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End Conditions for Improved Cubic Spline Derivative Approximations.

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ABSTRACT

We consider the problem of deriving accurate end conditions for cubic spline interpolation at equally spaced knots. In particular we derive a number of end conditions which lead to derivative approximations of high accuracy.

1. Introduction

Let s be a cubic spline on $[a,b]$ with equally spaced knots

$$x_i = a + ih; \qquad i = 0,1,\ldots,k, \tag{1.1}$$

where $h = (b-a)/k$. Then $s \in C^2[a,b]$ and in each of the intervals $[x_{i-1},x_i]$; $i = 1,2,\ldots,k$, s is a cubic polynomial.

Given the set of values y_i; $i = 0,1,\ldots,k$, where

$$y_i = y(x_i); \quad y \in C^n[a,b], \ n \geq 4,$$

we consider the problem of constructing an interpolatory s such that

$$s(x_i) = y_i; \qquad i = 0,1,\ldots,k. \tag{1.2}$$

To simplify the presentation we use throughout the abbreviations

$$m_i = s^{(1)}(x_i), \quad M_i = s^{(2)}(x_i) \quad \text{and} \quad y_i^{(r)} = y^{(r)}(x_i); \quad r = 1,2,3,4.$$

If the values m_i; $i = 0,1,\ldots,k$ are known, s can be constructed in each of the intervals $[x_{i-1},x_i]$ by use of Hermite's two point interpolation formula. Equivalently, if the values M_i; $i = 0,1,\ldots,k$ are known, s can be obtained in $[x_{i-1},x_i]$ by integrating

$$s^{(2)}(x) = \frac{1}{h}\{(x_i - x)M_{i-1} + (x - x_{i-1})M_i\}$$

twice with respect to x and using the interpolation conditions $s(x_{i-1}) = y_{i-1}$, $s(x_i) = y_i$ for the determination of the two constants of integration. To determine either of the $k+1$ parameters m_i or M_i the consistency relations

$$m_{i-1} + 4m_i + m_{i+1} = \frac{3}{h}\{y_{i+1} - y_{i-1}\}; \quad i = 1,2,\ldots,k-1, \tag{1.3}$$

or

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\{y_{i-1} - 2y_i + y_{i+1}\}\ ;\ i = 1,2,\ldots\ldots\ldots\ ,k-1, \tag{1.4}$$

are used, these being direct consequences of the continuity constraints on s. Since either (1.3) or (1.4) provide only k-1 linear equations, it follows that the interpolation conditions (1.2) are not sufficient to determine s uniquely. Two additional linearly independent conditions are always needed for this purpose. These are usually taken to be end conditions, i.e. conditions imposed on $s, s^{(1)}$ or $s^{(2)}$ near the two end ponts a and b.

As might be expected the choice of end conditions plays a critical role on the quality of the spline approximation. It is well known that the best order of uniform convergence that can be achieved by s and its derivatives is

$$\| s^{(r)} - y^{(r)} \| = 0(h^{4-r}); \qquad r = 0,1,2, \tag{1.5}$$

where $\| \quad \|$ denotes the uniform norm on [a,b]. It is also known that this order is obtained only if the end conditions of s are such that

$$m_i - y_i^{(1)} = 0(h^n)\ ; \qquad i = 0,1,\ldots.k\ , \tag{1.6}$$

with $n \geq 3$. This implies that the order of $\max\limits_{0\leq i\leq k} | mi - y_i^{(1)} |$ determines the quality of the end conditions of s; see e.g. Kershaw [6] and Behforooz and Papamichael [2], However, if y is sufficiently smooth then, as observed by Lucas [7], a better indication of the accuracy of s is provided by the order of $\max\limits_{0\leq i\leq k} | \lambda_i |$ where,

$$\lambda_i = y_i^{(2)} - \frac{h^2}{12} y_i^{(4)} + \frac{h^4}{360} y_i^{(6)} - M_i;\quad i = 0,1,\ldots\ldots\ldots\ \ldots\ldots.,\ k. \tag{1.7}$$

The reason for this emerges from the results (1.9) - (1.11), stated below.

Let s be an interpolatory cubic spline which agrees with $y \in C^8[a,b]$ at the equally spaced knots (1.1) and satisfies end conditions such that

$$\max_{0 \le i \le k} |\lambda_i| \le \alpha_n h^n \; ; \qquad 0 \le n \le 6 \; , \tag{1.8}$$

where α_n is a constant independent of h and the λ_i are defined by (1,7). Then, the following results are direct consequences of the results established in Lucas [7]:

(i) There exist constants A_n, B_n, C_n, D_n and E_n independent of h such that

$$\left.\begin{aligned}
&\max_{0\le i\le 3} | m_i - y_i^{(1)} | \le A_n \; h^{r+1} \; , \\
&\max_{0\le i\le k-1} | s^{(1)}(x_i + 0.5h) - y^{(1)}(x_i + 0.5h) | \le B_n \; h^{r+1} \; , \\
&\max_{0\le i\le k-1} | s^{(2)}(x_i + \mu h) - y^{(2)}(xi + \mu h) | \le C_n h^r , \\
&\qquad\qquad\qquad\qquad \text{with} \quad \mu = (3 + \sqrt{3})/6, \\
&\max_{0\le i\le k-1} | s^{(3)}(x_i + 0.5h) - y^{(3)}(x_i + 0.5h) | \le D_n \; h^{r-1} \; , \\
&\max_{1\le i\le k-1} | y_i^{(3)} - \frac{1}{2h}(M_{i+1} - M_{i-1}) | \le E_n h_{r-1} \; ,
\end{aligned}\right\} \tag{1.9}$$

where $r = \min(n,3)$

(ii) There exist constants F_n, G_n and H_n independent of h such that

$$\left.\begin{aligned}
&| y_0^{(2)} - \frac{1}{12}(14M_0 - 5M_1 + 4M_2 - M_3) | \le F_n \; h^r , \\
&\max_{1\le i\le k-1} | y_i^{(2)} - \frac{1}{12}(M_{i-1} + 10M_i + M_{i+1}) | \le G_n h_r \; , \\
&| y_k^{(2)} - \frac{1}{12}(14M_k - 5M_{k-1} + 4M_{k-2} - M_{k-3}) | \le H_n h^r \; ,
\end{aligned}\right\} \tag{1.10}$$

where $r = \min(n,4)$

(iii) There exist constants k_n and L_n independent of h such that

$$\left.\begin{aligned} &\max_{2\le i\le k-2} | y_i^{(3)} - \frac{1}{24h}(M_{i-2} - 14M_{i-1} + 14M_{i+1} - M_{i+2}) | \le k_n h^{r-1}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad r = \min\ (n,5), \\ &\max_{1\le i\le k-1} | y_i^{(4)} - \frac{1}{h^2}(M_{i-1} - 2M_i + M_{i+1}) | \le L_n h^{n-2} \end{aligned}\right\} \qquad (1.11)$$

The following conclusions can be drawn immediately from the above results. If, in (1.8), $n \ge 2$ then the cubic spline s has optimal $0(h^4)$ convergence uniformly on [a,b]. If $n \ge 3$ then the derivatives of s display the superconvergence properties (1.9), and the linear combinations of the M_i contained in (1.10) give more accurate approximations to $y_i^{(2)}$ than those obtained from $s^{(2)}$. Finally, if n = 6 then the linear combinations of the M_i contained in (1.11) give $0(h^4)$ approximations to $y_i^{(3)}$ and $y_i^{(4)}$ respectively.

