represent solutions with eigenvalues of negative real parts $(-, -)$, crossed squares ⊠ or circles ⊗ represent solutions with eigenvalues of positive real parts $(+, +)$ as white squares □ or circles ∘ are saddle points (real eigenvalues of opposite sign).

In the following figures we represent the solution set of (17). The discriminant is composed of three lines: $\mathcal{P}_\kappa$, $\mathcal{P}_\kappa^o$ and $\mathcal{B}_x^o$ [see (20)]. In $\mathrm{Fix}(\mathbb{Z}_2)$ the square on the left represents the trivial solution. In Fig. 1, we represent the solutions when $\epsilon_1\epsilon_2 = 1$ (containing the cases when F_0 is a gradient). The paths are linked to the example in [Wu, 1999]. When $\epsilon_1\epsilon_2 = -1$ we have possibilities for Hopf bifurcation points when crossing $\mathcal{H}$.

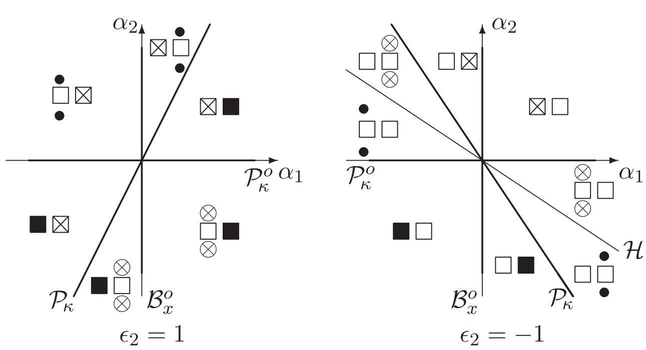

Fig. 2. Solution set of $F_0(z, \alpha) = 0$ as a function of α when $\epsilon_1\epsilon_2 = -1$.

5. Derlogs and Liftable Vector Fields

For the equivalence of the theories for finite codimension bifurcation germs and their associated paths, as well as explicit calculations, we need the generators of the module $\mathrm{Derlog}^*(F_0)$ of vector fields liftable via the projections π_{F_0}. To establish the results we use complexifying the situation [Furter & Sitta, 2004]. Nothing will be lost in finite codimension because we can work with germs equivalent to polynomials and we take care to preserve the real and complex algebras. The situation is even simpler here because the discriminants are actually the same for the generic core, with F_0 in (17). To help with the geometry, we complexify our situation using the map $(x, y) \to (z_1, z_2)$ to preserve the algebra between the real and complex context and work with analytic germs. The results can be readily moved between the two contexts. We denote by $\mathcal{O}$, instead of $\mathcal{E}$, the corresponding sets of complex germs. In coordinates, a vector field germ $\xi : (\mathbb{C}^a, 0) \to \mathbb{C}^a$ is *liftable* if there exists a vector field germ $\eta : (\mathbb{C}^{2+a}, 0) \to \mathbb{C}^2$ and a matrix map germ $T : (\mathbb{C}^{2+a}, 0) \to \mathrm{M}(2, \mathbb{C})$ such that

$$(F_0)_z(z, \alpha)\eta(z, \alpha) + (F_0)_\alpha(z, \alpha)\xi(\alpha) = T(z, \alpha)F_0(z, \alpha). \qquad (21)$$

This definition is also geometric in the sense that ξ lifts to the vector field (η, ξ) tangent to $F_0^{-1}(0)$ at its smooth points. It can also be defined as the kernel of an epimorphism of coherent modules (see [Furter *et al.*, 1998]). Let

$$F_0(z, \alpha) = f_0(z) + \sum_{i=1}^{a} \alpha_i h_i(z).$$

From the Malgrange Preparation Theorem, $\mathcal{N}_e\mathcal{K}_o^{\mathbb{Z}_2}(F_0) = \boldsymbol{\mathcal{O}}_{(z,a)}^{\mathbb{Z}_2}/\mathcal{T}_e\mathcal{K}_o^{\mathbb{Z}_2}(F_0)$ is freely generated as an $\mathcal{O}_\alpha$-module by $\{h_i\}_{i=1}^a$. The following formula

$$\varphi(p) = \sum_{i=1}^{a} p_i(\alpha) h_i(z)$$

defines a linear epimorphism [Mond & Montaldi, 1994; Furter & Sitta, 2004]:

$$\mathcal{O}_\alpha^a \xrightarrow{\varphi} \mathcal{N}_e\mathcal{G}(F_0) \to 0.$$

The kernel of φ is clearly $\mathrm{Derlog}^*(F_0)$.

