Azéma martingales for Bessel and CIR processes and the pricing of Parisian zero-coupon bonds

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Abstract

In this paper, we study the excursions of Bessel and CIR processes with dimensions $0 < \delta < 2$. We obtain densities for the last passage times and meanders of the processes. Using these results, we prove a variation of the Azéma martingale for the Bessel and CIR processes based on excursion theory. Furthermore, we study their Parisian excursions, and generalise previous results on the Parisian stopping time of Brownian motion to that of the Bessel and CIR processes. We obtain explicit formulas and asymptotic results for the densities of the Parisian stopping times, and develop exact simulation algorithms to sample the Parisian stopping times of Bessel and CIR processes. We introduce a new type of bond, the zero coupon Parisian bond. The buyer of such a bond is betting against zero interest rates, while the seller is effectively hedging against a period where interest rates fluctuate around 0. Using our results, we propose two methods for pricing these bonds and provide numerical examples.

Keywords: Azéma martingale, Parisian stopping time, Cox-Ingersoll-Ross process, Bessel process, Monte Carlo simulation .

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1 Introduction

Bessel processes are a class of diffusion processes introduced by McKean et al. (1960). The Squared Bessel process Y_t satisfies the following SDE

$$\mathrm{d}Y_t = 2(1-\alpha)dt + 2\sqrt{Y_t}\mathrm{d}W_t, \qquad Y_0 = y, \tag{1.1}$$

where we reparameterised with $\alpha := 1 - \frac{\delta}{2}$, which corresponds to the index. For dimensions $\delta > 2$, the process is transient and never reaches 0. We consider in particular dimensions $0 < \delta < 2$, which corresponds to $0 < \alpha < 1$. In this case, the process reaches 0 in finite time and is instantaneously reflecting at 0. Several papers have studied excursions of Bessel processes of dimensions $0 < \delta < 2$. Bertoin (1990) developed an excursion theory for Bessel processes of dimensions $0 < \delta < 1$, while Perman et al. (1992) and Pitman and Yor (1992) gave extensions of the arcsine law for the fraction of time spent positive by Brownian motion. Göing-Jaeschke and Yor (2003) derived Laplace transforms of the hitting times of Bessel processes and considered Bessel processes with negative dimensions. In mathematical finance, Bessel processes are often used to derive results for the Constant Elasticity of Variance (CEV) model, as they are related through a deterministic time change, see Delbaen and Shirakawa (2002) and Carr and Linetsky (2006).

We consider also the Cox-Ingersoll-Ross (CIR) process as a generalisation of the Bessel process. We denote it by R_t, and it satisfies the following SDE

$$\mathrm{d}R_t = \zeta(\theta - R_t)\mathrm{d}t + \sigma\sqrt{R_t}\mathrm{d}W_t, \qquad \zeta \in \mathbb{R}, \quad \theta, \sigma \in \mathbb{R}^+. \tag{1.2}$$

Without loss of generality, a simple time change $A(t) := 4t/\sigma^2$ and setting $\alpha = 1 - \frac{2\zeta\theta}{\sigma^2}$, $k = \frac{2\zeta}{\sigma^2}$ reduces the study of (1.2) to the following SDE

$$\mathrm{d}R_t = 2((1-\alpha) - kR_t)\mathrm{d}t + 2\sqrt{R_t}\mathrm{d}W_t, \qquad R_0 = r. \tag{1.3}$$

The CIR process was first considered by Cox et al. (1985) as an extension to the Vasicek model for interest rates. Its mean-reverting property and positivity makes it an attractive model for interest rates (Delsaen, 1993; Chen and Scott, 1992), stochastic volatility (Ball, 1993; Heston, 1993), and default intensity models (Jarrow et al., 2005; Brigo and Alfonsi, 2005). Similarly to the corresponding Bessel process, it hits 0 almost surely and is instantaneously reflecting at 0 for $0 < \alpha < 1$, and $k \geq 0$. The density of the first hitting time of level 0 for this process can be found in Elworthy et al. (1999); Pitman and Yor (1997) derived some results on the zero set of a CIR process using Girsanov transformation from a Bessel process. However, since interest rates have previously only been assumed to remain strictly positive, thus the excursions of the CIR process for $0 < \alpha < 1$ have not been widely studied.

In recent years, it has become more likely for interest rates to reach 0 or stay around 0 for a period of time. It thus makes practical sense to study the excursions of the CIR process as a model for interest rates. We introduce a new type of Parisian-type bond, called the zero coupon Parisian bond, which pays off an amount depending on the final interest rate, when the interest rate remains strictly positive for a consecutive length of time longer than a fixed window length D, if this happens before maturity time T. If the interest rate fluctuates around 0 until maturity, the bond expires worthless. The buyer of the bond is thus betting against zero interest rates. Likewise, the seller of the Parisian bond is effectively hedging against a period where interest rates fluctuate around 0. Alternatively, we can consider the shifted process $r^* + R_t$ as a model for interest rates, such that the minimum rate is set at r^* instead of 0. This can be useful when considering interest rates which are bounded by a floor rate away from 0. Let $U_t := t - \sup\{s < t | R_s = 0\}$ be the time elapsed since the last time R_t hits 0 for $R_0 = 0$. Then the Parisian stopping time of R_t starting at 0 is $\tau = \inf\{t > 0 | U_t = D\}$. This is the first time the duration of an excursion exceeds a certain threshold $D > 0$. The payoff of the bond will thus be $h(R_\tau)\mathbf{1}_{\{\tau<T\}}$ at time τ, where τ is the Parisian stopping time, and $h : \mathbb{R}^+ \to \mathbb{R}^+$ is the payoff function. If we consider interest rates which follow a CIR process with dynamics given by (1.3) under the risk neutral measure $\mathbb{Q}$, then denoting by $P(r, T)$ the no-arbitrage price of the bond, we have

$$P(r, T) = \mathbb{E}^r_{\mathbb{Q}} \left[\exp\left(-\int_0^\tau R_s \mathrm{d}s \right) h(R_\tau)\mathbf{1}_{\{\tau<T\}} \right], \tag{1.4}$$

where $\mathbb{E}^r_{\mathbb{Q}}$ denotes the expectation under the measure $\mathbb{Q}$, for a process starting at $R_0 = r$. By applying a Girsanov transformation, we show how our results can be used to compute the price of this option. We provide two pricing methods, one based on an explicit formula for the density of the Parisian stopping time, and the other based on Monte Carlo simulation.

To price these options, we need to study the excursions of the CIR and Squared Bessel processes. First, we derive the densities of the last passage time and meanders of the processes, which play an important part in our studies. The corresponding results for the Bessel process can also be easily obtained from that of the Squared Bessel process. We then look at the filtration generated by the zeroes of the process. Azéma martingales for the Brownian motion were discovered by Azéma (1985), and can be obtained by projecting martingales onto the slow filtration. A variation of the Azéma martingale are used to price Parisian options by Chesney et al. (1997), and an extension of it involving the local time is derived by Dassios and Lim (2016). Here, we use excursion theory to prove a variation of the Azéma martingale for the CIR and Bessel processes. Our martingale reduces to the two-sided version of the martingale used in Chesney et al. (1997) when $\alpha = \frac{1}{2}$.

The Azéma martingale enables us to study Parisian excursions of the CIR and Squared Bessel processes. Parisian stopping times are the first time that the process makes an excursion away from 0 that is of period longer than a fixed length D, and Parisian stopping times for Brownian motion has been studied extensively. Laplace transforms of the stopping times are obtained in Chesney et al. (1997); Dassios and Wu (2010). Analytical expressions and asymptotic behaviour of the density are derived in Dassios and Lim (2013, 2015). Here, using our Azéma martingale, we find the Laplace transform of the Parisian stopping time for the CIR and Squared Bessel processes, and obtain explicit recursive and asymptotic expressions for its density. One contribution of this paper is to generalise various results obtained for the Parisian stopping time of Brownian motion to the CIR and Squared Bessel processes, thus providing a detailed analysis of the law of the Parisian stopping times for CIR and Squared Bessel processes.

In addition, we obtain compound Geometric representations of the Laplace transforms of the Parisian stopping time for the CIR and Squared Bessel processes. From this, we develop exact simulation algorithms to sample from the stopping time distribution. This is a generalisation of the result for Brownian motion in Dassios and Lim (2017). Through the Laplace transform, we also observe that the Parisian stopping time of the Bessel process with index α is distributed according to a truncated stable process with index α taken at an exponential time, and the Parisian stopping time of the CIR process is distributed according to a truncated Lamperti stable process taken at exponential time. This distributional identity was observed in Dassios et al. (2017) for the Brownian motion, and in this paper, we show that it holds in a more general setting.

In the rest of this paper, we provide derivation of the results for the CIR process, and state the corresponding results for the Squared Bessel process. The paper will be structured as follows. Section 2 presents some preliminary results on the excursions of the Squared Bessel and CIR processes, which form an important part of our study. In Section 3, we prove the Azéma martingale for the Squared Bessel and CIR processes. Section 4 studies the Parisian stopping times of the processes. Explicit analytical formulas and the asymptotic distribution of the Parisian stopping time densities are obtained, as well as a compound Geometric representation for its Laplace transform. In Section 5, we present exact simulation algorithms for sampling from the Parisian stopping time distributions of the CIR process and the Squared Bessel process. We also establish several numerical comparisons with the analytical recursive densities of the Parisian stopping times. In Section 6, we provide details on the pricing of a zero coupon Parisian bond and present some numerical analysis of the results. Finally, Section 7 concludes the paper.

2 Excursions of the Squared Bessel and CIR processes

In this section, we prove some preliminary results on the last passage time densities and meanders of the Squared Bessel and CIR processes.

2.1 First Hitting Time and Transition Densities

The hitting time and transition densities of the Squared Bessel and CIR processes have been well-studied, and we state here the results which will be used in our computations. We denote by $T_{r\to 0} := \inf\{t > 0 | R_0 = r\}$ the first hitting time of level 0 of the CIR process R_t, and $T^Y_{y\to 0} := \inf\{t > 0 | Y_0 = y\}$ the first hitting time of level 0 of the Squared Bessel process Y_t.

Proposition 2.1 *The densities of the first hitting times of level 0 for the Squared Bessel and CIR processes starting at $Y_0 = y$ and $R_0 = r$ respectively, are given as*

$$\mathbb{P}\left(T^Y_{y\to 0} \in \mathrm{d}u\right) = \frac{1}{\Gamma(\alpha)}\left(\frac{y}{2}\right)^{\alpha} u^{-\alpha-1} e^{-\frac{y}{2u}}\mathrm{d}u, \tag{2.1}$$

$$\mathbb{P}\left(T_{r\to 0} \in \mathrm{d}u\right) = \frac{(2k)^{1+\alpha}\left(\frac{r}{2}\right)^{\alpha}}{\Gamma(\alpha)} e^{-\frac{kr}{e^{2ku}-1}+2ku}(e^{2ku}-1)^{-\alpha-1}\mathrm{d}u. \tag{2.2}$$

Proof. The first hitting time of the Squared Bessel process Y_t has a reciprocal gamma distribution, and its density satisfies (2.1), according to Jeanblanc et al. (2009). The corresponding first hitting time density for the CIR can be obtained using a time-reversal argument suggested in Elworthy et al. (1999) . ■

Proposition 2.2 *The transition density for Y_t, going from 0 to y, is*

$$\mathbb{P}\left(Y_{t+s} \in \mathrm{d}y | Y_s = 0\right) = \frac{(2t)^{-(1-\alpha)}}{\Gamma(1-\alpha)} y^{-\alpha} e^{-\frac{y}{2t}} \mathrm{d}y, \tag{2.3}$$

and for R_t, going from 0 to r, is given by

$$\mathbb{P}\left(R_{t+s} \in \mathrm{d}r | R_s = 0\right) = \frac{(2e^{-2kt}c(t))^{-(1-\alpha)}}{\Gamma(1-\alpha)} r^{-\alpha} e^{-\frac{re^{2kt}}{2c(t)}} \mathrm{d}r, \tag{2.4}$$

where $c(t) = \frac{1}{2k}(e^{2kt} - 1)$.