It should be observed that some of the results (1.9) - (1.11) hold under much weaker'requirements than y C^8[a,b]. Full details concerning these requirements can be found in Lucas [7]. (See also Behforooz and Papamichael [3], where an alternative interpretation to some of the results corresponding to the case n = 3 is established under the assumption $y \in C^5[a,b]$.)

The purpose of the present paper is to derive various classes of end conditions and to compare their quality by using as a criterion the order of

$$\max_{0\le i\le k} | y_i^{(2)} - \frac{h^2}{12} y_i^{(h)} + \frac{h^4}{360} y_i^{(6)} - M_i | \qquad (1.12)$$

In particular we derive a number of end conditions for which (1.12) achieves $0(h^n)$ with $n \geq 5$. Such end conditions are needed for computing accurate approximations to $y_i^{(3)}$ and $y_i^{(4)}$ by means of the formulae contained in (1.11). Although some of the results concerning the less accurate end conditions can be established under weaker continuity requirements, in order to simplify the presentation we assume throughout that $y \in C^8[a,b]$.

The following lemma is needed for the derivation of the results given in Section 2. It can be established easily, from (1.4), by Taylor series expansion about the point x_i; see Lucas [7, p.576].

<u>Lemma 1.1</u> Let

$$\lambda_i = y_i^{(2)} - \frac{h^2}{12} y_i^{(4)} + \frac{h^4}{360} y_i^{(6)} - M_i; \quad i = 0,1,\ldots\ldots\ldots \ldots, k.$$

If $y \in C^8[a,b]$ then

$$\lambda_{i-1} + 4\lambda_i + \lambda_{i+1} = E_i; \qquad i = 1,2,\ldots,k-1 \quad , \tag{1.13}$$

where

$$| E_i | \leq \frac{516}{8!} h^6 \, \| y^{(8)} \| \tag{1.14}$$

<u>2. End Conditions</u>

We let s be an interpolatory cubic spline which agrees with $y \in C^8[a,b]$ at the equally spaced knots (1.1) and satisfies end conditions of the form

$$\left.\begin{aligned}\alpha M_0+\beta M_1+\gamma M_2 &= \frac{1}{h^2}\left\{\sum_{i=0}^{4} a_i y_i + h\sum_{i=0}^{2} b_i y_i^{(1)} + h^2\sum_{i=0}^{2} c_i y_i^{(2)}\right\}\\ \gamma M_{k-2}+\beta M_{k\text{-}1}+\alpha M_k &= \frac{1}{h^2}\left\{\sum_{i=0}^{4} a_i y_{k-i} - h\sum_{i=0}^{2} b_i y_{k-i}^{(1)} + h^2\sum_{i=0}^{2} c_i y_{k-i}^{(2)}\right\}\end{aligned}\right\} \quad (2.1)$$

where we assume without loss of generality that $a \geq 0$. Our purpose is to examine the effect that various choices of the parameters α,β,γ, a_i, b_i and c_i have upon the quality of the spline approximation. We do this by using as a criterion the order of $\max\limits_{0\leq i\leq k} |\lambda_i|$ where,as before.

$$\lambda_i = y_i^{(2)} - \frac{h^2}{12} y_i^{(4)} + \frac{h^4}{360} y_i^{(6)} - M_i; \qquad i = 0,1,\ldots\ldots\ldots k. \qquad (2.2)$$

With this notation the equations (2.1) and (1.4) give

$$\left.\begin{aligned}&\alpha\lambda_0 + \beta\lambda_1 + \gamma\lambda_2 = E_0,\\ &\lambda_{i-1} + 4\lambda_i + \lambda_{i+1} = E_i\ ;\ i = 1,2,\ldots\ldots\ldots, k-1,\\ &\gamma\lambda_{k-2} + \beta\lambda_{k\text{-}1} + \alpha\lambda_k = E_k \quad ,\end{aligned}\right\} \quad (2.3)$$

where

$$\left.\begin{aligned}E_0 &= \frac{1}{h^2}\Bigg\{-\sum_{i=0}^{4} a_i y_i - h\sum_{i=0}^{2} b_i y_i^{(1)} - h^2\sum_{i=0}^{2} c_i y_i^{(2)} + h^2(\alpha y_0^{(2)} + \beta y_1^{(2)} + \gamma y_2^{(2)})\\ &\quad -\frac{h^4}{12}(\alpha y_0^{(4)} + \beta y_2^{(4)} + \gamma y_2^{(4)}) + \frac{h^6}{360}(\alpha y_0^{(6)} + \beta y_1^{(6)} + \gamma y_2^{(6)})\Bigg\},\\ E_k &= \frac{1}{h^2}\Bigg\{-\sum_{i=0}^{4} a_i y_{k-i} + h\sum_{i=0}^{2} b_i y_{k-i}^{(1)} - h^2\sum_{i=0}^{2} c_i y_{k-i}^{(2)} + h^2(\alpha y_k^{(2)} + \beta y_{k\text{-}1}^{(2)} + \gamma y_{k-2}^{(2)})\\ &\quad -\frac{h^4}{12}(\alpha y_k^{(4)} + \beta y_{k\text{-}1}^{(4)} + \gamma y_{k-2}^{(4)} + \frac{h^6}{360} + \frac{h^6}{360}(\alpha y_k^{(6)} + \beta y_{k\text{-}1}^{(6)} + \gamma y_{k-2}^{(6)}\Bigg\},\end{aligned}\right\} (2.4)$$

and, from (1.14),

$$|E_i| \le \frac{516}{8!} h^6 \, \| y^{(8)} \| \; ; \; i = 1,2,\ldots\ldots\ldots, k-1 \qquad (2.5)$$

Also, by Taylor series expansions about the points x_2 and x_{k-2}, we find that

$$\left.\begin{aligned} E_0 &= \sum_{j=0}^{7} \frac{B_j}{j!} h^{j-2} y_2^{(j)} + 0(h^6), \\ E_k &= \sum_{j=0}^{7} (-1)^j \frac{B_j}{j!} h^{j-2} y_{k-2}^{(j)} + 0(h^6), \end{aligned}\right\} \qquad (2.6)$$

where

$$\left.\begin{aligned} B_0 &= -a_0 - a_1 - a_2 - a_3 - a_4, \\ B_1 &= 2a_0 + a_1 - a_3 - 2a_4 - b_0 - b_1 - b_2, \\ B_2 &= -4a_0 - a_1 - a_3 - 4a_4 + 4b_0 + 2b_1 - 2c_0 - 2c_1 - 2c_2 + 2\alpha + 2\beta + 2\gamma\gamma \\ B_3 &= 8a_0 + a_1 - a_3 - 8a_4 + 12b_0 + 3b_1 + 12c_0 + 6c_1 - 12\alpha - 6\beta\beta \\ B_4 &= -16a_0 - a_1 - a_3 - 16a_4 + 32b_0 + 4b_1 - 48c_0 - 12c_1 + 46\alpha + 10\beta - 2\gamma\gamma \\ B_5 &= 32a_0 + a_1 - a_3 - 32a_4 - 80b_0 - 5b_1 + 160c_0 + 20c_1 - 140\alpha - 10\beta 0 \\ B_6 &= -64a_0 - a_1 = a_3 - 64a_4 + 192b_0 + 6b_1 - 480c_0 - 30c_1 + 362\alpha - 2\beta + 2\gamma\gamma \\ B_7 &= 128a_0 + a_1 - a_3 - 128a_4 - 448b_0 - 7b_1 + 1344c_0 + 42c_1 - 812\alpha + 14\beta 4 \end{aligned}\right\} \qquad (2.7)$$

