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A Green's function for diffraction by a rational wedge.

by

A. D. Rawlins

1. Introduction

In 1896 A. Sommerfeld obtained an exact closed form solution to the problem of plane wave diffraction by a half plane. He used the method of images on Riemann surfaces corresponding to multivalued solutions of the reduced wave equation, and indicated how this method could be used to obtain exact closed form solution for the problem of diffraction by a wedge. The method of constructing the required many-valued solution of the wave equation was simplified in a subsequent paper, Sommerfeld (1897). Subsequently Carslaw (1919) replaced the image method by a direct construction of the solution which yields simpler formulae for various type of wave sources. These solutions had been obtained independently by Macdonald (1902), (1915) by summing the Fourier series representation of the Green's function. Other methods of obtaining some or all of the principal results have since been put forward by various authors, including Bromwich (1915), Whipple (1917), Garnir (1952) and Oberhettinger (1958). The last two authors give extensive bibliographies.

Recently Rawlins (1986a,b,c) derived some new representations for the Green's function for plane, cylindrical and spherical wave diffraction by a rational wedge. The results for the cylindrical and spherical wave Green's function were based on the plane wave Green's function. Here we derive in a different, more direct, and simpler manner the Green's function for a spherical point source. The method used here can be applied without difficulty to deal with more general types of sources, in particular the plane wave and cylindrical wave source already considered.

In section 2 we shall give the geometry of the physical diffraction problems. In section 3 we shall define a periodic Green's function as the solution of a periodic boundary value problem for a wedge of open angle 2α. By means of this periodic Green's function we show in section 4 that one can obtain expressions for the Green's function for Dirichlet, Neumann or mixed Dirichlet/Neumann boundary value problems for a wedge of open angle α. In section 5 we give a general complex integral representation for the periodic Green's function. This complex integral representation for the periodic Green's function is not new. It has appeared in various guises in the papers already cited. In section 6 we show how this Green's function can be reduced to source terms and real integrals representing the diffracted waves for the special case of a rational wedge $\alpha = p\pi/q$ (p and q positive integers).

2. Geometry of diffraction problems.

We shall be interested in finding solution to diffraction problems in wedge shaped regions. To be specific we shall consider acoustic diffraction by time harmonic waves (with harmonic time variation e^{iwt} assumed, but not shown explicitly, in the rest of the paper).

We shall work in Euclidean space of three dimensions with cylindrical polar coordinates (r,θ,z). In this space we shall assume there is a wedge of open angle α with faces defined by the planes $\theta = 0$ and $\theta = \alpha$. For $\alpha = 0$ the wedge disappears; for $\alpha = \pi$ it becomes a half space. For $\alpha = 2\pi$ the wedge becomes a semi-infinite plane whose upper and lower faces are considered as distinct.

In physical problems α is necessarily not greater than 2π. However, in the mathematical analysis that follows we shall be dealing with functions that make mathematical sense for all real α. Therefore it is of interest to consider wedge angle α which are <u>virtual</u> with opening $\alpha > 2\pi$, their existence is purely mathematical.

We shall consider two points $P \equiv (r,\theta,z)$ and $P_0 \equiv (r_0,\theta_0,z_0)$ which both lie within the open angle region $0 < \theta < \alpha$, and let

$$R(P,P_0) = \sqrt{r_0^2 + r^2 + (z-z_0)^2 - 2rr_0 \cos(\theta-\theta_0)},$$
$$D(P,P_0) = \sqrt{r+r_0)^2 + (z-z_0)^2}.$$

$R(P,P_0)$ is the shortest distance (in the Euclidean plane) between P_0 and P; and $D(P,P_0)$ is the shortest distance of the broken line from P_0 to the edge of the wedge and thence to P .

We shall assume that a time harmonic point wave source is situated at P_0.

3. Periodic boundary value problem defining a Green's function.

We shall define a periodic Green's function for the reduced wave equation operator $\nabla^2 + k^2 (k = w/c$, where c is the velocity of sound) by means of a periodic boundary value problem posed on a wedge, real or <u>virtual,</u> of open angle 2α.