Proof. From Jeanblanc et al. (2009), we have that

$$\mathbb{P}\left(Y_{t+s} \in \mathrm{d}y | Y_s = x\right) = \frac{1}{2t}\left(\frac{y}{x}\right)^{-\frac{\alpha}{2}} e^{-\frac{x+y}{2t}} I_{-\alpha}\left(\frac{\sqrt{xy}}{t}\right) \mathrm{d}y,$$

and

$$\mathbb{P}\left(R_{t+s} \in \mathrm{d}r | R_s = x\right) = \frac{e^{2kt}}{2c(t)}\left(\frac{re^{2kt}}{x}\right)^{-\frac{\alpha}{2}} e^{-\frac{x+re^{2kt}}{2c(t)}} I_{-\alpha}\left(\frac{\sqrt{xre^{2kt}}}{c(t)}\right) \mathrm{d}r, \tag{2.5}$$

where $I_{-\alpha}$ is the usual modified Bessel function with index $-\alpha$. For $x = 0$, the transition densities of Y_t and R_t directly follow (2.3), and (2.4). ■

2.2 Last Passage Time Densities and Meanders

We study the last passage time densities of the processes. Let $U_t := t - \sup\{s < t | R_s = 0\}$, the time elapsed since the last time R_t hits 0, for $R_0 = 0$, and $U_t^Y := t - \sup\{s < t | Y_s = 0\}$ be the time elapsed since the last time the Squared Bessel process Y_t hits 0, $Y_0 = 0$.

Proposition 2.3 *The probability density function of U_t is*

$$\mathbb{P}(U_t \in \mathrm{d}u) = \frac{2k\sin(\alpha\pi)}{\pi(1-e^{-2k(t-u)})^{1-\alpha}(e^{2ku}-1)^{\alpha}} \mathrm{d}u, \qquad 0 < u < t, \tag{2.6}$$

and the joint distribution of (U_t, R_t) is given as

$$\mathbb{P}(U_t \in \mathrm{d}u, R_t \in \mathrm{d}r) = \frac{2k^2\sin(\alpha\pi)e^{2kt}e^{-2k(t-u)}}{\pi(1-e^{-2k(t-u)})^{1-\alpha}(e^{2ku}-1)^{1+\alpha}} e^{-\frac{kr}{1-e^{-2ku}}} \mathrm{d}u\mathrm{d}r, \tag{2.7}$$

for $0 < u < t$*,and* $0 < r < \infty$*. The conditional density of* $R_t | U_t = u$ *is*

$$\mathbb{P}(R_t \in \mathrm{d}r | U_t = u) = \frac{k}{1 - e^{-2ku}} e^{-\frac{kr}{1-e^{-2ku}}} \mathrm{d}r, \qquad 0 < r < \infty. \tag{2.8}$$

Proof. The Squared Bessel process Y_t starting at 0, satisfies the following time inversion property (Borodin and Salminen, 2012),

$$Y_t = t^2 Z_{1/t}, \qquad t > 0, \tag{2.9}$$

where $Z_t \overset{D}{\sim} \{Y_t : t \geq 0\}$ is a BESQ process with index α. Furthermore, the CIR process R_t satisfying the SDE (1.3) can be obtained from the Squared Bessel process Y_t via the following space-time change (Jeanblanc et al., 2009):

$$R_t = e^{-kt} Y_{\frac{1}{2k}(e^{2kt}-1)}. \tag{2.10}$$

Setting $c(t) := \frac{1}{2k}(e^{2kt} - 1)$, we thus have

$$\begin{aligned}
& \mathbb{P}\left(U_t > u\right) \\
= \; & \mathbb{P}\left(\inf_{t-u<s<t} R_s > 0\right) \\
= \; & \int_0^\infty \mathbb{P}(Z_{1/c(t)} \in dr) \mathbb{P}\left(T^Y_{r\to 0} > \frac{1}{c(t-u)} - \frac{1}{c(t)}\right) \\
= \; & \int_{\frac{1}{c(t-u)} - \frac{1}{c(t)}}^{\infty} \frac{s^{-\alpha} c(t)^{1-\alpha}}{\Gamma(1-\alpha)\Gamma(\alpha)} \frac{1}{c(t)s + 1} \mathrm{d}s,
\end{aligned} \tag{2.11}$$

where we have also used the Markov property of Z_t. Differentiating (2.11) with respect to u, we have

$$\begin{aligned}
\mathbb{P}(U_t \in \mathrm{d}u) &= \frac{\sin(\alpha\pi) e^{2k(t-u)}}{\pi c(t-u)^{1-\alpha} (c(t) - c(t-u))^{\alpha}} \mathrm{d}u \\
&= \frac{2k \sin(\alpha\pi)}{\pi (1 - e^{-2k(t-u)})^{1-\alpha} (e^{2ku} - 1)^{\alpha}} \mathrm{d}u.
\end{aligned}$$

For the joint distribution of (U_t, R_t), we have

$$\begin{aligned}
& \mathbb{P}\left(U_t > u, R_t < r\right) \\
= \; & \mathbb{P}\left(\inf_{t-u<s<t} R_s > 0, R_t < r\right)
\end{aligned}$$

$$= \mathbb{P}\left(\inf_{\frac{1}{c(t)}<v<\frac{1}{c(t-u)}} Z_v > 0, c^2(t)Z_{1/c(t)} < e^{2kt}r\right)$$

$$= \int_{\frac{1}{c(t-u)}-\frac{1}{c(t)}}^{\infty} \frac{s^{-\alpha}c(t)^{1-\alpha}}{\Gamma(1-\alpha)\Gamma(\alpha)} \frac{1}{c(t)s+1}\left(1-e^{-\frac{(c(t)s+1)}{2s}\frac{e^{2kt}r}{c^2(t)}}\right) \mathrm{d}s, \tag{2.12}$$

Differentiating (2.12) with respect to r and then u respectively, we have

$$\begin{aligned}\mathbb{P}(U_t \in \mathrm{d}u, R_t \in \mathrm{d}r) &= \frac{\sin(\alpha\pi)e^{2kt}e^{2k(t-u)}}{2\pi c(t-u)^{1-\alpha}(c(t)-c(t-u))^{1+\alpha}} e^{-\frac{e^{2kt}r}{2(c(t)-c(t-u))}} \mathrm{d}u\mathrm{d}r \\ &= \frac{2k^2\sin(\alpha\pi)e^{2kt}e^{-2k(t-u)}}{\pi(1-e^{-2k(t-u)})^{1-\alpha}(e^{2ku}-1)^{1+\alpha}} e^{-\frac{kr}{1-e^{-2ku}}} \mathrm{d}u\mathrm{d}r.\end{aligned}$$

Then, the density of $R_t|U_t$ immediately follows (2.8). ∎

Corollary 2.4 *For the Squared Bessel process Y_t starting at 0, the distribution of the time elapsed since its last 0, is $U_t^Y \sim Beta(\alpha, 1-\alpha)$, and its probability density function is*

$$\mathbb{P}(U_t^Y \in \mathrm{d}u) = \frac{\sin(\alpha\pi)}{\pi(t-u)^{1-\alpha}u^{\alpha}} \mathrm{d}u, \qquad 0 < u < t. \tag{2.13}$$

The joint distribution of (U_t^Y, Y_t) is

$$\mathbb{P}\left(U_t^Y \in \mathrm{d}u, Y_t \in \mathrm{d}y\right) = \frac{\sin(\alpha\pi)}{2\pi u^{1+\alpha}(t-u)^{1-\alpha}} e^{-\frac{y}{2u}} \mathrm{d}u\mathrm{d}y, \tag{2.14}$$

for $0 < u < t, 0 < y < \infty$. The conditional distribution of Y_t given U_t^Y is

$$\mathbb{P}(Y_t \in \mathrm{d}y | U_t^Y = u) = \frac{1}{2u} e^{-\frac{y}{2u}} \mathrm{d}y, \qquad 0 < y < \infty. \tag{2.15}$$

Proof. According to the time inversion property (2.9), we apply similar techniques to derive (2.13), (2.14), and (2.15) and replacing $c(t)$ by t. They can also be obtained by letting $k \to 0$ in (2.6), (2.7) and (2.8). ∎

Remark 2.5 *Since the zeroes of the Squared Bessel process are the same as those of the corresponding Bessel process, (2.13) is the last passage time density of a Bessel process. Furthermore, from (2.15), we note that the Bessel meander, which is the Bessel process starting at 0 and conditioned not to hit 0 before time u, has a distribution independent of α, namely the Rayleigh distribution with parameter $\sqrt{u}$.*

The following gives results for the first hitting times of the meander processes starting at R_t, given only information about U_t, the time elapsed since the last 0. Let $T_{R_t\to 0}$ denote the first time a CIR process starting at R_t hits 0, and $T_{Y_t\to 0}$ the first time a squared Bessel process starting at Y_t hits 0.

Proposition 2.6 *We have the following results for the CIR process R_t,*

$$\mathbb{P}\left(T_{R_t\to 0} \in \mathrm{d}s | U_t = u\right) = \frac{2\alpha k e^{2ks}(1-e^{-2ku})^{\alpha}}{(e^{2ks}-e^{-2ku})^{\alpha+1}}\mathrm{d}s, \tag{2.16}$$

and

$$\mathbb{P}\left(T_{R_t\to 0} > h | U_t = u\right) = \frac{(1-e^{-2ku})^{\alpha}}{(e^{2kh}-e^{-2ku})^{\alpha}}. \tag{2.17}$$

Similarly, for the squared Bessel process Y_t, we have

$$\mathbb{P}(T_{Y_t\to 0} \in \mathrm{d}s | U_t^Y = u) = \frac{\alpha u^{\alpha}}{(u+s)^{\alpha+1}}\mathrm{d}s, \tag{2.18}$$

and

$$\mathbb{P}(T_{Y_t\to 0} > h | U_t^Y = u) = \frac{u^{\alpha}}{(u+h)^{\alpha}}. \tag{2.19}$$

Proof. According to Proposition 2.1 and Proposition 2.3, and using the Markov property of R_t, we have

$$\begin{aligned}
&\mathbb{P}\left(T_{R_t\to 0} \in \mathrm{d}s | U_t = u\right) \\
=\ & \int_0^{\infty} \frac{k}{1-e^{-2ku}} e^{-\frac{kr}{1-e^{-2ku}}} \frac{2k^{1+\alpha} r^{\alpha}}{\Gamma(\alpha)} e^{-\frac{kr}{e^{2ks}-1}} e^{2ks}(e^{2ks}-1)^{-\alpha-1}\mathrm{d}s\mathrm{d}r \\
=\ & \frac{2\alpha k e^{2ks}(1-e^{-2ku})^{\alpha}}{(e^{2ks}-e^{-2ku})^{\alpha+1}}\mathrm{d}s.
\end{aligned}$$

Then, the associated survival function is given as

$$\begin{aligned}
&\mathbb{P}\left(T_{R_t\to 0} > h | U_t^R = u\right) \\
=\ & \int_h^{\infty} \frac{2\alpha k e^{2ks}(1-e^{-2ku})^{\alpha}}{(e^{2ks}-e^{-2ku})^{\alpha+1}}\mathrm{d}s \\
=\ & \frac{(1-e^{-2ku})^{\alpha}}{(e^{2kh}-e^{-2ku})^{\alpha}}.
\end{aligned}$$

And similarly, based on the distribution of the hitting time of Y_t in (2.1) and the conditional distribution of $Y_t|U_t^Y$ in (2.15), we obtain (2.18) and (2.19). ■

We now consider a CIR process R_t with index α and parameter k, starting at initial point $R_0 = r$. Then we have the following transition density of R_t conditioned to stay strictly positive.

Proposition 2.7 *The transition density of a CIR process R_t starting at $R_0 = r$ conditioned to stay positive is*

$$\mathbb{P}_r\left(R_t \in \mathrm{d}x; T_{r\to 0} > t\right) = \frac{e^{2kt}}{2c(t)} \left(\frac{xe^{2kt}}{r}\right)^{-\frac{\alpha}{2}} e^{-\frac{r+xe^{2kt}}{2c(t)}} I_\alpha \left(\frac{\sqrt{xre^{2kt}}}{c(t)}\right) \mathrm{d}x, \tag{2.20}$$

where $c(t) := \frac{1}{2k}(e^{2kt} - 1)$ and I_α is the modified Bessel function of the first kind.

