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By

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1. Introduction

The purpose of this note is to explain the result that a convex combination of two parametrically defined C^2 surfaces is not necessarily C^2. This result, which may seem surprising, has serious implications for the development of C^2 surface patches for computer aided geometric design (c.a.g.d) based on convex combination ideas. In particular, in Section 3, we show that the construction of a C^2 polygonal surface patch based on a simple generalization of a scheme in Charrot and Gregory '84, see also Gregory '86 , is not valid. In a later paper we will describe a construction which allows the development of C^2 polygonal surface patches for c.a.g.d.

In order to describe the problem, the concept of a C^2 surface and the C^2 join of two surface patches must first be explained. The basic idea is that two patches have a C^2 join if their union is a C^2 surface in the sense of differential topology. In this setting C^2 continuity presents rather a simple problem and the interested reader should note that the concept of 'geometric continuity', now being considered within the field of c.a.g.d., has already been addressed by the subject of differential topology. However, our concern here is to make the development meaningful to the non-specialist reader and hence we find it more convenient to describe the C^2 join of two surface patches without first needing to define a C^2 surface. A C^2 surface can then be considered as a collection of such 'C^2 joining' surface patches. This, in essence, is how the problem is treated in differential topology, although in that subject the topic is made mathematically simpler by having a collection of overlapping patches, each defined on an open domain in $\mathbb{R}^2$. In the practical application of c.a.g.d. our patches meet, but do not overlap, and are defined on closed domains in $\mathbb{R}^2$.

The problem of defining the C^2 join of two surface patches has already been addressed by a number of authors in the context of c.a.g.d., see for example Veron et al '76, Herron '85, DeRose '85, Höllig '86. Our development here builds on this earlier work. We begin, in Section 2, by making a basic mathematical definition of a C^2 join and then develop a practical test for verifying such a C^2

join of two surface patches (Theorem 2.1). This test proves most appropriate for the purposes of this paper, although there are many possible equivalents. We then consider the problem posed by taking a convex combination of two C^2 surface patches. An important point to note is that we are treating parametric rather than implicitly defined surface representations.

The problems raised by taking a convex combination of parametrically defined representations can be illustrated by considering the simpler problem of curves in $\mathbb{R}^2$. Consider two regular parametric curves $q : [0,1] \rightarrow R^2$ and $p : [0,1] \rightarrow \mathbb{R}^2$ which are such that

$$q(1) = p(0)$$

$$\dot{q}(1) = \mu \dot{q}(0), \mu > 0 \tag{1.1}$$

$$\dot{q}(1) = \mu^2 \ddot{p}(0) + \nu \dot{p}(0)$$

for real numbers μ and ν. Then the curves meet C^2 with respect to arc length, i.e. their union is a C^2 curve. (This condition seems to have appeared first in the context of c.a.g.d. in Manning '74. It was used by Nielson '74 in the development of his parametric ν - spline and later developed by Barsky '81, and generalized by Goodman '85, in connection with local support bases for parametric splines.)

Now consider two unions, of curves q_1 and p_1 and curves q_2 and p_2, both of which are C2 curves but with different transition values μ_1, ν_1 and μ_2, ν_2 in their respective conditions (1.1). Then the union of q and p , where

$$q = \alpha_1 q_1 + \alpha_2 q_2, p = \alpha_1 p_1 + \alpha_2 p_2, \alpha_1, \alpha_1 \in \mathbb{R} \tag{1.2}$$

is not necessarily C^2 or indeed C^1. In the context of this paper, we are essentially concerned with the case where $q_1 = q_2 = q$ say, is given and $\alpha_1 + \alpha_2 = 1$ $\alpha_1, \alpha_2 1, \geq 0$. (In fact the α_i will be scalar functions of the parametric variable.) In this case the union of q and p is C^1 but not, in general, C^2.

2. Geometric Continuity between Patches

Notation: Given the twice continuously differentiable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, we use the convenient and concise notation $\partial f\big|_X \text{and} \partial^2 f\big|_X$ to respectively represent the first derivative linear map and the second derivative symmetric bilinear map at a point $X=(x_1,x_2)\in \mathbb{R}^2$. Thus, given any vectors $U=(u_1,u_2)$, $V=(v_1,v_2)\in \mathbb{R}^2$, we have that

$$\partial f\,\big|\,X^{(U)=\frac{d}{dt}f(x+tU)}\Big|_{t=0}$$

$$u_1 f_{1,0}(x)+u_2 f_{0,1}(x) \tag{2.1}$$

$$\partial^2 f\Big|(U,V)=\frac{\partial^2}{\partial s\partial t}f(X+tU+sV)\Big|_{s=t=0}$$

$$=u_1v_1f_{2,0}(x)\ +\ (u_1v_2+u_2v_1)f_{1,1}(x)\ +\ u_2v_2(x) \tag{2.2}$$

where f $f_{\nu,\mu}=\partial^{\nu+\mu}f/\partial x_1^{\nu}\partial x_2^{\mu}$ denotes a partial derivative. If $\phi:\mathbb{R}^2\rightarrow\ \mathbb{R}^2$ and $f:\mathbb{R}^2\rightarrow\ \mathbb{R}^3$ are continuously differentiable, we note that the composed map $f\ {}^{\circ}\,\phi:\mathbb{R}^2\rightarrow\ \mathbb{R}^3$ is differentiable and that the chain rule

$$\partial(f\circ\phi)\big|_X=\partial f\big|_{\phi(X)}{}^{\circ}\partial\phi\,\big|\,X \tag{2.3}$$

applies.