We shall denote this periodic Green's function by $G_\alpha(r,\theta,z;r_0,\theta_0,z_0;k)$ or $G_\alpha(P,P_0;k)$ for an angular opening 2α. $G_\alpha(P,P_0;k)$ which is a function of P and P_0, is defined uniquely by the following periodic boundary value problem.

(a) It is a solution of

$$(\nabla^2 + k^2)G_\alpha(P,P_0;k) = \delta(P,P_0),$$
$$0 < r < \infty,\ 0 < \theta < 2\alpha,\ -\infty < z < \infty.$$

(b) It satisfies the periodic boundary conditions

$$G_\alpha(r,0,z;r_0,\theta_0,z_0;k) = G_\alpha(r,2\alpha,z;r_0,\theta_0,z_0;k),$$

(c) It satisfies the Sommerfeld radiation condition

$$\lim_{R(P,P_0)\to\infty} R(P,P_0)\left(\frac{\partial G_\alpha(P,P_0;k)}{\partial R(P,P_0)} + ikG_\alpha(P,P_0;k)\right) = 0.$$

(d) It satisfies the edge condition

$$G_\alpha(P,P_0;k) = 0(1), \quad \text{and} \quad |\text{grad } G_\alpha(P,P_0;k)| = o(1/r) \qquad \text{as} \quad r \to 0.$$

We remark that, if follows from the above properties that the Green's function will be symmetric i.e.

$$G_\alpha(P,P_0;k) = G_\alpha(P_0,P;k).$$

Once the above periodic Green's function has been obtained, it is a simple matter to obtain Green's functions for acoustically hard, soft or mixed hard/soft boundary value problems for wedges of half this open angle.

4. Green's functions for hard and soft bounderies for a wedge of open angle α.

To simplify notation in this section we shall let the periodic Green's function $G_\alpha(r,\theta,z;r_0,\theta_0,z_0;k)$ be denoted by $G_\alpha(\theta)$ or equivalently $G_\alpha(\theta_0)$. Then it is not difficult to show, (using the fact that $G_\alpha(2n\alpha+\theta) = G_\alpha(\theta)$ where n is any integer) that:

The Green's function $G^S_\alpha(P,P_0;k)$ for a soft wedge of open angle α is given by

$$G^S_\alpha(P,P_0;k) = G_\alpha(\theta) + G_\alpha(2\alpha-\theta);$$

the Green's function $G^h_\alpha(P,P_0;k)$ for a hard wedge of open angle α is given by

$$G^h_\alpha(P,P_0;k) = G_\alpha(\theta) + G_\alpha(2\alpha-\theta).$$

In the same way it can be shown that

$$G^{s/h}_\alpha(P,P;k) = G_{2\alpha}(\theta) + G_{2\alpha}(2\alpha-\theta) - G_{2\alpha}(2\alpha+\theta) - G_{2\alpha}(4\alpha-\theta)$$

is the appropriate Green's function for the wedge which is soft on the face θ = 0, and hard on the face θ = α. The corresponding Green's function for a hard boundary on the face θ = 0 and a soft boundary on the face θ = α is given by

$$G^{h/s}_\alpha(P,P;k) = G_{2\alpha}(\theta\theta - G_{2\alpha}(2\alpha-\theta) - G_{2\alpha}(2\alpha+\theta) + G_{2\alpha}(4\alpha-\theta).$$

For $\pi < \alpha \le 2\pi$ and $\pi/2 < \alpha \le 2\pi$ the functions $G_\alpha(\theta)$ and $G_{2\alpha}(\theta)$ respectively, mathematically relate to a virtual wedge angle, which explains the necessity to consider such wedges.