Proof. Denote by $\mathbb{P}_r^{-\alpha}$ the probability measure for a CIR process $R_t^{-\alpha}$ starting at $R_0^{-\alpha} = r$. Then using a change of measure result from Elworthy et al. (1999) Lemma 3.11, we have

$$\mathbb{P}_r(R_t \in \mathrm{d}x; T_{r\to 0} > t) \quad = \quad \left(\frac{xe^{kt}}{r}\right)^{-\alpha} \mathbb{P}_r^{-\alpha}(R_t^{-\alpha} \in \mathrm{d}x). \tag{2.21}$$

Then the result follows by using the transition density (2.5) with index $-\alpha$. ■

3 Azéma Martingale for the Squared Bessel and CIR processes

Let $\mathcal{F}_U := (\mathcal{F}_{U_t})_{t\geq 0}$ be the filtration generated by U_t containing the zeroes of the CIR process R_t, and $\mathcal{F}_{U^Y} := \left(\mathcal{F}_{U_t^Y}\right)_{t\geq 0}$ be the filtration containing the zeroes of the Squared Bessel process Y_t. We consider martingales for this filtration, which in the Brownian setting, are the celebrated Azéma martingales. Using the last passsage time results in the previous section, we prove an extension of the Azéma martingales to the Squared Bessel and CIR processes.

Theorem 3.1 *For the CIR process R_t, we have the following martingale. Let M_t be defined by*

$$M_t := e^{-\beta t} \left(1 + (e^{2kU_t} - 1)^\alpha \beta e^{\beta U_t} \int_0^{U_t} e^{-\beta s}(e^{2ks} - 1)^{-\alpha} \mathrm{d}s\right), \tag{3.1}$$

then M_t is an $\mathcal{F}_U$-martingale.

Proof. It is easy to see that M_t is integrable and adapted since $U_t \leq t$ always. The martingale property of M_t can be proved by directly applying the last passage time results from the previous section. However, we take another approach to give more insights into how the martingale is obtained. We start by considering a martingale of the form

$$e^{-\beta t} f(U_t),$$

and aim to find an integrable function f, with $f(0) = 1$ (without loss of generality), such that

$$\mathbb{E}\left[e^{-\beta(t+h)} f(U_{t+h}) | U_t = u\right] = e^{-\beta t} f(u), \tag{3.2}$$

for all $t \geq 0$ and $h > 0$. In particular, the following should hold

$$\mathbb{E}\left[e^{-\beta t} f(U_t) | U_0 = 0\right] = 1. \tag{3.3}$$

Using the density of U_t in (2.6), this is equivalent to finding f such that

$$\begin{aligned}
&\int_0^t \frac{f(u) 2k e^{2k(t-u)}}{\Gamma(\alpha)\Gamma(1-\alpha)(e^{2k(t-u)} - 1)^{1-\alpha}(e^{2kt} - e^{2k(t-u)})^{\alpha}} \mathrm{d}u \\
&= \frac{2k}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \frac{f(u)}{(1 - e^{-2k(t-u)})^{1-\alpha}(e^{2ku} - 1)^{\alpha}} \mathrm{d}u \quad = \quad e^{\beta t}.
\end{aligned} \tag{3.4}$$

Set $g(t) = \frac{f(t)}{(e^{2kt}-1)^{\alpha}}$, and taking Laplace transform over t with $\gamma < \beta$, the LHS of (3.4) becomes

$$\begin{aligned}
&\int_0^{\infty} e^{-\gamma t} \frac{2k}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \frac{f(u)}{(1 - e^{-2k(t-u)})^{1-\alpha}(1 - e^{-2ku})^{\alpha}} \mathrm{d}u\mathrm{d}t \\
= \quad & \frac{2k}{\Gamma(\alpha)\Gamma(1-\alpha)} \left(\int_0^{\infty} e^{-\gamma t} \frac{f(t)}{(e^{2kt} - 1)^{\alpha}} \mathrm{d}t\right) \left(\int_0^{\infty} e^{-\gamma t} (1 - e^{-2kt})^{\alpha-1} \mathrm{d}t\right) \\
= \quad & \frac{\hat{g}(\gamma)}{\Gamma(1-\alpha)} \frac{2k\Gamma\left(1 + \frac{\gamma}{2k}\right)}{\gamma\Gamma\left(\alpha + \frac{\gamma}{2k}\right)}.
\end{aligned}$$

As the Laplace transform of the RHS of (3.4) is $\frac{1}{\gamma-\beta}$, we then have

$$\hat{g}(\gamma) = \gamma \times \frac{\Gamma(1-\alpha)\Gamma\left(\alpha + \frac{\gamma}{2k}\right)}{2k\Gamma\left(1 + \frac{\gamma}{2k}\right)} \times \frac{1}{\gamma - \beta}.$$

Hence

$$
\begin{aligned}
g(u) &= \frac{\mathrm{d}}{\mathrm{d}u}\left(\mathcal{L}^{-1}\left\{\frac{1}{\gamma-\beta}\frac{\Gamma(1-\alpha)\Gamma\left(\alpha+\frac{\gamma}{2k}\right)}{2k\Gamma\left(1+\frac{\gamma}{2k}\right)}\right\}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}u}\left(\int_0^u e^{\beta(u-s)}(1-e^{-2ks})^{-\alpha}e^{-2k\alpha s}\mathrm{d}s\right) \\
&= (e^{2ku}-1)^{\alpha}+\beta e^{\beta u}\int_0^u e^{-\beta s}(e^{2ks}-1)^{-\alpha}\mathrm{d}s.
\end{aligned}
$$

Thus

$$
f(u) = 1+(e^{2ku}-1)^{\alpha}\beta e^{\beta u}\int_0^u e^{-\beta s}(e^{2ks}-1)^{-\alpha}\mathrm{d}s, \tag{3.5}
$$

satisfies (3.3). We now have a candidate for the martingale, and we now need to verify that (3.2) holds. More precisely, we need to prove

$$
\mathbb{E}\left[e^{-\beta h}f(U_{t+h})|U_t=u\right] = f(u).
$$

According to Corollary 2.6, Eq. (3.3), and the Markov property of R_t so that the process starts over again whenever it returns to 0, we have

$$
\begin{aligned}
&\mathbb{E}\left[e^{-\beta h}f(U_{t+h})|U_t=u\right] \\
&= \mathbb{P}\left(T_{R_t\to 0}>h|U_t=u\right)e^{-\beta h}f(u+h)+\int_0^h e^{-\beta s}\mathbb{P}\left(T_{R_t\to 0}\in \mathrm{d}s|U_t=u\right)\mathrm{d}s \\
&= (e^{2ku}-1)^{\alpha}e^{-\beta h}\frac{f(u+h)}{(e^{2k(u+h)}-1)^{\alpha}}+\int_0^h e^{-\beta s}\frac{2\alpha k e^{2k(s+u)}(e^{2ku}-1)^{\alpha}}{(e^{2k(s+u)}-1)^{1+\alpha}}\mathrm{d}s \\
&= (e^{2ku}-1)^{\alpha}e^{-\beta h}\frac{f(u+h)-1}{(e^{2k(u+h)}-1)^{\alpha}}+1-\beta e^{\beta u}\int_u^{u+h}\frac{(e^{2ku}-1)^{\alpha}}{(e^{2kz}-1)^{\alpha}}\mathrm{d}z \\
&= 1+\beta e^{\beta u}\int_0^u\frac{(e^{2ku}-1)^{\alpha}}{(e^{2kz}-1)^{\alpha}}\mathrm{d}z = f(u).
\end{aligned} \tag{3.6}
$$

Hence, $M_t = e^{-\beta t}f(U_t)$ with f defined in (3.5) is an $\mathcal{F}_U$-martingale. ■

Lemma 3.2 *The equivalent martingale for the Squared Bessel process Y_t is M_t^Y, defined by*

$$M_t^Y = e^{-\beta t}\left(1 + (U_t^Y)^\alpha \beta e^{\beta U_t^Y} \int_0^{U_t^Y} e^{-\beta s} s^{-\alpha} \mathrm{d}s\right).$$

Then M_t^Y is an $\mathcal{F}_U^Y$-martingale.

Proof. The proof follows in a similar way as in Theorem 3.1. ■

Remark 3.3 *When $\alpha = \frac{1}{2}$, Y_t becomes the squared Brownian motion and M_t^Y becomes the two-sided version of the Azéma martingale used in Chesney et al. (1997).*

4 Parisian Excursions of Squared Bessel and CIR Processes

In this section, we study the Parisian excursions of the CIR and Squared Bessel processes. Define the Parisian stopping times of R_t for a CIR process starting at 0, and correspondingly Y_t with $Y_0 = 0$ by

$$\tau = \inf\{t > 0 | U_t = D\}, \tag{4.1}$$

$$\tau_Y = \inf\{t > 0 | U_t^Y = D\}. \tag{4.2}$$

This is the first time the duration of an excursion exceeds a certain threshold $D > 0$. Without loss of generality we set $D = 1$. We obtain the Laplace transforms of the Parisian stopping time of both processes, and derive from it an explicit analytical expression for its density. Further, we study the asymptotic behaviour of the Parisian stopping times, and prove that they have exponential tails. We also present a compound geometric representation for the Laplace transforms, which we will use in the next section to develop efficient simulation algorithms for the stopping times.

4.1 Laplace transform of the Parisian stopping times

We apply optional stopping theorem on the martingale M_t to obtain the Laplace transform of the Parisian stopping times τ and τ_Y.

Lemma 4.1 *The Laplace transform of τ is*

$$\mathbb{E}\left[e^{-\beta\tau}\right] = \frac{e^{-\beta}}{1+2\alpha k(e^{2k}-1)^{\alpha}\int_0^1 (1-e^{-\beta x})e^{2kx}(e^{2kx}-1)^{-\alpha-1}\mathrm{d}x}, \tag{4.3}$$

and the Laplace transform of τ_Y is

$$\mathbb{E}\left[e^{-\beta\tau_Y}\right] = \frac{e^{-\beta}}{1+\alpha\int_0^1 (1-e^{-\beta u})u^{-\alpha-1}\mathrm{d}u}, \tag{4.4}$$

for $\beta \in \mathbb{R}^+$.

Proof. Since $U_{t\wedge\tau} \leq 1$, we have $|M_{t\wedge\tau}| \leq K$ for some constant K for all t. Thus optional stopping theorem and dominated convergence theorem applies, and we have

$$\begin{aligned}\mathbb{E}\left[e^{-\beta\tau}f(U_\tau)\right] &= \mathbb{E}\left[\lim_{t\to\infty} e^{-\beta(\tau\wedge t)}f(U_{\tau\wedge t})\right] \\ &= \mathbb{E}\left[e^{-\beta\tau}\left(1+(e^{2k}-1)^{\alpha}\beta e^{\beta}\int_0^1 e^{-\beta s}(e^{2ks}-1)^{-\alpha}\mathrm{d}s\right)\right] = 1,\end{aligned} \tag{4.5}$$

where we have used the function f as defined in (3.5) to ease notation. Hence, we have

$$\begin{aligned}\mathbb{E}\left[e^{-\beta\tau}\right] &= \frac{1}{1+(e^{2k}-1)^{\alpha}\beta e^{\beta}\int_0^1 e^{-\beta s}(e^{2ks}-1)^{-\alpha}\mathrm{d}s} \\ &= \frac{e^{-\beta}}{1+(e^{-\beta}-1)+(e^{2k}-1)^{\alpha}\beta\int_0^1 e^{-\beta s}(e^{2ks}-1)^{-\alpha}\mathrm{d}s} \\ &= \frac{e^{-\beta}}{1+2\alpha k(e^{2k}-1)^{\alpha}\int_0^1 (1-e^{-\beta u})e^{2ku}(e^{2ku}-1)^{-\alpha-1}\mathrm{d}u}.\end{aligned}$$

The Laplace transform of τ_Y can be derived in a similar way. It can also be obtained by letting $k \to 0$ in (4.3). ■

Remark 4.2 *It can be seen from its Laplace transform that τ_Y is distributed as $1 + X_{\tilde{T}}$, where X is a truncated stable process (Dassios et al., 2017) with Lévy measure $\nu(\mathrm{d}y) = \alpha y^{-\alpha-1}\mathbf{1}_{\{y<1\}}$ and $\tilde{T} \sim Exp(1)$.*

Likewise, τ is distributed as $1+X_{\tilde{T}}$, where X is a truncated Lamperti stable process with Lévy measure $\nu(\mathrm{d}y)=\frac{Ce^{2ky}}{(e^{2ky}-1)^{\alpha+1}}\mathbf{1}_{\{y<1\}}$, for some constant C, and $\tilde{T}\sim Exp(1)$.

