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Stable spike clusters on a compact two-dimensional Riemannian manifold

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Abstract

We consider the Gierer-Meinhardt system with small inhibitor diffusivity and very small activator diffusivity on a compact two-dimensional Riemannian manifold without boundary. We study steady state solutions which are far from spatial homogeneity. We construct two different spike clusters, each consisting of two spikes, which both approach the same nondegenerate local maximum point of the Gaussian curvature. We show that one of these spike clusters is stable, the other one is unstable.

MSC: 35K45; 53B20

Keywords: Pattern formation; Mathematical biology; Singular perturbation; Reaction-diffusion system; Riemannian manifold

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1. Introduction

1.1. The problem

Since the pioneering work of Turing in 1952 [39], many different reaction-diffusion system in biological modeling have been proposed and the occurrence of pattern formation has been investigated by studying what is now called Turing instability. One of the most popular models in biological pattern formation is the Gierer-Meinhardt system [14], see also [26]. In this paper, we consider the following Gierer-Meinhardt system on a compact two-dimensional Riemannian manifold $(\mathcal{M}, g)$ without boundary:

$$\begin{cases} \varepsilon^2 \Delta_g A - A + \frac{A^2}{H} = 0 \\ D\Delta_g H - H + A^2 = 0 \end{cases} \text{ in } \mathcal{M}. \tag{1.1}$$

Throughout the paper, we assume that

$$0 < \varepsilon << 1, \ 0 < D << 1.$$

We prove the existence and study the stability of a cluster of two spikes near a non-degenerate local maximum point p^0 of the Gaussian curvature of the manifold $\mathcal{M}$.

1.2. The geometric setting

Before stating the results, we first introduce the geometric setting of the problem. Let $T_p\mathcal{M}$ be the tangent place to $\mathcal{M}$ at p, and given an orthonormal basis $\{e_1(p), e_2(p)\}$ of $T_p\mathcal{M}$, we can obtain via the exponential map $\exp_p : T_p\mathcal{M} \to \mathcal{M}$, a natural correspondence $E_p(x) = x_1e_1(p)+x_2e_2(p) \to q = \exp_p(x_1e_1(p)+x_2e_2(p))$. Since $\mathcal{M}$ is a compact manifold, one knows that there exists a constant $i_g > 0$ such that

$$X_p := E_p^{-1} \circ \exp_p^{-1} : B_g(p, i_g) \to B(0, i_g)$$

is a diffeomorphism for every $p \in \mathcal{M}$. The values of this natural chart X_p are called normal coordinates about p.

We now define function spaces. Set

$$L^2(\mathcal{M}_\varepsilon) = \left\{ u \text{ measurable function defined on } \mathcal{M}_\varepsilon, \int_{\mathcal{M}_\varepsilon} u^2(q) dv_{g_\varepsilon} < \infty \right\},$$

where dv_{g_ε} denotes the Riemannian measure with respect to the metric g_ε. We further set

$$H^1(\mathcal{M}_\varepsilon) = \{u \in L^2(\mathcal{M}_\varepsilon), \nabla_{g_\varepsilon} u \in L^2(\mathcal{M}_\varepsilon)\}.$$

We will construct cluster solutions near a non-degenerate local maximum point of the Gaussian curvature function $K(p)$. In the rest of the paper, we assume that there is a local maximum of $K(p)$ is at $p^0 = 0$, i.e. we have

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$$\nabla K(0)=0,\ \nabla^2 K(0)=\begin{pmatrix} K_{11} & 0 \\ 0 & K_{22} \end{pmatrix}$$

where $K_{11},\ K_{22}<0$.

Let the local normal coordinates around p be $x=(x_1,x_2)$. Then we set $\chi=1$ for $|x|\leq \frac{i_g}{4}$ and $\chi=0$ for $|x|\geq \frac{i_g}{2}$, and introduce $\chi_\varepsilon=\chi(\frac{x}{\varepsilon})$.