<u>Definition</u> (C^2 Patch) Let $\Omega\in\mathbb{R}^2$ G R^2 be a closed polygon (for example a rectangle or triangle) or be diffeomorphic to a closed polygon. Then $p:\Omega\rightarrow\mathbb{R}^3$ defines a C^2 parametric patch if p is a C^2 map of rank 2 up to and including the boundary of Ω . (Thus p , ∂p and ∂^2p are well defined at all points including those on the boundary.)

Here, we are using a definition appropriate to c.a.g.d. and hence have specified a type of <u>closed</u> domain in $\mathbb{R}^2$ which describes those of practical interest. In particular cusps in the domain, where the derivative maps ∂ and ∂^2p are not well defined, are not allowed . We will consider derivatives on the boundary without adding the qualifying remark that limits to boundary points are approached from within Ω . (In any case, the existence of the C^2 map up to and including the boundary is equivalent to p having a C^2 extension to an <u>open</u> region $\Omega_E\,\Omega_E\supset\Omega$.) It should also be noted that, to avoid singularities, we require the parametric patch representation to be <u>regular</u>, i.e. ∂p has rank 2 for all points in Ω . However, self intersections of the patch are allowed.

<u>C^2 Continuity between Patches.</u> Consider two C^2 patches q and p with domains Ω_q and Ω_p . Let $e_q : [0,1] \to \mathbb{R}^2$ be a parametric representation of a boundary segment of Ω_q. Then we make the following basic definition.

<u>Definition</u> (C^2 join) The two patches q and p have a C^2 join if there exists a boundary segment $e_q : [0,1] \to \mathbb{R}^2$ of Ω_q and a C^2 diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, defined in a neighbourhood of e_q , such that the following properties hold:

(i) (domain continuation) $e_p = \phi \circ e_q$ is a boundary segment of Ω_p and ϕ is such that interior points of Ω_p are mapped from exterior points of Ω_p

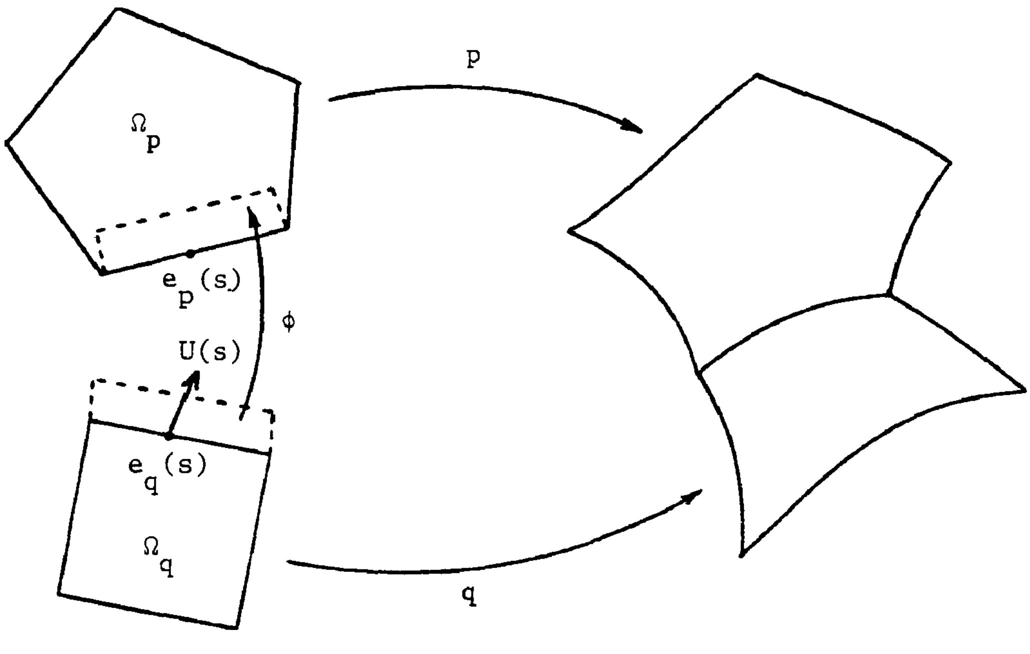
(ii) (patch continuity) For all s $\in$ [0,1]

$$q \circ e_q(s) = p \circ \phi \circ e_q(s) \tag{2.4}$$

$$\partial q\big|e_q(s) = \partial(p \circ \phi)\big|e_q(s), \tag{2.5}$$

$$\partial^2 q\big|e_q(s) = \partial^2(p \circ \phi)\big|e_q(s), \tag{2.6}$$

This definition is a consequence of the concept that q and $p \circ \phi$ should form a C^2 map on a domain containing the boundary sector e_q , see Figure 2.1



