On Entropy Flux of Anisotropic Elastic Bodies

J. Jaric
Faculty of Mathematics, University of Belgrade
Studentski trg 16, 11000 Beograd, Srbija

Corresponding author: R. Vignjevic (v.rade@brunel.ac.uk)
Structural Integrity Theme, Brunel University London
Kingston Lane, Uxbridge UB8 3PH, UK

D. Kuzmanovic
Faculty of Transport and Traffic, University of Belgrade
Vojvode Stepe 305, 11000 Beograd, Srbija

Abstract

The framework of irreversible thermodynamics is fundamental in development of constitutive models. One of the important aspects of the extended irreversible thermodynamics is the relationship between the entropy flux and the heat flux, especially for phenomena far from equilibrium.

In this paper, we demonstrate that the assumption that Lagrange multiplier (in the expression for the Second law of thermodynamics) is a function of temperature $\Lambda^{\varepsilon} = \Lambda^{\varepsilon}(\theta)$ is a sufficient condition to derive the entropy flux - heat flux relation for all isotropic materials as well as for a number of crystal classes including transverse isotropy, orthotropy, triclinic systems and rhombic systems. For all considered crystal classes the entropy flux - heat flux relation where derived explicitly. Further, we demonstrate that for some crystal classes heat flux is non-zero even when temperature gradient vanishes (as stated by Eringen). The anisotropic functions, with respect to the symmetry groups of the crystal classes, were expressed in terms of isotropic functions.

The proposed procedure is very general in the sense that it can be used with non-linear constitutive relations as demonstrated here. The presented results confirm that the all crystal elastic bodies considered, are hyperelastic.

Keywords: entropy flux, heat flux, costitutive models, anisotropy, symmetry groups, isotropic functions

1 Introduction

One of the important problems in thermodynamics is the relationship between the entropy flux and the heat flux for phenomena far from equilibrium.

The entropy principle based on the **Clausius-Duhem inequality**

$$\rho\dot{\eta} + \mathrm{div}\frac{\mathbf{q}}{\theta} - \rho\frac{r}{\theta} \geq 0, \tag{1}$$

where ρ is density, η is the specific entropy density, $\mathbf{q}$ the heat flux and r the heat supply, has been widely adopted in the development of modern rational thermodynamics after the fundamental work of Coleman and Noll [1]. The main assumptions, **motivated by the result of classical thermostatics**, are that the entropy flux $\mathbf{\Phi}$ and the entropy supply s are proportional to the heat flux and the heat supply, respectively. Moreover, both constants of proportionality. are assumed to be the reciprocal of the absolute temperature, i.e.

$$\mathbf{\Phi}_\kappa = \frac{1}{\theta}\mathbf{q}_\kappa, \qquad s = \frac{1}{\theta}r. \tag{2}$$

These main assumptions, while tacit in the classical theory of continuum mechanics, do not hold particularly well for materials in general. In fact, it is known that they are inconsistent with the kinetic theory of ideal gases and are also found to be inadequate to account for the thermodynamics of diffusion.

There is an extended formulation of the second law of thermodynamics which has been applied to nonequilibrium thermodynamics by Serrin [2] and Silhavy [3] and summarized by Truesdell and Bharatha [4]. See also Muschik [5] and Müller **Error! Reference source not found.**. A comprehensive review of related literature and detailed derivations can be found in Lebon at.al. [20] and Jou at. al. [21].

The extended form of the second law, usually called the entropy inequality, seems to be the most general formulation of the continuous second law of thermodynamics proposed so far . In this theory, the assumptions given by equation (2) were abandoned and the entropy flux $\mathbf{\Phi}$ and the heat flux vector $\mathbf{q}$ are treated as independent constitutive quantities and hence leaving the entropy inequality in its general form

$$\rho\dot{\eta} + \mathrm{div}\mathbf{\Phi} - \rho s \geq 0. \tag{3}$$

I-Shih Liu proposed in 1972 [6] a method, reminiscent of the classical method of Lagrange multipliers, for expanding the inequality (3). Instead of this inequality restricting the solution of field equations he considered solutions of an extended inequality which should hold for all fields. This can be done if one considers the field equations as constraints on solutions of the energy inequality.

Further, I-Shih Liu [7], analysed the thermodynamic theory of viscoelastic bodies and proved that for isotropic viscoelastic materials the results are identical to the classical results given by equation (2). In the same paper, he also proved that the body is hyperelastic because of the Lagrange multiplier $\Lambda^\varepsilon = \Lambda(\theta)$.

However, for anisotropic elastic materials in general, the validity of the classical entropy flux relation is yet to be demonstrated.

The first contribution in this direction has been given by I-Shih Liu [8][9], who proved by considering transversely isotropic elastic bodies and transversely isotropic rigid heat conductors that the classical entropy flux relation (2) need not be valid in general.

In this paper we investigate the functional dependence of the *Lagrange multiplier* $\Lambda^\varepsilon = \Lambda(\theta)$, *irrespective of whether the classical entropy flux relation is valid*. This enable us to derive the entropy flux - heat flux relations for a number of crystal classes including *transverse isotropy, orthotropy, triclinic systems, monoclinic systems and rhombic systems*.

The paper is organized as follows: In Section 2, the basic ideas and formulas typically used in this field are given as a starting point of our investigation. In Section 3 the entropy flux relation for viscoelastic bodies and transverse isotropic elastic materials is reconsidered assuming the *Lagrange multiplier dependency on temperature,* i.e. $\Lambda^\varepsilon = \Lambda(\theta)$. In Section 4 the procedure introduced in Section 3 is extended for the derivation of the entropy flux relation of anisotropic elastic materials defined by the crystal classes listed above. In Section 5 the entropy inequality for anisotropic bodies is examined further. Conclusions related to the outcome of this work are given in Section 6.

2 The entropy principle

In this section the basic framework of the entropy principle for viscoelastic materials is presented. The balance laws of mass, linear momentum and energy can be stated in current configuration as

$$\begin{aligned} &\dot{\rho} + \rho \operatorname{div}\dot{\mathbf{x}} = 0, \\ &\rho\ddot{\mathbf{x}} - \operatorname{div}\mathbf{T} = \rho\mathbf{b}, \\ &\rho\dot{\varepsilon} + \operatorname{div}\mathbf{q} - \mathbf{T}\cdot\operatorname{grad}\dot{x} = \rho r \end{aligned} \tag{4}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{b}$ is external body force and r is external heat supply.

Note that for **solid** bodies it is more convenient to use a referential description. Also, since constitutive relations do not depend on external supplies, it suffices to consider only supply-free bodies. Consequently, the balance laws can be rewritten as

$$\begin{aligned} &\rho = J^{-1}\rho_\kappa, \\ &\rho_\kappa\ddot{\mathbf{x}} - \operatorname{div}\mathbf{T}_\kappa = \mathbf{0}, \\ &\rho_\kappa\dot{\varepsilon} + \operatorname{Div}\mathbf{q}_\kappa - \mathbf{T}_\kappa\cdot\dot{\mathbf{F}} = 0, \end{aligned} \tag{5}$$

and the entropy inequality

$$\rho_\kappa\dot{\eta} + \operatorname{Div}\mathbf{\Phi}_\kappa \geq 0.$$

Here the first Piola-Kirchhoff stress tensor $\mathbf{T}_\kappa$, the material heat flux vector $\mathbf{q}_\kappa$ and the material entropy flux vector $\mathbf{\Phi}_\kappa$ are related to the Cauchy stress tensor $\mathbf{T}$, the heat flux vector $\mathbf{q}$ and the entropy flux vector $\mathbf{\Phi}$ by

$$\mathbf{T}_\kappa = J\mathbf{T}\mathbf{F}^{-T}, \quad \mathbf{q}_\kappa = J\mathbf{F}^{-1}\mathbf{q}, \quad \mathbf{\Phi}_\kappa = J\mathbf{F}^{-1}\mathbf{\Phi}, \tag{6}$$

where $\mathbf{F}$ is the deformation gradient in referential coordinates and $J = |\det\mathbf{F}|$. "Div" is the divergence operator with respect to the referential coordinates.

It is well known that the entropy principle imposes severe restrictions on constitutive functions and the **exploitation of such restrictions based on the Clausius Duhem inequality are relatively easy**. For elastic materials, in general, the thermodynamic restrictions can be easily obtained by the well-known Coleman-Noll procedure [1].

The derivation of the relation between the entropy flux and the heat flux based on the entropy principle, referred to as *entropy flux relation*, is a typical problem in this new theory.

Here we outline the consideration for isotropic viscoelastic materials with isotropic elasticity as a special case. Using the principle of equipresence, the constitutive relations for viscoelastic materials can be written as functions of the state variables

$$\left(\mathbf{F}, \dot{\mathbf{F}}, \theta, \mathbf{g}_\kappa\right), \tag{7}$$

i.e.

$$
\begin{aligned}
\mathbf{T}_\kappa &= \hat{\mathbf{T}}_\kappa\left(\mathbf{F}, \dot{\mathbf{F}}, \theta, \mathbf{g}_\kappa\right), \\
\mathbf{q}_\kappa &= \hat{\mathbf{q}}_\kappa\left(\mathbf{F}, \dot{\mathbf{F}}, \theta, \mathbf{g}_\kappa\right), \\
\varepsilon &= \hat{\varepsilon}\left(\mathbf{F}, \dot{\mathbf{F}}, \theta, \mathbf{g}_\kappa\right), \\
\eta &= \hat{\eta}\left(\mathbf{F}, \dot{\mathbf{F}}, \theta, \mathbf{g}_\kappa\right), \\
\mathbf{\Phi}_\kappa &= \hat{\mathbf{\Phi}}_\kappa\left(\mathbf{F}, \dot{\mathbf{F}}, \theta, \mathbf{g}_\kappa\right),
\end{aligned}
\tag{8}
$$

where $\dot{\mathbf{F}}$ is time derivative of the deformation gradient, ε is the specific internal energy, $\mathbf{g}_\kappa = \nabla\theta$ is temperature gradient, θ is an *empirical* temperature, which is some convenient measure of the hotness (or coldness) of the thermodynamic state. Note that the density field $\rho\left(\mathbf{X}, t\right)$ is completely determined by the motion $\mathbf{x}\left(\mathbf{X}, t\right)$ and the density $\rho_\kappa\left(\mathbf{X}\right)$ in the reference configuration. Therefore, the thermodynamic process is defined as the solution

$$
\left\{\mathbf{x}\left(\mathbf{X}, t\right),\ \theta\left(\mathbf{X}, t\right)\right\}
\tag{9}
$$

of the field equations (the balance laws of the linear momentum and energy) and integration of the constitutive relations for $\mathbf{T}_\kappa$, $\mathbf{q}_\kappa$, and ε.