Theorem 5. *Let F_0 given in (17), $\mathrm{Derlog}^*(F_0)$ is freely generated over $\mathcal{O}_\alpha$ by the vector fields*

$$\begin{aligned} \xi_1 &= (\alpha_1, \alpha_2), \\ \xi_2 &= (2\epsilon_2\alpha_1\alpha_2 - 2\alpha_1^2, 4\alpha_1\alpha_2 - \epsilon_2\alpha_2^2). \end{aligned} \tag{22}$$

Proof. Noting that

$$(F_0)_z Y_1 = [2vP_v, Q + 2vQ_v],$$

we can calculate explicitly the lifts using (21). We find that ξ_i, $i = 1, 2$, lift to (η_i, ξ_i), $i = 1, 2$, where $\eta(z) = (z_1, z_2)$ and $\eta_2 = (6x^2 + 4\alpha_1 x + 2\epsilon_2\alpha_2 x, 6xy + (\alpha_1 + 2\epsilon_2\alpha_2))$. Multiplying (η_i, ξ_i), $i = 1, 2$, by α_1 and α_2 we see that the resulting liftable vector fields can be decomposed in terms of ξ_1 and ξ_2 that are independent, and so $\mathrm{Derlog}^*(F_0)$ is freely generated over $\mathcal{O}_\alpha$ by ξ_1 and ξ_2. ■

As mentioned in Sec. 3.5, liftable vector fields must be tangent to Δ^{F_0} by projecting down along π_{F_0}. Note that the definition (19) of the discriminant for real F_0 should be amended. Actually, we should choose Δ^{F_0} as the *real* slice of the discriminant of the complexification of F_0 (see [Furter & Sitta, 2004]) but it is not important here because both are equal. In our problems, the liftable vector fields are exactly the vector fields tangent to the discriminant (see Proposition 6). Let $\mathcal{I}(\Delta^{F_0})$ denote the ideal of germs vanishing on Δ^{F_0}. Define

$$\mathrm{Derlog}(\Delta^{F_0}) = \{\xi \in \boldsymbol{\mathcal{O}}_a \,|\, \xi(\mathcal{I}(\Delta^{F_0})) \subset \mathcal{I}(\Delta^{F_0})\}.$$

The discriminant Δ^{F_0} is a *free* (*or Saito*) *divisor* if $\mathrm{Derlog}(\Delta^{F_0})$ is a locally free $\mathcal{O}_a$-module (of rank a).

Theorem 6 [Saito, 1980]. *If the vector fields $\{\xi_i\}_{i=1}^a$ are in $\mathrm{Derlog}(\Delta^{F_0})$ and the determinant $|\xi_1 \dots \xi_a|$ is a reduced defining equation for Δ^{F_0} then they generate freely $\mathrm{Derlog}(\Delta^{F_0})$.*

We therefore obtain the following result. For F_0 in (17), Δ^{F_0} is given by $h_0^{\mathbb{Z}_2}(\alpha) = 0$ where $h_0^{\mathbb{Z}_2}$ is (20). A *nilpotent basis* consists of an Euler field (like ξ_1 where $\xi_1(h_0^{\mathbb{Z}_2}) = 4h_0^{\mathbb{Z}_2}$) and a basis of the annihilator of $h_0^{\mathbb{Z}_2}$ ($\xi_2(h_0^{\mathbb{Z}_2}) = 0$).