<u>Figure 2.1</u>

In the subsequent work we assume that the domain segment e_q is a regular C^2 parametric representation. We then have the following lemma.

Lemma 2.1 (C2 join) Consider a (non-zero) transversal vector field U(s) defined on $e_q(s)$ and let

$$V(s) : \partial\phi\big|_{e_q(s)}(U(s)) \tag{2.7}$$

$$W(s) : \partial^2\phi\big|_{e_q(s)}(U(s), U(s)) \tag{2.8}$$

Define vector fields on $e_q(s) = \phi \circ e_q(s)$, where ϕ satisfies the domain Continuation property (i). Then the patches have a join if and only if

$$q(e_q(s)) = p(e_p(s)) \tag{2.9}$$

$$\partial q\big|_{e_q(s)}(U(s)) = \partial p\big|_{e_p(s)}(V(s)) \tag{2.10}$$

$$\partial^2 q\big|_{e_q(s)}(U(s), U(s)) = \partial^2 p\big|_{e_p(s)}(V(s), V(s)) + \partial p\big|_{e_p(s)}(W(s)) \tag{2.11}$$

for all s $\in$ [o,1].

Proof We must show that conditions (2.9)-(2.11) are equivalent to (2.4)-(2.6). The necessity is immediately apparent, applying the chain rule (2.3) to (2.5) and (2.6). To prove the sufficiency we must exhibit another vector field U(s), linearly independent of U(s), such that

$$\begin{aligned}
\partial q\big|_{e_q(s)}(\hat{U}(s)) &= \partial(p \circ \phi)\big|_{e_q(s)}(\hat{U}(s)) \\
\partial^2 q\big|_{e_q(s)}(\hat{U}(s), \hat{U}(s)) &= \partial^2(p \circ \phi)\big|_{e_q(s)}(\hat{U}(s), \hat{U}(s)) \\
\partial^2 q\big|_{e_q(s)}(U(s), \hat{U}(s)) &= \partial^2(p \circ \phi)\big|_{e_q(s)}(U(s), \hat{U}(s)).
\end{aligned} \tag{2.12}$$

Differentiating (2.9) twice and (2.10) once with respect to s shows that $\hat{U}(s) = \dot{e}_q(s)$ is such a vector field.

Lemma 2.1 leads to the following theorem on the C^2 join of two patches which, in effect, states conditions for the existence of the C^2 diffeomorphism ϕ.

Theorem 2.1 (C^2 join) Let $e_q : [0,1] \to \mathbb{R}^2$ and $e_q : [0,1] \to \mathbb{R}^2$ be regular C^2 representations of boundary segments of Ω_q and Ω_p respectively, such that the C° continuity constraint (2.9) holds. Let $U : [0,1] \to \mathbb{R}^2$ be a given non-zero C^2 vector field, transversal to e_q in an outward direction and suppose there exist a non-zero C^1 vector field $V : [0,1] \to \mathbb{R}^2$, transversal to e_p in an inward direction, and a C^0 vector field $W : [0,1 \to \mathbb{R}^2$, such that equations (2.10) and (2.11) hold. Then the patches q and p have a C^2 join.

Proof From Lemma 2.1 we require to exhibit a C^2 diffeomorphism ϕ such that

$$\begin{aligned} \phi(e_q(s)) &= e_q(s) \\ \partial\phi\big|_{e_q(s)}(U(s)) &= V(s) \\ \partial^2\phi\big|_{e_q(s)}(U(s), U(s)) &= W(s) \end{aligned} \tag{2.13}$$

Let $X : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$X(s,t) = e_q(s) + t\ U(s) \tag{2.14}$$

Then if $e_p(s)$, $V(s)$, and $W(s)$ are C^2, the C^2 diffeomorphism ϕ can be defined by the requirement that

$$\phi \circ x(s,t) = e_p(s) + tV(s) + \frac{t^2}{2}W(s) \tag{2.15}$$

(i.e. the conditions (2.13) are satisfied by Taylor interpolation) and then ϕ is well defined in a neighbourhood of e_q . The weaker continuity conditions of the theorem are possible by constructing ϕ such that

$$\begin{aligned} \phi \circ z(s,t) = e_p(s) &+ \int_s^{s+t} v(\theta)d\theta \\ &+ \int_s^{s+t}\int_s^{\theta}[W(\hat\theta) - \dot v(\hat\theta)]d\hat\theta d\theta. \end{aligned} \tag{2.16}$$

Remarks It should be noted that, with appropriate assumptions, conditions (2.9)-(2.11) are both necessary and sufficient for a C^2 join. Also the conditions are the exact surface equivalents of the conditions (1.1) for curves. Higher order C^k continuity conditions of this type can be derived by generalizing conditions (2.4)-(2.6) and making repeated use of the chain rule.