1.3. The main results

Let w be the unique solution of the problem

$$\Delta w-w+w^2=0,\ w>0 \text{ in } \mathbb{R}^2,\ w(0)=\max_{y\in\mathbb{R}^2} w(y),\ w(y)\to 0 \text{ as } |y|\to\infty. \tag{1.2}$$

In this paper, we shall prove results on the existence and stability of a spike cluster of (1.1) located around $p^0=0$ with two spikes. Our first result is on the existence:

Theorem 1.1. *Let p^0 be a non-degenerate local maximum point of the Gaussian curvature $K(p)$ of $\mathcal{M}$. Assume that*

$$0<\varepsilon<<\sqrt{D}<<1,\ 0<\sqrt{D}\log\frac{1}{\varepsilon^2 D\log\frac{\sqrt{D}}{\varepsilon}}<<1, \tag{1.3}$$

and

$$\frac{K_{22}}{K_{11}}\neq 1. \tag{1.4}$$

Then the Gierer-Meinhardt system (1.1) has at least two different 2-spike cluster solutions (A_i,H_i) for $i=1,2$, which both concentrate near p^0. In particular, each of these solutions satisfies

$$A\sim\frac{D\xi_\varepsilon}{\varepsilon^2}\left(w(\frac{x}{\varepsilon}+q_i)+w(\frac{x}{\varepsilon}-q_i)\right),\ H(\pm q_i)\sim\frac{D\xi_\varepsilon}{\varepsilon^2},$$

where $\varepsilon q_i\to 0$ as $\varepsilon\to 0$ and $\xi_\varepsilon\sim\frac{1}{\log\frac{\sqrt{D}}{\varepsilon}}$ for $i=1,2$.

Remark 1.2. The limit $\frac{\varepsilon}{\sqrt{D}}\to 0$ means that the diffusivity of the activator u is asymptotically smaller than the diffusivity of the inhibitor v. If this is not assumed, then the pattern will no longer have a spike profile. The second limit $\sqrt{D}\log\frac{1}{\varepsilon^2 D\log\frac{\sqrt{D}}{\varepsilon}}\to 0$ is the condition which guarantees that the spikes form a cluster, i.e. $\varepsilon q_i\to 0$ as $\varepsilon\to 0$.

Remark 1.3. As one will see from the proof, we will construct an approximate solution which concentrates on a regular k-polygon shrinking to the point 0 for general $k\geq 2$. But when solving the reduced problem we can only handle the case $k=2$. The condition (1.4) is imposed to make sure that the reduced problem is solvable.

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Next we study the stability of the 2-spike cluster constructed in Theorem 1.1. Our second result on the stability is the following:

Theorem 1.4. *Let p^0 be a non-degenerate local maximum point of the Gaussian curvature $K(p)$. Assume (1.3), (1.4) and let (A_i, H_i) for $i = 1, 2$ be the solutions constructed in Theorem 1.1. Then one of the solutions is stable and the other one is unstable.*

Using the transformation

$$x = \varepsilon y, \ u = \frac{\varepsilon^2}{D} A, \ v = \frac{\varepsilon^2}{D} H,$$

equation (1.1) becomes

$$\begin{cases} \Delta_{g_\varepsilon} u - u + \frac{u^2}{v} = 0 \\ \Delta_{g_\varepsilon} v - \sigma^2 v + u^2 = 0 \end{cases} \quad \text{in } \mathcal{M}_\varepsilon \tag{1.5}$$

where $\sigma = \frac{\varepsilon}{\sqrt{D}}$. In the rest of this paper, we will work with (1.5).

1.4. Related work and motivation

We now comment on some related work. Generally speaking, the Gierer-Meinhardt system is difficult to solve since it does neither have a variational structure nor a priori estimates. One way to study it is to examine the so-called shadow system. Namely, we let $D \to +\infty$ first. It is known (see [21,28,35]) that the study of the shadow system amounts to the study of the following single equation for $p = 2$:

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0, \quad u > 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Equation (1.6) has a variational structure and has been studied by numerous authors. It is known that equation (1.6) has both boundary spike solutions and interior spike solutions. For existence of boundary spike solutions, see [16,29–31,46,47] and the references therein. For existence of interior spike solutions, see [17,33] and the references therein. For stability of spike solutions see [32,44,45].

Next we review some results for bumps, spikes and related patterns in the Gierer-Meinhardt system. Ground states on the real line are studied in [9,11,12,58] and for the whole $\mathbb{R}^2$ in [10]. Spikes for an interval are studied in [18,19,25,37,43] and for bounded two-dimensional domains in [23,24,31,48–52]. Hopf bifurcation of spikes is investigated in [7,41,42]. For dynamics we refer to [5,6,13,20,36]. Steady states with spherical layers have been constructed in [25,34]. Stripes have been studied in [22]. Nonlocal eigenvalue problems related to the one in this paper have been studied in [44,45,57].

The existence of spikes for single semilinear elliptic PDEs on manifolds has been investigated in [4,8,27]. Existence and stability of a single spike solution for the Gierer-Meinhardt system on a Riemannian manifold has been shown in [38].