The determination of the restrictions imposed on the constitutive functions by the entropy principle is one of the major objectives in modern continuum thermodynamics.

2.1 Method of Lagrange multipliers

According to the entropy principle, there exist Lagrange multipliers Λ^v and Λ^ε which depend on the state variables, such that the inequality

$$
\rho_\kappa\dot{\eta} + \mathrm{Div}\mathbf{\Phi}_\kappa - \Lambda^v \cdot \left(\rho_\kappa\ddot{\mathbf{x}} - D\mathrm{iv}\mathbf{T}_\kappa\right) - \Lambda^\varepsilon\left(\rho_\kappa\dot{\varepsilon} + \mathrm{Div}\mathbf{q}_\kappa - \mathbf{T}_\kappa \cdot \dot{\mathbf{F}}\right) \geq 0
\tag{10}
$$

is valid under no additional constraints, i.e. valid for any field $\mathbf{x}\left(\mathbf{X}, t\right)$, $\theta\left(\mathbf{X}, t\right)$.

Further, we invoke the condition of **material objectivity**, which implies the following reduced constitutive equations for viscoelastic materials

$$
\begin{aligned}
\mathbf{T}_\kappa &= \hat{\mathbf{T}}_\kappa\left(\mathbf{C}, \dot{\mathbf{C}}, \theta, \mathbf{g}_\kappa\right), \\
\mathbf{q}_\kappa &= \hat{\mathbf{q}}_\kappa\left(\mathbf{C}, \dot{\mathbf{C}}, \theta, \mathbf{g}_\kappa\right), \\
\varepsilon &= \hat{\varepsilon}\left(\mathbf{C}, \dot{\mathbf{C}}, \theta, \mathbf{g}_\kappa\right), \\
\eta &= \hat{\eta}\left(\mathbf{C}, \dot{\mathbf{C}}, \theta, \mathbf{g}_\kappa\right), \\
\mathbf{\Phi}_\kappa &= \hat{\mathbf{\Phi}}_\kappa\left(\mathbf{C}, \dot{\mathbf{C}}, \theta, \mathbf{g}_\kappa\right),
\end{aligned}
\tag{11}
$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the right Cauchy-Green tensor.

Since the inequality (10) must hold for any $\mathbf{x}\left(\mathbf{X}, t\right)$ and $\theta\left(\mathbf{X}, t\right)$, the values of $\left\{\theta, \mathbf{g}_\kappa, \mathbf{C}, \dot{\mathbf{C}}\right\}$ and $\left\{\dot{\theta}, \ddot{\mathbf{x}}, \dot{\mathbf{g}}_\kappa, \ddot{\mathbf{C}}, \nabla\mathbf{g}_\kappa, \nabla\mathbf{C}, \nabla\dot{\mathbf{C}}\right\}$ in (10) can have arbitrarily values at any point and any instant.

First, note that (10) is linear with respect to $\ddot{\mathbf{x}}$. Consequently, $\rho_\kappa \Lambda^v$ the coefficient of $\ddot{\mathbf{x}}$ must be equal to zero, i.e.

$$\Lambda^v = 0. \tag{12}$$

Thus, (10) becomes

$$\rho_\kappa \dot{\eta} + \mathrm{Div}\boldsymbol{\Phi}_\kappa - \Lambda^\varepsilon \rho_\kappa \dot{\varepsilon} + \Lambda^\varepsilon \mathrm{Div}\mathbf{q}_\kappa - \Lambda^\varepsilon \mathbf{T}_\kappa \cdot \dot{\mathbf{F}} \geq 0 \tag{13}$$

Next, we consider terms in

$$\begin{aligned} \dot{\eta} - \Lambda^\varepsilon \dot{\varepsilon} = &\left(\frac{\partial \eta}{\partial \theta} - \Lambda^\varepsilon \frac{\partial \varepsilon}{\partial \theta} \right) \dot{\theta} + \left(\frac{\partial \eta}{\partial \mathbf{g}_\kappa} - \Lambda^\varepsilon \frac{\partial \varepsilon}{\partial \mathbf{g}_\kappa} \right) \cdot \dot{\mathbf{g}}_\kappa + \\ &+ \left(\frac{\partial \eta}{\partial \mathbf{C}} - \Lambda^\varepsilon \frac{\partial \varepsilon}{\partial \mathbf{C}} \right) \cdot \dot{\mathbf{C}} + \left(\frac{\partial \eta}{\partial \dot{\mathbf{C}}} - \Lambda^\varepsilon \frac{\partial \varepsilon}{\partial \dot{\mathbf{C}}} \right) \cdot \ddot{\mathbf{C}} \end{aligned} \tag{14}$$

$$\begin{aligned} \mathrm{Div}\boldsymbol{\Phi}_\kappa - \Lambda^\varepsilon \mathrm{Div}\mathbf{q}_\kappa = &\left(\frac{\partial \boldsymbol{\Phi}_\kappa}{\partial \theta} - \Lambda^\varepsilon \frac{\partial \mathbf{q}_\kappa}{\partial \theta} \right) \cdot \mathbf{g}_\kappa + \left(\frac{\partial \boldsymbol{\Phi}_\kappa}{\partial \mathbf{g}_\kappa} - \Lambda^\varepsilon \frac{\partial \mathbf{q}_\kappa}{\partial \mathbf{g}_\kappa} \right) \cdot \nabla \mathbf{g}_\kappa + \\ &+ \left(\frac{\partial \boldsymbol{\Phi}_\kappa}{\partial \mathbf{C}} - \Lambda^\varepsilon \frac{\partial \mathbf{q}_\kappa}{\partial \mathbf{C}} \right) \cdot \nabla \mathbf{C} + \left(\frac{\partial \boldsymbol{\Phi}_\kappa}{\partial \dot{\mathbf{C}}} - \Lambda^\varepsilon \frac{\partial \mathbf{q}_\kappa}{\partial \dot{\mathbf{C}}} \right) \cdot \nabla \dot{\mathbf{C}} \end{aligned} \tag{15}$$

Note that the term

$$\left(\frac{\partial \boldsymbol{\Phi}_\kappa}{\partial \mathbf{g}_\kappa} - \Lambda^\varepsilon \frac{\partial \mathbf{q}_\kappa}{\partial \mathbf{g}_\kappa} \right) \cdot \nabla \mathbf{g}_\kappa$$

in component form reads as

$$\left(\frac{\partial \boldsymbol{\Phi}_\kappa}{\partial \mathbf{g}_\kappa} - \Lambda^\varepsilon \frac{\partial \mathbf{q}_\kappa}{\partial \mathbf{g}_\kappa} \right) \cdot \nabla \mathbf{g}_\kappa = \left(\frac{\partial \Phi_{\kappa K}}{\partial \theta_{,L}} - \Lambda^\varepsilon \frac{\partial q_\kappa}{\partial \theta_{,L}} \right) \theta_{,LK}.$$

The other terms in the equation (15) have equivalent component forms.

After substituting (14) and (15) into (13), by inspection, we conclude that this inequality is also linear with respect to the following derivatives $\left\{ \dot{\theta}, \ddot{\mathbf{x}}, \dot{\mathbf{g}}_\kappa, \ddot{\mathbf{C}}, \nabla \mathbf{g}_\kappa, \nabla \mathbf{C}, \nabla \dot{\mathbf{C}} \right\}$.

As the inequality must hold for arbitrary fields we have eliminated the constraints imposed by the field equations. The coefficients of the above derivatives must vanish identically. Otherwise, we could choose the fields in such a way that one negative term would dominate all others and the inequality would be violated. Hence, we obtain the following equations

$$\begin{aligned} &\Lambda^v = 0 \\ &\frac{\partial \eta}{\partial \theta} - \Lambda^\varepsilon \frac{\partial \varepsilon}{\partial \theta} = 0 \\ &\frac{\partial \eta}{\partial \mathbf{g}_\kappa} - \Lambda^\varepsilon \frac{\partial \varepsilon}{\partial \mathbf{g}_\kappa} = 0 \\ &\frac{\partial \eta}{\partial \dot{\mathbf{C}}} - \Lambda^\varepsilon \frac{\partial \varepsilon}{\partial \dot{\mathbf{C}}} = 0 \end{aligned} \tag{16}$$

$$\left(\frac{\partial \mathbf{\Phi}_{\kappa}}{\partial \mathbf{g}_{\kappa}} - \Lambda^{\varepsilon} \frac{\partial \mathbf{q}_{\kappa}}{\partial \mathbf{g}_{\kappa}}\right)_{sym} = 0$$

$$\frac{\partial \mathbf{\Phi}_{\kappa}}{\partial \mathbf{C}} - \Lambda^{\varepsilon} \frac{\partial \mathbf{q}_{\kappa}}{\partial \mathbf{C}} = 0 \tag{17}$$

$$\frac{\partial \mathbf{\Phi}_{\kappa}}{\partial \dot{\mathbf{C}}} - \Lambda^{\varepsilon} \frac{\partial \mathbf{q}_{\kappa}}{\partial \dot{\mathbf{C}}} = 0$$