Equation (2.11) can be viewed as the requirement that

$$\partial^2 q\big|_{e_q(s)}(U(s), U(s)) = \partial^2 p\big|_{e_q(s)}(V(s), V(s))$$

should lie in the tangent plane of the point $q(e_q(s)) = p(e_p(s))$ along the join of the patches, since $\partial p|_{e_p(s)}$ (W(s) lies in this tangent plane. We thus have:

Corollary 2.1 Under the assumption that (2.9) and (2.10) hold, the third condition (2.11) can be replaced by the requirement that

$$<\partial^2 q|e_q(s)(U(s),(U(s)),N(s)> \; = \; <\partial^2 p|e_p(s)(V(s),V(s)),N(s)> \quad (2.17)$$

Where

$$N(s) = N_q(e_q(s)) = N_p(e_p)(s))$$

is the common surface normal along the edge.

Example Consider the simple case $\Omega_q = [0,1]\times[-1,0,]\Omega_p = [0,1]\times[0,1]$ with boundary segments e_q (s) = e_p(s) = (s,0) and transversal vector field U(s) = (0,1) Then with V(s) = (v_1(s),v_2(s)), W(s) = (w_1(s),w_2(s)),

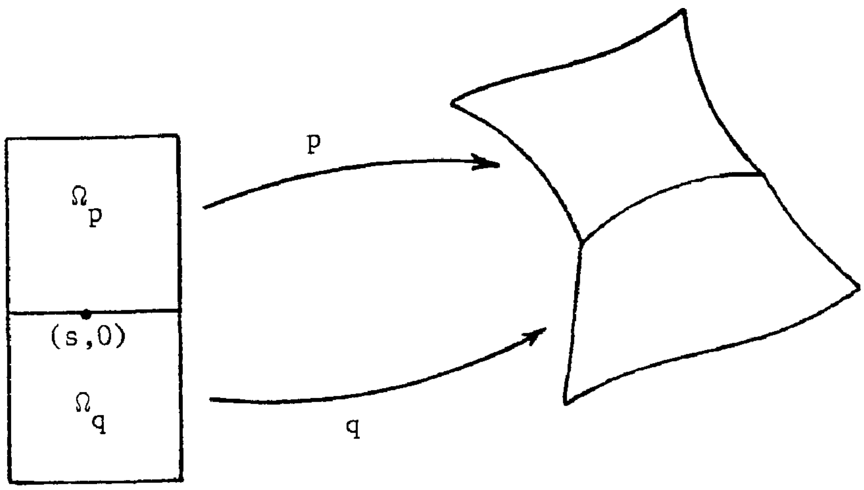


Figure 2.2

the continuity conditions (2.9)-(2.11) are stated as

$$q(s,0) = p(s,0) \quad (2.18)$$

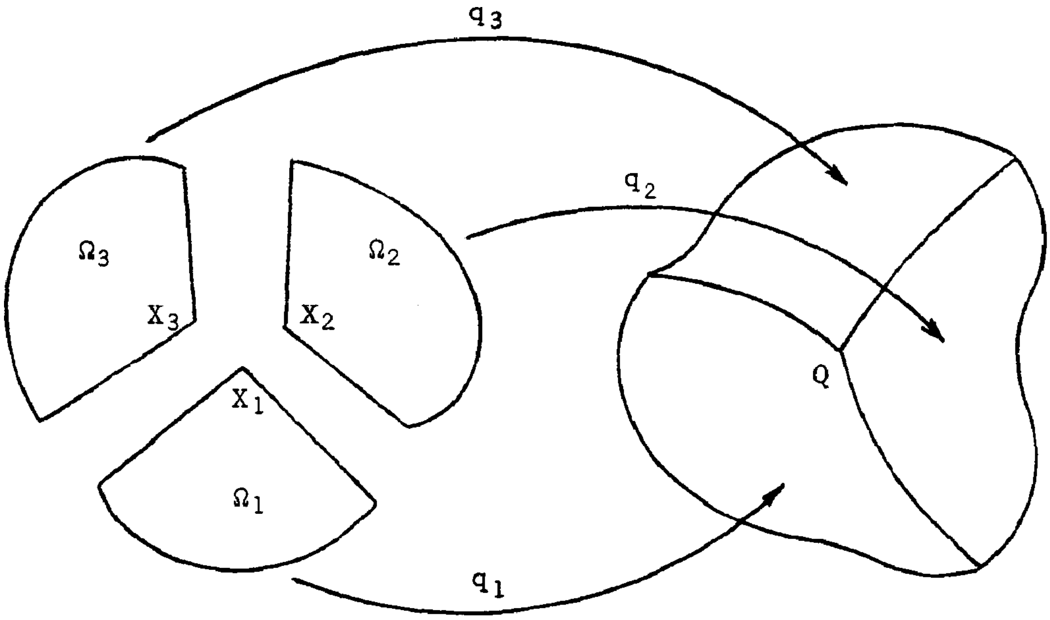
$$q_{0,1}(s,0) = v_1(s)p_{1,0}(s,0) + v_2(s)p_{0,1}(s,0) \quad (2.19)$$