Proposition 6. *The module*

$$\mathrm{Derlog}(\Delta^{F_0}) = \mathrm{Derlog}^*(F_0)$$

is freely generated over $\mathcal{O}_\alpha$ by the nilpotent basis $\{\xi_1, \xi_2\}$.

Proof. We conclude because $\xi_1, \xi_2 \in \mathrm{Derlog}(\Delta^{F_0})$ and $|\xi_1\xi_2| = 3h_0^{\mathbb{Z}_2}$. ■

Note that ξ_2 can be simplified to a non-nilpotent vector field $(0, 2\alpha_1\alpha_2 - \epsilon_2\alpha_2^2)$.

6. Classification of Bifurcation Maps

In this section, we classify bifurcation maps with one or two bifurcation parameters based on the generic core $f_0(z) = (x^2 + \epsilon_1 y^2, 2\epsilon_2 xy)$. Recall that the paths for (17) with l parameters are $\overline{\alpha} : (\mathbb{R}^l, 0) \to (\mathbb{R}^2, 0)$. In the following, the coefficients δ, δ_i are normalized to ± 1 and the β_i's are the unfolding parameters of the paths.

6.1. *One bifurcation parameter*

First we establish a classification of one-dimensional paths of low codimension. The difference with [Dangelmayer & Armbruster, 1983] is in the origin preserving property.

Theorem 7. *The one-dimensional paths based on the core F_0 in (17) of topological codimension up to three and their miniversal unfolding are in Table 1.*

Proof. Using explicit changes of coordinates, we cast the bifurcation germs into the path formulation structure $\overline{\alpha}^* F_0$ and use the unipotent tangent space to eliminate the higher order terms. Then, to determine the universal unfolding, we get an $\mathbb{R}$-basis for the normal space $\mathcal{N}_e\mathcal{K}_\lambda^*(\overline{\alpha})$. From Sec. 3.5, the unipotent tangent space for $\overline{\alpha}$ is equal to

$$\langle \lambda^2\overline{\alpha}_\lambda, \lambda\xi_1(\overline{\alpha}), \xi_2(\overline{\alpha})\rangle_{\mathcal{E}_\lambda}, \tag{23}$$

where ξ_1 and ξ_2 are given in (22), and the normal tangent space

$$\mathcal{N}_e\mathcal{K}_\lambda^*(\overline{\alpha}) = \frac{\boldsymbol{\mathcal{E}}_{\lambda,2}}{\langle \overline{\alpha}_\lambda, \xi_1(\overline{\alpha}), \xi_2(\overline{\alpha})\rangle_{\mathcal{E}_\lambda}}. \tag{24}$$

Table 1. One-dimensional paths of topological codimension up to three.

Normal Form	Top. Codim.	Smooth Codim.	Miniversal Unfolding
$(\delta\lambda, m\lambda)$ $m \neq 0, 2\epsilon_2\delta$	1	2	$(\delta\lambda, \beta + m\lambda)$
$(\delta_1\lambda, \delta_2\lambda^2)$	2	2	$(\delta_1\lambda, \delta_2\lambda^2 + \beta_1 + \beta_2\lambda)$
$(\delta_1\lambda, \delta_2\lambda^3)$	3	3	$(\delta_1\lambda, \delta_2\lambda^3 + \beta_1 + \beta_2\lambda + \beta_3\lambda^2)$
$(\delta_1\lambda^2, \delta_2\lambda^2)$	3	3	$(\delta_1^\lambda + \beta_1 + \beta_2\lambda, m\lambda + \beta_3)$
$(\delta_1\lambda^2, m\lambda^2 + \delta_2\lambda^3)$	3	4	$(\delta_1\lambda^2 + \beta_3, m\lambda^2 + \delta_2\lambda^3 + \beta_1 + \beta_2\lambda)$
$(\delta_1\lambda^2, m\lambda^2)$	3	5	$(\delta_1\lambda^2 + \beta_3, m\lambda^2 + n\lambda^3 + \beta_1 + \beta_2\lambda)$

