TR/14/90 November 1990

CONSISTENCY MEASURES

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Section O. Introduction

In this paper we discuss several measures of consistency as applied to judgements based upon pairwise comparisons between a set of n options. The methods discussed in Section 2 are Saaty's Eigenvector method (2.1) the so-called Geometric Method (2.2) and the Trace Index (2.3).

The first two methods also produce weights or scores for the options.

The consistency measures can be viewed as a measure of confidence in these weights.

We show in Section 2.4 that the above two methods of obtaining scores agree closely when the consistency is high as measured by the Trace Index.

Also we discuss the so-called instability of the relative weights of the remaining options when an option is deleted from consideration. We show in Section 3 that such behaviour is to be expected and is reasonable.

These measures have been used in several well known applications e.g. the decision support systems EXPERT CHOICE , PDS and PRIORITIES..

Section 1. Definitions

Let $\Omega = \{0_1,\ldots,0_n\}$ be a set of n options which are to be pairwise compared.

We shall abuse notation by referring to the options by their index number i rather than by 0_i.

Let $\mathbb{R}_* = \{x \in \mathbb{R} : x > 0\}$

If we have a function c: Ω x $\Omega \rightarrow$ such that $\mathbb{R}_*$ such that

1. $c(i,j) = \dfrac{1}{c(j,i)} \qquad \forall i, j \in \Omega$

2. $c(i,j) = 1, \forall i \in \Omega$

then c is called a comparison method and we think of c(i,j) giving the relative preference of option i to option j.

The comparisons are multiplicative in the sense that in a completely consistent set of pairwise comparisons between the options i, j, k we expect:

$$c(i,j)c(j,k) = c(i,k) \qquad (1).$$

This is the basic assumption underlying the treatment of consistency in the rest of this paper.

If there is a human decision-maker formimg these comparisons then it is likely that (1) is not satisfied for all i,j,k .

We shall measure the deviation from consistency in several ways , all applied to a fixed comparison method c.

1.1 Example of a comparison method

Consider the following scale :

Equal		Preferred		Strongly Preferred		v. Strong preferred		Absolutely preferred
1	2	3	4	5	6	7	8	9

Given distinct options i, j a decision—maker compares them by using the scale above and finds the numerical point c on the scale closest to her strength of preference of one over the other.

If i is preferred to j then c(i, j) = c , c(j, i) = 1/c.

(see [3], [5]).

Given the comparison method c as applied to the options Ω we then obtain a matrix $A = (a_{i,j})$ where $a_{i,j} = c(i,j)$, $1 \le i,j \le n$

A is called a reciprocal matrix as $a_{i,j} = a_{j,i}^{-1}$, $1 \le i,j \le n$.

If the comparisons are consistent i.e.

c(i,j)c(j,k> = c(i,k) for all triples i, j, k Є Ω we call the matrix A consistent.

Note that

<u>Lemma 1</u>

A consistent $\leftrightarrow \exists (w_1, \ldots, w_n), w_i \in \mathbb{R}_*, 1 \le i \le n$ such that

$$a_{i,j} = w_i / w_j \quad \forall i, j \in \Omega$$

<u>proof</u>

See [5].

A contains all the information needed to define a consistency measure and we now give some examples

Section 2. Consistency Measures

2.1 Saaty's Eigenvector Method

(see [5] for more details)

Let A be a reciprocal matrix. All entries of A are positive.

Hence the Perron-Froebenius Theorem [6] can be applied to show that A has a maximal eigenvalue λ with associated eigenspace E of dimension 1. The eigenvector $(w_1, \ldots, w_n)$, $w_i \in \mathbb{R}_*$, in E which satisfies $\sum_{i=1}^{n} w_i = 1$ is therefore unique.

The method then continues by defining

$\mu = \frac{\lambda - n}{n-1}$ and using μ as a measure of inconsistency.

Note that $\mu \geq 0$ as it is shown in [ ] that $\lambda \geq n$.

Also $\mu = 0$ iff A is consistent.