Then the entropy inequality (13) reduces to

$$\left(\frac{\partial \mathbf{\Phi}_{\kappa}}{\partial \theta} - \Lambda^{\varepsilon} \frac{\partial \mathbf{q}_{\kappa}}{\partial \theta}\right) \cdot \mathbf{g}_{\kappa} + \rho_{\kappa} \left(\frac{\partial \eta}{\partial \mathbf{C}} - \Lambda^{\varepsilon} \frac{\partial \varepsilon}{\partial \mathbf{C}}\right) \cdot \dot{\mathbf{C}} + \Lambda^{\varepsilon} \mathbf{T}_{\kappa} \cdot \dot{\mathbf{F}} \geq 0 \tag{18}$$

Making use of second Piola-Kirchhoff tensor $\mathbf{S}_{\boldsymbol{\kappa}} = \mathbf{F}^{-1}\mathbf{T}_{\boldsymbol{\kappa}} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$ and the right Cauchy-Green tensor $\dot{\mathbf{C}} = \dot{\mathbf{F}}^{T}\mathbf{F} + \mathbf{F}^{T}\dot{\mathbf{F}} = 2\left(\dot{\mathbf{F}}^{T}\mathbf{F}\right)_{sym}$ (which are symmetric tensors), the inequality (18) can be written in a more compact form as

$$\sigma = \left(\frac{\partial \mathbf{\Phi}_{\kappa}}{\partial \theta} - \Lambda^{\varepsilon} \frac{\partial \mathbf{q}_{\kappa}}{\partial \theta}\right) \cdot \mathbf{g}_{\kappa} + \rho_{\kappa} \left(\frac{\partial \eta}{\partial \mathbf{C}} - \Lambda^{\varepsilon} \frac{\partial \varepsilon}{\partial \mathbf{C}} + \frac{1}{2\rho_{\kappa}} \Lambda^{\varepsilon} \mathbf{S}_{\kappa}\right) \cdot \dot{\mathbf{C}} \geq 0 \tag{19}$$

Where σ is entropy production density. Moreover, from (16) we obtain

$$\frac{d\eta}{dt} - \Lambda^{\varepsilon} \frac{d\varepsilon}{dt} = \left(\frac{\partial \eta}{\partial \mathbf{C}} - \Lambda^{\varepsilon} \frac{\partial \varepsilon}{\partial \mathbf{C}}\right) \cdot \frac{d\mathbf{C}}{dt}$$

$$\mathrm{d}\eta = \Lambda^{\varepsilon}\, \mathrm{d}\varepsilon + \left(\frac{\partial \eta}{\partial \mathbf{C}} - \Lambda^{\varepsilon} \frac{\partial \varepsilon}{\partial \mathbf{C}}\right) \cdot \mathrm{d}\mathbf{C} \tag{20}$$

which has the form of **the thermostatic Gibbs relation**.

3 Entropy flux relation for viscoelastic materials

For further evaluation of the consequences of the entropy principle, particularly in connection with relations (17), we invoke **the material symmetry condition** that has to be satisfied by $\{\varepsilon, \mathbf{T}_{\kappa}, \mathbf{q}_{\kappa}, \mathbf{\Phi}_{\kappa}, \eta\}$ for **isotropic viscoelastic bodies**. For instance, this condition for heat flux can be expressed as

$$\hat{\mathbf{q}}_{\kappa}\left(\mathbf{QCQ}^{T}, \mathbf{Q}\dot{\mathbf{C}}\mathbf{Q}^{T}, \theta, \mathbf{Qg}_{\kappa}\right) = \mathbf{Q}\hat{\mathbf{q}}_{\kappa}\left(\mathbf{C}, \dot{\mathbf{C}}, \theta, \mathbf{g}_{\kappa}\right), \qquad \forall \mathbf{Q} \in O \tag{21}$$

where O is the full orthogonal group. Note that $\hat{\mathbf{q}}_{\kappa}$ is an isotropic vector-valued function of $\{\mathbf{C}, \dot{\mathbf{C}}, \theta, \mathbf{g}_{\kappa}\}$ and that (21) imposes restriction on its form. After a lengthy calculation starting from (15), I-Shih Liu [7] proved that the following entropy flux relation holds

$$\mathbf{\Phi}_{\kappa} = \Lambda^{\varepsilon}\left(\mathbf{C}, \dot{\mathbf{C}}, \theta, \mathbf{g}_{\kappa}\right)\mathbf{q}_{\kappa} \tag{22}$$

Further, based on (22) and (17) I-Shih Liu concluded that Λ^{ε} must be independent of $\mathbf{C}$, $\dot{\mathbf{C}}$ and $\mathbf{g}_{\kappa}$. Thus

$$\Lambda^{\varepsilon} = \Lambda^{\varepsilon}(\theta). \tag{23}$$

Accordingly (20) becomes

$$d\eta = \Lambda^{\varepsilon}\, d\varepsilon + \frac{\partial}{\partial \mathbf{C}}\left(\eta - \Lambda^{\varepsilon}\varepsilon\right)\cdot \mathrm{d}\mathbf{C}$$

or

$$\mathrm{d}\eta = \Lambda^{\varepsilon}\left(\mathrm{d}\varepsilon - \frac{\partial \psi}{\partial \mathbf{C}}\cdot \mathrm{d}\mathbf{C}\right)$$

where

$$\psi = \varepsilon - \eta / \Lambda^{\varepsilon} \tag{24}$$

By comparison with the classical Gibbs relation in thermostatics, the function Λ^{ε} can be identified as the reciprocal of the absolute temperature θ, i.e.

$$\Lambda^{\varepsilon} = \frac{1}{\theta} \tag{25}$$

Which leads to the classical entropy flux relation (2).

Consequently, one can come to the conclusion that, following I-Shih Liu's procedure for viscoelastic bodies outlined above, that for **isotropic elastic materials** with state variables $\left(\mathbf{F}, \theta, \mathbf{g}_{\kappa}\right)$ the relation (25) holds.

3.1 Entropy flux of anisotropic elastic materials

In this subsection the derivation of the relation between the entropy flux and the heat flux for anisotropic materials with state variables $\left(\mathbf{F},\theta,\mathbf{g}_{\kappa}\right)$ is considered. In this case the requirements (16) and (17) reduce to (26) and (27) respectively

$$\begin{aligned} &\frac{\partial \eta}{\partial \theta} - \Lambda^{\varepsilon}\frac{\partial \varepsilon}{\partial \theta} = 0, \\ &\frac{\partial \eta}{\partial \mathbf{g}_{\kappa}} - \Lambda^{\varepsilon}\frac{\partial \varepsilon}{\partial \mathbf{g}_{\kappa}} = 0, \end{aligned} \tag{26}$$

$$\begin{aligned} &\left(\frac{\partial \mathbf{\Phi}_{\kappa}}{\partial \mathbf{g}_{\kappa}} - \Lambda^{\varepsilon}\frac{\partial \mathbf{q}_{\kappa}}{\partial \mathbf{g}_{\kappa}}\right)_{sym} = 0, \\ &\frac{\partial \mathbf{\Phi}_{\kappa}}{\partial \mathbf{C}} - \Lambda^{\varepsilon}\frac{\partial \mathbf{q}_{\kappa}}{\partial \mathbf{C}} = 0. \end{aligned} \tag{27}$$

In addition, **the material symmetry condition for anisotropic elastic bodies** has to be satisfied by $\left\{\varepsilon, \mathbf{T}_{\kappa}, \mathbf{q}_{\kappa}, \mathbf{\Phi}_{\kappa}, \eta\right\}$.

The first paper considering entropy flux for transversely isotropic elastic bodies, was published by I-Shih Liu in 2009 [8], which relies on his paper [11]. In this, anisotropic materials properties in preferential directions were characterized by a number of unit vectors $\mathbf{m}_1,\ldots,\mathbf{m}_a$ and tensors $\mathbf{M}_1,\ldots,\mathbf{M}_b$. Ig g is a group of transformations which preserve these characteristics, i.e.

$$g = \left\{\mathbf{Q} \in G;\quad \mathbf{Q}\mathfrak{m} = \mathfrak{m},\quad \mathbf{Q}\mathfrak{M}\mathbf{Q}^{T} = \mathfrak{M}\right\},$$

where G is a subgroup of O the full orthogonal group, $\mathfrak{m} = \left(\mathbf{m}_1,\ldots,\mathbf{m}_a\right)$ and $\mathfrak{M} = \left(\mathbf{M}_1,\ldots,\mathbf{M}_b\right)$.

In other words g is characterized by the set $(\mathfrak{m},\mathfrak{M})$ and the group $G\in O$, i.e.

$$g=(G;\mathfrak{m},\mathfrak{M}). \tag{28}$$

Theorem 1 *A function* $f(\mathbf{v},\mathbf{A})$ *is invariant to* g *if and only if it can be represented by*

$$f(\mathbf{v},\mathbf{A})=\hat{f}(\mathbf{v},\mathbf{A},\mathfrak{m},\mathfrak{M}), \tag{29}$$

where $\hat{f}(\mathbf{v},\mathbf{A},\mathfrak{m},\mathfrak{M})$ *is invariant relative to* G.

Here **v** is a vector, **A** is a second order tensor and f is either scalar-valued, vector-valued or tensor valued function. Particularly, if $\hat{f}(\mathbf{v},\mathbf{A},\mathfrak{m},\mathfrak{M})$ is an isotropic function then:

- for a scalar-valued function
$$\hat{f}(\mathbf{Qv},\mathbf{QAQ}^T,\mathbf{Q}\mathfrak{m},\mathbf{Q}\mathfrak{M}\mathbf{Q}^T)=\hat{f}(\mathbf{v},\mathbf{A},\mathfrak{m},\mathfrak{M}),$$

- for a vector-valued function
$$\mathbf{Q}\hat{f}(\mathbf{v},\mathbf{A},\mathfrak{m},\mathfrak{M})=\hat{f}(\mathbf{Qv},\mathbf{QAQ}^T,\mathbf{Q}\mathfrak{m},\mathbf{Q}\mathfrak{M}\mathbf{Q}^T),$$

- and for a tensor-valued function
$$\mathbf{Q}\hat{f}(\mathbf{v},\mathbf{A},\mathfrak{m},\mathfrak{M})\mathbf{Q}^T=\hat{f}(\mathbf{Qv},\mathbf{QAQ}^T,\mathbf{Q}\mathfrak{m},\mathbf{Q}\mathfrak{M}\mathbf{Q}^T).$$

In the same paper I-Shih Liu gives a list of 14 such groups g for some crystal classes. He uses the following notation:

$\mathbf{n}_i$, $i=1,2,3$ where $\mathbf{n}_i$ are orthonormal vectors, i.e.

$$\mathbf{n}_i\cdot\mathbf{n}_j=\delta_{ij}. \tag{30}$$

$\mathbf{N}_i$ are skew-symmetric tensors defined by

$$\mathbf{N}_i=e_{ijk}\mathbf{n}_j\otimes\mathbf{n}_k,\quad i,j,k=1,2,3. \tag{31}$$

In order to write the exact form of the constitutive functions used in our further investigation we need several functional relations particularly among $\mathbf{n}_i$ and $\mathbf{N}_i$. These basic relations are given in the Appendix.