4.2 Densities of the Parisian stopping time

We obtain explicit analytical expressions for the densities of the Parisian stopping time. Just like in the case of the Brownian motion (Dassios and Lim, 2013), these expressions involve only a finite sum and thus can be computed easily.

Theorem 4.3 *For the CIR process R_t, let $f_\tau(t)$ be the density function of the Parisian stopping time of τ, we have*

$$f_\tau(t)=\sum_{i=0}^{n-1}(-1)^i L_i(t-1), \qquad \text{for } n<t\le n+1,\ n=1,2,...,$$

for $t>1$, where $L_i(t)$ is defined recursively as follows:

$$L_0(t) = \frac{2k\sin(\alpha\pi)}{\pi(e^{2k}-1)^\alpha}(1-e^{-2kt})^{\alpha-1}, \qquad \text{for } t>0, \tag{4.6}$$

$$L_{i+1}(t) = \int_1^{t-i} L_i(t-s)\frac{2k\sin(\alpha\pi)(1-e^{-2k(s-1)})^\alpha}{\pi(e^{2k}-1)^\alpha(1-e^{-2ks})}\mathrm{d}s, \qquad \text{for } t>i+1. \tag{4.7}$$

For the squared Bessel process Y_t, let $f_{\tau_Y}(t)$ be the density function of the Parisian stopping time of τ_Y, we have

$$f_{\tau_Y}(t)=\sum_{i=0}^{n-1}(-1)^i H_i(t-1), \qquad \text{for } n<t\le n+1,\ n=1,2,...,$$

for $t>1$, where $H_i(t)$ is defined recursively as follows:

$$H_0(t) = \frac{\sin(\alpha\pi)}{\pi}t^{\alpha-1}, \qquad \text{for } t>0, \tag{4.8}$$

$$H_{i+1}(t) = \int_1^{t-i} H_i(t-s)\frac{\sin(\alpha\pi)(s-1)^\alpha}{\pi s}\mathrm{d}s, \qquad \text{for } t>i+1. \tag{4.9}$$

Proof. The Laplace transform (4.3) can be written as

$$\mathbb{E}\left[e^{-\beta\tau}\right] = \frac{e^{-\beta}}{1+2\alpha k(e^{2k}-1)^\alpha\int_0^1(1-e^{-\beta x})e^{2kx}(e^{2kx}-1)^{-\alpha-1}\mathrm{d}x}$$

$$= \frac{e^{-\beta}}{(e^{2k}-1)^{\alpha}\beta\int_0^{\infty} e^{-\beta x}(e^{2kx}-1)^{-\alpha}\mathrm{d}x + \int_1^{\infty} e^{-\beta x}\frac{2\alpha k e^{2kx}(e^{2k}-1)^{\alpha}}{(e^{2kx}-1)^{\alpha+1}}\mathrm{d}x}$$

$$= \frac{e^{-\beta}}{(e^{2k}-1)^{\alpha}\frac{\beta}{2k}\frac{\Gamma(1-\alpha)\Gamma(\alpha+\frac{\beta}{2k})}{\Gamma(1+\frac{\beta}{2k})} + \int_1^{\infty} e^{-\beta x}\frac{2\alpha k e^{2kx}(e^{2k}-1)^{\alpha}}{(e^{2kx}-1)^{\alpha+1}}\mathrm{d}x}$$

$$= \frac{e^{-\beta}}{(e^{2k}-1)^{\alpha}}\frac{\Gamma(\frac{\beta}{2k})}{\Gamma(1-\alpha)\Gamma(\alpha+\frac{\beta}{2k})}$$

$$\times\sum_{i=0}^{\infty}(-1)^i\left(\frac{\Gamma(\frac{\beta}{2k})}{\Gamma(1-\alpha)\Gamma(\alpha+\frac{\beta}{2k})}\int_1^{\infty} e^{-\beta x}\frac{2\alpha k e^{2kx}}{(e^{2kx}-1)^{\alpha+1}}\mathrm{d}x\right)^i, \tag{4.10}$$

We denote

$$\hat{L}_i(\beta) = \frac{\Gamma(\frac{\beta}{2k})}{\Gamma(1-\alpha)\Gamma(\alpha+\frac{\beta}{2k})}\left(\frac{\Gamma(\frac{\beta}{2k})}{\Gamma(1-\alpha)\Gamma(\alpha+\frac{\beta}{2k})}\int_1^{\infty} e^{-\beta x}\frac{2\alpha k e^{2kx}}{(e^{2kx}-1)^{\alpha+1}}\mathrm{d}x\right)^i,$$

Then since $\hat{L}_1(\beta)\to 0$ as $\beta\to\infty$, and $\hat{L}_1(\beta)$ is continuous and decreasing in β, there exists some $\beta^* > 0$ such that the infinite series summation is valid for all $\beta > \beta^*$. Furthermore, we have the following Laplace inversions

$$\mathcal{L}_t^{-1}\left\{\frac{\Gamma(\frac{\beta}{2k})}{\Gamma(1-\alpha)\Gamma(\alpha+\frac{\beta}{2k})}\right\} = \frac{2k\sin(\alpha\pi)}{\pi}(1-e^{-2kt})^{\alpha-1},$$

and

$$\mathcal{L}_t^{-1}\left\{\frac{\Gamma(\frac{\beta}{2k})}{\Gamma(1-\alpha)\Gamma(\alpha+\frac{\beta}{2k})}\int_1^{\infty} e^{-\beta x}\frac{2\alpha k e^{2kx}}{(e^{2kx}-1)^{\alpha+1}}\mathrm{d}x\right\}$$

$$= \frac{4k^2\alpha}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_1^{t}\frac{e^{2ks}\mathbf{1}_{\{t>1\}}}{(1-e^{-2k(t-s)})^{1-\alpha}(e^{2ks}-1)^{1+\alpha}}\mathrm{d}s$$

$$= \frac{2k\alpha\mathbf{1}_{\{t>1\}}}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_{e^{2k}}^{e^{2kt}}\frac{1}{(1-e^{-2kt}x)^{1-\alpha}(x-1)^{1+\alpha}}\mathrm{d}x$$

$$= \frac{2k\sin(\alpha\pi)(1-e^{-2k(t-1)})^{\alpha}}{\pi(e^{2k}-1)^{\alpha}(1-e^{-2kt})}\mathbf{1}_{\{t>1\}}.$$

Hence, inverting the Laplace transform in each term of (4.10), we have that $f_\tau(t)$ is the sum of $L_i(t-1)$, where L_0 and L_i are as defined in (4.6) and (4.7). For the Squared Bessel process Y_t, recursions for the density $f_{\tau_Y}(t)$ are obtained in a similar way, or by letting $k \to 0$ in (4.6) and (4.7). ■

We state the following corollary for a CIR process starting at $R_0 = r$. We consider only the case when $T_{r\to 0} < 1$ since otherwise $\tau = 1$ and it is trivial.

Corollary 4.4 *Let $f^r_\tau(t; T_{r\to 0} < 1)$ denote the density of the Parisian stopping time for R_t starting at $R_0 = r > 0$, on the set $\{T_{r\to 0} < 1\}$. Then for $t > 1$, we have*

$$f^r_\tau(t; T_{r\to 0} < 1) = \int_0^1 \frac{(2k)^{1+\alpha}\left(\frac{r}{2}\right)^\alpha}{\Gamma(\alpha)} e^{-\frac{kr}{e^{2ku}-1}+2ku}(e^{2ku}-1)^{-\alpha-1} f_\tau(t-u)du. \tag{4.11}$$

Proof. Since by the strong Markov property,

$$\mathbb{E}^r\left[e^{-\beta\tau_r}\right] = \mathbb{E}^r\left[e^{-\beta T_{r\to 0}}\right]\mathbb{E}\left[e^{-\beta\tau}\right], \tag{4.12}$$

it follows immediately that the density $f^r_\tau(t; T_{r\to 0} < 1)$ is the convolution of (2.2) and $f_\tau(t)$. ■

4.3 Tail distribution of the Parisian stopping time

We derive the asymptotic distribution of the Parisian stopping times τ and τ_Y. In particular, they have exponential tails, and we find the associated constants and rates of decay.

Theorem 4.5 *Let $\bar{F}_\tau(t)$ be the tail of the distribution of the Parisian stopping time τ. As $t \to \infty$, we have*

$$\bar{F}_\tau(t) \sim C_R e^{-\beta^* t}, \tag{4.13}$$

where the constant C_R is

$$C_R = \frac{e^{\beta^*}}{\beta^*(e^{2k}-1)^\alpha \int_0^1 e^{\beta^* v}\frac{2k\alpha e^{2kv} v}{(e^{2kv}-1)^{\alpha+1}}\mathrm{d}v}, \tag{4.14}$$

and $\beta^* > 0$ such that $-\beta^*$ is the unique negative solution of the equation

$$1 + (e^{2k}-1)^\alpha \int_0^1 (1-e^{-\beta v})\frac{2k\alpha e^{2kv}}{(e^{2kv}-1)^{\alpha+1}}dv = 0.$$

Similarly, let $\bar{F}_{\tau_Y}(t)$ be the tail of the distribution of Parisian stopping time τ_Y. As $t \to \infty$, we have

$$\bar{F}_{\tau_Y}(t) \sim \frac{1}{\alpha} e^{-\gamma^* t}, \tag{4.15}$$

with $\gamma^* > 0$ such that $-\gamma^*$ is the unique negative solution of the equation

$$\int_0^1 \frac{e^{-\gamma s}}{s^\alpha} ds + \frac{e^{-\gamma}}{\gamma} = 0.$$

Proof. The results are generalisations of the asymptotic distribution of the two-sided Parisian stopping time with barrier 0. The details of its proof can be seen in Theorem 4.1 in Dassios and Lim (2015). The constants β^*, C_R and γ^* can be easily computed numerically. ∎

4.4 Compound Geometric representations for the Laplace transforms

Here, we provide compound Geometric representations[1] of the Laplace transforms of the Parisian stopping times of the CIR process R_t and squared Bessel Y_t, which immediately leads to the simulation algorithm.

Theorem 4.6 *The Laplace transform of the Parisian stopping time for R_t, namely τ, can be written as*

$$\mathbb{E}\left[e^{-\beta\tau}\right] = \frac{pe^{-\beta} \int_0^1 e^{-\beta t} f_{T_0}(t) \mathrm{d}t}{1 - (1-p)e^{-\beta} \int_0^1 e^{-\beta t} f_{T_i}(t) \mathrm{d}t}, \tag{4.16}$$

where we have defined

$$p = \frac{2kM}{\pi \csc(\pi\alpha)(e^{2k}-1)^\alpha}, \tag{4.17}$$

$$f_{T_0}(t) = \frac{(1-e^{-2kt})^{\alpha-1}}{M}, \tag{4.18}$$

$$f_T(t) = \frac{1}{E} \int_t^1 \frac{2\alpha k e^{2k(t+1-s)}(1-e^{-2ks})^{\alpha-1}(e^{2k}-1)^\alpha}{(e^{2k(t+1-s)}-1)^{\alpha+1}} ds, \tag{4.19}$$

[1] A random variable X has a compound Geometric representation if X can be expressed as $X = \sum_{i=1}^{N} J_i$, with $N \sim$ Geometric(p) for $0 < p < 1$, and $\{J_i\}_{i=1,2,\ldots,N}$ being i.i.d random variables.

and

$$M = \int_0^1 (1-e^{-2kt})^{\alpha-1} dt, \tag{4.20}$$

$$E = \int_0^1 \left(\frac{(e^{2k}-1)^\alpha}{(e^{2k(1-s)}-1)^\alpha} - 1 \right) (1-e^{-2ks})^{\alpha-1} \mathrm{d}s. \tag{4.21}$$

Furthermore, $f_{T_0}(t)$ and $f_T(t)$ are proper density functions over $t \in (0,1)$.