To simplify the presentation we assume that in (1-1) $k \geq 5$. Then a sufficient condition for the unique existence of s is that the parameters α,ß,and γ satisfy either

or

$$\left.\begin{aligned} &(i) \quad \alpha = \gamma \text{ and } \beta \neq 4\alpha \\ &(ii) \quad \alpha \neq \gamma \text{ and} \\ &\qquad 3\beta < 11\alpha + \gamma - \frac{2}{5}(\gamma - \alpha)_+ , \\ &\qquad \text{or} \\ &\qquad 5\beta > 19\alpha + \gamma + \frac{2}{3}(\gamma - \alpha)_+ , \end{aligned}\right\} \qquad (2.8)$$

where

$$(\gamma - \alpha)_+ = \begin{cases} 0, & \gamma < \alpha, \\ \gamma - \alpha > \alpha. \end{cases}$$

This follows easily from the results of Behforooz and Papamichael [2, p.358-59], by observing that the linear system (2.3) can he written in the tridiagonal form

$$\left.\begin{array}{ll} (\alpha - \gamma)\lambda_0 + (\beta - 4\gamma)\lambda_1 = E_0 - \gamma E_1, & \\ \lambda_{i-1} + 4\lambda_i + \lambda_{i+1} = E_i; & i = 1,2,\ldots\ldots\ldots \ldots, k-1, \\ (\beta - 4\gamma)\lambda_{k-1} + (\alpha - \gamma)\lambda_k = E_k - \gamma E_{k-1}, & \end{array}\right\} \quad (2.9)$$

and that the matrix in (2.9) is the matrix of the linear system which determines the parameters M_i of s. The results of [2] also show that if (2.8) holds and

$$E_i = 0(h^m); \qquad i = 0,k, \quad (2.10)$$

then

$$\max_{0 \le i \le k} |\lambda_i| = 0(h^n), \quad (2.11)$$

where n = min (m,6). This shows that the quality of end conditions of the form (2.1) is determined by the order of E_i; i = 0,k.

The remainder of the paper is concerned with examining various classes of end conditions of the form (2.1). In each case we consider only end conditions for which s attains the optimal order $0(h^4)$ of uniform convergence on [a,b]. This requires that $E_i = 0(h^m)$; i = 0,k, with $m \geq 2$, and implies that the parameters α, β, a_i,b_i and c_i must be chosen so that in (2.7) ,

$$B_i = 0; \qquad i = 0, 1, 2, 3. \quad (2.12)$$

To avoid unnecessary repetition, we point out now that all the results of subsequent sections are established under the assumption that the parameters α, β and γ satisfy the condition (2.8). This condition certainly holds for all the specific values of α, β and γ that occur in some of the results, considered in the following sections.

3. End conditions involving values of y only

We take $b_i = c_i = 0$; $i = 0,1,2$ and $\gamma = 0$ in (2.1) and consider end conditions of the form

$$\left.\begin{aligned} \alpha M_0 + \beta M_1 &= \frac{1}{h^2} \sum_{i=0}^{4} a_i y_i , \\ \beta M_{k-1} + \alpha M_k &= \frac{1}{h^2} \sum_{i=0}^{4} a_i y_{k-i} . \end{aligned}\right\} \qquad (3.1)$$

It should be observed that there is no loss of generality in assuming that $\gamma = 0$ The reason for this is that the terms γM_2 and γM_{k-2} can always be eliminated by means of the relations (1.4).

It can be shown easily from (2.7) that the requirement (2.12) is satisfied for any values of the parameters α, β and a_4 provided that the other four parameters in (3.1) satisfy the relations

$$\left.\begin{aligned} &a_0 - 2\alpha + \beta + a_4 , \quad a_1 = -5\alpha - 2\beta - 4a_4 , \\ &a_2 - 4\alpha + \beta + 6a_4 , \quad a_3 = -\alpha - 4a_4 . \end{aligned}\right\} \qquad (3.2)$$

When (3.2) hold then

$$\left.\begin{aligned} &B_i = 0 ; \ i = 0, 1, 2, 3, \\ &B_4 = 4(5\alpha - \beta - 6a_4), \ \ B_5 = 20(-4\alpha + \beta), \\ &B_6 = 60(4\alpha - \beta - 2a_4), \ B_7 = 140(-4\alpha + \beta), \end{aligned}\right\} \qquad (3.3)$$

and, by using (1.4), the end conditions (3.1) can be written as

$$\left.\begin{aligned} &a_4\Delta^4 M_0 - (\alpha - 6a_4)\Delta^3 M_0 - (5\alpha - \beta - 6a_4)\Delta^2 M_0 = 0, \\ &a^4\nabla^4 M_k + (\alpha - 6a_4)\nabla^3 M_k - (5\alpha - \beta - 6a_4)\nabla^2 M_k = 0 \end{aligned}\right\} \tag{3.4}$$

In particulars if $a_4 = 0$,i.e. if in (3.1)

$$\left.\begin{aligned} &a_0 = 2\alpha + \beta, \quad a_1 = -5\alpha - 2\beta, \\ &a_2 = 4\alpha + \beta, \quad a_3 = -\alpha, \quad a_4 = 0, \end{aligned}\right\} \tag{3.5}$$

then (3.4) gives the class of end conditions

$$\left.\begin{aligned} &\alpha\Delta^3 M_0 + (5\alpha - \beta)\Delta^2 M_0 = 0, \\ &\alpha\nabla^3 M_k - (5\alpha - \beta)\nabla M_k = 0, \end{aligned}\right\} \tag{3.6}$$

which is considered fully in Behforooz and Papamichael [2]. The special case $\alpha = 0$, $\beta = 1$ of (3.6) i.e. the conditions

$$\Delta^2 M_0 = \nabla^2 M_k = 0 , \tag{3.7}$$

have also been considered by De Boor [4] and [5, p.55] Kershaw [6] and Lucas [7].

For any values of α, β and a_4 the end conditions (3.4) are such that $E_i = 0(h^2)$; $i = 0,k$. However, it follows from (3.3) and (3.4) that when $a_4 = (5\alpha-\beta)/6$, i.e. when in (3.1),

$$\left.\begin{aligned} &a_0 = (17\alpha + 5\beta)/6, \quad a_1 = -(50\alpha + 8\beta)/6, \quad a_2 = 9\alpha, \\ &a_3 = (-25\alpha + 4\beta)/6, \quad a_4 = (5\alpha - \beta)/6, \end{aligned}\right\} \tag{3.8}$$

then

$$\left.\begin{aligned} &B_i = 0;\ i = 0,1,\dots\dots\dots 4, \quad B_5 = 20(4\alpha + \beta), \\ &B_6 = 20(7\alpha - 2\beta), \quad B_7 = 140(-4\alpha + \beta), \end{aligned}\right\} \tag{3.9}$$

and the end conditions (3.1) can be written as

$$\left.\begin{aligned}(5\alpha-\beta)\,\Delta^4 M_0 + 6(4\alpha-\beta)\,\Delta^3 M_0 = 0,\\ (5\alpha-\beta)\,\nabla^4 M_k + 6(4\alpha-\beta)\,\nabla^3 M_k = 0.\end{aligned}\right\} \tag{3.10}$$

This class of end conditions is considered in Behforooz [1].