I-Shih Liu [8] considered only **two different classes of transversally isotropic bodies**

$$g_2=\left(O;\ \mathbf{Qn}_1=\mathbf{n}_1\right),$$
$$g_5=\left(O;\ \mathbf{Qn}_1\otimes\mathbf{n}_1\mathbf{Q}^T=\mathbf{n}_1\otimes\mathbf{n}_1\right),$$

where $\mathbf{n}_1$ is the preferred direction of transverse isotropy.

In applying the isotropic representation of constitutive functions, instead of the Cauchy-Green strain tensor **C** he used the Green-St. Venant strain tensor **E**, i.e.

$$\mathbf{E}=\frac{1}{2}(\mathbf{C}-\mathbf{I})$$

which vanishes when there is no deformation, and considers constitutive functions $\mathbf{q}_\kappa$ and $\mathbf{\Phi}_\kappa$ of $\left(\mathbf{E},\theta,\mathbf{g}_\kappa\right)$ up to bilinear terms in **E** and $\mathbf{g}_\kappa$, i.e.

$$
\begin{aligned}
\mathbf{q}_\kappa = \; & \left(a_1 + a_2\,\mathrm{tr}\mathbf{E} + a_3\mathbf{n}\cdot\mathbf{En}\right)\mathbf{g}_\kappa + a_4\mathbf{E}\mathbf{g}_\kappa \\
& + \left(b_1 + b_2\,\mathrm{tr}\mathbf{E} + b_3\mathbf{n}\cdot\mathbf{En}\right)\left(\mathbf{n}\otimes\mathbf{n}\right)\mathbf{g}_\kappa + b_4\left(\mathbf{n}\otimes\mathbf{En}\right)\mathbf{g}_\kappa + b_5\left(\mathbf{En}\otimes\mathbf{n}\right)\mathbf{g}_\kappa + \\
& + \left(c_1 + c_2\,\mathrm{tr}\mathbf{E} + c_3\mathbf{n}\cdot\mathbf{En}\right)\mathbf{n} + c_4\mathbf{En}, \\
\mathbf{\Phi}_\kappa = \; & \left(\alpha_1 + \alpha_2\,\mathrm{tr}\mathbf{E} + \alpha_3\mathbf{n}\cdot\mathbf{En}\right)\mathbf{g}_\kappa + \alpha_4\mathbf{E}\mathbf{g}_\kappa \\
& + \left(\beta_1 + \beta_2\,\mathrm{tr}\mathbf{E} + \beta_3\mathbf{n}\cdot\mathbf{En}\right)\left(\mathbf{n}\otimes\mathbf{n}\right)\mathbf{g}_\kappa + \beta_4\left(\mathbf{n}\otimes\mathbf{En}\right)\mathbf{g}_\kappa + \beta_5\left(\mathbf{En}\otimes\mathbf{n}\right)\mathbf{g}_\kappa + \\
& + \left(\gamma_1 + \gamma_2\,\mathrm{tr}\mathbf{E} + \gamma_3\mathbf{n}\cdot\mathbf{En}\right)\mathbf{n} + \gamma_4\mathbf{En},
\end{aligned}
$$

where all the material coefficients are functions of the temperature θ only.

For the class of transversally isotropic bodies defined by

$$g_2 = \left(O;\ \mathbf{Q}\mathbf{n}_1 = \mathbf{n}_1\right),$$

he was able to prove that Λ^ε is a function of the temperature only, i.e.

$$\Lambda^\varepsilon = \Lambda^\varepsilon(\theta) = \frac{1}{\theta}.$$

Further, he obtained the entropy flux and the heat flux relation as

$$\mathbf{\Phi}_\kappa = \frac{1}{\theta}\mathbf{q}_\kappa + k(\theta)\mathbf{n}_1. \tag{32}$$

Therefore, for this class of transversally isotropic bodies the classical result does not hold in general.

For the class of transversally isotropic bodies defined by

$$g_5 = \left(O;\ \mathbf{Q}\mathbf{n}_1\otimes\mathbf{n}_1\mathbf{Q}^T = \mathbf{n}_1\otimes\mathbf{n}_1\right),$$

he obtained

$$\Lambda^\varepsilon = \frac{1}{\theta}, \qquad \mathbf{\Phi}_\kappa = \frac{1}{\theta}\mathbf{q}_\kappa, \tag{33}$$

Which is identical to the classical result (2).

For these cases a functional form of Λ^ε had to be found first to determine the relation between entropy flux and heat flux. It appears that $\Lambda^\varepsilon = \Lambda^\varepsilon(\theta)$ holds in all these cases, i.e. $\Lambda^\varepsilon = \Lambda^\varepsilon(\theta)$ is a necessary condition for the determination of a relation between the entropy flux and the heat flux.

This is where we pose the question whether this is sufficient condition? Having this in mind we proceed first by re-examining the above cases.

This assumption differs substantially from Green & Laws [15], as well as from Hutter [16] and Bargmann & Steinmann [17]. In Green & Laws [15] the entropy flux and heat flux relationship is defined as $\mathbf{\Phi}_\kappa = \frac{1}{\varphi}\mathbf{q}_\kappa$, where φ is a constitutive function which reduces to the absolute temperature θ in equilibrium. Hutter [16] postulated the classical entropy flux heat flux relation. In their contribution Bargmann & Steinmann [17] adopted the Green & Naghdi approach for non-classical theory of thermos-elasticity for isotropic materials and to obtain the entropy flux-heat flux relation.

4 The consequence of the assumption that Lagrange multiplier is function of temperature only $\Lambda^{\varepsilon} = \Lambda^{\varepsilon}(\theta)$

Eringen stated [19] that "it is always possible to express the entropy change as a sum of entropy flux and entropy source as"

$$\mathbf{\Phi} = \frac{1}{\theta}\mathbf{q} + \mathbf{\Phi}_1 \tag{34}$$

where $\mathbf{\Phi}_1$ is the entropy change due to all other effects except heat input.

In our consideration we do not use Eringen's postulate and do not make any assumption about the entropy flux relation. The only assumption used in our derivation is that Lagrange multiplier Λ^{ε} is a function of temperature θ only, i.e. $\Lambda^{\varepsilon} = \Lambda^{\varepsilon}(\theta)$. The implication of this assumption for isotropic viscoelastic bodies is considered first. In this, the starting point is the set of equations (17) which are restated below

$$\begin{aligned}
&\left(\frac{\partial \mathbf{\Phi}_{\kappa}}{\partial \mathbf{g}_{\kappa}} - \Lambda^{\varepsilon}\frac{\partial \mathbf{q}_{\kappa}}{\partial \mathbf{g}_{\kappa}}\right)_{sym} = 0\\
&\frac{\partial \mathbf{\Phi}_{\kappa}}{\partial \mathbf{C}} - \Lambda^{\varepsilon}\frac{\partial \mathbf{q}_{\kappa}}{\partial \mathbf{C}} = 0\\
&\frac{\partial \mathbf{\Phi}_{\kappa}}{\partial \dot{\mathbf{C}}} - \Lambda^{\varepsilon}\frac{\partial \mathbf{q}_{\kappa}}{\partial \dot{\mathbf{C}}} = 0
\end{aligned} \tag{35}$$

Let introduce a new variable $\mathbf{k} = \mathbf{\Phi}_{\kappa} - \Lambda^{\varepsilon}\mathbf{q}_{\kappa}$. Recall that $\mathbf{\Phi}_{\kappa} = \hat{\mathbf{\Phi}}_{\kappa}\left(\mathbf{C},\dot{\mathbf{C}},\theta,\mathbf{g}_{\kappa}\right)$ and $\mathbf{q}_{\kappa} = \hat{\mathbf{q}}_{\kappa}\left(\mathbf{C},\dot{\mathbf{C}},\theta,\mathbf{g}_{\kappa}\right)$, consequently $\mathbf{k} = \hat{\mathbf{k}}\left(\mathbf{C},\dot{\mathbf{C}},\theta,\mathbf{g}_{\kappa}\right)$. Using $\mathbf{k}$ equations (35) can be written as

$$\begin{aligned}
&\left(\frac{\partial \mathbf{k}_{\kappa}}{\partial \mathbf{g}_{\kappa}}\right)_{sym} = 0\\
&\frac{\partial \mathbf{k}_{\kappa}}{\partial \mathbf{C}} = 0\\
&\frac{\partial \mathbf{k}_{\kappa}}{\partial \dot{\mathbf{C}}} = 0
\end{aligned} \tag{36}$$

Thus $\mathbf{k} = \hat{\mathbf{k}}(\theta, \mathbf{g}_{\kappa})$ and the set of equations (36) reduces to

$$\left(\frac{\partial \mathbf{k}}{\partial \mathbf{g}_{\kappa}}\right)_{sym} = 0 \tag{37}$$

