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A CLASS OF ALGORITHMS FOR
RATIONAL APPROXIMATION OF
FUNCTIONS FORMALLY DEFINED
BY POWER SERIES

by

J.A, MURPHY and M.R. O'DONOHOE.

ABSTRACT

Corresponding sequence algorithms are defined and
shown to exist for a wide range of corresponding
continued fractions.   Particular examples of these
algorithms are given, including an algorithm for
forming Pade approximants, and an error analysis
is given in one case.

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1. Continued Fractions

1.1 Corresponding Fractions

We denote a continued fraction by

$$\frac{u_1}{v_1} + \frac{u_2}{v_2} = + \ldots + \frac{u_n}{v_n} \ldots\ldots \tag{1.1}$$

where u_n and v_n are numbers, real or complex. In this work
we shall consider u_n and v_n to be polynomials in the
complex variable z. The nth convergent of (1.1) is

$$\frac{U_n(z)}{V_n(z)} = \frac{u_1(z)}{v_1(z)} + \frac{u_2(z)}{v_2(z)} + \ldots + \frac{u_n(z)}{v_n(z)} \tag{1.2}$$

where U_n and V_n both satisfy the recurrence formula

$$W_n = u_n W_{n-2} + v_n W_{n-1} \tag{1.3}$$

with initial values $U_0 = 0, U_1 = u_1$ and $V_0 = 1$, $V_1 = v_1$.

Clearly, $U_n(z)$ and $V_n(z)$ are also polynomials so that (1.2)
is a rational approximation to (1.1).

We consider the function $f_0(z)$ formally defined by
the power series expansion

$$f_0(z) = a_0 + a_1 z + a_2 z^2 + \ldots , \tag{1.4}$$

convergent for $|z| < R$.

The continued fraction (1.1) is said to correspond
to the power series (1.4) if

$$f_0(z) - \frac{U_n(z)}{V_n(z)} = 0 \, (s^{\sigma(n)}), \tag{1.5}$$

for $|z| < R$, where $\sigma(n) > n$ for $n = 1, 2, 3, \ldots \ldots$

This is a slight generalisation of the usual definition

for corresponding fractions. Under this definition, given

$\sigma(n)$ for all n, there may be many different continued

fractions which correspond to the same power series but in

the remainder of this work we shall use the notation $f_o(z)$

to refer to the power series (1.4) or to any of its

corresponding fractions.

We now establish a form of continued fraction which

satisfies the definition (1.5). Most of the widely studied

corresponding fractions are particular forms of the fraction

$$f_0(z) = \frac{p_1(z)}{q_1(z)} + \frac{z^v(1)p_2(z)}{q_2(z)} + \frac{z^v(2)p_3(z)}{q_3(z)} + \ldots + \frac{z^v(n-1)p_n(2)}{q_n(z)} + \ldots \tag{1.6}$$

where $\{v(n)\}$ is a sequence of positive integers and

$P_n(z)$, $q_n(z)$ are polynomials such that $p_n(0) \neq 0$, $q_n(o) \neq 0$

and $P_n(z)$, $q_n(z)$ are both of degree $v(n)-1$ at most. Without

loss of generality, we normalise (1.6) by setting $q_n(0) = 1$

for all n. We also impose the following restrictions:

    (i) The sum of the numbers of non-zero coefficients

        in $P_n(z)$ and $q_n(z)$ shall he $v(n)$.

    (ii) If the polynomial $r_n(z) = p_n(z) + q_n(z)$ has degree

        $\mu(n) \leq v(n)-1$ then all the $\mu(n) + 1$ coefficients

        of $r_n(z)$ shall be non-zero.

If $P_n(z)/Q_n(z)$ is the nth convergent of (1.6) then we must

prove that

$$f_0(z) - \frac{p_n(z)}{Q_n(z)} = 0(z^{\sigma(n)}) \tag{1.7}$$

where

$$\sigma(n) = \sum_{i=1}^{n} v(i). \tag{1.8}$$

Now, (1.7) may be written

$$Q_n(z)f_0(z) - p_n(z) = z\sigma(n)s_n(z) \tag{1.9}$$

where S_n (z) is some power series of the form (1. 4). The identity (1.9) may be proved by induction. We first assume that (1.9) holds for both n - 1 and n, and using (1.3) we have

$$Q_{n+1}f_0 - P_{n+1} = z^{v(n)} P_{n+1}(Q_{n-1}f_0 - P_{n-1}) + q_{n+1}(Q_n f_0 - P_n).$$