$$\begin{aligned} q_{0,2}(s,0) = {} & v_1(s)^2 p_{2,0}(s,0) + 2v_1(s)v_2(s)p_{1,1}(s,0) + v_2(s)^2 p_{0,2}(s,0) \\ & + W_1(s)p_{1,0}(s,0) + W_2(s)p_{1,0}(s,0) \end{aligned} \quad (2.20)$$

These are similar to conditions derived by Höllig '86, following de Rose '85 and Barsky and de Rose '85, where the unisvariate functions v_1,v_2,w_1,and w_2 are called 'shape parameters'. If conditions (2.18)-(2.20) hold, then the two patches have a C^2 join, where from (2.13), the diffeomorphism ϕ is such that

$$\begin{aligned} \phi(s,0) &= (s,0) \\ \phi_{0,1}(s,0) &= (v_1(s), v_2(s)) \\ \phi_{0,2}(s,0) &= (w_1(s), w_2(s)) \end{aligned} \qquad (2.21)$$

<u>Continuity at a vertex.</u> We have so far considered edge continuity conditions across boundary sectors of two adjacent patches, consider now the situation where a number of such patches meet with a common “vertex point” in $\mathbb{R}^{\cdot}$ This is illustrated by the case of three parches $q_i : \Omega_I \to \mathbb{R}^3$, i=1,2,3, meeting at the vertex point $Q \in \mathbb{R}^3$ as in Figure 2.3 .



<u>Figure 2.3</u>

Let $q_i(X_i) = Q$ for given boundary points $X_i \in \Omega_i$, i = 1,2,3, and assume that edge continuity conditions of the form (2.9)-(2.11) are satisfied between q_1 and q_2, q_2 and q_3 , q_3 and q respectively. The C^1 edge continuity constraints imply that

$$\partial q_1|X_1(\mathbb{R}^2) = \partial q_2|X_2(\mathbb{R}^2) = \partial q_3|X_3(\mathbb{R}^3)$$

and hence the patches $q_i : \Omega_i \to \mathbb{R}^3$ have a common tangent plane at $q(X_i) = Q$, i = 1,2,3 Furthermore, the C^2 edge continuity constraints imply that the orthogonal projection of the composite surface onto the tangent plane, in a neighbourhood of Q , defines a map which is C^2 across each of three edges in the common tangent plane domain. Hence the map is C^2 across the vertex Q .

A C^2 Surface, We are now in a position to define a C^2 surface for c.a.g.d. as a composition of C^2 patches, where adjacent patches join with C^2 continuity. This accords with the idea of a "C^2" immersed surface" (with piecewise C^2 boundary) in the language of differential topology, where a "surface" is a 2-dimensional manifold. An introduction to the calculus of manifolds can be found in Spivak '65, although there the discussion is restricted to "embedded manifolds" for which self intersections are not allowed. The case of "immersed manifolds", where self intersections are allowed, is covered by Hirsch '76.

3. Convex Combination Patches

The General Problem. Let $q_i : \Omega_q \rightarrow IR^3$, and $p_i : \Omega_p \rightarrow IR^3$, i=1,2,. be such that the C^2 continuity constraints of Theorem 2.1 are satisfied across the boundary sectors $e_q : [0,1] \rightarrow IR^2$ of Ω_q and $e_p : [0,1] \rightarrow IR^2$ of Ω_p. Thus

$$\begin{aligned} & q_i(e_q(s)) = p_i(e_p(s)) \\ & \partial q_i\big|_{e_q(s)}(U)\ (s)\ ) = \partial p_i\big|_{e_p(s)}(V_i(s)\ ) \\ & \partial^2 q_i\big|_{e_q(s)}(U(s)\ ,\ U(s)\ ) = \partial^2 p_i\big|_{e_p(s)}(V_i(s)\ ,\ V_i(s)\ ) + \partial p_i\big|_{e_p(s)}(w_i(s)\ ) \end{aligned} \tag{3.1}$$

for appropriately defined vector fields. Consider the union of two surfaces $q : \Omega_q \rightarrow IR^3$, $p : \Omega_p \rightarrow IR^3$, defined by

$$\begin{aligned} p &= \alpha_1 p_1 + \alpha_2 p_2 \\ q &= \beta_1 q_1 + \beta_2 q_2 \end{aligned} \tag{3.2}$$

where $\alpha_i : \Omega_p \rightarrow IR$, $\beta_i : \Omega_q \rightarrow IR$ are given C^2 functions and $\alpha_1(x) + \alpha_2(x) = 1$, $x \in \Omega_p$, $\beta_1(x) + \beta_2(x) = 1$, $x \in \Omega_q$. Then the resulting surface is C^0 but not, in general, C^1.