For any values of α and β the end conditions (3.10) are such that $E_i = 0(h^3)$; $i = 0,k$ .However, if $\alpha = 1$ and $ß = 4$, i.e. when in (3.1)

$$\alpha=1,\ \beta=4,\ a_0=37/6,\ a_1=-82/6,\ a_2=9,\ a_3=-10/6,\ a_4=1/6, \tag{3.11}$$

then

$$B_i = 0 \ ; \quad i = 0,1,\ldots,5 \ , \quad B_6 = -20 \ , \quad B_7 = 0. \tag{3.12}$$

Thus, from (3,10) and (3.12) the end conditions

$$\Delta^4 M_0 = \nabla^4 M_k = 0 \ , \tag{3.13}$$

are such that $E_i = 0(h^4)$; $i = 0,k$. Furthermore, (3.13) are the most "accurate" end conditions of the class (3.1), in the sense that they are the only such end conditions for which $E_i = 0(h^4)$; $i = 0,k$.

The end conditions (3.13) are considered by Lucas [7] who also considers the conditions

$$\Delta^3 M_0 = \nabla^3 M_k = 0 \ , \tag{3.14}$$

i.e. the special case $\alpha = 1, \beta = 5$, of (3.10). It is interesting to observe that (3.14) are also the special case $\alpha = 1, ß = 5$ of the class (3.6), and that they are the only conditions of this class for which $E_i = 0(h^3)$; $i = 0,k$.

4. End conditions involving values of $y^{(1)}$ only

In this section we consider end conditions of the form

$$\left.\begin{aligned} \tilde{\alpha} m_0 + \tilde{\beta} m_1 + \tilde{\gamma} m_2 &= \tilde{b}_0 y_0^{(1)} + \tilde{b}_1 y_1^{(1)} + \tilde{b}_2 y_2^{(1)}, \\ \tilde{\gamma} m_{k-2} + \tilde{\beta} m_{k-1} + \tilde{\alpha} m_k &= \tilde{b}_2 y_{k-2}^{(1)} + \tilde{b}_1 y_{k-1}^{(1)} + \tilde{b}_0 y_k^{(1)}, \end{aligned}\right\} \tag{4.1}$$

where $\tilde{\alpha} \geq 0$ and, as before $m_i = s^{(1)}(x_i)$.

By using the cubic spline identities

$$m_i = -\frac{h}{3} M_i - \frac{h}{6} M_{i+1} + \frac{1}{h}(y_{i+1} - y_i); \quad i = 0,1,\ldots\ldots\ldots, k-1,$$

and

$$m_i = \frac{h}{6} M_{i-1} + \frac{h}{3} M_i + \frac{1}{h}(yi - y_{i-1}); \quad i = 1,2,\ldots\ldots\ldots, k,$$

the conditions (4.1) can be written in the form (2.1) with

$$\left.\begin{aligned} &\alpha = \tilde{\alpha}, \quad \beta = (\tilde{\alpha} + 2\tilde{\beta} - \tilde{\gamma})/2, \quad \gamma = (\tilde{\beta} - 2\tilde{\gamma})/2, \\ &a_0 = -3\tilde{\alpha}, \; a_1(\tilde{\alpha} - \tilde{\beta} - \tilde{\gamma}), \quad a_2 = 3(\tilde{\beta} + \tilde{\gamma}), \quad a_3 = a_4 = 0, \\ &b_i = -3\tilde{b}_i \; ; \quad i = 0, 1, 2, \\ &c_i = 0 \; ; \quad i = 0,1,2. \end{aligned}\right\} \tag{4.2}$$

It follows easily from (2.7) and (4.2) that the requirement (2.12) is satisfied provided that in (4.1)

$$\tilde{b}_0 = \tilde{\alpha}, \quad \tilde{b}_1 = \tilde{\beta} \text{ and } \tilde{b}_2 = \tilde{\gamma}. \tag{4.3}$$

Furthermore, it turns out that for the parameters defined by (4.2) and (4.3), $B_4 = 0$ also. More specifically (4.2), (4.3) and (2.7) show that for any values of $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma}$ the end conditions

$$\left.\begin{aligned} \tilde{\alpha} m_0 + \tilde{\beta} m_1 + \tilde{\gamma} m_2 &= \tilde{\alpha} y_0^{(1)} + \tilde{\beta} y_1^{(1)} + \tilde{\gamma} y_2^{(1)} , \\ \tilde{\gamma} m_{k-2} + \tilde{\beta} m_{k-1} + \tilde{\alpha} m_k &= \tilde{\gamma} y_{k-2}^{(1)} + \tilde{\beta} y_{k-1}^{(1)} + \tilde{\alpha} y_k^{(1)} , \end{aligned}\right\} \tag{4.4}$$

are such that

$$\left.\begin{aligned} & B_i = 0 \; ; \quad i = 0,1,\ldots\ldots,4, \\ & B_5 = 2(\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma}), \quad B_6 = -12(2\tilde{\alpha} + \tilde{\beta}), \quad B_7 = 158\,\tilde{\alpha} + 32\,\tilde{\beta} - 10\,\tilde{\gamma} \end{aligned}\right\} \tag{4.5}$$

Therefore, for any values of $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma}$ the end conditions (4.4) are such that $E_i = 0(h^3)$; $i = 0,k$, and if $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} = 0$ then $E_i = 0(h^4)$; $i = 0,k$.

The most "accurate" end conditions of the class (4.4) are those which correspond to the values $\tilde{\alpha} = 1$, $\tilde{\beta} = -2$ and $\tilde{\gamma} = 1$, For these values (4,5) gives

$$B_i = 0 \; ; \qquad i = 0,1,\ldots,6 , \qquad B_7 = 84 , \tag{4.6}$$

and thus the end conditions

$$\left.\begin{aligned} \Delta^2 m_0 &= \Delta^2 y_0^{(1)} , \\ \nabla^2 m_k &= \nabla^2 y_k^{(1)} , \end{aligned}\right\} \tag{4.7}$$

and such that $E_i = 0(h^5)$; $i = 0,k$.

The special case $\tilde{\alpha} = 1$, $\tilde{\beta} = -1$, $\tilde{\gamma} = 0$ of (4.4), (i.e). the end condtion

$$\left.\begin{aligned} \Delta m_0 &= \Delta y_0^{(1)} , \\ \nabla m_k &= \nabla y_k^{(1)} , \end{aligned}\right\} \tag{4.8}$$

are considered in Lucas [7]. For these end conditions (4.5) gives

$$B_i = 0; \quad i = 0, 1 \ldots.5, \qquad B_6 = -12, \qquad B_7 = 126, \qquad (4.9)$$

and thus $E_i = 0(h^4)$; $i = 0,k$.