5. A complex integral representation of the periodic Green's function,

A complex integral representation for the periodic Green's function defined in section 3 for $\delta(P,P_0) = -4\pi\delta(|P-P_0|)$, has been shown by Carslaw and others (1919) to be given by

$$G_\alpha(r,\theta,z;r_0,\theta_0,z_0;k) = \frac{1}{2i\alpha}\int_C G(\zeta)\frac{\sin(\zeta i n\,\alpha)}{\cos(\zeta\cos(\zeta - \cos((\theta-\theta)\,\pi/\alpha)}\,d\zeta, \qquad (1)$$

where

$$G(\zeta) = \frac{e^{-ikR(\zeta)}}{R(\zeta)},\quad R(\zeta) = (r^2+r_0^2+(z-z)^2-2rr_0\cos\zeta)^{\frac{1}{2}}. \qquad (2)$$

The square root is defined uniquely by $R(0)=D(P,P_0)$, so that $R(\zeta)$ is analytic everywhere except along branch cuts that extend vertically up and down from branch points above and below the real axis, respectively. These branch points are located at $2\pi\ell \pm i\beta$, where $\beta = \cosh^{-1}\{(r^2+r_0^2+(z-z_0)^2/(2rr_0)\}$, and ℓ is any integer, see fig.1. In the chosen cut plane $-\pi/2 \le \arg R(\zeta) \le \pi/2$. The contour of integration C in the expression (1) is such that the starting point is given by $c_1+i\infty$, and the termination point is given by $c_2+i\infty$, where $-\pi \le c_1 \le 0, \pi \le c_2 \le 2\pi$; and the contour lies between the branch points $\zeta = \pm i\beta$ and does not intersect any of the other branch cuts which run parallel to the imaginary axis, see fig.1.

We shall now use the complex integral representation (1) to obtain a different representation for the special case of α being a rational multiple of π, i.e. $\alpha = p\pi/q$ (p and q positive integers). This new representation will be in terms of point source terms and <u>real</u> integrals. From the results we have already given in section 4 this means we can represent the acoustic field scattered by a soft, hard, or mixed soft/hard wedge with an open angle $\alpha = p\pi/q$ in terms of point sources and real integrals representing the diffracted field.

6. Green's function for a rational wedge.

If the angle $\alpha = p\pi/q$, where p and q are positive integers the expression (1) becomes

$$G_{\frac{p\pi}{q}}(P,P_0;k) = \frac{1}{2ip\pi}\int_C G(\zeta)\,\frac{q\,\sin(q/p)}{\cos(q/p)-\cos((\theta-\theta_0)q/p)}\,d\zeta \qquad (3)$$

By using the identity

$$\frac{q\,\sin(\zeta p/q)}{\cos(\zeta/q)-\cos((\theta-\theta_0)p/q)} = \sum_{m=0}^{q-1}\frac{\sin(\zeta/p)}{\cos(\zeta/p)-\cos\left(\frac{\theta-\theta_0}{p}+\frac{2\pi m}{q}\right)} \qquad (4)$$

see Rawlins (1986), we can rewrite (3) in the form

$$G_{\frac{p\pi}{q}}(P,P_0;k) = \sum_{m=0}^{q-1} I_p(\theta-\theta_0+2\pi mp/q), \qquad (5)$$

where

$$I_p(\psi) = \frac{2}{2\pi i}\int_C G(\zeta)\,\frac{\frac{1}{p}\sin(\zeta/p)}{\cos(\zeta/p)-\cos(\psi/p)}\,d\zeta. \qquad (6)$$

We now distort the path of integration C, in the integral (6), so that it takes up the new path C', as shown in fig.1. The new path of integration C^1 lies

along regions where the integral is uniformly convergent. In distorting the contour C to take up the contour C' the line $0 \le \mathrm{Re}\zeta \le \pi$, $\mathrm{Im}\zeta = 0$, is crossed, and therefore if any zeros of $\cos(\zeta/p) - \cos(\psi/p)$, (viz $\zeta = \pm(\psi + 2\pi pN)$ where N is an integer) are captured then they will give rise to pole contributions.