For clarity reasons the following few steps are written in index notation or component notation. Equation (37) when written in component form is $\frac{\partial k_i}{\partial g_j} + \frac{\partial k_j}{\partial g_i} = 0$, where k_i and g_i are components of vectors $\mathbf{k}$ and $\mathbf{g}_{\kappa}$, respectively. Differentiation of (37) with respect to $\mathbf{g}_{\kappa}$ yields

$$\frac{\partial^2 k_p}{\partial g_r \partial g_q} + \frac{\partial^2 k_q}{\partial g_r \partial g_p} = 0$$

The remaining two relations obtained by cyclic index permutation are

$$\frac{\partial^2 k_q}{\partial g_p \partial g_r} + \frac{\partial^2 k_r}{\partial g_p \partial g_q} = 0$$

$$\frac{\partial^2 k_r}{\partial g_q \partial g_p} + \frac{\partial^2 k_p}{\partial g_q \partial g_r} = 0$$

From them we have

$$\frac{\partial^2 k_p}{\partial g_r \partial g_q} = 0$$

The solution of this simple set of differential equations is

$$k_p = A_{pq}(\theta) g_q + a_p(\theta), \qquad A_{(pq)} = 0$$

This solution can be rewritten, using symbolic notation, as

$$\mathbf{k}(\theta, \mathbf{g}_\kappa) = \mathbf{A}\left(\theta\right) \mathbf{g}_\kappa + \mathbf{a}(\theta) \tag{38}$$

where $\mathbf{A}(\theta)$ is skew symmetric. Since we are dealing with isotropic viscoelastic bodies $\mathbf{k}(\theta, \mathbf{g}_\kappa)$ must be a vector-valued isotropic function. Thus

$$\mathbf{Q}\mathbf{k}(\theta, \mathbf{g}_\kappa) = \mathbf{k}(\theta, \mathbf{Q}\mathbf{g}_\kappa) \tag{39}$$

must hold for all $\mathbf{Q} \in O$ and an arbitrary $\mathbf{g}_\kappa$. Equivalently, (39) can be restated as

$$\mathbf{Q}\mathbf{A}\left(\theta\right)\mathbf{g}_\kappa + \mathbf{Q}\mathbf{a}(\theta) = \mathbf{A}\left(\theta\right)\mathbf{Q}\mathbf{g}_\kappa + \mathbf{a}(\theta) \tag{40}$$

Particularly for the case of $\mathbf{Q} = -\mathbf{I}$ it follows that $\mathbf{a}(\theta) = 0$ and consequently (40) reduces to

$$\mathbf{Q}\mathbf{A}\left(\theta\right)\mathbf{g}_\kappa = \mathbf{A}\left(\theta\right)\mathbf{Q}\mathbf{g}_\kappa \tag{41}$$

Which can be rewritten as

$$\mathbf{Q}\mathbf{A}(\theta)\mathbf{Q}^T\mathbf{Q}\mathbf{g}_\kappa = \mathbf{A}(\theta)\mathbf{Q}\mathbf{g}_\kappa \tag{42}$$

Since (42) must hold for all $\mathbf{Q} \in O$ and an arbitrary $\mathbf{g}_\kappa$ we have that

$$\mathbf{Q}\mathbf{A}(\theta)\mathbf{Q}^T = \mathbf{A}(\theta) \tag{43}$$

since $\mathbf{A}(\theta)$ is skew symmetric, only $\mathbf{A}(\theta) = 0$ satisfies (43), and

$$\mathbf{k} = \mathbf{\Phi}_\kappa - \Lambda^\varepsilon \mathbf{q}_\kappa = \mathbf{0} \tag{44}$$

This result is identical to I-Shih Liu's [7] for isotropic viscoelastic bodies, which validates the new procedure and led to its application to the anisotropic materials considered below. It is important to observe that the assumption $\Lambda^\varepsilon = \Lambda^\varepsilon(\theta)$ significantly simplifies and shortens the procedure.

4.1 Entropy flux relation for anisotropic elastic materials in general

This section considers **anisotropic bodies** characterised with

$$g = \left\{ \mathbf{Q} \in O; \quad \mathbf{Q}\mathfrak{m} = \mathfrak{m}, \quad \mathbf{Q}\mathfrak{M}\mathbf{Q}^T = \mathfrak{M} \right\}. \tag{45}$$

In other words g comprises the set $(\mathfrak{m},\mathfrak{M})$ and the group O, i.e. $g = (O;\ \mathfrak{m},\ \mathfrak{M})$. Consequently

$$\mathbf{k} = \hat{\mathbf{k}}(\theta,\mathbf{g}_\kappa,\mathfrak{m},\mathfrak{M}) = \mathbf{A}(\theta,\mathfrak{m},\mathfrak{M})\mathbf{g}_\kappa + \mathbf{a}(\theta,\mathfrak{m},\mathfrak{M}) \tag{46}$$

is isotropic vector-valued function, i.e.

$$\mathbf{Q}\mathbf{k}(\theta,\mathbf{g}_\kappa,\mathfrak{m},\mathfrak{M}) = \mathbf{k}(\theta,\mathbf{Q}\mathbf{g}_\kappa,\mathbf{Q}\mathfrak{m},\mathbf{Q}\mathfrak{M}\mathbf{Q}^T), \tag{47}$$

or

$$\mathbf{Q}\mathbf{A}(\theta,\mathfrak{m},\mathfrak{M})\mathbf{g}_\kappa + \mathbf{Q}\mathbf{a}(\theta,\mathfrak{m},\mathfrak{M}) = \mathbf{A}(\theta,\mathbf{Q}\mathfrak{m},\mathbf{Q}\mathfrak{M}\mathbf{Q}^T)\mathbf{Q}\mathbf{g}_\kappa + \mathbf{a}(\theta,\mathbf{Q}\mathfrak{m},\mathbf{Q}\mathfrak{M}\mathbf{Q}^T) \tag{48}$$

which must hold for all $\mathbf{Q} \in O$ and arbitrary $\mathbf{g}_\kappa$. Particularly for $\mathbf{g}_\kappa = 0$, we have

$$\mathbf{Q}\mathbf{a}(\theta,\mathfrak{m},\mathfrak{M}) = \mathbf{a}(\theta,\mathbf{Q}\mathfrak{m},\mathbf{Q}\mathfrak{M}\mathbf{Q}^T), \tag{49}$$

i.e. $\mathbf{a}(\theta,\mathfrak{m},\mathfrak{M})$ is vector valued isotropic function of its arguments. Moreover,

$$\mathbf{Q}\mathbf{A}(\theta,\mathfrak{m},\mathfrak{M})\mathbf{g}_\kappa = \mathbf{A}(\theta,\mathbf{Q}\mathfrak{m},\mathbf{Q}\mathfrak{M}\mathbf{Q}^T)\mathbf{Q}\mathbf{g}_\kappa \tag{50}$$

or

$$\mathbf{Q}\mathbf{A}(\theta,\mathfrak{m},\mathfrak{M})\mathbf{Q}^T\mathbf{Q}\mathbf{g}_\kappa = \mathbf{A}(\theta,\mathbf{Q}\mathfrak{m},\mathbf{Q}\mathfrak{M}\mathbf{Q}^T)\mathbf{Q}\mathbf{g}_\kappa.$$

This must hold for all $\mathbf{Q} \in O$, and hence

$$\mathbf{Q}\mathbf{A}\left(\theta,\mathfrak{m},\mathfrak{M}\right)\mathbf{Q}^T = \mathbf{A}\left(\theta,\mathbf{Q}\mathfrak{m},\mathbf{Q}\mathfrak{M}\mathbf{Q}^T\right) \tag{51}$$

i.e. $\mathbf{A}(\theta,\mathfrak{m},\mathfrak{M})$ is skew symmetric tensor-valued isotropic function of its arguments.

For vector-valued and skew-symmetric tensor-valued isotropic functions in $\mathbf{R}^3$ see Smith [12] and Spencer [18]. All groups of crystal classes given by I-Shih Liu [11] are considered below.

4.2 Transversally isotropic material bodies

In this section we consider transversely isotropic material bodies divided into four cases. The cases a) characterised by $g_2 = (O;\mathbf{n}_1)$ and the case b) characterised by $g_5 = (O;\mathbf{n}_1 \otimes \mathbf{n}_1)$ were selected in order to validate the proposed procedure against the results obtained by I-Shih Liu [8] for the same materials. The results for the cases c) characterised by $g_1 = (O;\mathbf{n}_1,\mathbf{N}_1)$ and d) characterised $g_1 = (O;\mathbf{n}_1,\mathbf{N}_1)$ are new and, to the best of our knowledge, not available in published literature.

a) transversally isotropic bodies with group symmetry $g_2 = (O;\mathbf{n}_1)$

In this case

$$\mathbf{k} = \hat{\mathbf{k}}\left(\theta,\mathbf{g}_\kappa,\mathbf{n}_1\right) = \mathbf{A}\left(\theta,\mathbf{n}_1\right)\mathbf{g}_\kappa + \mathbf{a}\left(\theta,\mathbf{n}_1\right) \tag{52}$$

Where the following must hold

$$\mathbf{Q}\mathbf{a}\left(\theta,\mathbf{n}_1\right) = \mathbf{a}\left(\theta,\mathbf{Q}\mathbf{n}_1\right) \tag{53}$$

i.e. $\mathbf{a}(\theta,\mathbf{n}_1)$ is vector-valued function which can be expressed as

$$\mathbf{a}(\theta,\mathbf{n}_1)=\lambda(\theta)\mathbf{n}_1 \tag{54}$$

where $\lambda(\theta)$ is an arbitrary scalar function. Further

$$\mathbf{QA}(\theta,\mathbf{n}_1)\mathbf{Q}^T=\mathbf{A}(\theta,\mathbf{Qn}_1) \tag{55}$$

Thus $\mathbf{A}(\theta,\mathbf{n}_1)$ is skew-symmetric tensor-valued isotropic function. Consequently, $\mathbf{A}=\mathbf{0}$ (see Smith [12]). In this case

$$\mathbf{k}=\mathbf{\Phi}_\kappa-\Lambda^\varepsilon\mathbf{q}_\kappa=\lambda(\theta)\mathbf{n}_1. \tag{56}$$