The eigenvector $(w_1, \ldots, w_n)$, $w_i \in \mathbb{R}_*$, $\sum_{i=1}^{n} w_i = 1$

is then taken to represent the normalised scores given to the options in Ω.

2.2 Geometric Method

This method of measuring consistency was developed independently by Crawford and Williams [1] and by Foster [2].

This method, for want of a better terminology , is called Geometric for two reasons

1. It uses the geometric mean
2. Vector space techniques are used to analyse it.

Basic Definitions

All matrices will be n x n. All logs are to base e and denoted by

1n(x) , for $x \in \mathbb{R}_*$. R*.

Skew-Symmetric Matrices

Let $\Re$ denote the set of reciprocal matrices

Let $A = (a_{i,j}) \in \Re$

Let $B = (b_{i,j})$ be defined by $b_{i,j} = 1n(a_{i,j})$

Then B is skew—symmetric i.e. $b_{i,j} = - b_{j,i}$,.

Let m be the vector space of all $n \times n$ matrices over $\mathbb{R}$.

Let $G \subseteq M$ be the subspace of skew—symmetric matrices .

G has dimension $\frac{1}{2}n(n-1)$.

Hence In defines a map $Ln : \Re \to G$ which is 1-1 and onto.

The inverse map is given by

$Ex((b_{i,j})) = (e^{b_{i,j}})$

The zero matrix in M will be denoted by 0 , without we hope, any confusion.

If $@ \subseteq \Re$ is the set of all consistent matrices then we let

$\Re = Ln@$.

<u>Lemma 2</u>

$B = (b_{i,j}) \in \Re$ iff $\exists b_1,....,b_n$ such that $b_{i,j} = b_i - b_j \ \forall$ i,j.

$\Re \subseteq G$ is a subspace of dimension n-1.

<u>Proof</u>

Follows from Lemma 1

E.O.P.

Note that Ln and Ex together with Lemma 2 allows us to analyse reciprocal and consistent matrices in the category of vector spaces.

In order to analyse these further we need the idea of distance in M.

<u>Metrics</u>

Let $B,C \in M$ where $B = (b_{i,j})$, $C = (c_{i,j})$.

Define

$$d(B,C) = \left(\sum_{j=1}^{n} \sum_{i=1}^{n} \left(b_{i,j} - c_{i,j}\right)^2 \right)^{1/2}$$

Thus m is identified with $\mathbb{R}^{n^2}$ and the metric is the standard Euclidean Distance on. $\mathbb{R}^{n^2}$

Also we have the norm $\|B\| = d(B,0)$ and associated inner product

$$< B,C >= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{i,j} c_{i,j} .$$

Using this inner product allows us to define orthogonality i.e.

B,C are orthogonal iff $<B,C> = 0$.

Subspaces U, B $\in$ G are <u>orthogonal</u> iff A $\in$ U , B $\in$ B $\Rightarrow$ $<A,B> = O$.

We say orthogonal subspaces $U,B \in G$ form an <u>orthogonal decomposition</u> iff U+B=G

In this case we write $U \oplus B = G$ and we have

$C \in G \to \exists A \in U, B \in B$ such that C = A + B and A and B are unique, (see [4]).

We now describe an orthogonal description of G.

First we need:

Let 1 be the 1 x n vector $\frac{1}{n}(1,1,....,1)$.

We have a linear map:

$w : G \to \mathbb{R}^n$ defined by w(A) = A1.

Thus w(A) is the column vector with ith entry the mean of the sum of the entries of the ith row of A.

$$w(A)_i = \frac{1}{n}\left(\sum_{j=1}^{n} a_{i,j}\right)$$

Note that $\sum_{i=1}^{n} w(A_j) = 0$ as A is skew—symmetric.

Now let $S = \ker(w) \subseteq G$.

Note that dimension $S = \frac{1}{2} n(n-1) - (n-1) = \frac{1}{2}(n-1)(n-2)$

<u>Lemma 3</u>

<u>$S \oplus \Re = G$</u>

<u>Proof</u>

First we show that the two subspaces Are orthogonal.