More particularly we are concerned with the case where $q_1 = q_2 = q$ and p is a convex combination of p_1 and p_2. Here the union of $q : \Omega_q \rightarrow IR^3$ and $p : \Omega_p \rightarrow IR^3$, where

$$p = \alpha_1 p_1 + \alpha_2 p_2 \tag{3.3}$$

$$\alpha_1(x) + \alpha_2(x) = 1, \alpha_1(x), \alpha_2(x) \geq 0, x \in \Omega_p \tag{3.4}$$

is C^1 but not necessarily C^2. To argue this in the general case it is easier to reverse the roles of the continuity constraints and also make use of Corollary 2.1 Hence for i = 1,2, we assume that

$$p_i(e_q(s)) = q(e_q(s)), \tag{3.5}$$

$$\partial p_i\Big|_{e_p(s)}(U(s)) = \partial q\Big|_{e_q(s)}(V_i(s)), \tag{3.6}$$

$$< \partial^2 p_i\Big|_{e_{p((s)}}(U(s)) - \partial^2 q\Big|_{e_q(s)}(V_i(s), V_i(s)), N(s) > = 0 \tag{3.7}$$

where U(s) is a vector field transversal to e_p (s) in an outward direction, $V_i(s)$, i = 1,2, are transversal to e_q (s) in inward directions and N(s) is the common surface normal. It can then be shown that

$$p(e_p(s)) = q(e_q(s)) \tag{3.8}$$

$$\partial p\Big|_{e_{p(s)}}(U(s)) = \partial q\Big|_{e_{q(s)}}(v(s)) \tag{3.9}$$

where

$$V(s) = \alpha_1(s)V_1(s) + \alpha_2(s)V_2(s) \tag{3.10}$$

is transversal to $e_q(s)$ in an inward direction since the weights $\alpha_i(s) := \alpha_i(e_p(s))$ are non-negative. Thus the union of q and p is C^1. However

$$
\begin{aligned}
&< \partial^2 p\Big|_{e_p(s)}(U(s), U(s)) - \partial^2 q\Big|_{e_p(s)}(V(s), V(s)), N(s) > \\
&=< \alpha_1(s)\partial^2 q\Big|_{e_q(s)}(V_1(s), V_1(s)) + \alpha_2(s)\partial^2 q\Big|_{e_q(s)}(V_2(s), V_2(s)) \\
&\quad - \partial^2 q\Big|_{e_q(s)}(\alpha_1(s)V_1(s) + \alpha_2(s)V_2(s), \alpha_1(s)V_1(s) + \alpha_2(s)V_2(S)), N(s) >,\Big|
\end{aligned} \tag{3.11}
$$

which is not, in general, zero. Hence the union of q and p is not, in ${}_2$ general, C^2.

The union of q and p could be C^2 in particular cases. In the following subsection we consider the particular case of a polygonal interpolant due to Charrot and Gregory.

The Gregory-Charrot Scheme for Polygonal Patches. The Gregory-Charrot interpolant, see Charrot and Gregory '84 and Gregory '86, solves the problem of fitting a C^1 patch defined on a polygonal domain into a C^1 surface of rectangular patches. Let Ω_p be a regular N-sided polygon with sides of length unity and vertices $x_i, i = 0,....,n-1$. Then the Gregory-Charrot patch $p : \Omega_p \to IR^3$ takes the form

$$p(x) = \sum_{i=0}^{N-1} \alpha_i(x) p_i(x), (x), x \in \Omega_p \tag{3.12}$$

Here, $p_i : \Omega_p \to IR^3$ is a surface patch which has a C^1 join with adjacent rectangular patches along the two boundary sectors of the polygon with X_i as a common vertex. The weights $\alpha_i : IR^3 \to IR$ are C^1 functions such that

$$\sum_{i=0}^{n-1} \alpha_i(x) = 1, \alpha_i(x) \geq 0, x \in \Omega_p \tag{3.13}$$

and for boundary points X not on the two boundary sectors with the common vertex X.