Proof. The Laplace transform of the Parisian stopping times are given in Lemma 4.1. Multiplying both the numerator and denominator of (4.6) with

$$\int_0^1 e^{-\beta u}(1-e^{-2ku})^{\alpha-1}\mathrm{d}u,$$

we have

$$\mathbb{E}\left[e^{-\beta\tau}\right] = \frac{e^{-\beta}\int_0^1 \frac{e^{-\beta u}}{(1-e^{-2ku})^{1-\alpha}}\mathrm{d}u}{\int_0^1 \frac{e^{-\beta u}}{(1-e^{-2ku})^{1-\alpha}}\mathrm{d}u + \int_0^1 \frac{(1-e^{-\beta s})e^{2ks}}{(e^{2ks}-1)^{\alpha+1}}\mathrm{d}s \int_0^1 \frac{2\alpha k(e^{2k}-1)^\alpha e^{-\beta u}}{(1-e^{-2ku})^{1-\alpha}}\mathrm{d}u}. \tag{4.22}$$

For the denominator of (4.22), we have

$$\begin{aligned}
&\int_0^1 \frac{e^{-\beta u}}{(1-e^{-2ku})^{1-\alpha}}\mathrm{d}u + \int_0^1 (1-e^{-\beta s})\frac{2\alpha k e^{2ks}}{(e^{2ks}-1)^{\alpha+1}}\mathrm{d}s \int_0^1 \frac{e^{-\beta u}(e^{2k}-1)^\alpha}{(1-e^{-2ku})^{1-\alpha}}\mathrm{d}u \\
=\ & \int_0^1 \frac{e^{-\beta u}}{(1-e^{-2ku})^{1-\alpha}}\mathrm{d}u \left(1+(e^{2k}-1)^\alpha \left(-\frac{1-e^{-\beta}}{(e^{2k}-1)^\alpha} + \beta\int_0^1 \frac{e^{-\beta s}}{(e^{2ks}-1)^\alpha}\mathrm{d}s\right)\right) \\
=\ & e^{-\beta}\int_0^1 \frac{e^{-\beta u}}{(1-e^{-2ku})^{1-\alpha}}\mathrm{d}u + (e^{2k}-1)^\alpha \beta \int_0^1 \frac{e^{-\beta s}}{(e^{2ks}-1)^\alpha}\mathrm{d}s \int_0^1 \frac{e^{-\beta u}}{(1-e^{-2ku})^{1-\alpha}}\mathrm{d}u.
\end{aligned} \tag{4.23}$$

In the second term of (4.23), we have

$$
\begin{aligned}
&\beta \int_0^1 \frac{e^{-\beta u}}{(e^{2ku}-1)^\alpha} \mathrm{d}u \int_0^1 e^{-\beta u}(1-e^{-2ku})^{\alpha-1} \mathrm{d}u \\
= \;& \beta \int_0^1 e^{-\beta t} \int_0^t \frac{(1-e^{-2ks})^{\alpha-1}}{(e^{2k(t-s)}-1)^\alpha} \mathrm{d}s\mathrm{d}t + \beta \int_1^2 e^{-\beta t} \int_{t-1}^1 \frac{(1-e^{-2ks})^{\alpha-1}}{(e^{2k(t-s)}-1)^\alpha} \mathrm{d}s\mathrm{d}t \\
= \;& \frac{\pi \csc(\pi\alpha)(1-e^{-\beta})}{2k} + e^{-\beta} \frac{\pi \csc(\pi\alpha)}{2k} + \int_1^2 e^{-\beta t} \frac{\partial}{\partial t} \left\{ \int_{t-1}^1 \frac{(1-e^{-2ks})^{\alpha-1}}{(e^{2k(t-s)}-1)^\alpha} \mathrm{d}s \right\} \mathrm{d}t \\
= \;& \frac{\pi \csc(\pi\alpha)}{2k} - e^{-\beta} \int_0^1 \frac{e^{-\beta t}(1-e^{-2kt})^{\alpha-1}}{(e^{2k}-1)^\alpha} \mathrm{d}t \\
& \qquad - e^{-\beta} \int_0^1 e^{-\beta t} \int_t^1 \frac{2k\alpha e^{2k(t+1-s)}(1-e^{-2ks})^{\alpha-1}}{(e^{2k(t+1-s)}-1)^{\alpha+1}} \mathrm{d}s\mathrm{d}t,
\end{aligned}
$$

Hence, the Laplace transform of τ can be expressed as the Laplace transform of a compound Geometric distribution as follows

$$
\begin{aligned}
&\mathbb{E}\left[e^{-\beta\tau}\right] \\
= \;& \frac{e^{-\beta}}{1 + 2\alpha k (e^{2k}-1)^\alpha \int_0^1 (1-e^{-\beta x}) e^{2kx} (e^{2kx}-1)^{-\alpha-1} \mathrm{d}x} \\
= \;& \frac{p e^{-\beta} \int_0^1 \frac{(1-e^{-2kt})^{\alpha-1}}{M} \mathrm{d}t}{1 - (1-p) e^{-\beta} \int_0^1 e^{-\beta t} \frac{1}{E} \int_t^1 \frac{2\alpha k e^{2k(t+1-s)}(1-e^{-2ks})^{\alpha-1}(e^{2k}-1)^\alpha}{(e^{2k(t+1-s)}-1)^{\alpha+1}} \mathrm{d}s\mathrm{d}t},
\end{aligned}
$$

where p, M and E are given as (4.17), (4.20) and (4.21). It is then easy to check that

$$
\begin{aligned}
f_{T_0}(t) &= \frac{(1-e^{-2kt})^{\alpha-1}}{M}, \\
f_T(t) &= \frac{1}{E} \int_t^1 \frac{2\alpha k e^{2k(t+1-s)}(1-e^{-2ks})^{\alpha-1}(e^{2k}-1)^\alpha}{(e^{2k(t+1-s)}-1)^{\alpha+1}} \mathrm{d}s,
\end{aligned}
$$

are proper density functions over $t \in (0,1)$. ■

Lemma 4.7 *The Laplace transform of the Parisian stopping time of Y_t, namely τ_Y, can be written as*

$$\mathbb{E}\left[e^{-\beta\tau_Y}\right] = \frac{p' e^{-\beta} \int_0^1 e^{-\beta t} g_{T_0}(t)\mathrm{d}t}{1 - (1-p')\, e^{-\beta} \int_0^1 e^{-\beta t} g_T(t)\mathrm{d}t}, \tag{4.24}$$

where we defined

$$p' = \frac{\sin(\alpha\pi)}{\alpha\pi}, \tag{4.25}$$

$$g_{T_0}(t) = \alpha t^{\alpha-1}, \tag{4.26}$$

$$g_T(t) = \frac{t^{-\alpha} - t^{\alpha}}{(\frac{\pi}{\sin(\alpha\pi)} - \frac{1}{\alpha})(t+1)}, \tag{4.27}$$

and $g_{T_0}(t)$ and $g_T(t)$ are proper density functions over $t \in (0,1)$.

Proof. We can obtain the result using a similar argument as above, this time multiplying both the numerator and denominator of (4.4) by

$$\int_0^1 e^{-\beta s} s^{-(1-\alpha)}\mathrm{d}s.$$

Setting $k \to 0$ in each term of (4.16) will also produce the desired result. ■

Since the zeros of Squared Bessel process and Bessel are the same, the distributions of the Parisian stopping times for these two processes are the same. Hence, the compound Geometric representation for the Laplace transform of the Parisian stopping time for Bessel process also satisfies (4.24).

5 Simulation

In this section, we develop exact simulation schemes for the Parisian stopping times based on the compound Geometric Laplace transforms we obtained in Theorem 4.6 and Lemma 4.7. We also propose a modified simulation algorithm to improve the simulation speed. In addition, we present several numerical experiments to illustrate the performance and effectiveness of our exact simulation schemes in Section 5.2.

5.1 Simulation Algorithms

Algorithm 5.1 *The simulation algorithm for the Parisian stopping time of the CIR process τ is given as follows:*

1. *Generate a Geometric random variable N with*

$$\mathbb{P}(N=n)=\frac{2kM}{\Gamma(1-\alpha)\Gamma(\alpha)(e^{2k}-1)^{\alpha}}\left(1-\frac{2kM}{\Gamma(1-\alpha)\Gamma(\alpha)(e^{2k}-1)^{\alpha}}\right)^{n},$$

where $n=0,1,2,...$, and M is given in (4.20).

2. *Generate a random variable T_0 using an A/R scheme*[2] *via the following steps*

   (I) Generate $\overline{T}_0$ by setting

$$\overline{T}_0=U_1^{\frac{1}{\alpha}},\qquad U_1\sim\mathcal{U}[0,1];$$

   (II) Generate a standard uniform random variable $V_1\sim\mathcal{U}[0,1]$, if

$$V_1\leq\frac{(1-e^{-2k\overline{T}_0})^{\alpha-1}\overline{T}_0^{1-\alpha}}{(1-e^{-2k})^{\alpha-1}},\tag{5.1}$$

   then, return $\overline{T}_0\leftarrow T_0$ and set $\tau_0=T_0+1$; Otherwise, reject this candidate and go back to step (I).

3. *For $N=n$, generate the sequence of independent and identical distributed random variables $\{T_i\}_{i=1,2,...,n}$ via the following steps,*

   (1) Numerically maximising

$$C(s)=\left(\frac{(e^{2k}-1)^{\alpha}}{(e^{2k(1-s)}-1)^{\alpha}}-1\right)(1-e^{-2ks})^{\alpha-1}(1-s)^{\alpha},$$

   record the optimal s^* and set $C=C(s^*)$.

   (2) Generate $\overline{S}_i$ by setting

$$\overline{S}_i=1-U_2^{\frac{1}{1-\alpha}},\qquad U_2\sim\mathcal{U}[0,1].$$

   (3) Generate a standard uniform random variable $V_2\sim\mathcal{U}[0,1]$, if

$$V_2\leq\frac{1}{C}\frac{\left(\frac{(e^{2k}-1)^{\alpha}}{(e^{2k(1-\overline{S}_i)}-1)^{\alpha}}-1\right)(1-e^{-2k\overline{S}_i})^{\alpha-1}}{(1-\overline{S}_i)^{-\alpha}},\tag{5.2}$$

[2] The acceptance and rejection scheme is a type of exact simulation method, see Glasserman (2013) for further information on the A/R scheme.

then, accept and set $\overline{S}_i \leftarrow S_i$; Otherwise, reject this candidate and go back to step (1).

(4) With the accepted S_i, we generate T_i by setting

$$T_i = S_i - 1 + \frac{1}{2k} \ln \left(\left[\frac{(e^{2k}-1)^\alpha}{\frac{(e^{2k}-1)^\alpha}{(e^{2k(1-S)}-1)^\alpha} - \left(\frac{(e^{2k}-1)^\alpha}{(e^{2k(1-S)}-1)^\alpha} - 1 \right) U_3} \right]^{\frac{1}{\alpha}} + 1 \right), \tag{5.3}$$

with $U_3 \sim \mathcal{U}[0,1]$, and then set $\tau_i = T_i + 1$.

4. Return $\tau = \tau_0 + ... + \tau_n$.

Proof. From Theorem 4.6, we know that τ follows a compound Geometric distribution. In particular, we have

$$\tau \stackrel{D}{=} \tau_0 + \sum_{i=1}^{N} \tau_i,$$

where

- N is a Geometric distributed random variable with parameter p given in (4.17);
- $\tau_0 = T_0 + 1$, the density of T_0 is given in (4.18);
- $\tau_i = T_i + 1$, the density of independent and identically distributed random variables $\{T_i\}_{i=1,2,...N}$ is given in (4.19).