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In [52] the existence and stability of N-peaked steady states for the Gierer-Meinhardt system with precursor inhomogeneity has been explored. The spikes in the patterns can vary in amplitude. In particular, the results imply that a precursor inhomogeneity can induce instability. Single-spike solutions for the Gierer-Meinhardt system with precursor including spike dynamics have been studied in [40].

For more background, modeling, analysis and computation on the Gierer-Meinhardt system, we refer to [54] and the references therein.

Previous results on stable spike clusters include a stable spike cluster for a consumer chain model [53]. For the Gierer-Meinhardt system spike clusters have been established in the following situations: stable interior spike clusters for the one-dimensional Gierer-Meinhardt system with precursor inhomogeneity [55], stable interior spike clusters for the two-dimensional Gierer-Meinhardt system with precursor inhomogeneity [56] and stable boundary spike clusters for the two-dimensional Gierer-Meinhardt system [2]. In the last paper the boundary curvature plays the role of the precursor in the previous papers. In the current paper we will see that the Gaussian curvature takes over that role for the Gierer-Meinhardt system on a compact two-dimensional Riemannian manifold without boundary. We would like to summarize this role as follows: the spikes in the cluster are mutually repelling and also each spike is attracted to a local maximum point of the Gaussian curvature (or to a local minimum of the precursor gradient or local maximum of the boundary curvature, respectively). This balance between attracting and repelling interactions can lead to a stable spike cluster.

This paper is organized as follows. In Section 2, we give some preliminaries and describe the construction of the approximate cluster solution. In Section 3, we use the Liapunov-Schmidt method to reduce the existence problem to finite dimensions. In Section 4 we solve this reduced problem. In Sections 5-6, we study the stability of the spike cluster steady states. In Section 5 we consider large eigenvalues. In Section 6 we study small eigenvalues. In Section 7 we discuss the results of the paper. In the appendix we give some identities needed in the main part of the paper and we calculate the eigenvalues of the reduced matrix in main order for a general number of spikes.

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2. Preliminaries and construction of the approximate solution

2.1. Expansion of the Laplacian

Let the local normal coordinates around point p be x. For a function u in the rescaled coordinates $y = \frac{x}{\varepsilon}$, one has the following expansion of the Laplace-Beltrami operator (see appendix A of [38] and also [1]):

$$\Delta_{g_\varepsilon} u(y) = \Delta_y u(y) + \Big[\frac{1}{3}K(p)\varepsilon^2 + \frac{1}{6}(\nabla K(p)\cdot y)\varepsilon^3 + \frac{1}{20}(y\nabla^2 K(p)y^t)\varepsilon^4\Big](Q[u] - 2P[u])$$

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$$+\frac{1}{45}K^2(p)|y|^2\varepsilon^4(3Q[u]-4P[u])$$
$$+\frac{1}{6}\varepsilon^3R_1[u]+\frac{1}{10}\varepsilon^4R_2[u]+O(\varepsilon^5) \tag{2.1}$$

where

$$Q[u]=y_2^2\frac{\partial^2 u}{\partial y_1^2}-2y_1y_2\frac{\partial^2 u}{\partial y_1\partial y_2}+y_1^2\frac{\partial^2 u}{\partial y_2^2},$$
$$P[u]=y_1\frac{\partial u}{\partial y_1}+y_2\frac{\partial u}{\partial y_2},$$
$$R_1[u]=\frac{y_2^2-y_1^2}{2}[\frac{\partial K}{\partial x_1}(p)\frac{\partial u}{\partial y_1}-\frac{\partial K}{\partial x_2}(p)\frac{\partial u}{\partial y_2}]$$
$$-y_1y_2[\frac{\partial K}{\partial x_2}(p)\frac{\partial u}{\partial y_1}+\frac{\partial K}{\partial x_1}(p)\frac{\partial u}{\partial y_2}],$$
$$R_2[u]=[\frac{y_2^2-y_1^2}{2}\frac{\partial u}{\partial y_1}-y_1y_2\frac{\partial u}{\partial y_2}][y_1\frac{\partial^2 K}{\partial x_1^2}(p)+y_2\frac{\partial^2 K}{\partial x_1\partial x_2}(p)]$$
$$-[\frac{y_2^2-y_1^2}{2}\frac{\partial u}{\partial y_2}+y_1y_2\frac{\partial u}{\partial y_1}][y_2\frac{\partial^2 K}{\partial x_2^2}(p)+y_1\frac{\partial^2 K}{\partial x_1\partial x_2}(p)].$$

Note that $\nabla K(p)=(\frac{\partial K}{\partial x_1},\frac{\partial K}{\partial x_2})(p)$, and $\nabla^2K(p)=\begin{pmatrix}\frac{\partial^2 K}{\partial x_1\partial x_1} & \frac{\partial^2 K}{\partial x_1\partial x_2}\\ \frac{\partial^2 K}{\partial x_1\partial x_2} & \frac{\partial^2 K}{\partial x_2\partial x_2}\end{pmatrix}(p)$ are not rescaled.