Moreover,

$$\lambda(\theta)=\left(\mathbf{\Phi}_\kappa-\Lambda^\varepsilon\mathbf{q}_\kappa\right)\Big|_{\mathbf{C}=\mathbf{I},\mathbf{g}_\kappa=\mathbf{0}}\cdot\mathbf{n_1} \tag{57}$$

Where **C=I** implies that there is no deformation and $\mathbf{g}_\kappa=\mathbf{0}$ implies that there is no temperature gradient.
The result (57) agrees with I-Shih Liu's result for **transversally isotropic bodies** with group symmetry group g_2 [8].

b) transversally isotropic bodies with group symmetry $g_5=(O;\mathbf{n}_1\otimes\mathbf{n}_1)$.
In this case

$$\mathbf{k}(\theta,\mathbf{n}_1\otimes\mathbf{n}_1,\mathbf{g}_\kappa)=\mathbf{A}(\theta,\mathbf{n}_1\otimes\mathbf{n}_1)\mathbf{g}_\kappa+\mathbf{a}(\theta,\mathbf{n}_1\otimes\mathbf{n}_1) \tag{58}$$

where $\mathbf{a}(\theta,\mathbf{n}_1\otimes\mathbf{n}_1)$ and $\mathbf{A}(\theta,\mathbf{n}_1\otimes\mathbf{n}_1)$, as isotropic functions, must vanish, resulting in

$$\mathbf{k}=\mathbf{\Phi}_\kappa-\Lambda^\varepsilon\mathbf{q}_\kappa=\mathbf{0}, \tag{59}$$

The result (59) agrees with I-Shih Liu's result for **transversally isotropic bodies** with group symmetry group g_5 [8].

Therefore, for the above cases we demonstrated that $\Lambda^\varepsilon=\Lambda^\varepsilon(\theta)$ is not only a necessary but also a sufficient condition to determine the entropy flux-heat flux relation.

To demonstrate the generality of the proposed procedure we consider the other crystal classes for which representation of anisotropic invariants is given by I-Shih Liu [11]. Note, in all these cases representations of anisotropic invariant function are obtained using the tables for isotropic functions. In addition, we do not see any physical reason that Λ^ε would have a different form to $\Lambda^\varepsilon=\Lambda^\varepsilon(\theta)$ for these crystal classes.

The result for the following cases are new (to the best of our knowledge not available in published literature).

c) transversally isotropic bodies with group symmetry $g_1=(O;\mathbf{n}_1,\mathbf{N}_1)$.
In this case

$$\mathbf{k}=\hat{\mathbf{k}}(\theta,\mathbf{g}_\kappa,\mathbf{n}_1,\mathbf{N}_1)=\mathbf{A}(\theta,\mathbf{n}_1,\mathbf{N}_1)\mathbf{g}_\kappa+\mathbf{a}(\theta,\mathbf{n}_1,\mathbf{N}_1) \tag{60}$$

Here and in what follows, we use particularly the representation formulae ((2.41), (4.2), (4.3), (4.6), (4.7)) given by Smith [12] (see also I-Shih Liu [11], Wilmanski [14]) for vector-valued isotropic function and skew symmetric tensor-valued function of their arguments. We strictly apply these formulae for the vector function $\mathbf{a}\left(\theta,\mathbf{n}_1,\mathbf{N}_1\right)$ and a skew-symmetric tensor-valued function $\mathbf{A}\left(\theta,\mathbf{n}_1,\mathbf{N}_1\right)$. Their scalar invariant functions are always functions of θ. Therefore, we must find the basis invariants of the set $\left(\mathbf{n}_1,\mathbf{N}_1\right)$. They are

$$\begin{aligned}&\mathbf{n}_1\cdot\mathbf{n}_1\\ &\mathrm{tr}\mathbf{N}_1^2\\ &\mathbf{n}_1\cdot\mathbf{N}_1^2\mathbf{n}_1\end{aligned} \tag{61}$$

The generator of the set $\left(\mathbf{n}_1,\mathbf{N}_1\right)$ for $\mathbf{a}\left(\theta,\mathbf{n}_1\otimes\mathbf{n}_1\right)$ is $\mathbf{n}_1$ and consequently $\mathbf{a}=\lambda(\theta)\mathbf{n}_1$. The generator of the set $\left(\mathbf{n}_1,\mathbf{N}_1\right)$ for $A\left(\theta,\mathbf{n}_1,\mathbf{N}_1\right)$ is $\mathbf{N}_1$ and, accordingly $\mathbf{A}(\theta,\mathbf{n}_1,\mathbf{N}_1)=\mu(\theta)\mathbf{N}_1$. Thus

$$\begin{aligned}\mathbf{k}\ &=\mathbf{\Phi}_\kappa-\Lambda^\varepsilon\mathbf{q}_\kappa=-\mu(\theta)\mathbf{N}_1\mathbf{g}_\kappa+\lambda(\theta)\mathbf{n}_1\\ &=\mu(\theta)\mathbf{n}_1\times\mathbf{g}_\kappa+\lambda(\theta)\mathbf{n}_1.\end{aligned} \tag{62}$$

d) transversally isotropic bodies with group symmetry $g_3=\left(O;\mathbf{N}_1\right)$.
In this case

$$\mathbf{k}=\mathbf{\Phi}_\kappa-\Lambda^\varepsilon\mathbf{q}_\kappa=\mu(\theta)\mathbf{n}_1\times\mathbf{g}_\kappa \tag{63}$$

4.2.1 Orthotropic material bodies

This section considers anisotropic bodies characterised with group symmetry $g_6=\left(O;\mathbf{n}_1\otimes\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2\right)$. The basis of invariants for $\left(\theta,\mathbf{n}_1\otimes\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2\right)$ are functions only of θ (see Smith [12] and the Appendix). There are no generators for $\mathbf{a}\left(\theta,\mathbf{n}_1\otimes\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2\right)$, i.e. $\mathbf{a}\left(\theta,\mathbf{n}_1\otimes\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2\right)=\mathbf{0}$ and $\mathbf{A}\left(\theta,\mathbf{n}_1\otimes\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2\right)=\mathbf{0}$.
Thus

$$\mathbf{k}=\mathbf{\Phi}_\kappa-\Lambda^\varepsilon\mathbf{q}_\kappa=\mathbf{0} \tag{64}$$

4.2.2 Triclinic system

a) **Predial class** characterised with group symmetry $g_7=\left(O;\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\right)$. For this class there are no invariants of $\left(O;\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\right)$. Generators for $\mathbf{a}\left(\theta,\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\right)$ are $\mathbf{n}_1,\mathbf{n}_2$ and $\mathbf{n}_3$. Thus $a\left(\theta,\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\right)=\lambda(\theta)\mathbf{n}_1+\mu(\theta)\mathbf{n}_2+\nu(\theta)\mathbf{n}_3$. Generators for $\mathbf{A}\left(\theta,\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\right)$ are $\mathbf{n}_1\otimes\mathbf{n}_2-\mathbf{n}_2\otimes\mathbf{n}_1$, $\mathbf{n}_2\otimes\mathbf{n}_3-\mathbf{n}_3\otimes\mathbf{n}_2$ and $\mathbf{n}_3\otimes\mathbf{n}_1-\mathbf{n}_1\otimes\mathbf{n}_3$. Thus

$$
\begin{aligned}
\mathbf{A}(\theta,\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3) &= -p(\theta)(\mathbf{n}_1\otimes\mathbf{n}_2-\mathbf{n}_2\otimes\mathbf{n}_1)\\
&\quad -q(\theta)(\mathbf{n}_2\otimes\mathbf{n}_3-\mathbf{n}_3\otimes\mathbf{n}_2)-r(\theta)(\mathbf{n}_3\otimes\mathbf{n}_1-\mathbf{n}_1\otimes\mathbf{n}_3)\\
&= -p(\theta)\mathbf{N}_3-q(\theta)\mathbf{N}_1-r(\theta)\mathbf{N}_2
\end{aligned}
$$

and consequently

$$
\begin{aligned}
\mathbf{k}=\mathbf{\Phi}_\kappa-\Lambda^\varepsilon\mathbf{q}_\kappa &= \left[p(\theta)\mathbf{n}_3+q(\theta)\mathbf{n}_1+r(\theta)\mathbf{n}_2\right]\times\mathbf{g}_\kappa\\
&\quad +\lambda(\theta)\mathbf{n}_3+\mu(\theta)\mathbf{n}_1+\nu(\theta)\mathbf{n}_2
\end{aligned}
\tag{65}
$$

b) **Pinacoidal class** characterised with group symmetry $g_8=(O;\mathbf{N}_1,\mathbf{N}_2)$. For this class there are no invariants and no generators for $\mathbf{a}(\theta,\mathbf{N}_1,\mathbf{N}_2)$, i.e. $\mathbf{a}=\mathbf{0}$. Generators for $\mathbf{A}(\theta,\mathbf{N}_1,\mathbf{N}_2)$ are $\mathbf{N}_1,\quad \mathbf{N}_2$ and $\mathbf{N}_1\mathbf{N}_2-\mathbf{N}_2\mathbf{N}_1=-\mathbf{N}_3$. Thus

$$
\begin{aligned}
\mathbf{k}=\mathbf{\Phi}_\kappa-\Lambda^\varepsilon\mathbf{q}_\kappa &= \left[-p(\theta)\mathbf{N}_3-q(\theta)\mathbf{N}_1-r(\theta)\mathbf{N}_2\right]\mathbf{g}_\kappa\\
&= \left[p(\theta)\mathbf{n}_3+q(\theta)\mathbf{n}_1+r(\theta)\mathbf{n}_2\right]\times\mathbf{g}_\kappa
\end{aligned}
\tag{66}
$$