The probability mass function of the Geometric random variable N follows from Theorem 4.6 and the identity

$$\pi \csc(\pi\alpha) = \Gamma(1-\alpha)\Gamma(\alpha).$$

To generate T_0, we choose an envelop $\overline{T}_0$ with density

$$f_{\overline{T}_0}(t) = \frac{\alpha}{t^{1-\alpha}}, \qquad 0 < t < 1.$$

The associated A/R decision directly follows (5.1). To generate T_i for $i \neq 0$, we develop a two-dimensional simulation scheme. Since the density T_i is given as (4.19), and

$$\int_0^s \frac{2\alpha k e^{2k(t+1-s)}(e^{2k}-1)^\alpha}{(e^{2k(t+1-s)}-1)^{\alpha+1}} \mathrm{d}t = \left[-\frac{(e^{2k}-1)^\alpha}{(e^{2k(t+1-s)}-1)^\alpha} \right]_0^s = \frac{(e^{2k}-1)^\alpha}{(e^{2k(1-s)}-1)^\alpha} - 1,$$

then the integrand in (4.19) can simulated as the joint density of (T_i, S_i),

$$\begin{aligned} & f_{T,S}(t,s) \\ = \quad & \frac{1}{E}\frac{2\alpha k e^{2k(t+1-s)}(e^{2k}-1)^{\alpha}}{(e^{2k(t+1-s)}-1)^{\alpha+1}}(1-e^{-2ks})^{\alpha-1} \\ = \quad & \frac{\frac{2\alpha k e^{2k(t+1-s)}(e^{2k}-1)^{\alpha}}{(e^{2k(t+1-s)}-1)^{\alpha+1}}}{\frac{(e^{2k}-1)^{\alpha}}{(e^{2k(1-s)}-1)^{\alpha}}-1}\frac{1}{E}\left(\frac{(e^{2k}-1)^{\alpha}}{(e^{2k(1-s)}-1)^{\alpha}}-1\right)(1-e^{-2ks})^{\alpha-1}, \end{aligned}$$

with $0 < t < s < 1$. We use an A/R scheme to sample S_i by choosing an envelop $\overline{S}_i$ with the following density

$$f_{\overline{S}}(s) = (1-\alpha)(1-s)^{-\alpha},$$

and

$$\frac{f_S(s)}{f_{\overline{S}}(s)} = \frac{1}{E(1-\alpha)}\frac{\left(\frac{(e^{2k}-1)^{\alpha}}{(e^{2k(1-s)}-1)^{\alpha}}-1\right)(1-e^{-2ks})^{\alpha-1}}{(1-s)^{-\alpha}} \leq \frac{C}{E(1-\alpha)},$$

where C can be found via numerical optimisation. Given S_i, the CDF of T_i is given as

$$F_{T|S}(t|s) = \frac{\frac{(e^{2k}-1)^{\alpha}}{(e^{2k(1-s)}-1)^{\alpha}} - \frac{(e^{2k}-1)^{\alpha}}{(e^{2k(t+1-s)}-1)^{\alpha}}}{\frac{(e^{2k}-1)^{\alpha}}{(e^{2k(1-s)}-1)^{\alpha}}-1},$$

which can be inverted explicitly by

$$F^{-1}_{T|S}(t|s) = s-1+\frac{1}{2k}\ln\left(\left[\frac{(e^{2k}-1)^{\alpha}}{\frac{(e^{2k}-1)^{\alpha}}{(e^{2k(1-S)}-1)^{\alpha}}-\left(\frac{(e^{2k}-1)^{\alpha}}{(e^{2k(1-S)}-1)^{\alpha}}-1\right)t}\right]^{\frac{1}{\alpha}}+1\right).$$

Hence, T_i can be exactly simulated via (5.3). ■

Although the Parisian stopping time for the Squared Bessel/Bessel process is the limit of the Parisian stopping time for the CIR process with $k \to 0$, we cannot directly simulate the Parisian stopping time for the Squared Bessel/Bessel process using Algorithm 5.1 as we can only set k close to 0 but not equal to 0. Hence we develop a separate Algorithm 5.2 to generate the associated Parisian stopping time.

Algorithm 5.2 *The simulation algorithm for the Parisian stopping time of the squared Bessel process τ_Y is given as follow:*

1. *Generate a Geometric random variable N with*

$$\mathbb{P}(N=n)=\frac{\sin(\alpha\pi)}{\alpha\pi}\left(1-\frac{\sin(\alpha\pi)}{\alpha\pi}\right)^n, \qquad n=0,1,2,...$$

2. *Generate a random variable T_0 with density via inverse transformation by setting*

$$T_0=\sqrt[\alpha]{U_1}, \qquad U_1\sim\mathcal{U}[0,1],$$

   and set $\tau_0=T_0+1$.

3. *For $N=n$, generate the sequence of $\{T_i\}_{i=1,2,...,n}$ using an A/R scheme via the following steps,*

   (1) Numerically maximising

$$C(t)=\frac{1}{\frac{\pi}{\sin(\alpha\pi)}-\frac{1}{\alpha}}\frac{t^{-\alpha}-t^{\alpha}}{1+t}\frac{B(\theta,2)}{t^{\theta-1}(1-t)},$$

   where $\theta=0.59-0.01\alpha-0.60\alpha^2$ and $B(\cdot,\cdot)$ is the standard Beta function, record the optimal t^* and set $C=C(t^*,\theta)$.

   (2) Generate $\overline{T}_i$ by setting

$$\overline{T}_i=Beta(\theta,2),$$

   (3) Generate a standard uniform random variable $V\sim\mathcal{U}[0,1]$, if

$$V\leq\frac{1}{C}\frac{1}{\frac{\pi}{\sin(\alpha\pi)}-\frac{1}{\alpha}}\frac{\overline{T}_i^{\,-\alpha}-\overline{T}_i^{\,\alpha}}{1+\overline{T}_i}\frac{B(\theta,2)}{\overline{T}_i^{\,\theta-1}(1-\overline{T}_i)},$$

   then, accept and set $T_i=\overline{T}_i$; Otherwise, reject this candidate and go back to Step (2).

   with the accepted T_i, set $\tau_i=T_i+1$.

4. *Return $\tau_Y=\tau_0+...+\tau_n$.*

Proof. As before, from Lemma 4.7, we can see that τ_Y has a compound Geometric distribution $\tau_Y\stackrel{D}{=}\tau_0+\sum_{i=1}^{N}\tau_i$, but this time with parameters $N\sim$ *Geometric* (p'), and the density of T_i is given in (4.26) for $i=0$ and (4.27) for $i\geq1$. For T_0, it can be simulated directly via inverse transformation. And for the i.i.d T_i with $i\geq1$, it can be simulated via an A/R scheme. We choose an envelop $\overline{T}_i$ which follows a Beta distribution with density

$$g_{\overline{T}}(t) = \frac{t^{\theta-1}(1-t)}{B(\theta,2)}, \tag{5.4}$$

where $B(\theta,2) = \frac{\Gamma(\theta)\Gamma(2)}{\Gamma(\theta+2)}$. We have

$$\frac{g_T(t)}{g_{\overline{T}}(t)} = \frac{1}{\frac{\pi}{\sin(\alpha\pi)} - \frac{1}{\alpha}} \frac{t^{-\alpha} - t^{\alpha}}{1+t} \frac{B(\theta,2)}{t^{\theta-1}(1-t)} = C(t) \leq C \tag{5.5}$$

In order to find the optimal parameter θ, we numerically approximate θ which minimises the A/R constant. This is different for each α, but since α is given, each time we only need to do the numerical optimisation once. Hence, the entire simulation efficiency will not be affected. ■

We know that it takes longer to generate the Parisian stopping time when α increases, as the mean of N increases with respect to α. Furthermore, the main computation cost comes from applying the A/R schemes to generate T_i. Hence, we design a modified approach, by changing the distribution of T_i slightly, so that we still have a compound Geometric representation with different parameters, but this allows us to vectorise the sampling of T_i. This might not help much for low level languages, but it is extremely helpful in high level programming which is very much in use these days. The modified framework is illustrated in Algorithm 5.3.

Algorithm 5.3 *The modified simulation algorithm for the Parisian stopping time is given as follow:*

1. *Generate a Geometric random variable $N \sim Geometric(q)$, where*

$$q = \frac{p}{1 + (C-1)(1-p)}.$$

2. *Generate a random variable τ_0 by setting*

$$\tau_0 = T_0 + 1,$$

where T_0 can be simulated via step 2 in Algorithm 5.1 and Algorithm 5.2.

3. *For $N = n$, generate $\overline{T}_i$ and $V_i \sim \mathcal{U}[0,1]$ for $i = 1, 2, ..., n$. Set*

$$\bar{\tau}_i = (\overline{T}_i + 1)\mathbf{1}_{\left\{V_i \leq \frac{1}{C}\frac{f_T(\overline{T}_i)}{f_{\overline{T}}(\overline{T}_i)}\right\}}, \tag{5.6}$$

where C is the A/R constant to generate T_i.

4. *Return* $\tau = \tau_0 + \sum\limits_{i=1}^{n} \bar{\tau}_i$.

Proof. We can modify the distribution of T_i slightly and design a more efficient simulation algorithm based on the Laplace transform of the compound Geometric distribution. We start from the compound Geometric representation for the Laplace transform (4.16) of the Parisian stopping time for the CIR. Its denominator can be written as

$$\begin{aligned} & 1-(1-p)e^{-\beta}\int_0^1 e^{-\beta t} f_T(t)\mathrm{d}t \\ = & 1-(1-p)Ce^{-\beta}\int_0^1 e^{-\beta t} f_{\overline{T}}(t)\frac{f_T(t)}{Cf_{\overline{T}}(t)}\mathrm{d}t \\ = & 1+(1-p)\left(C-1\right)-C(1-p) \\ & \times\left(e^{-\beta}\int_0^1 e^{-\beta t} f_{\overline{T}}(t)\frac{f_T(t)}{Cf_{\overline{T}}(t)}\mathrm{d}t+\int_0^1 f_{\overline{T}}(t)\left(1-\frac{f_T(t)}{Cf_{\overline{T}}(t)}\right)\mathrm{d}t\right), \end{aligned} \tag{5.7}$$

where C is the same A/R constant used in (5.2). Then we have the modified compound Geometric representation

$$\begin{aligned} & \mathbb{E}\left[e^{-\beta\tau}\right] \\ = & \frac{pe^{-\beta}\int_0^1 e^{-\beta t} f_{T_0}(t)\mathrm{d}t}{1-(1-p)e^{-\beta}\int_0^1 e^{-\beta t} f_T(t)\mathrm{d}t} \\ = & \frac{qe^{-\beta}\int_0^1 e^{-\beta t} f_{T_0}(t)\mathrm{d}t}{1-(1-q)\left(e^{-\beta}\int_0^1 e^{-\beta t} f_{\overline{T}}(t)\frac{f_T(t)}{Cf_{\overline{T}}(t)}\mathrm{d}t+\int_0^1 f_{\overline{T}}(t)\left(1-\frac{f_T(t)}{Cf_{\overline{T}}(t)}\right)\mathrm{d}t\right)}, \end{aligned} \tag{5.8}$$

where $q = \frac{p}{1+(1-p)(C-1)}$. In fact, (5.8) is the Laplace transform of a compound Geometric distribution, $\tau \overset{D}{=} \tau_0 + \sum_{i=1}^{N} \bar{\tau}_i$, where N is Geometric with parameter q, the density of τ_0 is $f_{T_0}(t)$ and $\{\bar{\tau}_i\}_{i=1,2,\ldots,N}$ can be simulated as in (5.6). Likewise, we can modify the simulation scheme for the Parisian stopping time for the Squared Bessel/Bessel process by replacing p, f_T,$f_{\overline{T}}$ by p', g_T,$g_{\overline{T}}$, respectively. ■

Table 1: CPU Time under Algorithm 5.2 v.s Algorithm 5.3 based on parameter setting $\alpha = \{0.1, 0.2, ..., 0.8, 0.9\}$; each point of values is produced from 100,000 replications.

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Algorithm 5.2	0.88	1.14	1.39	1.59	2.55	2.92	3.89	5.41	9.72
Algorithm 5.3	1.44	1.34	1.27	1.37	1.59	1.75	1.68	1.95	2.16

5.2 Numerical Verification

In this section, we illustrate the performance and effectiveness of our simulation schemes via various numerical analysis. The simulation experiments are all implemented on a common laptop with Intel Core i7-6500 CPU@2.50GHz processor, 8.00GB RAM,Windows 10, 64-bit Operating System and performed in R 3.4.0. To verify the accuracy of our algorithms, we compare the estimated densities of the Parisian stopping times for Squared Bessel/Bessel and CIR processes based on 100,000 samples generated from Algorithm 5.1 and 5.2 with the recursive form densities derived in Theorem 4.3. In particular, we carry out the comparison for the Parisian stopping time of the CIR process under the parameter settings $\alpha = 0.4, 0.6$ and $k = 0.5, 1$. The associated density plot is illustrated in Figure 1. The comparison of the simulated and recursive density for the Parisian stopping time of the Squared Bessel/Bessel process under the parameter settings $\alpha = 0.4, 0.6$ is illustrated in Figure 2. Since for $\alpha = 0.5$, results were obtained for the Brownian motion in Dassios and Lim (2015), we also establish a comparison of the simulation results for Algorithm 5.1 with k close to 0, and Algorithm 5.2 against the simulation algorithm for the drawdown stopping time of Brownian motion described in Dassios and Lim (2017). The density plot is given in Figure 3. In general, these algorithms produce the same sample means and the simulated densities are more or less the same. The slight difference between Algorithm 5.1 and the other two is only due to the fact that k is only close to 0 but not equal to 0.