2.2. *The Green's function*

Now we introduce a Green's function G_σ which is needed for our analysis. Let G_σ be the Green's function given by

$$\Delta_{g_\varepsilon}G_\sigma(p,q)-\sigma^2G_\sigma(p,q)+\delta_q=0 \text{ in } \mathcal{M}_\varepsilon. \tag{2.2}$$

For properties of this Green's function please see [3]. From (2.2), one has

$$\int_{\mathcal{M}_\varepsilon} G_\sigma dv_{g_\varepsilon}(p)=\frac{1}{\sigma^2}.$$

Setting $G_\sigma(p,q)=\frac{1}{\sigma^2|\mathcal{M}_\varepsilon|}+\bar{G}_\sigma(p,q)$, then $\bar{G}_\sigma$ satisfies

$$\begin{cases}\Delta_{g_\varepsilon}\bar{G}_\sigma-\sigma^2\bar{G}_\sigma-\frac{1}{|\mathcal{M}_\varepsilon|}+\delta_q=0 \text{ in } \mathcal{M}_\varepsilon\\ \int_{\mathcal{M}_\varepsilon}\bar{G}_\sigma dv_{g_\varepsilon}=0.\end{cases} \tag{2.3}$$

Let $\tilde{G}_\sigma$ be defined by

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$$\Delta\tilde{G}_\sigma - \sigma^2\tilde{G}_\sigma + \delta_0 = 0 \text{ in } \mathbb{R}^2.$$

By the expansion of the Laplace Beltrami operator, one has

$$\bar{G}_\sigma(q,r) = \tilde{G}(y,z) + \varepsilon^2 G_1(y,z) + O(\varepsilon^3) \tag{2.4}$$

where $y = X_p(q),\ z = X_p(r)$ and $G_1(y,z)$ is even function in $|y-z|$.

For $\tilde{G}_\sigma(y,z) := \tilde{G}_1(\sigma y, \sigma z)$, one has

Lemma 2.1. *If $|y-z| << 1$,*

$$\tilde{G}_1(y,z) = \frac{1}{2\pi}\log\frac{1}{|y-z|} + \tilde{H}_1(y,z)$$

where $\tilde{H}_1$ is the regular part of the Green's function and $\nabla_y\tilde{H}_1(y,z)|_{y=z} = 0$.

If $|y-z| >> 1$,

$$\tilde{G}_1(y,z) = c|y-z|^{-\frac{1}{2}}e^{-|y-z|}(1+o(1)),\ |\nabla_y\tilde{G}_1(y,z)| = \tilde{G}_1(y,z)(1+o(1))$$

for some constant $c > 0$.

2.3. The construction of the approximate solutions

In this subsection, we describe the approximate solution we will use. Given $k \geq 2$, define

$$q_j^0 = (R\cos\theta_j, R\sin\theta_j) \text{ for } j = 1, \cdots, k$$

where $\theta_j = \alpha + \frac{2\pi}{k}(j-1)$ in geodesic normal coordinates. Here α is the parameter for the angle representing the degeneracy due to rotations. The constant R for the radius will be determined later in the leading order of the reduced problem. Since our manifold is not rotationally symmetric α will be derived below in a higher order of the reduced problem.

Next we introduce suitable coordinates in a neighborhood of $\mathbf{q}^0 = (q_1^0, \cdots, q_k^0)$. Let $\tilde{f}_i, \tilde{g}_i \in \mathbb{R},\ i = 1, \cdots, k$, we define

$$q_i = q_i^0 + \tilde{f}_i\vec{n}_i + \tilde{g}_i\vec{t}_i \tag{2.5}$$

where

$$\vec{t}_i = (-\sin\theta_i, \cos\theta_i),\ \vec{n}_i = (\cos\theta_i, \sin\theta_i).$$

So $\tilde{f}_i, \tilde{g}_i$ measure the displacements in the normal and tangential directions, respectively. Denote

$$\mathbf{Q}_\varepsilon = \{q_i,\ i = 1, \cdots, k,\ \sigma|\tilde{f}_i| + \sigma|\tilde{g}_i| \leq C\}. \tag{2.6}$$

Now we introduce w_j to be the unique radially symmetric solution of the equation

$$\Delta_y w_j - w_j - \frac{1}{3}K(\varepsilon q_j)\varepsilon^2 r w_j'(r) + w_j^2(r) = 0 \text{ in } \mathbb{R}^2 \tag{2.7}$$

where $K(q)$ is the Gaussian curvature at $q \in \mathcal{M}$.