Let $A \in S$, $B \in \Re$; $A = (a_{i,j})$, $B = (b_{i,j})$.

Then $\exists b_1,....b_n$ such that $b_{i,j} = b_i - b_j$, $\forall i, j$.

$$Now < A, B >= \sum_{j=1}^{n}\sum_{i=1}^{n} a_{i,j} b_{i,j} = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{i,j}(b_i - b_j)$$

$$= \sum_{i=1}^{n} b_i \sum_{j=1}^{n} a_{i,j} - \sum_{j=1}^{n} b_j \sum_{i=1}^{n} a_{i,j}$$

But $\sum_{j=1}^{n} a_{i,j} = \sum_{i=1}^{n} a_{i,j} = 0$

Hence <A,B> = 0. so $\Re$ and S are orthogonal.

We now note that

$$\dim(S) + \dim(R) = \frac{1}{2}(n-1)(n-2) + (n-1) = \frac{1}{2} n(n-1) = \dim(G)$$

Hence we have $R \oplus S = G$

E.O.P.

Lemma 4

Let $A \in G$ Let B = ($b_{i,j}$) where $b_{i,j} = w(A)_i. - w(A)_j$

Then d(A, $\Re$> = d(A,B).

Thus B is the unique closest matrix in $\Re$ to A.

Proof

It is easy to show that w(A-B) = 0 .

Hence A - B $\in S$ which implies the result.

E.O.P.

We use d (A,B) as a measure of the consistency of A.

Thus d (A,B) = 0 iff A$\in$ R iff Ex(A) $\in$ iff Ex (A) is consistent

We note that w :G $\rightarrow \mathbb{R}^n$ is onto the subspace of R^n given by

$\{(x_1,....x_n): \sum_{i=1}^{n} x_i = 0\}$ and w| $\Re$ is an isomorphism.

In the above discussion a reciprocal matrix A = ($a_{i,j}$) was linearised by considering Ln(A) = (ln($a_{i,j}$)) which is a skew symmetric matrix.

By Lemma 4 the matrix C = ($c_{i,j}$) =($w(Ln(A))_i - w((Ln(A))_j$) is then the closest matrix in $\Re$ = Ln(@) to A

Note that

$$c_{i,j} = \frac{1}{n}\left(\sum \ln(a_{i,k}) - \ln(a_{j,k})\right) = \ln\left(\frac{a_{i,1}a_{i,2}...a_{i,n}}{a_{j,1}a_{j,2}...a_{j,n}}\right)^{\frac{1}{n}}$$

Thus on converting back into reciprocal matrices using Ex we see that C converts to the consistent matrix Ex(C) = (r_i/r_j) where

$$r_i = (a_{i,1}a_{i,2}...a_{i,n})^{\frac{1}{n}}, i = 1,...n.$$

r_j is the geometric mean of the ith row.

Thus we have:

1. ($r_1,...,r_n$) is an estimate of the relative weights of the options i.e. Ex(C) gives relative values of the options.

2. The measure of consistency of A is given by

$$\mu(A) = d(A,B) = \left(\sum_{j=1}^{n}\sum_{i=1}^{n}(\log(a_{i,j}) - C_{i,j})^2\right)^{\frac{1}{2}}$$

where B is as in Lemma 4 and

$$c_{i,j} = \ln\left(\frac{a_{i,1}a_{i,2}...a_{i,\mathrm{n}}}{a_{j,1}a_{j,2}...a_{i,\mathrm{n}}}\right)^{\frac{1}{\mathrm{n}}}$$

i.e.

$$\mu(A) = {}^{1}_{n}\left[\sum_{j=1}^{n}\sum_{i=1}^{n} (\ln(a_{i,1}\,a_{1,j}\,a_{j,i}\,a_{i,z}\,a_{z,j}\,a_{j,i}\,......a_{i,\mathrm{n}}\,a_{\mathrm{n},i}\,a_{j,i}))^{2}\right]^{\frac{1}{2}}$$

2.3 Trace Index

If $M = (m_{i,j})$ is a matrix then $\mathrm{tr}(M) = \sum_{i=1}^{n} m_{i,i}$ is the trace of M.