By our hypothesis we have

$$Q_{n+1}f_0 - P_{n+1} = z^{v(n)} P_{n+1} z^{\sigma(n-1)} s_{n-1} + q_{n+1} z^{\sigma(n)} s_n.$$

Using (1.8) this gives

$$Q_{n+1}f_0 - P_{n+1} = r_{n+1} z^{\sigma(n)} (s_{n-1} + s_n). \tag{1.10}$$

Since, by restriction (ii), none of the coefficients of r_{n+1} (z) are missing it can be shown that we may choose the v(n+1) non-zero coefficients of $p_{n+1}(z)$ and $q_{n+1}(z)$ such that the first v(n+1) terms of. r_{n+1} $(s_{n+1} + S_n)$ vanish. We can then write (1.10) in the form

$$Q_{n+1}f_0 - p_{n+1} = z^{\sigma(n+1)} s_{n+1}$$

so that (1.9) holds for n+1 provided that it holds for

n-1 and n. If we choose σ(0) =0 then the result holds

trivially for n = 0 so that to complete the proof we need

only verify (1.9) for n=1. In this case we have

$$Q_1 \, f_0 - P_1 \equiv q_1 \, f_0 - P_1.$$

Once again restrictions (i) and (ii) are clearly sufficient

to ensure that we can choose the $v(1)$ coefficients of

$p_1 (z)$ and $q_1 (z)$ so that the first $v(1)$ terms vanish. Thus we

have proved that the successive convergents of the continued

fraction (1.6) correspond to σ(1), σ(2), σ(3) . . . . terms

of the power series (1. 4).

1.2   The Corresponding Sequence

We consider a function $f_0 (z)$ formally defined by

the power series (1.4). We state without proof that if a

corresponding fraction (1.1) exists then it may be represented

by the set of recurrence relations

$$
\left.
\begin{aligned}
f_1(z) &= u_1(z) & &- & & v_1(z)f_0(z) \\
f_2(z) &= u_2(z)f_0(z) & &- & & v_2(z)f_1(z) \\
f_3(z) &= u_3(z)f_1(z) & &- & & v_3(z)f_2(z) \\
& \quad - - - - - \\
& \quad - - - - - \\
f_n(z) &= u_n(z)f_{n-2}(z) & &- & & v_n(z)f_{n-1}(z) \\
& \quad - - - - - \\
& \quad - - - - -
\end{aligned}
\right\}
\qquad (1.11)
$$

where the sequence $[f_n(z)]$ is a sequence of functions which,

like $f_0(z)$, may be represented either as series or as

corresponding fractions. We shall refer to $[f_n(z)]$ as the

corresponding sequence associated   with the continued

fraction (1.1) and we will use the series representations

of the corresponding sequence to form algorithms to compute

the coefficients of the continued fraction (1.6).

The basic similarity transformation of continued

fractions is defined by

$$f_0(z) = \frac{k_1 u_1}{k_1 v_1} + \frac{k_1 k_2 u_2}{k_2 v_2} + \frac{k_2 k_3 u_3}{k_3 v_3} + \ldots + \frac{k_{n-1} k_n u_n}{k_n v_n} + \ldots \tag{1.12}$$

where k_1, k_2, k_3, ...   are arbitrary non-zero numbers.

Although the value of the continued fraction does not change

under this transformation the corresponding sequence is

altered.   The transformation is equivalent to multiplying the

nth equation of the set (1.11) by K_n, where $k_n = \prod_{r=1}^{n} k_r$, and

forming the new corresponding sequence $\{F_n\}$, where

$$\left. \begin{array}{l} F_0 = f_0 , \\ F_n = k_n f_n . \end{array} \right\} \tag{1.13}$$

Under this transformation the nth equation of (1 . 11) becomes

$$F_n(z) = k_n \{k_{n-1} u_n(z) F_{n-2}(z) - v_n(z) F_{n-1}(z)\} \tag{1.14}$$

We shall use this transformation to modify some of the

algorithms we shall develop in Section 2.