Existence and uniqueness of w_j can be derived using the implicit function theorem and the non-degeneracy of the positive solution w to the equation $\Delta w - w + w^2 = 0$. Moreover, one has $\|w_j - w\|_{H^2(\mathbb{R}^2)} = O(\varepsilon^2)$ if $|\varepsilon q_j|$ is bounded. The readers are referred to [38] for more details.

Then we set our approximate solution to be

$$U = \sum_{i=1}^{k} \xi_{\varepsilon,q_i} w_i(y - q_i)\chi_\varepsilon(y - q_i) \tag{2.8}$$

where $\chi = 1$ for $|x| \le \frac{i_g}{4}$ and $\chi = 0$ for $|x| \ge \frac{i_g}{2}$ and $\chi_\varepsilon = \chi(\frac{x}{\varepsilon})$, the height ξ_{ε,q_j} is to be determined in the following subsection.

2.4. *Calculating the height of the peaks*

In this subsection, we formally calculate the height of the peaks. It turns out that the height of the peaks does not depend on the spike location in leading order but only in higher order.

For a function $u \in H^2(\mathcal{M}_\varepsilon)$, let $T[u]$ be the unique solution to the equation

$$\Delta_{g_\varepsilon} T[u] - \sigma^2 T[u] + u = 0.$$

Then from the equation satisfied by v, one can choose the approximate solution as

$$u = U, v = T(U^2) = V. \tag{2.9}$$

Next we calculate the height of the peaks

$$\begin{aligned}
\xi_{\varepsilon,q_j} &= V(q_j) \\
&= \int_{\mathcal{M}_\varepsilon} G_\sigma(q_j, q) U^2(q) dv_{g_\varepsilon}(q) \\
&= \xi_{\varepsilon,q_j}^2 \int_{\mathcal{M}_\varepsilon} G_\sigma(q_j, y) w_j(y - q_j)^2 \chi_{\varepsilon,j} dv_{g_\varepsilon} \\
&\quad + \sum_{i \ne j} \xi_{\varepsilon,q_i}^2 \int_{\mathcal{M}_\varepsilon} G_\sigma(q_j, y) w_i(y - q_i)^2 \chi_{\varepsilon,i} dv_{g_\varepsilon} \\
&\quad + \sum_i O(\xi_{\varepsilon,q_i}^2 e^{-2R \sin \frac{\pi}{k}}) \\
&= \xi_{\varepsilon,q_j}^2 \frac{1}{2\pi} \log \frac{1}{\sigma} \int_{\mathbb{R}^2} w_j^2 dy + O(\sum_i \xi_{\varepsilon,q_i}^2)
\end{aligned}$$

one has

$$\frac{1}{\xi_{\varepsilon,q_j}} = \frac{1}{2\pi}\log\frac{1}{\sigma}\left(\int_{\mathbb{R}^2} w_j^2 dy + O\left(\frac{1}{\log\sigma}\right)\right). \tag{2.10}$$

Denote

$$\xi_\varepsilon = \left(\frac{1}{2\pi}\log\frac{1}{\sigma}\int_{\mathbb{R}^2} w^2 dy\right)^{-1}.$$

Then one has $\xi_{\varepsilon,q_j} = \xi_\varepsilon(1 + O(\frac{1}{\log\sigma}))$.

3. Existence: reduction to finite dimension

3.1. Error of the approximate solution

Let us start to prove Theorem 1.1. The first step is choosing a good approximate solution which was done in the last section. The second step is to use the Liapunov-Schmidt reduction to reduce the problem to a finite dimension problem which we do in this section. First we need to calculate the error of the approximate solution (U, V) given in (2.9).