$$\alpha_i(x) = 0,\ \partial\alpha_i|_X = 0 \tag{3.14}$$

The patches p_i have C^1 joins in the sense of Theorem 2.1, being defined in terms of local 'radial' coordinate systems about each vertex. Thus

$$p_i = P_i \circ \phi_i, \tag{3.15}$$

where $p_i : IR^2 \to IR^3$ is the local coordinate patch which matches the adjoining rectangular patch with parametric C^1 continuity. The diffeomorphism $\phi_i : \Omega_p \to IR^2$ is defined by the radial construction

$$(u_i, v_i) = \phi_i(x), \tag{3.16}$$

where

$$\begin{aligned} E_i &= (1-u_i)x_i + u_i x_{i+1} \\ E_{i-1} &= (1-v_i)x_i + v_i x_{i+1} \end{aligned} \tag{3.17}$$

define points of intersection of radial lines through $x \in \Omega_p$ with the boundary sectors as in Figure 3.1.

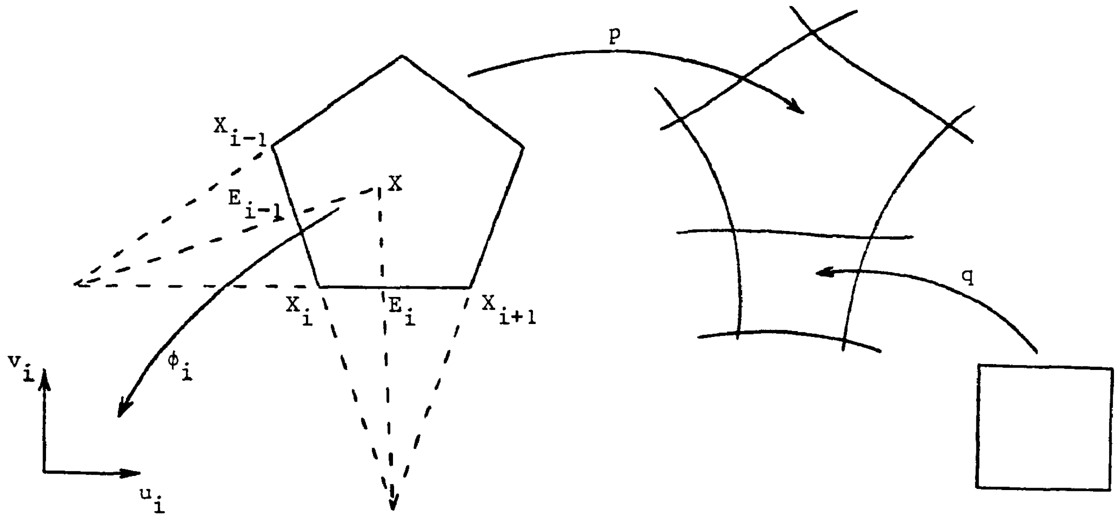


Figure 3.1

A full description of the Gregory-Charrot patch, including details of the construction of P_i using Boolean sum Taylor interpolation, is given in Gregory '86. Our purpose here is to show that the scheme cannot be extended in an obvious way to treat the case of C^2 surfaces. For this it suffices to consider the simpler construction

$$p(x) = \alpha_1(x)p_1(x) + \alpha_2(x)p_2(x) \tag{3.18}$$

where

$$\alpha_1(x) + \alpha_2(x) = 1 \tag{3.19}$$

Let

$$e(s) = (1-s)x_1 + sx_2 \tag{3.20}$$

Then from (3.16) and (3.17) it follows that

$$\phi_1(e(s)) = (s,0),\ \phi_2(e(s)) = (0,1-s). \tag{3.21}$$

Also, if $V_i(s)$ and $V_2(s)$ are vector fields transversal to e(s) such that

$$\partial\phi_1\Big|_{e(s)}(V_1(s)) = (0,1), \partial\phi_2\Big|_{e(s)}(V_2(s)) = (1.0) \tag{3.22}$$

then the radial construction implies that

$$V_1(s) = \lambda_1(s)(e(s)-z)\ ,\ \ V_2(s) = \lambda_2(s)(e(s)-z) \tag{3.23}$$

for some positive scalar functions λ_1, λ_2 where Z is the intersect point in Figure 3.2. Thus V_i (s) and $V_2(s)$ are vector fields pointing inwards to the polygon, along the same radial line but with different magnitudes.