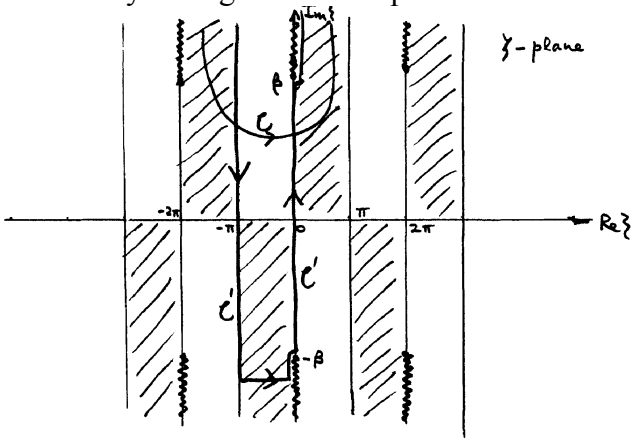

Diagram of the paths of integration in the ζ-plane. The shaded regions show where the integral (6) is uniformly convergent. The squiggly lines are the branch cuts for the function $G(\zeta)$.

fig.1.

Thus

$$I(\psi(= \sum_N H[\pi - |\psi + 2\pi pN|]\ G(\psi + 2\pi pN)$$

$$+ \frac{1}{2\pi i}\left\{ \int_{-\beta}^{\beta} + \int_{\beta}^{\infty} + \int_{-\infty}^{-\beta} \right\} G(iy) \frac{\frac{i}{p}\sinh(y/p)\, i\ dy}{\cosh(y/p) - \cos(\psi/p)}$$

$$+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(\pi + iy)\ \frac{\frac{1}{p}\sin((\pi + iy)/p)\, i\, dy}{\cos\left(\frac{iy+\pi}{p}\right) - \cos\left(\frac{\psi}{p}\right)}$$

$$+ \lim_{\rho\to\infty} \frac{1}{2\pi i}\int_{0}^{\pi} G(x - i\rho)\ \frac{\frac{1}{p}\sin((x - i\rho)/p)\ dx}{\cos\left(\frac{x-i\rho}{p}\right) - \cos\left(\frac{\psi}{p}\right)}$$

where $H[x] = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 1 \\ 1 & \text{for } x > 1 \end{cases}$ is the Heaviside step function,

and where the symbol $\sum_N$ means summation is performed for all integer values of N which satisfy the inequality $-\pi < \psi + 2\pi pN < \pi$.

In the above expression the first integral has had its limits of integration broken up, because the path traverses branch cuts. It can be shown, using the appropriately defined values of G (iy) on the branch cuts that this whole term vanishes because the integrand is odd in y. The last integral in the above expression also gives zero contribution because $\lim_{\rho\to\infty} |G(x-i\rho)| \to 0,\ 0 \le x \le \pi$. Thus

$$I_p(\psi) = \sum_N H[\pi - |\psi + 2\pi p N|]\, G(\psi + 2\pi p N)$$

$$+ \frac{2}{2\pi i} \int_{-\infty}^{\infty} G(\pi + iy)\, \frac{\frac{1}{p}\sin((\pi+iy)/p)\, i\, dy}{\cos\left(\frac{iy+\pi}{p}\right) - \cos\left(\frac{\psi}{p}\right)} . \qquad (7)$$

By using the identity (see Rawlins 1986)

$$\frac{\frac{1}{p}\sin(\zeta/p)}{\cos(\zeta/p) - \cos(\psi/p)} = \frac{a_{p-1}(\psi)\sin(\zeta/p) + \sum_{n=0}^{p-2} a_n(\psi)\{\sin(\zeta(p-n)/p) - \sin(\zeta(p-2-n)/p)\}}{\cos\zeta - \cos\psi}$$

where $a_n(\psi) = \sin((n+1)\psi/p)/\{p\sin(\psi/p)\}$, in the expression (7) we have

$$I_p(\psi) = \sum_N H[\pi - |\psi + 2\pi pN|]G(\psi + 2\pi pN)$$

$$-\frac{1}{2\pi} a_{p-1}(\psi) \int_{-\infty}^{\infty} \frac{G(\pi+iy)\ \sin\ ((\pi+iy)/p)\ dy}{\cosh y + \cos\psi}$$

$$-\frac{1}{2\pi}\sum_{n=0}^{p-2} a_n(\psi \int_{-\infty}^{\infty} \frac{G(\pi+iy)}{\cosh y + \cos\psi}\left\{\sin\left((\pi+iy)\frac{(p-n)}{p}\right) - \sin\left((\pi+iy)\frac{(p-2-n)}{p}\right)\right\}\ dy.$$