In addition, we carry out separate numerical experiments for Algorithm 5.2 and 5.3. Table 1 reports the simulation times for these two algorithms under various α with 100,000 replications. We see that the time needed for the two algorithms are more or less the same for small α. However, when α is large, Algorithm 5.3 outperforms Algorithm 5.2. The out-performance becomes even more substantial when α is close to 1. In particular, it is nearly 4 times faster when $\alpha = 0.8, 0.9$.

We also plot histograms of the Parisian stopping time for a CIR process with different values of α and k in Figure 4. The first three histograms are based on parameter settings $k = 1$ and $\alpha = 0.25, 0.5, 0.75$, and the last three histograms are based on parameter settings $\alpha = 0.25$ and $k = 0.5, 2, 3.5$.

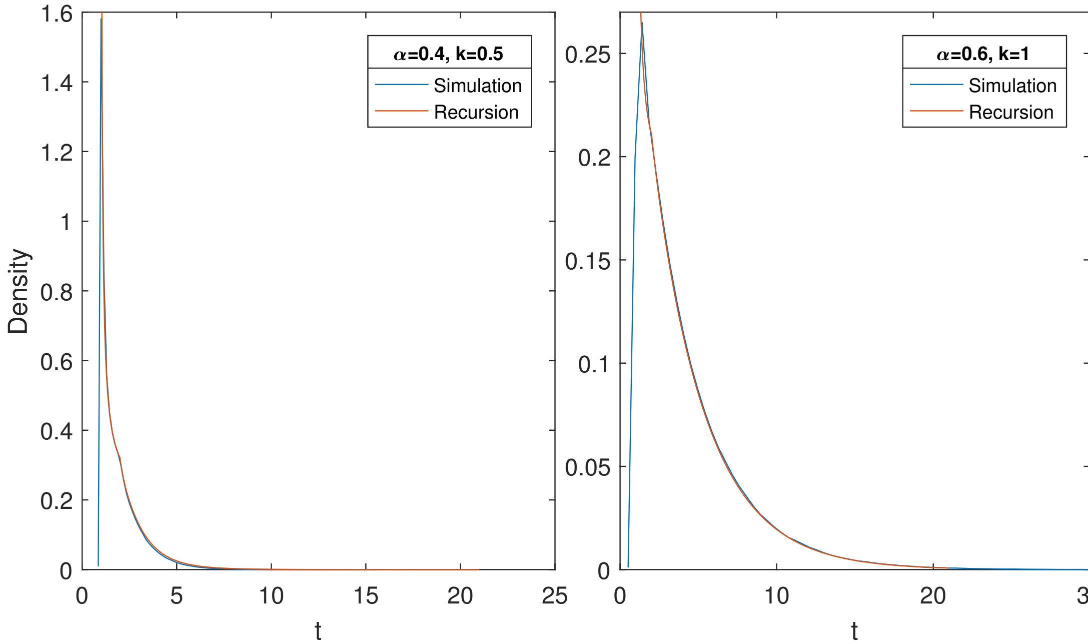

Figure 1: Comparison of simulated and analytical densities with recursive form of Parisian stopping time for CIR process with parameter settings $\alpha = 0.4$, $k = 0.5$ and $\alpha = 0.6$, $k = 1$, respectively.

It is clear that as we increase α and k, the mean for the Parisian hitting time also increases.

6 Application: Pricing a zero-coupon Parisian bond

In this section, we use our results to price a zero-coupon Parisian bond. We assume that the dynamics of the interest rate follows a CIR process under the risk neutral probability measure $\mathbb{Q}$, given in (1.2). Following a simple time change, the model reduces to one following the dynamics given in (1.3), which is a CIR process with index α and parameter $k > 0$. We define the zero-coupon Parisian bond as the bond which pays off $h(R_\tau)$ at time τ for some function h, if this happens before maturity time T. We also use $\mathbb{E}^r_{\mathbb{Q}}$ to denote the expectation under a measure $\mathbb{Q}$, for a process with initial value r.

Proposition 6.1 *Denote by $P(r, T)$ the price of a bond with payoff $h(R_\tau)$ at time τ, if this happens before maturity time T, with interest rate following dynamics* (1.3)*, starting at $R_0 = r$. Then the risk neutral price of the bond is*

$$\begin{aligned} P(r,T) &= \mathbb{E}^r_{\mathbb{Q}}\left[h(R_\tau)\exp\left(-\int_0^\tau R_t \mathrm{d}t\right)\mathbf{1}_{\{\tau<T\}}\right] \\ &= e^{-\eta r}\mathbb{E}^r_{\mathbb{Q}^*}\left[h(R_\tau)e^{-2(1-\alpha)\eta\tau}e^{\eta R_\tau}\mathbf{1}_{\{\tau<T\}}\right], \end{aligned} \tag{6.1}$$

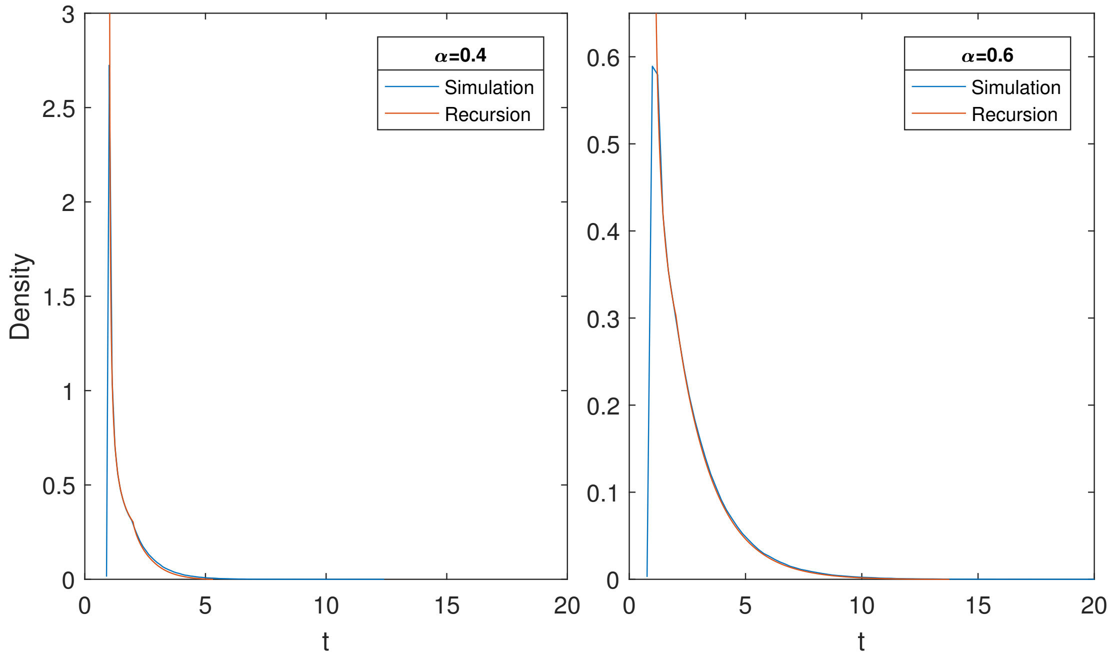

Figure 2: Comparison of simulated and analytical densities with recursive form of Parisian stopping time for Squared Bessel/Bessel process with parameter settings $\alpha = 0.4$ and $\alpha = 0.6$, respectively.

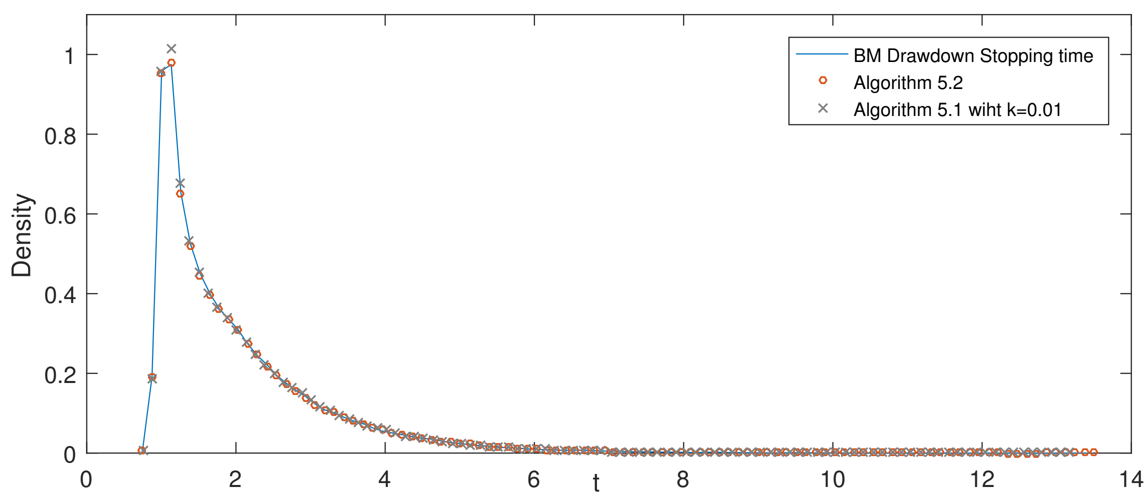

Figure 3: Comparison of simulated density of the Parisian stopping time of Squared Bessel/Bessel process with $\alpha = 0.5$ v.s. the simulated density of the Drawdown stopping time of Brownian motion derived in Dassios and Lim (2017).

where we have $\eta := \frac{\sqrt{k^2+2}-k}{2}$, *and* $\mathbb{Q}^*$ *denotes the probability measure defined by*

$$\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{Q}}|\mathcal{F}_\tau = e^{-2(1-\alpha)\eta\tau} e^{\eta(R_\tau - R_0)} \exp\left(\int_0^\tau R_t \mathrm{d}t\right). \tag{6.2}$$

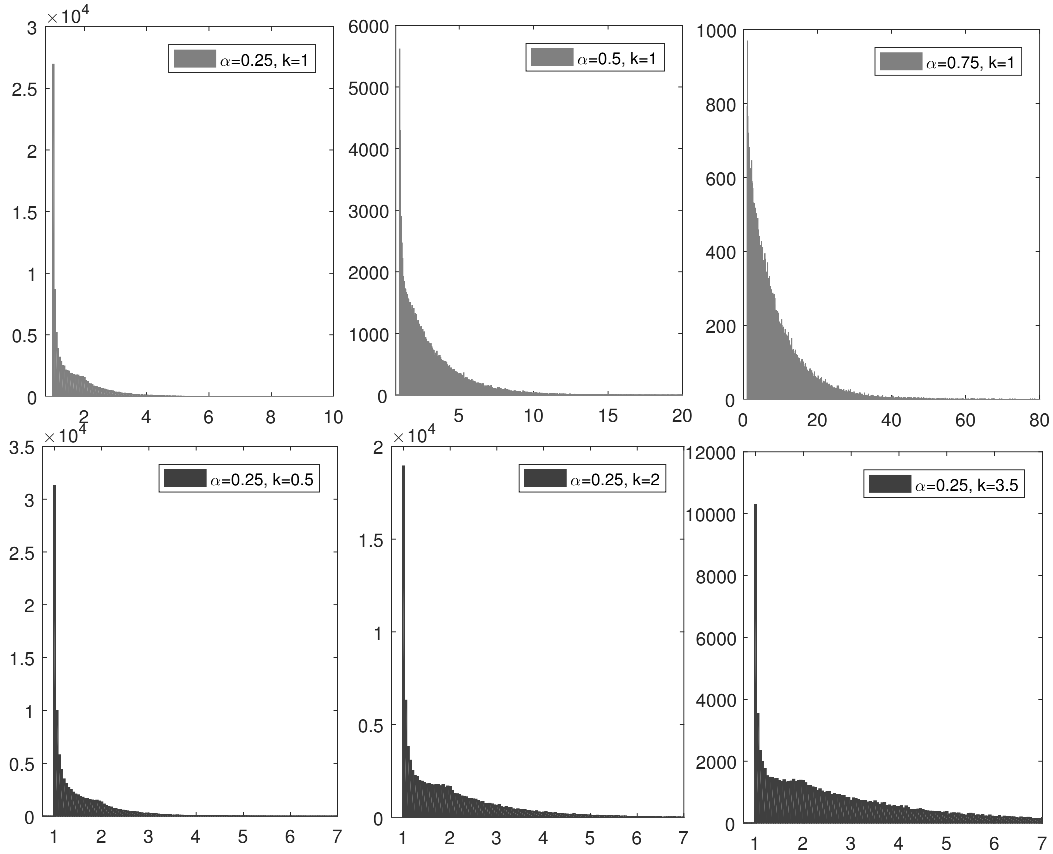

Figure 4: Histograms of Parisian Stopping time of CIR process with $\alpha = 0.25, 0.5, 0.75$, and $k = 0.5, 1, 2, 3.5$, respectively.