Let A be a reciprocal matrix.

Then $\mathrm{tr}(A^3) = \sum_k \sum_j \sum_i a_{i,j} a_{j,k} a_{k,i}$

Let $c_{ijk} = a_{i,j}\, a_{j,k}\, a_{k,i}$.

A is consistent iff $C_{ijk} = 1 \quad \forall i, j, k$.

There are 6 possible permutations of a fixed i,j,k.

Testing the values of each such permutation gives b) and c) in the following Lemma.

<u>Lemma 5</u>

a) $c_{ijk} = 1$ if any two of i,j,k are the same.
b) $c_{ijk} = c_{abc}$ if a,b,c is an even permutation of i,j,k.
c) $c_{ijk} = c^{-1}_{abc}$ if a,b,c is an odd permutation of i,j,k.

Given i,j,k where $i < j < k$ then there are three permutations which are odd, three even.

Also the total contribution to $\mathrm{tr}(A^3)$ from triples where at least two are the same is by a) above

$$n^3 - 6.\frac{1}{6}n(n-1)(n-2) = n^3 - n(n-1)(n-2)$$

Hence we see that $\mathrm{tr}(A^3) = 3 \sum_{i<j<k} (c_{ijk} + c^{-1}_{ijk} + 2) + \qquad n^3$

We let

$$\tau(A) = \frac{\mathrm{tr}(A^3) - n^3}{n^2} = 3\sum_{i<j<k}(c_{ijk} + c^{-1}_{ijk} - 2)/n^2 \,.$$

We call $\tau(A)$ the <u>trace</u> <u>index</u> of A.

Properties of τ (A)

1. $\tau(A) \geq 0$

2. τ (A) = 0 iff A is consistent

Proof

1. follows as $c_{ijk} + c_{ijk}^{-1} - 2 \geq 0$ for all i, j, k

2. A is consistent $\rightarrow c_{ijk} = 1$ for all i,j,k

$$\rightarrow \tau(A) = 0$$

$$\tau(A) = 0 \rightarrow \sum_{i<j<k} (c_{ijk} + c_{ijk}^{-1} - 2) = 0$$

$$\rightarrow \quad c_{ijk} + c_{ijk}^{-1} - 2 = 0 \qquad \text{for all i,j,k}$$

$\rightarrow$ $c_{ijk} = 1$ for all i.j.k as the mi minimum value of

$c_{ijk} + c^{-1}{}_{ijk} - 2$ at $c_{ijk} = 1$.

Hence A is consistent.

E.O.P.

Note that in order to compute $\tau(A)$ we need only consider the

$\frac{1}{6}$n n(n-l)(n-2) triples i,j,k where i<j<k .

Further if we consider $A^3 = (d_{i,j})$ then we call

$$\tau_i(A) = \frac{d_{i,i} - n^2}{n^2} \quad \text{the i - index} \ , \ 1 \leq i \leq n$$

Lemma 6

a) $\tau_i(A) \geq 0 \quad ,i = 1,..,n$

b) $\tau(A) = \sum_{i=1}^{n} \tau_i(A)$

c) A is consistent iff $\exists$ i such that $\tau_i(A) = 0$

Proof

a) and b) follow easily from the definition..

For c) we observe that $c_{jkl} = c_{ijk} c_{ikl} c_{ijl}^{-1}$

Hence $\tau(A) = 0 \Leftrightarrow c_{ijk} = 1$ for all j,k $\Leftrightarrow c_{jkl} = 1$ for all j,k,l

$\Leftrightarrow$ A is consistent.

The i— indices $\tau(A)$ i = l,..,n are useful in indicating which option is linked to the most inconsistent decisions.

We examine this in more detail in the next section.