To check our results on the paths $(\delta_1\lambda^2, m\lambda^2 + n\lambda^3)$, (23) is generated by

$$(2\delta_1\lambda^3, 2m\lambda^3 + 3n\lambda^4), (\delta_1\lambda^2, m\lambda^3 + n\lambda^4),$$
$$(0, 2\delta_1\lambda^2(m\lambda^2 + n\lambda^3) - \epsilon_2(m\lambda^2 + \mathcal{M}_\lambda^3)^2)$$

using the simpler generator $(0, 2\alpha_1\alpha_2 - \epsilon_2\alpha_2^2)$ for ξ_2. Simplifying, we see that the unipotent tangent space is generated by

$$(\delta_1\lambda^3, 0), (0, n\lambda^4), (0, m(2\delta - \epsilon_2 m)\lambda^4 + \mathcal{M}_\lambda^5).$$

If $n \neq 0$, or if $m \neq 0, 2\epsilon_2\delta_1$ when $n = 0$, $(\mathcal{M}_\lambda^3, \mathcal{M}_\lambda^4)$ contains the higher order terms and can be eliminated on the path. To calculate the codimension, $\mathcal{T}_e\mathcal{K}_\lambda^*(\overline{\alpha})$ is generated by

$$(2\delta_1\lambda, 2m\lambda + 3n\lambda^2), (\delta_1\lambda, m\lambda^2 + n\lambda^3),$$
$$(0, 2\delta_1\lambda^2(m\lambda^2 + n\lambda^3) - \epsilon_2(m\lambda^2 + \mathcal{M}_\lambda^3)^2).$$

Simplifying we get

$$(\delta_1\lambda, 0), (0, n\lambda^3), (0, \mathcal{M}_\lambda^4).$$

And the conclusion, $(1, 0)$, $(0, \lambda^j)$, $j = 0, 1, 2$, are always in the extended normal space and $(0, \lambda_1^3)$ when $n = 0$. ■

6.2. *Two bifurcation parameters*

Recall the nondegeneracy conditions

$$\text{ND0} : p_{xx}^o \cdot p_v^o \cdot q_x^o \cdot (p_{x\lambda_1}^o q_{\lambda_2}^o - q_{\lambda_1}^o p_{x\lambda_2}^o) \neq 0,$$
$$\text{ND1a} : p_{x\lambda_1}^o \cdot q_{\lambda_1}^o \neq 0.$$

Actually, ND0 is enough if we do not distinguish between the two parameters (λ_1, λ_2) using the bifurcation equivalence group $\mathcal{K}_{o,\lambda}^{\mathbb{Z}_2}$ (see Sec. 3.2). With ND0 and ND1a we can preserve (z, λ_1)-slices using the group $\mathcal{K}_{o,\lambda_1/\lambda_2}^{\mathbb{Z}_2}$ (see Sec. 3.2). In the next result, we assume that $p_{xx}^o > 0$, otherwise we can simply multiply f by -1 and then the stability assignments will be reversed.

Theorem 8

(1) *Under ND0 only, the* $\mathcal{K}_{o,\lambda}^{\mathbb{Z}_2}$*-normal form of* $f \in \boldsymbol{\mathcal{F}}_{(z,\lambda)}^{\mathbb{Z}_2}$ *is*

$$\begin{pmatrix} x^2 + \epsilon_1 y^2 + \lambda_1 x \\ 2\epsilon_2 xy + \delta\lambda_2 y \end{pmatrix}, \tag{25}$$

of codimension-0 (note that $p^o \cdot p_x^o \cdot q^o = 0$*),* $\delta = \text{sign}(p_{x\lambda_1}^o q_{\lambda_2}^o - q_{\lambda_1}^o p_{x\lambda_2}^o)$*.*

(2) *Under ND0 and ND1a, the* $\mathcal{K}_{o,\lambda_1/\lambda_2}^{\mathbb{Z}_2}$*-normal form of* $f \in \boldsymbol{\mathcal{F}}_{(z,\lambda)}^{\mathbb{Z}_2}$ *is*

$$\begin{pmatrix} x^2 + \epsilon_1 y^2 + \lambda_1 x \\ 2\epsilon_2 xy + (m\lambda_1 + \lambda_2)y \end{pmatrix}, \tag{26}$$

of topological codimension-0 (m is a modal parameter).