Furthermore, under $\mathbb{Q}^*$, R_t is a CIR process with index α and parameter $k^* = k + 2\eta$.

Proof. First, note that

$$Z_t = e^{-2(1-\alpha)\eta\tau} e^{\eta(R_\tau - R_0)} \exp\left(\int\limits_0^\tau R_t \mathrm{d}t\right), \tag{6.3}$$

for $\eta = \frac{\sqrt{k^2+2}-k}{2}$ is a martingale with expectation 1. We can then define the change of measure up to the stopping time τ by the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{Q}} | \mathcal{F}_\tau = Z_\tau$. Also let $X_t = \int\limits_0^t R_s \mathrm{d}s$. We consider a function $g : \mathbb{R}^+ \to \mathbb{R}$, and let $f(t, r, x) = e^{-x+\gamma t - \eta r} g(r)$. Then $f(t, R_t, X_t)$ is a local martingale if it satisfies the following PDE

$$\frac{\partial f}{\partial t} + r\frac{\partial f}{\partial x} + 2((1-\alpha) - kr)\frac{\partial f}{\partial r} + 2r\frac{\partial^2 f}{\partial x^2} = 0. \tag{6.4}$$

This is equivalent to solving the following PDE

$$2((1-\alpha)-k^*r)g'(r)+2rg''(r)=0, \tag{6.5}$$

which is the infinitesimal generator of $g(R_t^*)$, where R_t^* is a CIR process with index α and parameter $k^*=k+2\eta$. We also have that $f(t,R_t,X_t)=\frac{1}{Z_t}g(R_t)$ is a $\mathbb{Q}$-local martingale if and only if $g(R_t)$ is a $\mathbb{Q}^*$-local martingale, which implies that under $\mathbb{Q}^*$, R_t is a CIR process with index α and parameter k^*. ■

Thus, we propose two methods to price the zero coupon Parisian bond with a certain payoff $h(R_\tau)$. The first is using Monte Carlo simulation based on (6.1), which we can write as

$$\begin{aligned} & P(r,T) \\ = \; & e^{-\eta r}\mathbb{E}^r_{\mathbb{Q}^*}\left[h(R_1)e^{-2(1-\alpha)\eta}e^{\eta R_1}\mathbf{1}_{\{T_{r\to 0}>1\}}\right] \qquad (6.6)\\ & +e^{-\eta r}\mathbb{E}^r_{\mathbb{Q}^*}\left[e^{-2(1-\alpha)\eta T_{r\to 0}}\mathbf{1}_{\{T_{r\to 0}<1\}}\mathbb{E}^0_{\mathbb{Q}^*}\left[h(R_\tau)e^{-2(1-\alpha)\eta\tau}e^{\eta R_\tau}\mathbf{1}_{\{\tau<T-T_{r\to 0}\}}\right]\right] \qquad (6.7) \end{aligned}$$

For a interest rate process starting at r, we first simulate the first hitting time $T_{r\to 0}$ with density (2.2). If this is greater than 1, τ is hit and we approximate (6.6) using the density for $R_1|T_{r\to 0}$ given in (2.20). This $R_1|T_{r\to 0}$ can be simulated via an A/R scheme. If it is less than 1, we obtain the Monte Carlo estimate of (6.7) by simulating τ using Algorithm 5.3, and in which case R_τ is an exponentially distributed random variable based on (2.8). In Table 2, we present numerical examples of the digital zero coupon Parisian bond ($h(x)=1$) and the zero coupon Parisian call ($h(x)=(x-K)^+$), for a range of parameters α and k. In general, the price for the zero coupon Parisian bond is higher than the zero coupon Parisian call for all α and k. We observe that the price decreases when the dimension of the CIR process decreases, i.e. α increases. We also compare the preformance of the prices under different α and k, with details provided in Figure 5.

Alternatively, we can use explicit expressions for the expectation (6.1) to obtain numerical prices for the zero coupon Parisian bond. We have

$$P(r,T)$$

Table 2: Price of zero coupon Parisian bond and zero coupon Parisian call with $K = 0.15$ under parameter setting $r_0 = 0.05, 0.2$, $\alpha = 0.4, 0.6$ and $k = 0.5, 1$.

Payoff	$h(x) = 1$			
T	$\alpha = 0.4, k = 0.5$	$\alpha = 0.4, k = 1$	$\alpha = 0.6, k = 0.5$	$\alpha = 0.6, k = 1$
	$r_0 = 0.05$		$r_0 = 0.05$	
2	0.3318	0.3141	0.1797	0.1501
3	0.4139	0.4150	0.2563	0.2255
4	0.4414	0.4568	0.3018	0.2760
5	0.4493	0.4718	0.3269	0.3071
6	0.4550	0.4800	0.3393	0.3244
7	0.4560	0.4824	0.3458	0.3407
8	0.4566	0.4826	0.3513	0.3470
	$r_0 = 0.2$		$r_0 = 0.2$	
2	0.3430	0.3282	0.1937	0.1635
3	0.4241	0.4288	0.2670	0.2370
4	0.4525	0.4691	0.3100	0.2849
5	0.4621	0.4858	0.3328	0.3160
6	0.4627	0.4939	0.3440	0.3336
7	0.4647	0.4970	0.3518	0.3449
8	0.4658	0.4982	0.3565	0.3538
Payoff	$h(x) = (x-K)^+$			
T	$\alpha = 0.4, k = 0.5$	$\alpha = 0.4, k = 1$	$\alpha = 0.6, k = 0.5$	$\alpha = 0.6, k = 1$
	$r_0 = 0.05$		$r_0 = 0.05$	
2	0.2675	0.1780	0.1421	0.0836
3	0.3240	0.2348	0.2098	0.1296
4	0.3500	0.2578	0.2330	0.1554
5	0.3536	0.2694	0.2518	0.1742
6	0.3624	0.2704	0.2685	0.1851
7	0.3615	0.2778	0.2699	0.1913
8	0.3658	0.2815	0.2713	0.1963
	$r_0 = 0.2$		$r_0 = 0.2$	
2	0.2712	0.1852	0.1519	0.0904
3	0.3405	0.2433	0.2116	0.1343
4	0.3583	0.2693	0.2435	0.1634
5	0.3645	0.2744	0.2605	0.1812
6	0.3658	0.2791	0.2734	0.1878
7	0.3670	0.2583	0.2871	0.1953
8	0.3639	0.2837	0.2824	0.1996

$$= e^{-\eta r} \int_0^\infty h(x) \frac{k^* e^{\eta x}}{1-e^{-2k^*}} e^{-\frac{k^* x}{1-e^{-2k^*}}} \mathrm{d}x \int_0^T e^{-2(1-\alpha)\eta t} f_\tau^{r^*}(t; T_{r\to 0} < 1) \mathrm{d}t$$
$$+ e^{-\eta r} \int_0^\infty h(x) e^{\eta x} \frac{k^* e^{2k^*}}{e^{2k^*}-1} \left(\frac{x e^{2k^*}}{r} \right)^{-\frac{\alpha}{2}} e^{-\frac{k^*(r + x e^{2k^*})}{e^{2k^*}-1}} I_\alpha \left(\frac{2k^* \sqrt{x r e^{2k^*}}}{e^{2k^*}-1} \right) \mathrm{d}x,$$

where $f_\tau^{r^*}$ denotes the probability and Parisian stopping time density for a CIR process under the measure $\mathbb{Q}^*$. As before, we split into the cases when $T_{r\to 0} < 1$ and $T_{r\to 0} > 1$. In the first case, observe that τ and R_τ are independent, and R_τ is the CIR meander starting at 0, with density given by (2.8). For $T_{r\to 0} > 1$, we have $\tau = 1$ and the density of R_τ is the transition density of R_1, conditioned to stay positive throughout, and is given by the transition density in (2.20).

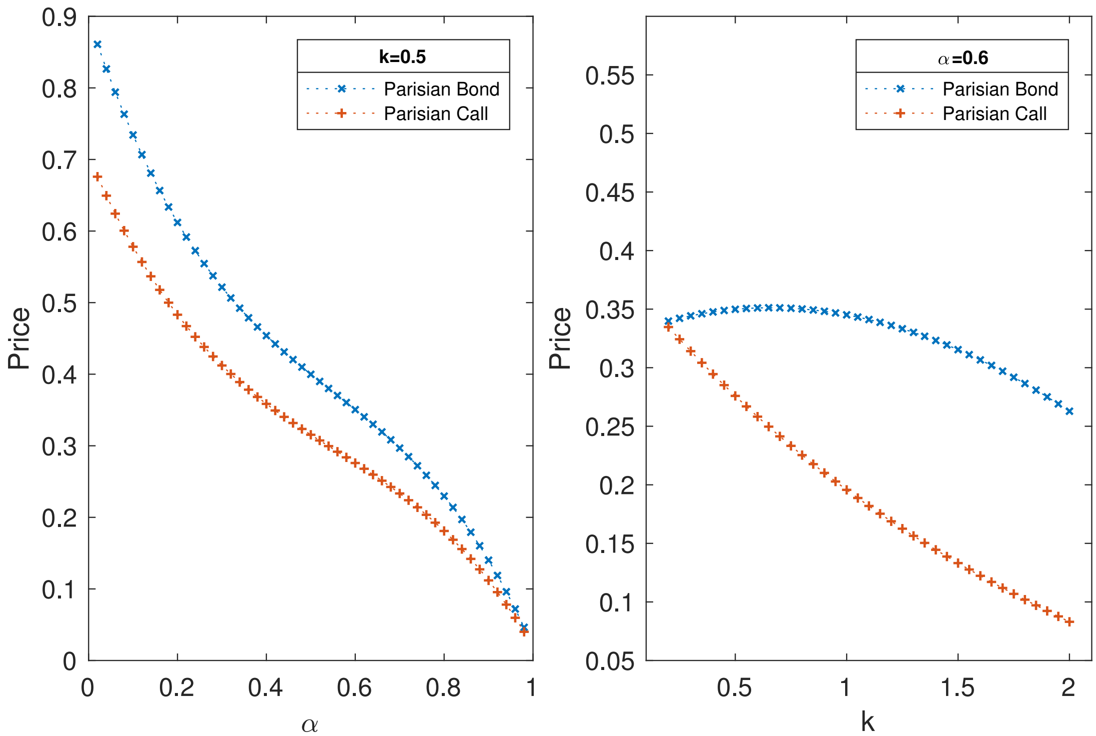

Figure 5: Price of zero coupon Parisian bond and call with respect to α and k under the parameter setting $r_0 = 0.05$, $T = 8$, and $K = 0.15$.

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7 Conclusion

In this paper, we derive various results that extend excursion theories of the CIR and Squared Bessel/Bessel processes. We also obtain Azéma martingales for these processes. Furthermore, we study the law of the Parisian stopping times based on these Azéma martingales. We develop exact simulation algorithms to sample these Parisian times based on a compound Geometric representation. Extensive numerical experiments and tests are established in order to demonstrate the accuracy of our results. Finally, we introduce the zero coupon Parisian bond and propose two approaches for its pricing. We give numerical examples of the prices for bonds with different payoff structures.

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