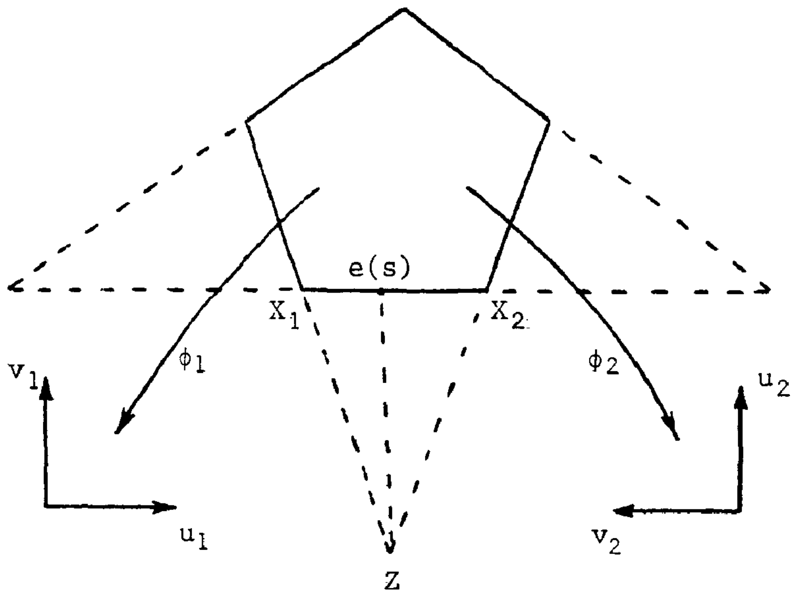


Figure 3.2

The local coordinate patches P_1 and P_2 are constructed such that

$$P_1(s,0) = P_2(0,1-s) = f(s), \tag{3.24}$$

$$\partial P_1\Big|_{(s,0)}(0,1) = \partial P_2\Big|_{(0,1-s)}(1,0) = t(s), \tag{3.25}$$

$$\partial^2 P_1\Big|_{(s,0)}((0,1),(0,1)) = \partial^2 P_2\Big|_{(0,1-s)}((1,0),(1,0)) = c(s), \tag{3.26}$$

where we assume that $q\colon [0,1]^2 \to IR^3$ is the adjoining rectangular patch with its domain $[0,1]^2$ orientated such that

$$f(s) := q(s,1), t(s) := q_{0,1}(s,1), c(s) := q_{0,2}(s,1) \tag{3.27}$$

Noting (3.21)-(3.23) we thus have, for $p_i = P_i o \varphi_i, i = 1,2,$

$$P_i(e(s) = f(s) \tag{3.28}$$

$$\partial p_i\Big|_{e(s)}(V_i(s)) = t(s) \tag{3.29}$$

Hence, for $p = \alpha_1 p_1 + \alpha_2 p_2$, where $\alpha_1(X) + \alpha_2(X) = 1$ and writing $\alpha_i(s)\colon \alpha_i(e(s))$ for brevity, we obtain

$$p(e(s)) = f(s) \tag{3.30}$$

$$\partial p\big|_{e(s)}(V(s)) = \alpha_1(s)\partial p_1\big|_{e(s)}(v(s)) + \alpha_2(s)\partial p_2\big|_{e(s)}(V(s))$$

$$= t(s), \tag{3.31}$$

the latter equality holding for a vector field

$$V(s) = \lambda(s)(e(s) - z) \tag{3.32}$$

where $\lambda(s)$ is such that

$$\lambda(s)[\alpha_1(s)/\lambda_1(s) + \alpha_2(s)/\lambda_2(s)] = 1 \tag{3.33}$$

Thus C^1 continuity is attained.

The C^2 continuity requirement of Corollary 2.1, namely

$$< \partial^2 p\big|_{e_p(s)}(V(s), v(s)), N(s) >=< c(s), N(s) > \tag{3.34}$$

leads to the condition

$$< \alpha_1(s)\partial^2 p_1\big|_{e(s)}(V(s), V(S)) + \alpha_2(s)\partial^2 p_2\big|_{e(s)}(V(s), V(s)), N(s) >$$

$$=< c(s), N(s) > \tag{3.35}$$

where, in general, $< c, N > \neq 0$. Since

$$\partial^2 p_i\big|_{e(s)}(V_i(s), V_i(s)) = c(s) + \partial p_i\big|_{(s,0)}(\partial^2 \phi_i\big|_{e(s)}(V_i(s), V_i(s)))$$

condition (3.35) then gives the requirement that

$$[\alpha_1(s)/\lambda_1(s)^2 + \alpha_2(s)/\lambda_2(s)^2]\lambda(s)^2 = 1 \tag{3.36}$$

Eliminating $\lambda(s)$ between (3.33) and (3.36) and noting $\alpha_1(s) + \alpha_2(s) = 1$ gives, after some calculation

$$[\lambda_1(s)^2 - \lambda_2(s)^2]\alpha_1(s)[1 - \alpha_1(s)] = 0 \tag{3.37}$$

Since $\lambda_1 \neq \lambda_2$ for $N \neq 4$ we obtain only the trivial solutions

$$\alpha_1(s) = 0 \text{ or } \alpha_1(s) = 1 \tag{3.38}$$

However, these conditions are not consistent with the requirements (3.13) and (3.14) of the convex combination, which imply that $\alpha_1(0) = 1$ and $\alpha_1(0) = 0$. Hence the C^2 continuity requirement does not hold for the Gregory-Charrot scheme.

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