Now by using the fact that $G(\pi+iy) = G(\pi-iy)$ we can write the above expression as

$$I_p(\psi) = \sum_N H[\pi - |\psi + 2\pi pN|]G(\psi + 2\pi pN)$$

$$-\frac{1}{\pi} a_{p-1}(\psi)\sin\left(\frac{\pi}{p}\right) \int_0^{\infty} \frac{G(\pi-iy)\cosh(y/p)dy}{\cosh y + \cos\psi}$$

$$-\frac{1}{\pi}\sum_{n=1}^{p-2}\left\{a_n(\psi)\sin\left(\pi\frac{(p-n)}{p}\right)\int_0^{\infty} \frac{G(\pi-iy)\cosh(y(p-n)/p)dy}{\cosh y + \cos\psi}\right.$$

$$\left. - a_{n-1}(\psi)\sin\left(\pi\frac{(p-1-n)}{p}\right)\int_0^{\infty} \frac{G(\pi-iy)\cosh(y(p-1-n)/p)dy}{\cosh y + \cos\psi}\right\} \qquad (8)$$

So that finally on substituting (8) into (5) we get

$$G_{\frac{p\pi}{q}}(P,P_0;k) = \sum_{m=0}^{q-1} \sum_{N} H[\pi - |\theta - \theta_0 + 2\pi mp/q + 2\pi pN|]G(\theta - \theta_0 + 2\pi mp/q + 2\pi pN)$$

$$-\frac{1}{P\pi}\sum_{m=0}^{q-1} \frac{\sin(\theta - \theta_0 + 2\pi mp/q)\sin(\pi/p)}{\sin((\theta - \theta_0 + 2\pi mp/q)/p)} \int_0^{\infty} \frac{\cosh(y/p)G(\pi - iy)\quad dy}{\cosh y + \cos(\theta - \theta_0 + 2\pi mp/q)}$$

$$-\frac{1}{P\pi}\sum_{m=0}^{q-1}\sum_{n-1}^{p-2}\left\{\frac{\sin((n+1)(\theta - \theta_0 + 2\pi mp/q)\sin(\pi/p)}{\sin((\theta - \theta_0 + 2\pi mp/q)/p)}\int_0^{\infty} \frac{\cosh(p-n)y/p)G(\pi - iy)\,dy}{\cosh y + \cos(\theta - \theta_0 + 2\pi mp/q)}\right.$$

$$\left.-\frac{\sin(n(\theta - \theta_0 + 2\pi mp/q)\sin((n+1)\pi/p)}{\sin((\theta - \theta_0 + 2\pi mp/q)/p)} \int_0^{\infty} \frac{\cosh(p-1-n)y/p)G(\pi - iy)\,dy}{\cosh y + \cos(\theta - \theta_0 + 2\pi mp/q)}\right\}, \qquad (9)$$

where the summation over N is for all integer values of N which can make the argument of the Heaviside step function non negative; and $G(\zeta)$ is given by equation (2).

In conclusion we remark that if we replace $G(\zeta)$ in the expression (9) by $e^{ikr\cos\zeta}$ or $H_0^{(2)}(kR(\zeta))$ we get the appropriate Green's function for a plane or cylindrical wave source, respectively. If we replace $G(\zeta)$ by $1/R(\zeta)$ we obtain the Green's function for Laplace's equation, which has applications in incompressible fluid mechanics, and electrostatics. Inversion of this problem will give the Green's function for a len's problem as well. The author hopes to pursue this aspect in a later paper.

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