The most frequently used end conditions of the class (4.4) are those which correspond to $\tilde{\alpha} = 1$, $\tilde{\beta} = \tilde{\gamma} = 0$, i.e. the conditions

$$m_0 = y_0^{(1)}, \; m_k = y_k^{(1)} \; . \qquad (4.10)$$

For these end conditions (4.5) gives

$$B_i = 0; \quad i = 0,1,\ldots, 4 \;, \qquad B_5 = 2\;, \; B_6 = -24\;, B_7 = 158\;, \qquad (4.11)$$

and thus $E_i = 0(h^3)$; $i = 0,k$.

5. End conditions involving values of $y^{(2)}$ only

We take $a_l = 0$; $i = 0,1,\ldots,4$, and $b_i = 0$, $i = 0,1,2$, in (2.1) and consider end conditions of the form

$$\left.\begin{aligned} &\alpha M_0 + \beta M_1 + \gamma M_2 = c_0 y_0^{(2)} + c_1 y_0^{(2)} + c_2 y_2^{(2)}, \\ &\gamma M_{k-2} + \beta M_{k\text{-}1} + \alpha M_k = c_2 y_{k-2}^{(2)} + c_1 y_{k-2}^{(2)} + c_0 y_k^{(2)}, \end{aligned}\right\} \qquad (5.1)$$

Then the requirement (2.12) is satisfied for any values of α, β, γ and c_2 provided that the other two parameters in (5.1) satisfy the relations

$$c_0 = \alpha = \gamma + c_2\;, \qquad c_1 = \beta + 2\gamma - 2c_2\; . \qquad (5.2)$$

When (5.2) hold then, from (2.7),

$$\left.\begin{array}{l} B_i = 0;\ i = 0,1,2,3, \\ B_4 = -2(\alpha + \beta - 11\gamma + 12c_2),\quad B_5 = 10(\alpha + \beta - 12\gamma + 12c_2) \\ B_6 = -118\alpha - 28\beta + 422\gamma - 420c_2 \quad B_7 = 532\alpha + 56\beta - 1260\gamma + 1260c_2, \end{array}\right\} \quad (5.3)$$

and the end conditions (5.1) can be written as

$$\left.\begin{array}{l} \alpha M_0 + \beta M_1 + \gamma M_2 = \alpha y_0^{(2)} + \beta y_1^{(2)} + \gamma y_2^{(2)} + (c_2 - \gamma)\Delta^2 y_0^{(2)}, \\ \gamma M_{k-2} + \beta M_{k-1} + \alpha M_k = \gamma y_{k-2}^{(2)} + \beta y_{k-1}^{(2)} + \alpha y_k^{(2)} + (c_2 - \gamma)\nabla^2 y_k^{(2)}. \end{array}\right\} \quad (5.4)$$

For any values of α, β, γ and C_2 the end conditions (5.4) are such that $E_i = 0(h^2)$; $i = 0,k$. However, if $c_2 = -(\alpha+\beta = 11\gamma)/12$, i.e. if in (5.1),

$$c_0 = (11\alpha-\beta-\gamma)/12, \quad c_1 = (2\alpha+14\beta+2\gamma)/12, \quad c_2 = -(\alpha+\beta-11\gamma)/12, \quad (5.5)$$

then, from (5.3) ,

$$\left.\begin{array}{l} B_i = 0;\ i = 0,1,\ldots\ 4,\quad B_5 = 10(\alpha - \gamma), \\ B_6 = -83\alpha + 7\beta + 37\gamma,\quad B_7 = 427\alpha - 49\beta - 105\gamma. \end{array}\right\} \quad (5.6)$$

Therefore, for any values of α, β and γ the end conditions

$$\left.\begin{array}{l} \alpha M_0 + \beta M_1 + \gamma M_2 = \alpha y_0^{(2)} + \beta y_1^{(2)} + \gamma y_2^{(2)} - (\alpha + \beta + \gamma)\Delta^2 y_0^{(2)}/12, \\ \gamma M_{k-2} + \beta M_{k-1} + \alpha M_k = \gamma y_{k-1}^{(2)} + \beta y_{k-1}^{(2)} + \alpha y_k^{(2)} - (\alpha + \beta + \gamma)\nabla^2 y_k^{(2)}/12, \end{array}\right\} \quad (5.7)$$

are such that $E_i = 0(h^3)$; $i = 0,k$. In particular if $\alpha = \gamma$ then, from (5.6) ,

$$B_i = 0;\quad i = 0,1,\ldots,5\ ,\qquad B_6 = -46\alpha + 7\beta\ ,\qquad B_7 = 7(46_\alpha - 7_\beta)\ , \qquad (5.8)$$

and therefore, for any values of α and β, the end conditions

$$
\left.\begin{aligned}
&\alpha M_0 + \beta M_1 + \alpha M_2 = \alpha y_0^{(2)} + \beta y_1^{(2)} + \alpha y_2^{(2)} - (2\alpha+\beta)\nabla^2 y_0^{(2)}/12, \\
&\alpha M_{k-2} + \beta M_{k\text{-}1} + \alpha M_k = \alpha y_{k-2}^{(2)} + \beta y_{k\text{-}2}^{(2)} + \alpha y_k^{(2)} - (2\alpha+\beta)\nabla^2 y_k^{(2)}/12,
\end{aligned}\right\} \qquad (5.9)
$$

are such that $E_i = O(h^4)$; $i = 0 , k$

The special case $\alpha = 0$, $\beta = 1$ of (5.9), i.e. the end conditions

$$
\left.\begin{aligned}
&12\, M_1 = -y_0^{(2)} + 14\, y_1^{(2)} - y_2^{(2)}, \\
&12\, M_{k-1} = -y_{k-2}^{(2)} + 14\, {}_{k-1}^{(2)} - y_k^{(2)},
\end{aligned}\right\} \qquad (5.10)
$$

is considered in Lucas [7]. Lucas also considers the case $\alpha = 1$, $\beta = 10$ of (5.9), i.e. the conditions

$$
\left.\begin{aligned}
&M_0 + 10\, M_1 + M_2 = 12\, y_1^{(2)}, \\
&M_{k-2} + 10\, M_{k-1} + M_k = 12\, y_{k-1}^{(2)}
\end{aligned}\right\} \qquad (5.11)
$$

The conditions (5.11) are of special interest because they require knowledge of $y^{(2)}$ only at the two points x_1 and x_{k-1}

As it is clear from (5.8), the most accurate end conditions of the class (5.1) are those which correspond to the values $\alpha = 7$, $\beta = 46$ in (5.9), i.e. the conditions

$$
\left.\begin{aligned}
&7M_0 = 46M_1 + 7M_2 = 2y_0^{(2)} + 56y_1^{(2)} + 2y_2^{(2)}, \\
&7M_{k-2} + 46M_{k-1} + 7M_k = 2y_{k-2}^{(2)} + 56y_{k-2}^{(2)} + 2y_k^{(2)}.
\end{aligned}\right\} \qquad (5.12)
$$

For these end conditions $B_i = 0$; $i = 0, 1, \ldots, 7$ and the E_i; $i = 0, k$, achieve the best possible order $0(h^6)$. The conditions (5.12) are also considered in Lucas [7].