$$\begin{aligned}
S_1(U,V) &= \Delta_{g_\varepsilon} U - U + \frac{U^2}{V} \\
&= \frac{U^2}{V} - \sum_{i=1}^k \xi_{\varepsilon,q_i} w_i^2(y-q_i)\chi_{\varepsilon,i} \\
&\quad + \sum_{i=1}^k \varepsilon^3\Big[\frac{1}{6}\nabla K(\varepsilon q_i)\cdot(y-q_i)(Q[w_i]-2P[w_i]) + \frac{1}{6}R_1[w_i]\Big] \\
&\quad + \sum_{i=1}^k \varepsilon^4\Big[\frac{1}{20}(y-q_i)\nabla^2 K(\varepsilon q_i)(y-q_i)^t(Q[w_i]-2P[w_i]) \\
&\quad + \frac{1}{45}K^2(\varepsilon q_i)|y-q_i|^2(3Q[w_i]-4P[w_i]) + \frac{1}{10}R_2[w_i]\Big] + O(\varepsilon^5).
\end{aligned}$$

Next we calculate for $j = 1, \cdots, k$, and $y = q_j + z$ with $|\varepsilon z| \le \frac{i_g}{2}$,

$$\begin{aligned}
&V(q_j+z) - V(q_j) \\
&= \int_{\mathcal{M}_\varepsilon} [G_\sigma(q_j+z,p) - G_\sigma(q_j,p)]U^2(p)dv_{g_\varepsilon} \\
&= \xi_{\varepsilon,q_j}^2 \int_{\mathcal{M}_\varepsilon} [G_\sigma(q_j+z,p) - G_\sigma(q_j,p)]w_j(y-q_j)^2\chi_j^2 dv_{g_\varepsilon}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l\neq j}\xi_{\varepsilon,q_l}^2\int_{\mathcal{M}_\varepsilon}[G_\sigma(q_j+z,p)-G_\sigma(q_j,p)]w_l(y-q_l)^2\chi_l^2 dv_{g_\varepsilon}+O(\xi_\varepsilon^2 e^{-2R\sin\frac{\pi}{k}})\\
=&\,\xi_{\varepsilon,q_j}^2[\int_{\mathcal{M}_\varepsilon}(\tilde G_\sigma(q_j+z,p)-\tilde G_\sigma(q_j,p))w_j(y-q_j)^2\chi_j^2 dv_{g_\varepsilon}]\\
&+\sum_{l\neq j}\xi_{\varepsilon,q_l}^2\int_{\mathcal{M}_\varepsilon}\nabla_{q_j}\tilde G_\sigma(q_j,q_l)\cdot z w_l(y-q_l)^2\chi_l^2\, dv_{g_\varepsilon}\\
&+\sum_{l\neq j}\xi_{\varepsilon,q_l}^2\frac{1}{2}\int_{\mathcal{M}_\varepsilon} z\nabla_{q_j}^2\tilde G_\sigma(q_j,q_l)z^t w_l(y-q_l)^2\chi_l^2\, dv_{g_\varepsilon}\\
&+O(\xi_\varepsilon^2 R^{-\frac12}e^{-2R\sin\frac{\pi}{k}}+\xi_\varepsilon^2\varepsilon^2 R_\sigma^{-\frac12}e^{-R_\sigma}|z|^2+\xi_\varepsilon^2\sigma^3\sum_{j\neq l}\tilde G_\sigma(q_j,q_l)|z|^3)\\
=&\,\xi_\varepsilon^2\Big[\int_{\mathbb{R}^2}\log\frac{|y|}{|y-z|}w_j^2(y)dy+\nabla_{q_j}F(\mathbf{q})\cdot z\int_{\mathbb{R}^2}w_j^2 dy+\frac12 z\nabla_{q_j}^2F(\mathbf{q})z^t\int_{\mathbb{R}^2}w_j^2dy\Big]\\
&+O(\xi_\varepsilon^2[\sigma^3|z|^3+\varepsilon^2|z|^2]R_\sigma^{-\frac12}e^{-R_\sigma})
\end{aligned}
$$

where

$$
F(\mathbf{q})=\sum_{i=1}^k\tilde H_\sigma(q_i,q_i)+\sum_{i\neq j}\tilde G_1(\sigma q_i,\sigma q_j),\;\; R_\sigma=2R\sigma\sin\frac{\pi}{k}. \tag{3.1}
$$