4.2.3 Monoclinic system

a) **Domatic class** characterised with group symmetry $g_9=(O;\mathbf{n}_2,\mathbf{n}_3)$. In the same way as in the case of the **predial class** we obtain

$$
\mathbf{k}=\mathbf{\Phi}_\kappa-\Lambda^\varepsilon\mathbf{q}_\kappa=\left[q(\theta)\mathbf{n}_2+r(\theta)\mathbf{n}_3\right]\times\mathbf{g}_\kappa+\lambda(\theta)\mathbf{n}_2+\mu(\theta)\mathbf{n}_3 \tag{67}
$$

b) **Sphenoidal class** characterised with group symmetry $g_{10}=(O;\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1)$. For this class there are no invariants for $\mathbf{a}(\theta,\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1)$ and $\mathbf{A}(\theta,\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1)$. Generator of $\mathbf{a}(\theta,\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1)$ is $\mathbf{n}_1$ and $\mathbf{a}=\lambda(\theta)\mathbf{n}_1$. Generator of $\mathbf{A}(\theta,\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1)$ is $\mathbf{N}_1$, and $\mathbf{A}(\theta,\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1)=-\mu(\theta)\mathbf{N}_1$. Therefore

$$
\mathbf{k}=\mathbf{\Phi}_\kappa-\Lambda^\varepsilon\mathbf{q}_\kappa=\mu(\theta)\mathbf{n}_1\times\mathbf{g}_\kappa+\lambda(\theta)\mathbf{n}_1. \tag{68}
$$

c) **Prismatic class** characterised with group symmetry $g_{11}=(O;\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1)$. For this class there are no invariants for $\mathbf{a}(\theta,\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1)$ and $\mathbf{A}(\theta,\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1)$. There are no generator of $\mathbf{a}(\theta,\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1)$, thus $\mathbf{a}=\mathbf{0}$. Generator of $\mathbf{A}(\theta,\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1)$ is $\mathbf{N}_1$ and $\mathbf{A}(\theta,\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1)=-\mu(\theta)\mathbf{N}_1$. Therefore,

$$
\mathbf{k}=\mathbf{\Phi}_\kappa-\Lambda^\varepsilon\mathbf{q}_\kappa=\mu(\theta)\mathbf{n}_1\times\mathbf{g}_\kappa. \tag{69}
$$

4.2.4 Rhombic systems

a) **Pyramidal class** characterised with group symmetry $g_{12} = \left(O;\mathbf{n}_1,\mathbf{n}_2 \otimes \mathbf{n}_2\right)$. For this class there are no invariants of $\mathbf{a}\left(O;\mathbf{n}_1,\mathbf{n}_2 \otimes \mathbf{n}_2\right)$ and $\mathbf{A}\left(O;\mathbf{n}_1,\mathbf{n}_2 \otimes \mathbf{n}_2\right)$. Generator of $\mathbf{a}\left(\theta,\mathbf{n}_1,\mathbf{n}_2 \otimes \mathbf{n}_2\right)$ is $\mathbf{n}_1$ and $\mathbf{a} = \lambda(\theta)\mathbf{n}_1$. There are no generators of $\mathbf{A}\left(\theta,\mathbf{n}_1,\mathbf{n}_2 \otimes \mathbf{n}_2\right)$, i.e. $\mathbf{A}\left(O;\mathbf{n}_1,\mathbf{n}_2 \otimes \mathbf{n}_2\right) = \mathbf{0}$.

Hence

$$\mathbf{k} = \mathbf{\Phi}_\kappa - \Lambda^\varepsilon \mathbf{q}_\kappa = \lambda(\theta)\mathbf{n}_1. \tag{70}$$

b) **Dipyramidal class** characterised with group symmetry $g_{14} = g_6 = \left(O;\mathbf{n}_2 \otimes \mathbf{n}_2,\mathbf{n}_3 \otimes \mathbf{n}_3\right)$ and consequently

$$\mathbf{k} = \mathbf{\Phi}_\kappa - \Lambda^\varepsilon \mathbf{q}_\kappa = \mathbf{0}. \tag{71}$$

5 The entropy inequality for anisotropic bodies

So far we did not investigate the entropy inequality

$$\sigma = \left(\frac{\partial \mathbf{\Phi}_\kappa}{\partial \theta} - \Lambda^\varepsilon \frac{\partial \mathbf{q}_\kappa}{\partial \theta}\right)\cdot \mathbf{g}_\kappa + \rho_\kappa \left(\frac{\partial \eta}{\partial \mathbf{C}} - \Lambda^\varepsilon \frac{\partial \varepsilon}{\partial \mathbf{C}} + \frac{1}{2\rho_\kappa}\Lambda^\varepsilon \mathbf{S}_\kappa\right)\dot{\mathbf{C}} \geq 0 \tag{72}$$

for general anisotropic materials under the assumption that $\Lambda^\varepsilon = \Lambda^\varepsilon(\theta)$. Making use of the free energy function $\psi = \varepsilon - \theta\eta$ the expression for σ reduces to

$$\sigma = \left(\frac{\partial \mathbf{\Phi}_\kappa}{\partial \theta} - \Lambda^\varepsilon \frac{\partial \mathbf{q}_\kappa}{\partial \theta}\right)\cdot \mathbf{g}_\kappa + \frac{\Lambda^\varepsilon}{2}\left(\mathbf{S}_\kappa - 2\rho_\kappa \frac{\partial \psi}{\partial \mathbf{C}}\right)\dot{\mathbf{C}} \geq 0 \tag{73}$$

which is linear in $\dot{\mathbf{C}}$. Thus

$$\mathbf{S}_\kappa = 2\rho_\kappa \frac{\partial \psi}{\partial \mathbf{C}} \tag{74}$$

Equation (74) leads to the conclusion that *all anisotropic elastic materials are hyperelastic* as a consequence of $\Lambda^\varepsilon(\theta)$, irrespective of whether the classical entropy flux relation is valid. The remaining inequality now reads

$$\sigma = \left(\frac{\partial \mathbf{\Phi}_\kappa}{\partial \theta} - \Lambda^\varepsilon \frac{\partial \mathbf{q}_\kappa}{\partial \theta}\right)\cdot \mathbf{g}_\kappa \geq 0 \tag{75}$$

or

$$\sigma = \left(\frac{\partial \mathbf{k}}{\partial \theta} + \frac{\partial \Lambda^\varepsilon}{\partial \theta}\mathbf{q}_\kappa\right)\cdot \mathbf{g}_\kappa \geq 0 \tag{76}$$

where $\mathbf{k} = \mathbf{\Phi}_\kappa - \Lambda^\varepsilon \mathbf{q}_\kappa$. Now from (46) we have

$$\frac{\partial \mathbf{k}}{\partial \theta} = \frac{\partial \mathbf{A}\left(\theta,\mathfrak{m},\mathfrak{M}\right)}{\partial \theta}\mathbf{g}_\kappa + \frac{\partial \mathbf{a}\left(\theta,\mathfrak{m},\mathfrak{M}\right)}{\partial \theta} \tag{77}$$

Since $\mathbf{A}(\theta,\mathfrak{m},\mathfrak{M})$ is skew-symmetric $\mathbf{g}_\kappa \cdot \frac{\partial \mathbf{A}(\theta,\mathfrak{m},\mathfrak{M})}{\partial \theta}\mathbf{g}_\kappa = 0$, hence

$$\frac{\partial \mathbf{k}}{\partial \theta}\cdot \mathbf{g}_\kappa = \frac{\partial \mathbf{a}(\theta,\mathfrak{m},\mathfrak{M})}{\partial \theta}\cdot \mathbf{g}_\kappa \tag{78}$$

Therefore (75) becomes

$$\sigma = \left(\frac{\partial \mathbf{a}(\theta,\mathfrak{m},\mathfrak{M})}{\partial \theta} - \frac{1}{\theta^2}\mathbf{q}_\kappa(\mathbf{C},\theta,\mathbf{g}_\kappa,\mathfrak{m},\mathfrak{M})\right)\cdot \mathbf{g}_\kappa \geq 0 \tag{79}$$

The non-negative entropy production density σ attains its minimum, which is in fact zero, when $\mathbf{g}_\kappa = \mathbf{0}$. A necessary condition for an extremum at $\mathbf{g}_\kappa = \mathbf{0}$ is

$$\left.\frac{\partial \sigma}{\partial \mathbf{g}_\kappa}\right|_{\mathbf{g}_\kappa=\mathbf{0}} = \mathbf{0} \tag{80}$$

or

$$\mathbf{q}_\kappa(\mathbf{C},\theta,\mathbf{g}_\kappa = \mathbf{0},\mathfrak{m},\mathfrak{M}) = \theta^2 \frac{\partial \mathbf{a}(\theta,\mathfrak{m},\mathfrak{M})}{\partial \theta}, \tag{81}$$

at equilibrium.