The most frequently used end conditions of the class (5.4) are those which correspond to $\alpha = 1$, $\beta = \gamma = c_2 = 0$, i.e. the conditions

$$M_0 = y_0^{(2)}, \quad M_k = y_k^{(2)} \tag{5.13}$$

For (5.13),

$$B_i = 0 ; \quad i = 0,1,2,3, \quad B_4 = -2, \quad B_5 = 20, \quad B_6 = -118, B_7 = 532, \tag{5.14}$$

and thus $E_i = 0(h^2)$; $i = 0,k$

6. Other end conditions

Of the end conditions considered in Sections 3, 4 and 5 the most accurate are (4.7) and (5.12). These conditions give $E_i = 0(h^n)$; $i = 0,k$, with $n = 5$ and $n = 6$ respectively. However, (4.7) and (5.12) require knowledge of $y^{(1)}$ and $y^{(2)}$ respectively, at the six knots x_i ; $i = 0,1,2,k-2,k-1,k$, and it is unlikely that this additional information would be available in an interpolation problem. End conditions of the class (3.1) do not require any additional information, but the most accurate of these, i.e. the conditions (3.13), give $E_i = 0(h^4)$; $i = 0,k$. In this sect ion we show that it is possible to construct end conditions which require derivative information only at the two end points x_0 and x_k and which, like (4.7) and (5.12), give $E_i = 0(h^n)$; $i=0,k$, with $n \geq 5$. This is done by forming linear combinations of end conditions derived in earlier sections.

Let ECO, EC1 and EC2 denote end conditions which belong respectively in the three classes defined by (3.1), (4.1) and (5.1). Assume that the E_i; $i = 0,k$, corresponding to ECO, EC1 and EC2 are given by (2.6) with $B_j = B_j^{(r)}$; $r = 0,1,2$, respectively. Let EC denote the linear combination of EC0, EC1 and EC2, in the proportion d_0 parts of EC0 to d_1 parts of EC1 to d_2 parts of EC2, i.e. symbolically

$$EC \equiv d_0 (EC0) + d_1 (ECl) + d_2 (EC2).. \tag{6.1}$$

Then clearly the E_i ; $i = 0,k$, for the end condition EC are given by (2.6) with

$$B_j = d_0 = B_j^{(0)} + d_1 B_j^{(1)} + d_2 B_j^{(2)}; \quad j = 0, 1,...7, \tag{6.2}$$

This observation leads to a simple technique for constructing accurate end conditions of the form (2.1). We illustrate the technique by deriving three such end conditions which are of greater practical value than (4.7) and (5.12), in the sense that they require derivative information only at the two points x_0 and x_k.

Let

$$EC = ECO + d_i (ECl), \tag{6.3}$$

where ECO are conditions of the class (3.1) with parameters (3.8) and EC1 are the conditions (4.10). Then, from (3.9) (4.11) and (6.2), the B. corresponding to the conditions EC are given by

$$\left.\begin{array}{l} B_i = 0 \;;\; i = 0,1,......, 4, \qquad B_5 = 20(-4\alpha + \beta) + 2d_1 , \\ B_6 = 20(7\alpha - 2\beta) - 24d_1, \; B_7 = 140(-4\alpha + \beta) = 158d_1. \end{array}\right\} \tag{6.4}$$

Therefore, if the parameters α, β of ECO and d_1 of (6.3) are chosen so that

$$\beta/\alpha = 41/10 \qquad \text{and} \qquad d_1/\alpha = -1 , \tag{6.5}$$

then

$$B_i, = 0 \;; \quad i=0,1,...,6, \qquad B_7 = -144\alpha , \tag{6.6}$$

and the conditions EC defined by (6.3) are such that $E_i = 0(h^5)$; $i = 0,k$. In particular, when $\alpha = 20/72$ then (6.5) and (6.3) give the end conditions

$$\left.\begin{aligned} M_1 &= \frac{1}{72h^2}\{185y_0 - 336y_1 + 180y_2 - 32y_3 + 3y_4 + 60hy_0^{(1)}\}, \\ M_{k-1} &= \frac{1}{72h^2}\{185y_k - 336y_{k-1} + 180y_{k-2} - 32y_{k-3} + 3y_{k-4} - 60hy_k^{(1)}\} \end{aligned}\right\} \tag{6.7}$$

for which

$$B_i = 0; \quad i = 0,1,\ldots,6, \qquad B_7 = -40, \tag{6.8}$$

In a similar manner it can be shown that the end conditions

$$\left.\begin{aligned} 144M_0 + 876M_1 &= \frac{1}{h^2}\{1313y_0 - 288y_1 + 1866y_2 - 320y_3 - 29y_4 - 60h^2y_0^{(2)}\}, \\ 876M_{k-1} + 144M_k &= \frac{1}{h^2}\{1313y_k - 2888y_{k-1} + 1866y_{k-2} - 320y_{k-3} + 29y_{k-4} - 60h^2y_k^{(2)}\}, \end{aligned}\right\} \tag{6.9}$$

are such that $E_i = 0(h^5)$; $i = 0,k$. More specifically, the B_i corresponding to (6.9) are

$$B_i = 0; \quad i = 0,1,\ldots,6, \quad B_7 = -23520. \tag{6.10}$$

This result is obtained by taking ECO to be conditions of the form (3.1) with parameters (3.2), EC2 to be the conditions (5.13) and determining the parameters α, β and a_4 of ECO and the constant of proportionality d_2 so that

$$EC = ECO + d_2(EC2)$$

gives $B_i = 0$; $i = 0,1,\ldots,6$. Finally, by taking EC01 and EC02 to be respectively the conditions (6.7) and (6.9) and determining the constant d so that

$$EC \,^{o}\; EC01 + d(EC02)$$

gives $B_7=0$, we find that the end conditions

$$\left.\begin{aligned} M_0+2M_1 &= \frac{1}{864h^2}\{-1187y_0-864y_1+237y_2-352y_3+27y_4 \\ &\qquad -2940hy_0^{(1)}-360h^2y_0^{(2)}\} \\ 2M_{k-1}+M_k &= \frac{1}{864h^2}\{-1187y_k-864y_{k-1}+2376y_{k-2}-352y_{k-3}+27y_{k-4} \\ &\qquad +2940hy_k^{(1)}-360h^2{}_k^{(2)}\}, \end{aligned}\right\} \quad (6.11)$$

are such that $E_i = 0(h^6)$; $i = 0, k$.

7. Numerical Results

In this section we present numerical results obtained by taking

$$y(x) = \exp(x) \ ; \quad x_i = 0.05i \ ; \quad i = 0,1,\ldots,20 \ ,$$

and computing the parameters M_i ; $i = 0,1,\ldots,20$, of the cubic splines with end conditions (3.13), (4.10), (5.13), (6.7), (6.9) and (6.11). We denote these six splines respectively by S_I ,S_{II} , S_{III},S_{IV},S_V, and S_{VI}.

As was remarked earlier (3.13) are the most accurate end condtions of the class (3.1), whilst (4.10) and (5.13) are respectively those most frequently used from the classes (4.1) and (5.1). The three new "accurate" end conditions (6.7), (6.9) and (6.11), like (4.10) and (5.13), require derivative information only at the two endpoints and for this reason are of greater practical interest than (4.7) and (5.12).