Using this estimate and the expansion (2.1), we have the following estimate for the error:

$$
\begin{aligned}
&S_1(U,V)(z)\\
=&-\xi_\varepsilon^2 w_j^2(z)\Big[\int_{\mathbb{R}^2}\log\frac{|y|}{|y-z|}w_j^2(y)dy+\nabla_{q_j}F(\mathbf{q})\cdot z\int_{\mathbb{R}^2}w^2dy+\frac12 z\nabla_{q_j}^2F(\mathbf{q})z^t\int_{\mathbb{R}^2}w^2dy\Big]\\
&+\sum_{i=1}^k\xi_\varepsilon\varepsilon^4\Big[\frac16 q_j\nabla^2K(0)z^t(Q[w]-2P[w])+\frac16\tilde R_1[w]\Big]\\
&+\sum_{i=1}^k\xi_\varepsilon\varepsilon^4\Big[\frac{1}{20}z\nabla^2K(\varepsilon q_i)z^t(Q[w]-2P[w])\\
&+\frac{1}{45}K^2(\varepsilon q_i)|z|^2(3Q[w]-4P[w])+\frac{1}{10}R_2[w]\Big]\\
&+O\Big(\xi_\varepsilon^2([\sigma^3|z|^3+\varepsilon^2|z|^2]R_\sigma^{-\frac12}e^{-R_\sigma})+\xi_\varepsilon\varepsilon^5\Big)
\end{aligned}
$$

where

$$\tilde{R}_1[u] = \frac{z_2^2 - z_1^2}{2}\left(\frac{\partial}{\partial x_1}\nabla K(0)\cdot q_j \frac{\partial u}{\partial z_1} - \frac{\partial}{\partial x_2}\nabla K(0)\cdot q_j \frac{\partial u}{\partial z_2}\right)$$
$$+ z_1 z_2\left(\frac{\partial}{\partial x_2}\nabla K(0)\cdot q_j \frac{\partial u}{\partial z_1} - \frac{\partial}{\partial x_1}\nabla K(0)\cdot q_j \frac{\partial u}{\partial z_2}\right).$$

It is easy to see from the above estimate that for $y = q_j + z$, and $|\varepsilon z| \le \frac{i_g}{2}$.

Lemma 3.1.

$$S_1(U, V)(z) = S_{11} + S_{12}$$

where S_{11} is an even function in z given by

$$S_{11} = \xi_\varepsilon^2 w_j^2(z)\mathcal{R}_1(z) + \xi_\varepsilon \varepsilon^4 \mathcal{R}_2(z) w_j(z)$$

and $\mathcal{R}_1(z) = O(\log(1 + |z|))$, $\mathcal{R}_2(z) = O(|z|^2)$, while

$$S_{12} = -\xi_\varepsilon^2 w_j^2(z)\Big[\nabla_{q_j} F(\mathbf{q})\cdot z \int_{\mathbb{R}^2} w^2 dy + \frac{1}{2} z \nabla_{q_j}^2 F(\mathbf{q}) z^t \int_{\mathbb{R}^2} w^2 dy\Big]$$
$$+ \sum_{i=1}^{k} \xi_\varepsilon \varepsilon^4 \Big[\frac{1}{6} q_j \nabla^2 K(0) z^t (Q[w] - 2P[w]) + \frac{1}{6}\tilde{R}_1[w]\Big]$$
$$+ O\Big(\xi_\varepsilon^2([\sigma^3 |z|^3 + \varepsilon^2 |z|^2] R_\sigma^{-\frac{1}{2}} e^{-R_\sigma}) + \xi_\varepsilon \varepsilon^5\Big).$$

Furthermore, $S_1(U, V) = O(\xi_\varepsilon e^{-\frac{\delta}{\sigma}})$ for $|z| > \frac{\delta}{\sigma}$.

3.2. Linear theory

In this section, we study the linearized operator $L_{\varepsilon,\mathbf{q}} : H^2(\mathcal{M}_\varepsilon) \times H^2(\mathcal{M}_\varepsilon) \to L^2(\mathcal{M}_\varepsilon) \times L^2(\mathcal{M}_\varepsilon)$ defined by

$$L_{\varepsilon,\mathbf{q}} = DS_1 \begin{pmatrix} U \\ V \end{pmatrix}.$$

To denote the dependence on ε and $\mathbf{q}$ we will also use the notation $S_1 = S_{\varepsilon,\mathbf{q}}$.

First define

$$Z_{i,j}(y) = \frac{\partial w_i}{\partial y_j}(y - q_i)\chi_\varepsilon(y - q_i)$$

where the coordinates are the geodesic normal coordinates.