2.   The Corresponding Sequence Algorithms

2.1   The General Algorithm.

We first apply result (1.14) to the continued

fraction (1.6) to obtain

$$F_n^{(z)} = k_n \{ k_{n-1} z^{v(n-1)} p_n^{(z)} F_{n-2}^{(z)} - q_n^{(z)} F_{n-1}^{(z)} \}$$
(2.1)

for n = 1,2,3, ... where $k_o =$ v(0) $= F_{-1}(z) =$ 1. If we
write

$$F_n(z) = z^{\sigma(n)} (b_0^{(n)} + b_1^{(n)} z + b_2^{(n)} z^2 + ... + b_r^{(n)} z^r + ...)$$
(2.2)

then we can equate coefficients of powers of z in (2.1) and,

in general , obtain

$$b_r^{(n)} = k_n E^{v(n)} \{ k_{n-1} p_n(E^{-1}) b_r^{(n-2)} - q_n(E^{-1}) b_r^{(n-1)} \} .$$
(2.3)

The shift operator E is defined by

$$E^m b_r^{(n)} = b_{r+m}^{(n)}$$
(2.4)

and we choose $b_r^{(n)} = 0$ for $r < 0, b_r^{(-1)} = 1,$ and $b_r^{(-1)} = 0$ for

$r \neq 0$. The expression (2.3) then holds for n = 1,2,3,.... and

r = -v(n),-v(n)+1 ,....-2, -1,0,1,2,3,.... and we note that, in

particular $b_r^{(0)} = a_r$ for   r = 0,1,2,3...... The equation (2.3)

summarises an algorithm for obtaining the coefficients of

the continued fraction (1.6) from the sequence $\{a_r\}$,   We call

this algorithm the corresponding sequence algorithm, or CS

algorithm, for the continued fraction (1.6).

Although not generally true, the equations summarised
by (2.3) often form a triangular system so that the problem
of solution is simple.   The class of continued fractions for
which this is true includes most of the widely studied
corresponding fractions. Many of these fractions are such that
$p_n(z)$ is a constant, λ_n say, for all n.   In this case we also
define the modified CS algorithm in which we choose

$$k_n = \lambda_{n+1}^{-1} \tag{2.5}$$

for n= 1,2,3, ... and we adjust our series such that
$a_0 = .\lambda_1 = 1.$ The modified CS algorithm may then be summarized
by

$$b_r^{(n)} = k_n E^{v(n)}\left\{ b_r^{(n-2)} - q_n(E^{-1})b_r^{(n-1)} \right\} \tag{2.6}$$

We   also note that we need not store the values $\left\{ b_0^{(n)} \right\}$ as they
are   all unity.

The   essential difference between the   alternative
algorithms is that the sequence $\{k_n\}$ is arbitrarily chosen
in the ordinary algorithm   (2.3) but is computed in the
modified algorithm (2.6).   However, the modified algorithm
is usually simpler for computational purposes. In the   remaining
sections we discuss particular examples of CS algorithms.

2.2 S-Fractions and Padé Approximants

The continued fraction

$$f_0(z) = \frac{c_0}{1+} \frac{c_1 z}{1+} \frac{c_2 z}{1+} \cdots + \frac{c_n z}{1+} \cdots \tag{2.7}$$

is called an S-fraction if the sequence $\{c_n\}$ is chosen such that the fraction corresponds, term for term, with the power series (1.4). Such a fraction exists provided that the Hankel determinants

$$H_{2n} = \begin{vmatrix} a_0 & a_1 & \cdots a_n \\ a_1 & a_2 & \\ ----- \\ ----- \\ a_n & a_{n+1} \cdots a_{2n} \end{vmatrix} \quad \text{and,} \quad H_{2n+1} = \begin{vmatrix} a_1 & a_2 \cdots a_{n+1} \\ a_2 & a_3 \cdots a_{n+2} \\ ----- \\ ----- \\ a_{n+1} a_{n+2} \cdots a_{2n+1} \end{vmatrix} \tag{2.8}$$

are non-zero for $n = 0,1,2,3, \ldots$ We shall assume this condition is satisfied and to simplify the computation we adjust our series so that $a_0 = c_0 = 1$.

Choosing $k_n = 1$ for all n and applying result (2.3) to the continued fraction (2.7) we obtain the CS algorithm

$$b_r^{(n)} = c_{n-1} b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)} \tag{2.9}$$

for all r, n or, written in full,

$$\begin{cases} c_0 = 1 \\ b_r^{(1)} = -a_{r+1} \ , \ r = 0,1,2,3, \ldots, \\ c_n = \dfrac{b_0^{(n)}}{b_0^{(n-1)}} \ , \ n = 1,2,3, \ldots, \\ b_r^{(n)} = c_{n-1} b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)} \ \ r = 0,1,2,3,\ldots, \ n = 2,3,4 \ldots \end{cases}$$