The results in Tables 1 - 5 are computed values of

$$|\lambda_i| = \left| y_i^{(2)} - \frac{h^2}{12} y_i^{(4)} + \frac{h^4}{360} y_i^{(6)} - M_i \right|,$$

$$|m_i - y_i^{(1)}|, |\tilde{y}_i^{(2)} - y_i^{(2)}|, |\tilde{y}_i^{(3)} - y_i^{(3)}| \text{ and } |\tilde{y}_i^{(4)} - y_i^{(4)}|,$$

corresponding to $s_I, s_{II}, \ldots, s_{VI}$, where $y_i^{(r)}$; $r = 2,3,4$ denote the approximations to $y_i^{(r)}$ obtained by using the formulae contained in (1.10) and (1.11). The results illustrate clearly that the use of accurate end conditions, like (6,7), (6.9) and (6.11), leads to significant improvement in the accuracy of the approximations $\tilde{y}_i^{(r)}$; $r = 2,3,4$, especially near the two ends of the interval of interpolation.

An important observation concerns the results corresponding to the end conditions (3.13) and (4.10). Although for these conditions the theory gives $l_i = O(h^4)$ and $l_i = 0(h^3)$ respectively, the numerical results of s_I are slightly less accurate than those of s_{II}. The reason for this is that the theoretical results of the present paper concern orders of convergence only. In fact a more detailed analysis similar to that used in Behforooz and Papamichael [2] gives, for (3.13) and (4.10),

$$\left.\begin{aligned} &\max |\lambda_i| \leq .5834\, h^4 \exp(1) + 0(h^6); \quad i = 0, k, \\ \text{and} \quad &\max |\lambda_i| \leq (.0203 + .0405\, h + .031\, h^2) h^3 \exp(1) + 0(h^6)\ ; \ i = 0, k. \end{aligned}\right\} \quad (7.1)$$

respectively. With $h = 0.05$ and the $0(h^6)$ terms ignored (7.1) gives

$$\max |\lambda_i| \leq 0.0225 x(0.05)^3 \exp(l),$$

and

$$\max |\lambda_i| \leq 0.0291 x(0,05)^3 \exp(l).$$

TABLE 1

Values of $|\lambda_i|$

	S_I	S_{II}	S_{III}	S_{IV}	S_V	S_{VI}
x_0	$.267x10^{-5}$	$.240x10^{-5}$	$.208x10^{-3}$	$.997x10^{-8}$	$.173x10^{-7}$	$.873x10^{-10}$
x_1	$.716x10^{-6}$	$.644x10^{-6}$	$.558x10^{-4}$	$.268x10^{-8}$	$.465x10^{-8}$	$.175x10^{-10}$
x_2	$.192x10^{-6}$	$.173x10^{-6}$	$.150x10^{-4}$	$.702x10^{-9}$	$.123x10^{-8}$	$.196x10^{-10}$
x_4	$.138x10^{-7}$	$.124x10^{-7}$	$.107x10^{-5}$	$.357x10^{-10}$	$.735x10^{-10}$	$.162x10^{-10}$
x_6	$.978x1^{-9}$	$.879x1^{-9}$	$.771x10^{-7}$	$.732x1^{-11}$	$.460x10^{-11}$	$.110x10^{-10}$
x_8	$.662x10^{-10}$	$.574x10^{-10}$	$.562x10^{-8}$	$.536x10^{-11}$	$.519X10^{-11}$	$.565x10^{-11}$
x_{10}	$.183x10^{-11}$	$.261x10^{-10}$	$.150x10^{-8}$	$.183xI0^{-10}$	$.183x10^{-10}$	$.183x10^{-10}$
x_{12}	$.156x1^{-9}$	$.176x10^{-9}$	$.151x10^{-7}$	$.335x10^{-11}$	$.379x10^{-11}$	$.274x10^{-11}$
x_{14}	$.219x10^{-8}$	$.243x10^{-8}$	$.210x10^{-6}$	$.188x10^{-10}$	$.252x10^{-10}$	$.102xI0^{-10}$
x_{16}	$.306x10^{-7}$	$.337x10^{-7}$	$.292xI0^{-5}$	$.153x10^{-9}$	$.241x10^{-9}$	$.331x10^{-10}$
x_{18}	$.427x10^{-6}$	$.469x10^{-6}$	$.407x10^{-4}$	$.172xI0^{-8}$	$.295x10^{-8}$	$.517x10^{-10}$
x_{19}	$.159x10^{-5}$	$.175x10^{-5}$	$.152x10^{-3}$	$.627x10^{-8}$	$.109x10^{-7}$	$.418x10^{-10}$
x_{20}	$.595x10^{-5}$	$.634x10^{-5}$	$.566x10^{-3}$	$.235x10^{-7}$	$.406x10^{-7}$	$.214x10^{-9}$

TABLE 2

Values of $|m_i - y_i^{(1)}$

	S_I	S_{II}	S_{III}	S_{IV}	S_V	S_{VI}
x_0	$.387 \times 10^{-8}$	—	$.304 \times 10^{-5}$	$.346 \times 10^{-7}$	$.345 \times 10^{-7}$	$.347 \times 10^{-7}$
x_1	$.468 \times 10^{-7}$	$.458 \times 10^{-7}$	$.769 \times 10^{-6}$	$.365 \times 10^{-7}$	$.366 \times 10^{-7}$	$.365 \times 10^{-7}$
x_2	$.356 \times 10^{-7}$	$.359 \times 10^{-7}$	$.254 \times 10^{-6}$	$.384 \times 10^{-7}$	$.383 \times 10^{-7}$	$.384 \times 10^{-7}$
x_4	$.422 \times 10^{-7}$	$.422 \times 10^{-7}$	$.579 \times 10^{-7}$	$.424 \times 10^{-7}$	$.424 \times 10^{-7}$	$.424 \times 10^{-7}$
x_6	$.468 \times 10^{-7}$	$.468 \times 10^{-7}$	$.480 \times 10^{-7}$	$.469 \times 10^{-7}$	$.469 \times 10^{-7}$	$.469 \times 10^{-7}$
x_8	$.518 \times 10^{-7}$	$.518 \times 10^{-7}$	$.519 \times 10^{-7}$	$.518 \times 10^{-7}$	$.518 \times 10^{-7}$	$.518 \times 10^{-7}$
x_{10}	$.572 \times 10^{-7}$	$.572 \times 10^{-7}$	$.572 \times 10^{-7}$	$.572 \times 10^{-7}$	$.572 \times 10^{-7}$	$.572 \times 10^{-7}$
x_{12}	$.633 \times 10^{-7}$	$.632 \times 10^{-7}$	$.630 \times 10^{-7}$	$.632 \times 10^{-7}$	$.632 \times 10^{-7}$	$.632 \times 10^{-7}$
x_{14}	$.699 \times 10^{-7}$	$.699 \times 10^{-7}$	$,669 \times 10^{-7}$	$.699 \times 10^{-7}$	$.699 \times 10^{-7}$	$.699 \times 10^{-7}$
x_{16}	$.777 \times 10^{-7}$	$.768 \times 10^{-7}$	$.351 \times 10^{-7}$	$.773 \times 10^{-7}$	$.772 \times 10^{-7}$	$.773 \times 10^{-7}$
x_{18}	$.915 \times 10^{-7}$	$.786 \times 10^{-7}$	$.501 \times 10^{-6}$	$.854 \times 10^{-7}$	$.853 \times 10^{-7}$	$.854 \times 10^{-7}$
x_{19}	$.667 \times 10^{-7}$	$.115 \times 10^{-6}$	$.228 \times 10^{-5}$	$.898 \times 10^{-7}$	$.899 \times 10^{-7}$	$.898 \times 10^{-7}$
x_{20}	$.180 \times 10^{-6}$	—	$.808 \times 10^{-5}$	$.940 \times 10^{-7}$	$.938 \times 10^{-7}$	$.944 \times 10^{-7}$