2.4 .The Saaty , Geometric Weights and the Trace Index

We now show for sufficiently small trace index that the Saaty eigenvector and the geometric weights agree to within a given tolerance.

Using the Saaty method we have the maximum eigenvalue λ and the associated eigenvector $(w_1, \ldots, w_n)$ such that $w_i > 0$ for all i and $\lambda \geq n$.

Note that $\lambda = \sum_{j=1}^{n} a_{i,j} \frac{v_j}{v_i}$

Using the geometric method we have geometric weights

$(v_1, \ldots v_n)$ where $v_i = (a_{i1} a_{i2} \ldots a_{in})^{\frac{1}{n}}$, i = 1,..,n.

$$\text{Let } S_i = \sum_{j=1}^{n} a_{i,j} \frac{v_j}{v_i}$$

$$= \sum_{j=1}^{n} (a_{i,j} a_{j,1} a_{1,i} \ldots a_{i,j} a_{j,n} a_{n,i})^{\frac{1}{n}}$$

$$\text{Now } (a_{i,j} a_{j,1} a_{1,j} \ldots a_{i,j} a_{j,n} a_{n,i})^{\frac{1}{n}} \leq \frac{1}{n} \sum_{k=1}^{n} a_{i,j} a_{j,k} a_{k,i}$$

This follows by the usual arithmetic mean - geometric mean inequality.

Hence we obtain for i = 1,..,n

$$S_i \leq \frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{i,j} a_{j,k} a_{k,i}$$

$$\text{Let } \sigma = \frac{1}{n} \sum_{i=1}^{n} S_i .$$

$$\text{Then } \sigma \leq \frac{1}{n} 2 \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{i,j} a_{j,k} a_{k,i} \right] = \frac{tr(A^s)}{n^2}$$

Let us now assume that $\tau(A) = \frac{tr(A^3) - n^3}{n^2} = \varepsilon > 0$

Hence we have that $tr(A^3) = \varepsilon n^2 + n^3$

And it follows that $\sigma \leq \frac{\varepsilon n^2 + n^3}{n^2} = n + \varepsilon.$

Lemma 7

Assume $\tau(A) = \varepsilon$, then

1. $\sigma \le n + \varepsilon$.

2. $n \le S_i \le n + n\varepsilon$.

Proof

1. is shown above

Proof of 2.

We have $$S_i = \sum_{j=1}^{n} a_{i,j} \frac{v_j}{v_i}$$

$$\ge \frac{n}{v_j} (a_{i1} a_{i2} \ldots a_{in})^{\frac{1}{n}} (v_1 v_2 \ldots v_n)^{\frac{1}{n}}$$

$$= \frac{n}{v_i} v_i = n$$

as $v_1 v_2 \ldots v_n = 1$

Hence $S_i \ge n$, $i = 1,..,n$

Also $S_i \le n + n\varepsilon$, $i = 1,..,n$

follows from $\sigma = \frac{1}{n}\sum_{i=1}^{n} S_i$ and $\sigma \le n + \varepsilon$..

E.O.P.

Now let V= $(v_1 , v_2 ,\ldots v_n)$. Since $S_i = \sum_{j=1}^{n} a_{i,j} \frac{v_j}{v_i}$ we have

$AV' = (S_1 V_1, S_2 V_2, \ldots, S_n v_n)'$.

Thus Lemma 7 implies that the smaller $\tau(A) = \varepsilon$ is the closer that V' is to an eigenvector associated to the maximum eigenvalue i.e. the Saaty eigenvector.

Note that V' is not in general an eigenvector and that the variance of the S_i , i = 1,..,n , measures to some extent how far V' is from being an eigenvector with associated eigenvalue σ.

Thus in the presence of "large" values of $\tau(A)$, or major inconsistencies in the pairwise comparisons ,the geometric weights and the Saaty weights derived from the eigenvector will differ.

This was investigated in [1] where it was also -found that the geometric weights were better than the eigenvector weights in the following sense.