Also, from (2.6) we obtain the modified CS algorithm by choosing $k_n = c_n^{-1}$ so that

$$b_r^{(n)} = k_n \left\{ b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)} \right\}$$

(2.10)

for all r, n. We note that the values $\left\{ b_r^{(n)} \right\}$ are not the same in each algorithm. In computational form the modified algorithm is

$$
\left\{
\begin{array}{l}
c_0 = 1, \quad c_1 = -a_1, \\[2mm]
b_r^{(1)} = -a_{r+1}, \quad r = 1,2,3, \ldots, \\[2mm]
c_n = b_1^{(n-2)} - b_1^{(n-1)}, \quad n = 2,3,4, \ldots, \\[2mm]
b_r^{(n)} = \dfrac{1}{c_n} \left\{ b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)} \right\} \quad r = 1,2,3, \ldots, \ n = 2,3,4, \ldots
\end{array}
\right.
$$

As an example we perform each algorithm on the power series

$$e^{-z} = 1 - + \frac{1}{2} z^2 - \frac{1}{6} z^3 + \frac{1}{24} z^4 - \frac{1}{120} z^5 + \ldots$$

10.

a_5	$-\dfrac{1}{120}$					
a_4	$\dfrac{1}{24}$	$\dfrac{1}{120}$				
a_3	$-\dfrac{1}{6}$	$-\dfrac{1}{24}$	$\dfrac{1}{30}$			
a_2	$\dfrac{1}{2}$	$\dfrac{1}{6}$	$-\dfrac{1}{8}$	$-\dfrac{1}{80}$		
a_1	-1	$-\dfrac{1}{2}$	$\dfrac{1}{3}$	$\dfrac{1}{24}$	$-\dfrac{1}{120}$	
a_0	1	1	$-\dfrac{1}{2}$	$-\dfrac{1}{12}$	$\dfrac{1}{72}$	$\dfrac{1}{720}$
	1	1	$-\dfrac{1}{2}$	$\dfrac{1}{6}$	$-\dfrac{1}{6}$	$\dfrac{1}{10}$
	c_0	c_1	c_2	c_3	c_4	c_5

ordinary CS algorithm

a_5	$-\dfrac{1}{120}$				
a_4	$\dfrac{1}{24}$	$\dfrac{1}{120}$			
a_3	$-\dfrac{1}{6}$	$-\dfrac{1}{24}$	$-\dfrac{1}{15}$		
a_2	$\dfrac{1}{2}$	$\dfrac{1}{6}$	$\dfrac{1}{4}$	$\dfrac{3}{20}$	
a_1	-1	$-\dfrac{1}{2}$	$-\dfrac{2}{3}$	$-\dfrac{1}{2}$	$-\dfrac{3}{5}$
a_0	1	1	$-\dfrac{1}{2}$	$\dfrac{1}{6}$	$-\dfrac{1}{6}$ $\dfrac{1}{10}$
	c_0	c_1	c_2	c_3	c_4 c_5

Modified CS algorithm

Each algorithm indicates the S-fraction expansion

$$e^{-z} = \frac{1}{1} + \frac{z}{1} - \frac{\frac{1}{2}z}{1} + \frac{\frac{1}{6}z}{1} - \frac{\frac{1}{6}z}{1} + \frac{\frac{1}{10}z}{1} - \dots -$$

The modified algorithm (2. 10) commends itself for hand calculation as it is simple to use and easy to remember. Also, the coefficients $\left\{b_r^{(n)}\right\}$ in the ordinary algorithm (2.9) become small very quickly and are more inconvenient to use than those of (2.10).

We now consider the effect of rounding errors in the series coefficients $[a_n|$ on the continued fraction coefficients $\{c_n\}$. As the effect of any particular choice of - the constants $\{k_n\}$ is ultimately cancelled out, the build-up of rounding errors in the continued fraction coefficients is independent of these constants. Consequently the modified algorithm is equivalent to the ordinary algorithm in this respect, so we perform a numerical error analysis on the modified algorithm only.