TABLE 3

Values of $|\tilde{y}_i^{(2)} - y_i^{(2)}|$.

	s_I	s_{II}	s_{III}	s_{IV}	s_V	s_{VI}
x_0	$.402x10^{-5}$	$.367x10^{-5}$	$.271x10^{-3}$	$.543x10^{-6}$	$.553x10^{-6}$	$.530x10^{-6}$
x_1	$.376x10^{-6}$	$.340x10^{-6}$	$.279x10^{-4}$	$.196x10^{-7}$	$206x10^{-7}$	$.182x10^{-7}$
x_2	$.768x10^{-7}$	$.671x10^{-7}$	$.750x10^{-5}$	$.188x10^{-7}$	$.186x10^{-7}$	$.192x10^{-7}$
x_4	$.143x10^{-7}$	$.150x10^{-7}$	$.558x10^{-6}$	$.212x10^{-7}$	$.212x10^{-7}$	$.212x10^{-7}$
x_6	$.229x10^{-7}$	$.230x10^{-7}$	$.620x10^{-7}$	$.234x10^{-7}$	$.234x10^{-7}$	$.234x10^{-7}$
x_8	$.259x10^{-7}$	$.259x10^{-7}$	$.287x10^{-7}$	$.259x10^{-7}$	$.259x10^{-7}$	$.259x10^{-7}$
x_{10}	$.286x10^{-7}$	$.286x10^{-7}$	$.294x10^{-7}$	$.286x10^{-7}$	$.286x10^{-7}$	$.286x10^{-7}$
x_{12}	$.315x10^{-7}$	$.317x10^{-7}$	$.392x10^{-7}$	$.316x10^{-7}$	$.316x10^{-7}$	$.316x10^{-7}$
x_{14}	$.339x10^{-7}$	$.362x10^{-7}$	$.140x10^{-6}$	$.350x10^{-7}$	$.350x10^{-7}$	$.350x10^{-7}$
x_{16}	$.233x10^{-7}$	$.555x10^{-7}$	$.150x10^{-5}$	$.387x10^{-7}$	$.388x10^{-7}$	$.387x10^{-7}$
x_{18}	$.171x10^{-6}$	$.277x10^{-6}$	$.204x10^{-4}$	$.436x10^{-7}$	$.442x10^{-7}$	$.427x10^{-7}$
x_{19}	$.842x10^{-6}$	$.831x10^{-6}$	$.758x10^{-4}$	$.418x10^{-7}$	$.395x10^{-7}$	$.449x10^{-7}$
x_{20}	$.906x10^{-5}$	$.722x10^{-5}$	$.737x10^{-3}$	$.127x10^{-5}$	$.125x10^{-5}$	$.130x10^{-5}$

TABLE 4

Values of $|\tilde{y}_i^{(3)} - y_i^{(3)}|$.

	s_I	s_{II}	s_{III}	s_{IV}	s_V	s_{VI}
x_2	$.101x10^{-4}$	$.909x10^{-5}$	$.777x10^{-3}$	$.152x10^{-6}$	$.180x10^{-6}$	$.115x10^{-6}$
x_4	$.843x10^{-6}$	$.771x10^{-6}$	$.557x10^{-4}$	$.130x10^{-6}$	$.132x10^{-6}$	$.127x10^{-6}$
x_6	$,192x10^{-6}$	$.187x10^{-6}$	$.387x10^{-5}$	$.141x10^{-6}$	$.141x10^{-6}$	$.141x10^{-6}$
x_8	$.159x10^{-6}$	$.159x10^{-6}$	$.128x10^{-6}$	$.155x10^{-6}$	$.155x10^{-6}$	$.155x10^{-6}$
x_{10}	$.172x10^{-6}$	$.172x10^{-6}$	$.207x10^{-6}$	$.172x10^{-6}$	$.172x10^{-6}$	$.172x10^{-6}$
x_{12}	$.181x10^{-6}$	$.199x10^{-6}$	$.970x10^{-6}$	$.190x10^{-6}$	$.190x10^{-6}$	$.190x10^{-6}$
x_{14}	$.954x10^{-7}$	$.336x10^{-6}$	$.111x10^{-4}$	$.210x10^{-6}$	$.211x10^{-6}$	$.210x10^{-6}$
x_{16}	$.136x10^{-5}$	$.198x10^{-5}$	$.152x10^{-3}$	$.239x10^{-6}$	$.243x10^{-6}$	$.232x10^{-6}$
x_{18}	$.219x10^{-4}$	$.246x10^{-4}$	$.211x10^{-2}$	$.343x10^{-6}$	$.407x10^{-6}$	$.257x10^{-6}$

TABLE 5

Values of $|\tilde{y}_i^{(4)} - y_i^{(4)}|$.

	s_I	s_{II}	s_{III}	s_{IV}	s_V	s_{VI}
x_1	.172x10^{-2}	.155x10^{-2}	.134	.642x10^{-5}	.111x10^{-4}	. 476x10^{-7}
x_2	.461x10^{-3}	.414x10^{-3}	.359x10^{-1}	.170x10^{-5}	.296x10^{-5}	.343x10^{-7}
x_4	.330x10^{-4}	.297x10^{-4}	.258x10^{-2}	.984x10^{-7}	.189x10^{-6}	.260x10^{-7}
x_6	.236x10^{-5}	.212x10^{-5}	.185x10^{-3}	.984x10-^{8}	.332x10^{-8}	.188x10^{-7}
x_8	.161x10^{-6}	.140x10^{-6}	.135x10^{-4}	.104x10^{-7}	.995x10^{-8}	.110x10^{-7}
x_{10}	.154x10^{-7}	.430x10^{-7}	.357x10^{-5}	.242x10^{-7}	.242x10^{-7}	.241x10^{-7}
x_{12}	.369x10^{-6}	.428x10^{-6}	.362x10^{-4}	.130x10^{-7}	.141x10^{-7}	.116x10^{-7}
x_{14}	.527x10^{-5}	.582x10^{-5}	.503x10^{-3}	.359x10^{-7}	.510x10^{-7}	.152x10^{-7}
x_{16}	.735x10^{-4}	.809x10^{-4}	.701x10^{-2}	. 339x10^{-6}	.550x10^{-6}	.508x10^{-7}
x_{18}	.103x10^{-2}	.113x10^{-2}	.976x10^{-1}	.410x10^{-5}	.704x10^{-5}	.891x10^{-7}
x_{19}	.383x10^{-2}	.420x10^{-2}	.364	.151x10^{-4}	,261x10^{-4}	.117x10^{-6}

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