A perfectly consistent matrix A gives a set of weights which are the same for both methods, geometric or eigenvector.

The matrix was then perturbed by changing some of the entries, although these changes were constrained by the perturbed matrix remaining reciprocal. The geometric and eigenvector weights were then calculated for the perturbed matrix and compared with the original weights. It was found over a large number of trials that the geometric weights were statistically significantly closer to the original weights than the eigenvector weights when the inconsistency was high.

Section 3. Deletion and Addition of Options

Let $A^{(i)}$ be the (n-1) × (n-1) matrix obtained by deleting the ith row and column. Note that $A^{(i)}$ is still a reciprocal matrix. Consider the following definition:

The i - inconsistency of option i is the trace index of $A^{(i)}$.

i.e. $\tau(A^{(i)})$.

Lemma 7

$$\tau(A^{(i)} = \frac{n^2}{(n-1)^2}(\tau(A) - 3\tau_i(A)) \text{ , } i = 1,\dots,n$$

Proof

Follows by an easy computation. E.O.P.

Thus we note that if we delete the option with largest i – index $\tau_i(A)$ then we obtain the most consistent (as measured by the trace index) set of n-1 options.

For this reason we can consider τ_i (A) as a reasonable measure of the contribution that a particular option has on the inconsistency of the set of decisions.

If we calculate the geometric or Saaty weights of $A^{(i)}$,i.e. after deleting option i , we do not in general preserve the relativities between the other options as calculated from A. (Although they are preserved for A consistent.)

For example:

Suppose there are 4 options and the -following comparison matrix A is formed using the 9 point scale described in Section 1.1 . Also we have the normalised geometric weights calculated from this matrix:

Decision Matrix A				Geometric Weights
1	1/3	½	½	0.1427
3	1	1	2	0.3531
2	1	1	1	0.2695
1	1/2	1	1	0.2347

Table 1

Now let us add another option to be compared to the other 4 and we obtain the following matrix B, where the fifth options' comparisons are included in the fifth row and column and the weights have been recalculated as shown..

Decision Matrix B					**Geometric Weights**
1	1/3	1/2	1/2	7	0.1833
3	1	1	2	9	0.3422
2	1	1	1	7	0.2612
2	1/2	1	1	7	0.2274
1/7	1/9	1/7	1/7	1	0.0309

Table 2

If we consider the first 4 options then their relative weights have changed e.g. if $r_{i,j}$ is the ratio of option i to option j weights we have for the above matrices

	$r_{1,2}$	$r_{2,3}$	$r_{3,4}$
A	.3384	1.3171	1.1944
B	.4073	1.3237	1.1487

This has been considered a problem (see [1]) and as a result these methods of obtaining weights from a reciprocal matrix have been criticised .

However these criticisms are readily answered by considering the effect of adding the fifth option.

Matrix A provides a set of normalised weights for the first 4 options as in Table 1 .
Also the fifth column of matrix B gives weights w_i , i=1,...4 for the first four options where $w_i = b_{i5}$. On normalising these weights we now have two normalised lists of weights for the first four options.

From Matrix A	From Last Column of B
0.1427	0.2333
0.3531	0.3000
0.2695	0.2333
0.2347	0.2333

Thus there are two differing sets of weights (not too dissimilar due to the reasonable overall consistency of this example as meaured by the trace index).
Thus it is not reasonable to expect that the weights of the first four options as calculated from A should remain the same in the presence of the extra comparisons given by B.
The methodologies adopted by Saaty or in the Geometric Method have within them implicit methods of combining these conflicting lists of weights.
For example , in the Geometric Method the two lists above are combined by taking the weighted geometric mean of the entries of the lists and then renormalising. The weights used are 4/5 for the weights from A and 1/5 for the weights from the last column of B. This is reasonable given the relative amounts of information provided by A and the last column of B as far as the first four options are concerned.
The above argument clearly demonstrates that in the presence of significant inconsistency it would be invalid for the deletion of an item to lead , in general , to the same relative weights for the other options.

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