We consider rounding errors $\in_r^{(n)}$ in $b_r^{(n)}$ and η_n inn $\iota\iota\iota$ c_n and, by substitution in (2.10), we get

$$b_r^{(n)} + \in_r^{(n-2)} = \frac{\left\{b_{r+1}^{(n-2)} + \in_{r+1}^{(n-2)}\right\} - \left\{b_{r+1}^{(n-1)} + \in_{r+1}^{(n-1)}\right\}}{c_n + \eta_n}$$

so that, for small errors,

$$\in_r^{(n)} = \cdot\frac{1}{c_n}\left\{\in_{r+1}^{(n-2)} - \in_{r+1}^{n-1}\right\} - \frac{\eta_n}{c_n}\left\{b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)}\right\}. \tag{2.11}$$

In computational form this is

$$
\begin{cases}
\eta_1 \doteq - \dot{\in}_1^{(o)} \\[1em]
\dot{\in}_r^{(1)} \doteq - \left\{ \dfrac{\dot{\in}_{r+1}^{(o)}}{c_1} + \dfrac{\in_1^{(o)} a_{r+1}}{c_1^2} \right\}, \qquad r = 1,2,3,\ldots, \\[1em]
\eta_n \doteq \dot{\in}_1^{(n-2)} - \dot{\in}_1^{(n-1)} , \quad n = 2,3,4,\ldots , \\[1em]
\dot{\in}_r^{(n)} \doteq \dfrac{1}{c_n} \left\{ \in_{r+1}^{(n-2)} - \in_{r+1}^{(n-1)} \right\} - \dfrac{\eta_n}{c_n^2} \left\{ b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)} \right\} \\
\hspace{4em} r = 1,2,3, \ldots \quad n = 2,3,4, \ldots
\end{cases}
$$

We now perform two error analyses for the example e^{-z}

already given . First, we suppose an error e in each a_n .

$\in_5^{(0)}$ $\in$					
$\in_4^{(0)}$ $\in$	$-\dfrac{119}{120}\in$				
$\in_3^{(0)}$ $\in$	$-\dfrac{25}{24}\in$	$-\dfrac{259}{60}\in$			
$\in_2^{(0)}$ $\in$	$-\dfrac{5}{6}$	$-\dfrac{17}{6}\in$	$\dfrac{147}{10}\in$		
$\in_1^{(0)}$ $\in$	$-\dfrac{3}{6}$	$-7\in$	$\dfrac{57}{2}\in$	$233\in$	
0	$\in$	$\dfrac{5}{2}\in$	$\dfrac{11}{2}\in$	$-\dfrac{71}{2}\in$	$-\dfrac{409}{2}\in$
	η_1	η_2	η_3	η_4	η_5

Now, we suppose an error $(-1)^n \epsilon$ in each a_n

$\epsilon_5^{(o)}$	$-\epsilon$					
$\epsilon_4^{(o)}$	ϵ	$\dfrac{119}{120}\epsilon$				
$\epsilon_3^{(o)}$	$-\epsilon$	$-\dfrac{23}{24}\epsilon$	$\dfrac{1}{20}\epsilon$			
$\epsilon_2^{(o)}$	ϵ	$\dfrac{5}{6}\epsilon$	$-\dfrac{1}{6}\epsilon$	$-\dfrac{53}{10}\epsilon$		
$\epsilon_1^{(o)}$	$-\epsilon$	$-\dfrac{1}{2}$	$\dfrac{1}{3}\epsilon$	$\dfrac{7}{2}\epsilon$	$-\dfrac{97}{5}$	
0		ϵ	$-\dfrac{1}{2}\epsilon$	$-\dfrac{5}{6}\epsilon$	$-\dfrac{19}{6}\epsilon$	$\dfrac{229}{10}\epsilon$
		η_1	η_2	η_3	η_4	η_5

Given a maximum rounding error $|\epsilon|$, these examples may be used to estimate the number of terms for which the continued fraction expansion may be computed with a desired degree of accuracy. Clearly, if implemented on a computer the algorithm will accurately evaluate more continued fraction coefficients if higher precision working is used.

We now define the [M / N] Padé approximant to the series (1.4) to be $A_M(z) / B_N(z)$ where $A_M(z)$, $B_N(z)$ are polynomials of degree M,N respectively such that

$$B_N(z)\, f_o(z) \ - \ A_M(z) \ = \ 0(z^{M+N+1}).$$

(2.12)

The sequence of Padé approximants

[L-1/0]   [L/O]

[L/1]   [L+1/1]

[L+1/2]   [L+2/2] . . . .
. . . .

is given by the successive convergents of the corresponding

fraction

$$f_0(z) = a_0 + a_1 z + a_2 z^2 + ... + a_{L-1} z^{L-1} + a_L z^L \left\{ \frac{1}{1+} \frac{c_1^{(L)}}{1+} \frac{c_2^{(L)} z}{1+} .... \right\}$$

(2.13)

and the sequence

[O/L-1]

[0/L]   [1/L]

[1/L+1]   [2/L+1]

[2/L+2] . . . .