Proof. Rescale (4) and (5) and if $A < 0$ multiply both equations by -1. This has the effect of interchanging the stability assignments between the eigenvalues pairs of the linearization $++$ and $--$, the saddle points $+-$ remain unchanged. Thus we cast $\in \boldsymbol{\mathcal{F}}_{(z,\lambda)}^{\mathbb{Z}_2}$ into

$$\begin{pmatrix} x^2 + \epsilon y^2 + \overline{\alpha}_1(\lambda)x + h_1(z, \lambda) \\ 2\epsilon xy + \overline{\alpha}_2(\lambda)y + h_2(z, \lambda) \end{pmatrix} \tag{27}$$

where $\epsilon = \text{sign}(AC)$, $\overline{\alpha}_1(\lambda_1, \lambda_2) = \bar{a}_1\lambda_1 + \bar{b}_1\lambda_2$ and $\overline{\alpha}_2(\lambda_1, \lambda_2) = \bar{a}_2\lambda_1 + \bar{b}_2\lambda_2$.

(1) Using (11), f is strongly $\mathcal{K}_{o,\lambda}^{\mathbb{Z}_2}$-equivalent to $\tilde{\alpha}^* F_0$ with a path with 1-jet $(\overline{\alpha}_1, \overline{\alpha}_2)$. In this case we do not really need to use all the algebra of path formulation because we can use the inverse function theorem to determine L from the system

$$\lambda_1 = \tilde{\alpha}_1(L(\lambda_1, \lambda_2)),$$
$$\delta\lambda_2 = \tilde{\alpha}_2(L(\lambda_1, \lambda_2)),$$

with $|L^o_\lambda| > 0$. This last condition allows us to determine δ. The path $(\lambda_1, \delta\lambda_2)$ is of codimension-0 using its extended tangent space.

(2) Let $\delta_1 = \mathrm{sign}(p^o_{x\lambda_1})$, $\delta_2 = \mathrm{sign}(p^o_{x\lambda_1}(p^o_{x\lambda_1}q^o_{\lambda_2} - q^o_{\lambda_1}p^o_{x\lambda_2}))$ and $m = \frac{2q^o_{\lambda_1}\cdot p^o_{xx}}{|p^o_{x\lambda_1}|\cdot|q^o_x|}$. Similarly, f is $\mathcal{K}^{\mathbb{Z}_2}_{o,\lambda_1/\lambda_2}$-equivalent to $\tilde{\alpha}^*F_0$ with 1-jet equal to $(\delta_1\lambda_1, m\lambda_1 + \delta_2\lambda_2)$. We use path equivalence to eliminate the higher order terms. We proceed as before, the only difference is that now we have a structure of system of rings on the paths $\{\mathcal{E}_\lambda, \mathcal{E}_{\lambda_2}\}$. The unipotent tangent space is $\langle\tilde{\alpha}_\lambda\rangle_{\mathcal{M}^2_\lambda} + \tilde{\alpha}^*(\langle\lambda\xi_1, \xi_2\rangle)$. Straightforward calculations give it equal to $(\mathcal{M}^2_\lambda, \mathcal{M}^2_\lambda)$ under ND0 and ND1a. The extended tangent space is given by (13). Then we can conclude using similar techniques as before that (26) is of topological codimension 0 when ND0 and ND1a hold true. ■