Since equation (81) holds *for all anisotropic elastic bodies* it can be used to define the heat flux vector at equilibrium for specific crystal classes. For instance, for transversally isotropic material bodies, considered by I-Shih Liu [8], with the following group symmetries:

a) $g_2 = (O;\mathbf{n}_1)$ and $\mathbf{a}(\theta,\mathbf{n}_1) = \lambda(\theta)\mathbf{n}_1$, where $\lambda(\theta)$ is an arbitrary scalar function, we have

$$\mathbf{q}_\kappa(\mathbf{C},\theta,\mathbf{0},\mathfrak{m},\mathfrak{M}) = \theta^2 \frac{d\lambda}{d\theta}\mathbf{n}_1 \tag{82}$$

b) $g_5 = (O;\mathbf{n}_1 \otimes \mathbf{n}_1)$

$$\mathbf{q}_\kappa(\mathbf{C},\theta,\mathbf{0},\mathfrak{m},\mathfrak{M}) = \mathbf{0} \tag{83}$$

And similarly, for other group symmetries investigate here the following results hold:

$$
\begin{aligned}
&g_1 = (O;\mathbf{n}_1,\mathbf{N}_1): && \mathbf{q}_\kappa\left(\mathbf{C},\theta,\mathbf{0},\mathbf{n}_1,\mathbf{N}_1\right) && = \theta^2 \frac{d\lambda}{d\theta}\mathbf{n_1} \\
&g_2 = (O;\mathbf{n}_1): && \mathbf{q}_\kappa\left(\mathbf{C},\theta,\mathbf{0},\mathbf{n}_1\right) && = \theta^2 \frac{d\lambda}{d\theta}\mathbf{n_1} \\
&g_{10} = (O;\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1): && \mathbf{q}_\kappa\left(\mathbf{C},\theta,\mathbf{0},\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1\right) && = \theta^2 \frac{d\lambda}{d\theta}\mathbf{n_1} \\
&g_{12} = \left(O;\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2\right): && \mathbf{q}_\kappa\left(\mathbf{C},\theta,\mathbf{0},\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2\right) && = \theta^2 \frac{d\lambda}{d\theta}\mathbf{n_1} \\
&g_3 = \left(O;\mathbf{N}_1\right): && \mathbf{q}_\kappa\left(\mathbf{C},\theta,\mathbf{0},\mathbf{N}_1\right) && = \mathbf{0} \\
&g_5 = (O;n_1\otimes n_1): && \mathbf{q}_\kappa\left(\mathbf{C},\theta,\mathbf{0},\mathbf{n}_1\otimes\mathbf{n}_1\right) && = \mathbf{0} \\
&g_6 = \left(O;\mathbf{n}_1\otimes\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2\right): && \mathbf{q}_\kappa\left(\mathbf{C},\theta,\mathbf{0},\mathbf{n}_1\otimes\mathbf{n}_1,\mathbf{n}_2\otimes\mathbf{n}_2\right) && = \mathbf{0} \\
&g_8 = \left(O;\mathbf{N}_1,\mathbf{N}_2\right): && \mathbf{q}_\kappa\left(\mathbf{C},\theta,\mathbf{0},\mathbf{N}_1,\mathbf{N}_2\right) && = \mathbf{0} \\
&g_{11} = \left(O;\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1\right): && \mathbf{q}_\kappa\left(\mathbf{C},\theta,\mathbf{0},\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{N}_1\right) && = \mathbf{0} \\
&g_{14} = \left(O;\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{n}_3\otimes\mathbf{n}_3\right): && \mathbf{q}_\kappa\left(\mathbf{C},\theta,\mathbf{0},\mathbf{n}_2\otimes\mathbf{n}_2,\mathbf{n}_3\otimes\mathbf{n}_3\right) && = \mathbf{0}
\end{aligned}
\tag{84}
$$

Note, $g_{14} = g_6$.

$$
\begin{aligned}
&g_7 = \left(O;\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\right): && \mathbf{q}_\kappa(\mathbf{C},\theta,\mathbf{0},\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3) && = \frac{d\lambda}{d\theta}\mathbf{n}_1 + \frac{d\mu}{d\theta}\mathbf{n}_2 + \frac{d\nu}{d\theta}\mathbf{n}_3 \\
&g_9 = (O;\mathbf{n}_2,\mathbf{n}_3): && \mathbf{q}_\kappa(\mathbf{C},\theta,\mathbf{0},\mathbf{n}_2,\mathbf{n}_3) && = \frac{d\mu}{d\theta}\mathbf{n}_2 + \frac{d\nu}{d\theta}\mathbf{n}_3
\end{aligned}
\tag{85}
$$

A necessary condition that entropy production σ has minimum at $\mathbf{g}_\kappa = 0$ is that the second gradient of σ with respect to $\mathbf{g}_\kappa$ be semi-positive, i.e.

$$
\left.\frac{\partial^2\sigma}{\partial g_i \partial g_j}\right|_{\mathbf{g}_\kappa=\mathbf{0}} \geq 0 \tag{86}
$$

Note that $\frac{\partial^2\sigma}{\partial g_i \partial g_j}$ is a symmetric tensor. Using (76) and (79) it is easy to show that

$$
\left.\frac{\partial^2\sigma}{\partial g_i \partial g_j}\right|_{\mathbf{g}_\kappa=\mathbf{0}} = 2\frac{\partial\Lambda^\varepsilon}{\partial\theta}\left.\frac{\partial q_i}{\partial g_j}\right|_{(i,j)|\mathbf{g}_\kappa=\mathbf{0}} = -2\frac{1}{\theta^2}\left.\left(\frac{\partial q_i}{\partial g_j}\right)\right|_{(i,j)|\mathbf{g}_\kappa=\mathbf{0}} \geq 0
$$

having in mind that $\Lambda^\varepsilon = 1/\theta$.

Finally (86) can be written as

$$
\left.\left(\frac{\partial\mathbf{q}_\kappa}{\partial\mathbf{g}_\kappa}\right)\right|_{sym|\mathbf{g}_\kappa=\mathbf{0}} \leq 0 \tag{87}
$$

which holds for *all anisotropic elastic materials*. The constraints that (87) imposes must be investigated for particular anisotropic materials. For example, for the transversely isotropic elastic materials where

$$
\mathbf{q}_\kappa = a_0\mathbf{g}_\kappa + \left(b_0 + b_1\left(\mathbf{n}\cdot\mathbf{g}_\kappa\right)\right)\mathbf{n} + c_0\mathbf{n}\times\mathbf{g}_\kappa
$$

we have

$$\left(\frac{\partial \mathbf{q}_\kappa}{\partial \mathbf{g}_\kappa}\right)\Bigg|_{sym|\mathbf{g}_\kappa=\mathbf{0}} = a_0\mathbf{I} + b_1\mathbf{n}\otimes\mathbf{n} \le 0$$

with the constraints

$$a_0 \le 0, \quad a_0 + b_1 \le 0$$

6 Conclusion

This paper revisits entropy flux and heath flux relations for isotropic and several anisotropic elastic materials. More specifically we investigated the consequence of the assumption that $\Lambda^\varepsilon = \Lambda(\theta)$ irrespective of validity of the classical entropy flux relation. This assumption is used to derive the relationship between the entropy flux and heat flux for all isotropic elastic materials as well as for some crystal classes including transverse isotropy, orthotropy, triclinic systems and rhombic systems. First re-examined the entropy flux-heat flux relation for viscoelastic materials, isotropic elastic materials and transversely isotropic elastic bodies and demonstrate that our results agree with the results obtained by I-Shih Liu [6] to [10]. All these cases confirm that $\Lambda^\varepsilon = \Lambda(\theta)$ is a necessary and sufficient condition for the determination of the entropy flux-heat flux relation.

Furthermore, we derived the entropy flux-heat flux relations for all the following crystal classes: transverse isotropy, orthotropy, triclinic systems, monoclinic systems and rhombic systems for which representations of anisotropic functions with respect to their symmetry groups can be expressed in terms of isotropic functions. Our derivation is very general in the sense that the constitutive relations are non-linear. One of our main results is the proof that all crystal elastic bodies, we considered, are hyperelastic. This represents a generalization of I-Shih Liu's finding for transversely isotropic bodies, the only case he analysed.

We would like to draw attention to the following three points:

i. The vector function $\mathbf{a}$ and skew-symmetric function $\mathbf{A}$ are isotropic functions depending only on the set $(\theta, \mathfrak{m}, \mathfrak{M})$ which simplifies the procedure

ii. Generally, the classical entropy flux-heat flux relation does not hold; it is true for all crystal classis investigated here except for $g_2, g_6 = g_{14}$

iii. The heat flux in the absence of a temperature gradient is not zero for all crystal classes. This confirms Eringen's statement that $\mathbf{\Phi} = \dfrac{1}{\theta}\mathbf{q} + \mathbf{\Phi}_1$ where $\mathbf{\Phi}_1$ is the entropy change due to all other effects except heat input

Of course, all our predictions have to be verified by demonstrating that $\Lambda^\varepsilon = \Lambda(\theta)$ is also a necessary condition, at least for all crystal classes investigated above. This is a task for future investigation.

8 Appendix

For orthonormal vectors $\mathbf{n}_i \cdot \mathbf{n}_j = \delta_{ij}$ and $\mathbf{N}_i$ the skew-symmetric tensors defined by $\mathbf{N}_i = e_{ijk}\mathbf{n}_j \otimes \mathbf{n}_k, \quad i, j, k = 1,2,3.$ it is easy to show that

$$
\begin{aligned}
&\mathbf{n}_i \otimes \mathbf{n}_j = \left(\mathbf{n}_i \otimes \mathbf{n}_j\right)^n \quad \text{for any natural number } n,\\
&\mathbf{N}_i\mathbf{v} = -\mathbf{n}_i \times \mathbf{v},\\
&\mathbf{N}_i\mathbf{N}_j = \mathbf{n}_i \otimes \mathbf{n}_j - \delta_{ij}\mathbf{I},\\
&\mathbf{N}_i\mathbf{N}_j\mathbf{v} = \left(\mathbf{n}_i \cdot \mathbf{v}\right)\mathbf{n}_j - \delta_{ij}\mathbf{v},\\
&\mathbf{N}_i^2\mathbf{N}_j = -\delta_{ij}\mathbf{N}_i,\\
&\mathbf{w} \cdot \mathbf{N}_i\mathbf{v} = \mathbf{w} \cdot \left(\mathbf{v} \times \mathbf{n}_i\right),\\
&\mathbf{w} \cdot \mathbf{N}_i\mathbf{N}_j\mathbf{v} = \left(\mathbf{n}_i \cdot \mathbf{v}\right)\left(\mathbf{n}_j \cdot \mathbf{v}\right) - \delta_{ij}\left(\mathbf{v} \cdot \mathbf{w}\right),\\
&\mathrm{tr}\mathbf{N}_i = 0\\
&\mathrm{tr}\mathbf{N}_i\mathbf{N}_j = -2\delta_{ij}\\
&\mathbf{n}_i \otimes \mathbf{n}_j - \mathbf{n}_j \otimes \mathbf{n}_i = e_{ijk}\mathbf{N}_k,\\
&\mathbf{N}_i\mathbf{N}_j - \mathbf{N}_j\mathbf{N}_i = -e_{ijk}\mathbf{N}_k,
\end{aligned}
$$

Where $\mathbf{v}$ and $\mathbf{w}$ are arbitrary vectors

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