.. . .

is given by the successive convergents of the corresponding

fraction

$$f_0(z) = \cfrac{1}{d_0 + d_1 z + ...d_{L-1} z^{L-1} + d_L z^L \left\{ \frac{1}{1+} \frac{g_1^{(L)} z}{1+} \frac{g_2^{(L)} z}{1+} \right\}}$$ (2.14)

By suitable choice of L, we can express any Padé approximant as a convergent of one of the fractions (2.13) and (2.14). In (2.13) the first $(L + 1)$ coefficients are identical to those of the series (1.4) and the coefficients $c_1^{(L)}, c_2^{(L)}, c_3^{(L)},.....$ may be obtained by applying the modified CS algorithm to the sequence $a_{L+1} / a_L, a_{L+2} / a_L, a_{L+3} / a_L, ...$ In (2.14) the series $d_0 + d_1 z + d_2 z^2 + ...$ is the power series of the reciprocal of $f_o(z)$ and its coefficients may be computed from the recurrence relation

$$d_n = -d_0 \sum_{r=1}^{n} d_{n-r} a_r \tag{2.15}$$

for $n = 1, 2, 3, ...$ and where $d = a_o^{-1}$. The coefficients $g_1^{(L)}, g_2^{(L)}, g_3^{(L)}, ...$ are then obtained by applying the modified CS algorithm to the sequence $d_{L+1} / d_L, d_{L+2} / d_L, d_{L+3} / d_L, ...$

In particular, we find that the [2/3] Pade approximant is given by

$$\cfrac{1}{d_0 + d_1 z +} \cfrac{d_2 z^2}{1 +} \cfrac{g_1(2)z}{1 +} \cfrac{g_2(2)z}{1 +} \cfrac{g_3^{(2)} z}{1}$$

As an example we shall consider the [2/3] Padé approximant to e^{-z}. We already know the reciprocal series

$$e^z = 1 + z + \frac{1}{2} z^2 + \frac{1}{6} z^3 + \frac{1}{24} z^4 + \frac{1}{120} z^5 + ...$$

so that we have

$$\frac{d_3}{d_2} = \frac{1}{3}, \frac{d_4}{d_2} = \frac{1}{12}, \frac{d_5}{d_2} = \frac{1}{60}$$

and we apply the modified CS algorithm to these values

$$
\begin{array}{c|ccc}
d_5/d_2 & \dfrac{1}{60} & & \\
d_4/d_2 & \dfrac{1}{12} & \dfrac{1}{20} & \\
d_3/d_2 & \dfrac{1}{3} & \dfrac{1}{4} & \dfrac{2}{5} \\
\hline
1 & -\dfrac{1}{3} & \dfrac{1}{12} & -\dfrac{3}{20} \\
& g_1^{(2)} & g_2^{(2)} & g_3^{(2)}
\end{array}
$$

Thus, the [2/3] Padé approximant to e^{-z} is

$$e^{-z} = \cfrac{1}{1 + z + \cfrac{\frac{1}{2}z^2}{1 - \cfrac{\frac{1}{3}z}{1 + \cfrac{\frac{1}{12}z}{1 - \frac{3}{20}z}}}} + 0(z^6).$$

This is one of the simplest methods for obtaining a

Padé approximant and is easily accomplished by a minimum

of hand calculation.

It is interesting to compare this algorithm with

that of Longman (1971). Longman's algorithm computes the

coefficients of Padé approximants in the more usual rational

form. The advantage of the continued fraction form is that,

by computing just one more coefficient, we can progress from

one approximant to another. Whilst Longman's algorithm is more useful for computing the whole Padé table, we can use the CS algorithm to calculate high order approximants without computing the whole of the preceding table. As fewer computational steps are necessary we may suppose that there is less build-up of rounding error with the CS algorithm.

2.3 J-Fractions.

The continued fraction

$$f_0(z) = \cfrac{c_0}{1 + d_0 z} + \cfrac{c_1 z^2}{1 + d_1 z} + \cfrac{c_2 z^2}{1 + d_2 z} + \cdots \cfrac{C_n Z^2}{1 + d_n z} + \cdots \qquad (2.16)$$

is called a J-fraction and corresponds to the power series (1.4), successive convergents corresponding to 2,4,6 ....terms of the series. If $c_0 = 1$ the modified CS algorithm for this fraction is

$$b_r^{(n)} = k_n \left\{ b_{r+2}^{(n-2)} - b_{r+2}^{(n-1)} - d_{n-1} b_{r+1}^{(n-1)} \right\} \qquad (2.17)$$