7. Cylindrical Panel

The abstract results of [Wu, 1999] were applied to the bifurcation of a cylindrical panel subjected to axial compression near mode interaction. The compression load λ and the aspect ratio of the panel μ are two main bifurcation parameters. Next we recall some of the details of the calculation of the coefficients of (4) and (5) in [Wu, 1999]. The configuration domain is $\Omega = [0, \mu] \times [0, 1]$ where μ represents the aspect ratio of the rectangle. We represent the coordinates in Ω by (s, t). The functions u and f represent the nondimensional values of the vertical displacement and the Airy tension. From the von Kármán–Donnell's shell theory, u and f satisfy

$$\Delta^2 u = -\lambda u_{ss} + [u, f] + df_{ss}, \tag{28}$$

$$\Delta^2 f = -\frac{1}{2}[u, u] - du_{ss}, \tag{29}$$

where partial derivatives are $u_s = \frac{\partial u}{\partial s}$. The operator Δ represents the usual Laplacian in the plane and $[u, v] = u_{ss}v_{tt} + u_{tt}v_{ss} - 2u_{st}v_{st}$ is the Monge–Ampère's operator. The parameter d is proportional to the inverse of the radius of curvature of the panel and λ to the external force exercised on the t-side. There are many boundary conditions for u and f in the literature. In [Wu, 1999] the simply supported panel is used: $u = \Delta u = f = \Delta f = 0$. The choice of the best boundary conditions for a particular experiment is a subject of much discussion, practically, deciding the mixture of boundary conditions to put on u and f.

The following results can be found in [Wu, 1999]. The first bifurcation values from the unstressed plate are $\lambda_c = \frac{9}{2}\pi^2 + d^2(\frac{9}{2}\pi^2)^{-1}$ and $\mu_c = \sqrt{2}$. The kernel of the linearization is two-dimensional, generated by $\phi_1(s, t) = \sin\frac{\pi s}{\sqrt{2}}\sin\pi t$ and $\phi_2(s, t) = \sin\sqrt{2}\pi s\sin\pi t$. We can solve uniquely (29) for f as a function of any given u with our boundary conditions. Replacing in (28) we find a fourth-order PDE in u:

$$\Delta^2 u = -\lambda u_{ss} + [u, f(u)] + d(f(u))_{ss}. \tag{30}$$

The equation is invariant with respect to the reversor $S : (s, t) \mapsto (\mu - s, 1 - t)$ and that ϕ_1 is S-invariant and ϕ_2 is S-equivariant. Following a classical Lyapunov–Schmidt process [Golubitsky & Schaeffer, 1985] we find the following.

Proposition 7 [Wu, 1999]. *The bifurcation function for (30) is (4) and (5) where $x\phi_1 + y\phi_2$ is the component of u in the kernel of the linearization,* $A = -\frac{162^{\frac{3}{4}}}{9}d$, $C = -\frac{3642^{\frac{3}{4}}}{45}d$, $a_1 = -\frac{\pi^2}{2}$, $a_2 = 4a_1$, $b_1 = \frac{3\sqrt{2}}{4}\pi^4 - \frac{\sqrt{2}}{27}d^2$ *and* $b_2 = -3\sqrt{2}\pi^4 + \frac{4\sqrt{2}}{27}d^2$.

As a consequence, the coefficients for the normal forms of Theorem 8 are $\epsilon = \delta_1 = \delta_2 = 1$ and $\mu = \frac{80}{91}$. In the normal forms, the bifurcation parameters are the difference to the critical values (λ_c, μ_c), so we define the initial bifurcation parameters as $\lambda_1 = \lambda - \lambda_c$, $\lambda_2 = \mu - \mu_c$. The (λ_1, λ_2)-diagrams are found above the lines $\alpha_2 = \frac{80}{91}\alpha_1 + \lambda_2$ in Fig. 1 (path_1 is for $\lambda_2 > 0$, path_2 for $\lambda_2 < 0$) where the stable solutions (local minima of the energy) are represented by ⊠ ($A < 0$). Because of the change of coordinates for the normal form, (λ_1, λ_2) in (26) represents some combination of the compression load and the aspect ratio (but μ is an invariant of the change of coordinates). From the modeling point of view, if the parameters we consider do not all preserve the trivial branch, like the presence of dead loads, we either need to take one of those parameters as one of the bifurcation parameters or consider only one bifurcation parameter problem.

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