for all r, n. In computational form this algorithm is

$$
\begin{cases}
c_0 = 1, \qquad d_0 = -a_1 \quad , \\
c_1 = - a_2 + a_1^2 \quad , \\
b_r^{(1)} = \dfrac{1}{c_1} \{-a_{r+2} + a_1 a_{r+1}\} \quad , \quad r = 1,2,3, \quad ....., \\
d_n = b_1^{(n-1)} - b_1^{(n)} \quad , \quad n = 1,2,3,.... ., \\
d_n = b_1^{(n-1)} - b_1^{(n)}, \quad n = 1,2,3,.... , \\
c_n = b_2^{(n-2)} - b_2^{(n-1)} - d_{n-1} b_1^{(n-1)}, \quad n = 2,3,4,.... , \\
b_r^{(n)} = \dfrac{1}{c_n} \{b_{r+2}^{(n-2)} - b_{r+2}^{(n-1)} - b_{r+1}^{(n-1)}\} \quad ,
\end{cases}
$$

$$r = 1,2,3 ...... n = 2,3,4, ...$$

If we replace z by 1/z in (2.16) and use the basic similarity transformation we obtain the alternative form of the J-fraction

$$f_0(z) = \frac{c_0}{d_0 + z +} \; \frac{c_1}{d_1 + z +} \; \frac{c_2}{d_2 + z +} \; \cdots + \frac{c_n}{d_n + z +} \; \cdots \tag{2.18}$$

This fraction corresponds to the asymptotic series

$$f_0(z) = \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{a_2}{z^3} + \cdots \tag{2.19}$$

for $|z|$ large. Clearly, if we choose $a_o = c_o = 1$ then the algorithm (2.17) may also be applied to the fraction (2.18).

2.4. M-Fractions.

The continued fraction

$$f_0(z) = \frac{c_0}{1 + d_0 z +} \; \frac{c_1 z}{1 + d_1 z +} \; \frac{c_2 z}{1 + d_2 z +} \; \cdots \; \frac{c_n z}{+ 1 + d_n z +} \; \cdots \tag{2.20}$$

is called an M-fraction [see McCabe (1971)] if its coefficients are chosen such that it corresponds simultaneously to the power series (1.4) and to the asymptotic series

$$f_0(z) = \frac{\alpha_0 +}{z} \; \frac{\alpha_1 +}{z^2} \; \frac{\alpha_2 +}{z^3} \cdots \tag{2.21}$$

for $|z|$ large, successive convergents of (2.20) corresponding to $1, 2, 3, \ldots$ terms of each series. The ordinary CS algorithm

for converting the series (1.4) to the M-fraction (2.20)

is

$$b_r^{(n)} = c_{n-1} b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)} - d_{n-1} b_{)r}^{(n-1)}$$

(2.22)

for all r,n.   Using the basic similarity transformation

we see that the fraction (2.20) may be written

$$f_0(z) = \cfrac{c_0 \frac{1}{z}}{d_0 + \frac{1}{z} +} \cfrac{c_1 \frac{1}{z}}{d_1 + \frac{1}{z} +} \cfrac{C_2 \frac{1}{z}}{d_n + \frac{1}{z} +} \cdots \cfrac{c_n \frac{1}{z}}{+ d_n + \frac{1}{2} +} \cdots$$

(2.23)

By writing the fraction in this form we see that there is

also a CS algorithm for converting the series (2. 21) to

the fraction (2. 23).   This algorithm is

$$\beta_r^{(n)} = c_{n-1} \beta_{r+1}^{(n-2)} - d_{n-1} \beta_{r+1}^{(n-1)} - \beta_r^{(n-1)}$$

(2.24)

for all r,n where   the values $\{ \beta_r^{(n)} \}$ are analogous to the

values $\left| b_r^{(n)} \right|$ in (2. 22).   The relations (2. 22) and (2. 24)

taken together form the CS algorithm for   the M-fraction (2. 20).

Written in computational form this algorithm

is

$$
\begin{cases}
c_0 \quad a_0, d_0 = \dfrac{a_0}{\alpha_0}, \\[2mm]
b_r^{(1)} = -\{a_{r+1} + d_0 a_r\}, r = 0,1,2,3,\ldots, \\[2mm]
\beta_r^{(1)} = -\{d_0 \alpha_{r+1} + \alpha_r\}, r = 0,1,2,3,\ldots, \\[2mm]
c_n = \dfrac{b_0^{(n)}}{b_0^{(n-1)}}, \; n = 1,2,3,\ldots,n \\[4mm]
d_n = c_n \dfrac{\beta_0^{(n-1)}}{\beta_0^{(n)}}, \; n = 1,2,3,\ldots, \\[4mm]
b_r^{(n)} = c_{n-1} b_{r+1}^{(n-2)} - b_{r+1}^{(n-1)} - d_{n-1} b_r^{(n-1)}, r = 0,1,2,3,\ldots n = 2,3,4,\ldots, \\[2mm]
\beta_r^{(n)} = c_{n-1} \beta_{r+1}^{(n-2)} - d_{n-1} \beta_{r+1}^{(n-1)} - \beta_r^{(n-1)}, r = 0,1,2,3,\ldots n = 2,3,4,\ldots,
\end{cases}
$$

we can form a modified CS algorithm but as we cannot choose $a_o = \alpha_o = 1$ without rescaling z the resulting algorithm is no simpler than that above.

We now give a simple numerical example to illustrate a suitable layout for hand computation. We consider Dawson's integral

$$
f_0(z) = e^{-2} \int_0^z e^{t^2} \, dt
$$

which has the two series expansions

$$f_0(z) = 1 - \frac{z}{3} + \frac{z^2}{15} - \frac{z^3}{105} + \frac{z^4}{945} - \dots$$

for $|z|$ small and

$$f_0(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{3}{z^3} + \frac{15}{z^4} + \frac{105}{z^5} + \dots$$

for $|z|$ large.

$$
\begin{array}{c|ccccc}
a_4 & \frac{1}{945} & & & & \\
a_3 & -\frac{1}{105} & \frac{8}{945} & & \text{b}-\text{array} & \\
a_2 & \frac{1}{15} & -\frac{2}{35} & \frac{16}{945} & & \\
a_1 & -\frac{1}{3} & \frac{4}{15} & -\frac{8}{105} & \frac{64}{4725} & \\
a_0 & 1 & -\frac{2}{3} & \frac{8}{45} & -\frac{16}{525} & \frac{128}{33075} \\
\hline
& 1 & -\frac{2}{3} & -\frac{4}{15} & -\frac{6}{35} & -\frac{8}{63} \\
& c_0 & c_1 & c_2 & c_3 & c_4
\end{array}
$$

$$
\begin{array}{c|ccccc}
\alpha_4 & 105 & & & & \\
\alpha_3 & 15 & -120 & & \beta-\text{array} & \\
\alpha_2 & 3 & -18 & 48 & & \\
\alpha_1 & 1 & -4 & -8 & -\frac{64}{5} & \\
\alpha_0 & 1 & -2 & \frac{8}{3} & -\frac{16}{5} & \frac{128}{35} \\
\hline
& 1 & -\frac{1}{3} & -\frac{1}{5} & -\frac{1}{7} & -\frac{1}{9} \\
& d_0 & d_1 & d_2 & d_3 & d_4
\end{array}
$$

The resulting M-fraction is thus

$$f_0(z) = \cfrac{1}{1 + z -} \cfrac{\frac{2}{3z}}{1 + \cfrac{1}{3z} -} \cfrac{\frac{4}{15}z}{1 + \cfrac{1}{5}z -} \cfrac{\frac{6}{35}z}{1 + \cfrac{1}{7}z -} \cfrac{\frac{8}{63}z}{1 + \cfrac{1}{9}z -} \cdots$$

or, using the basic similarity transformation,

$$f_0(z) = \cfrac{1}{1 + z -} \cfrac{2z}{3 + z -} \cfrac{4z}{5 + z -} \cfrac{6z}{7 + z -} \cfrac{8z}{9 + z -} \cdots$$

3.   Conclusion

We have shown how CS algorithms may be applied in the case of S-fractions, Padé approxiaants, J- fractions and M- fractions and indicated the broad class or continued fractions for which such an algorithm exists. Many of these algorithms are simple and require a minimal number of computational steps. They appear to provide one of the most straightforward and accurate means for obtaining continued fraction approximations to functions defined by power series.

REFERENCES

LONGMAN, I. M. 1971   Computation of the Padé Table, Intern. J. Computer Math. Section B, Vol. 3, 53- 64.

McCABE, J. H. 1971   Continued Fraction Methods, with Applications to First Order Linear Differential Equations. Ph. D. Thesis, Brunel University.

WALL, H.S. 1948   Analytic Theory of Continued Fractions. Chelsea.