Set

$$K_{\varepsilon,\mathbf{q}} = \{Z_{i,j}, i = 1, \cdots, k,\ j = 1, 2\} \subset H^2(\mathcal{M}_\varepsilon),$$
$$C_{\varepsilon,\mathbf{q}} = \{Z_{i,j}, i = 1, \cdots, k,\ j = 1, 2\} \subset L^2(\mathcal{M}_\varepsilon).$$

We define our approximate kernels and cokernels as

$$\mathcal{K}_{\varepsilon,\mathbf{q}} := K_{\varepsilon,\mathbf{q}} \times \{0\} \subset H^2(\mathcal{M}_\varepsilon) \times H^2(\mathcal{M}_\varepsilon),$$
$$\mathcal{C}_{\varepsilon,\mathbf{q}} := C_{\varepsilon,\mathbf{q}} \times \{0\} \subset L^2(\mathcal{M}_\varepsilon) \times L^2(\mathcal{M}_\varepsilon).$$

Then we let $K^\perp_{\varepsilon,\mathbf{q}}$ and $C^\perp_{\varepsilon,\mathbf{q}}$ denote the orthogonal complement with respect to the scalar product $L^2(\mathcal{M}_\varepsilon)$ in $H^2(\mathcal{M}_\varepsilon)$ and $L^2(\mathcal{M}_\varepsilon)$, respectively.

Define

$$\mathcal{K}^\perp_{\varepsilon,\mathbf{q}} := K^\perp_{\varepsilon,\mathbf{q}} \times \{0\} \subset H^2(\mathcal{M}_\varepsilon) \times H^2(\mathcal{M}_\varepsilon),$$
$$\mathcal{C}^\perp_{\varepsilon,\mathbf{q}} := C^\perp_{\varepsilon,\mathbf{q}} \times \{0\} \subset L^2(\mathcal{M}_\varepsilon) \times L^2(\mathcal{M}_\varepsilon).$$

Let $\pi_{\varepsilon,\mathbf{q}}$ denote the projection in $L^2(\mathcal{M}_\varepsilon)$ onto $\mathcal{C}^\perp_{\varepsilon,\mathbf{q}}$. We are going to show that the equation

$$\pi_{\varepsilon,\mathbf{q}} \circ S_{\varepsilon,\mathbf{q}} \begin{pmatrix} U+\phi \\ V+\psi \end{pmatrix} = 0 \tag{3.2}$$

has a unique solution $\Sigma = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathcal{K}^\perp_{\varepsilon,\mathbf{q}}$.

Set

$$\mathcal{L}_{\varepsilon,\mathbf{q}} = \pi_{\varepsilon,\mathbf{q}} \circ L_{\varepsilon,\mathbf{q}} : \mathcal{K}^\perp_{\varepsilon,\mathbf{q}} \to \mathcal{C}^\perp_{\varepsilon,\mathbf{q}}. \tag{3.3}$$

The following proposition shows the invertibility of $\mathcal{L}_{\varepsilon,\mathbf{q}}$. The proofs are quite standard now and so we omit the details here. We refer to [2] for details.

Proposition 3.2. *Let $\mathcal{L}_{\varepsilon,\mathbf{q}}$ be defined in (3.3). Then there exists a positive constant δ_0 such that for $\frac{\varepsilon}{\sqrt{D}} < \delta_0$, there is a constant $C > 0$ such that*

$$\|\mathcal{L}_{\varepsilon,\mathbf{q}}\Sigma\|_{L^2(\mathcal{M}_\varepsilon)} \geq C\|\Sigma\|_{H^2(\mathcal{M}_\varepsilon)}, \tag{3.4}$$

for any $\mathbf{q} \in \mathbf{Q}_\varepsilon$, $\Sigma \in \mathcal{K}^\perp_{\varepsilon,\mathbf{q}}$. Moreover, the map $\mathcal{L}_{\varepsilon,\mathbf{q}}$ is surjective.

3.3. Solving the nonlinear problem module the cokernel

From the above proposition, we know that $\mathcal{L}_{\varepsilon,\mathbf{q}}$ is invertible (denote the inverse by $\mathcal{L}^{-1}_{\varepsilon,\mathbf{q}}$). Then we can rewrite the equation (3.2) as

$$\Sigma = -(\mathcal{L}^{-1}_{\varepsilon,\mathbf{q}} \circ \pi_{\varepsilon,\mathbf{q}})\left(S_{\varepsilon,\mathbf{q}}\begin{pmatrix} U \\ V \end{pmatrix}\right) - (\mathcal{L}^{-1}_{\varepsilon,\mathbf{q}} \circ \pi_{\varepsilon,\mathbf{q}})N_{\varepsilon,\mathbf{q}}(\Sigma) := M_{\varepsilon,\mathbf{q}}(\Sigma)$$

where

$$\Sigma = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \ N_{\varepsilon,\mathbf{q}} = S_{\varepsilon,\mathbf{q}}\begin{pmatrix} U+\phi \\ V+\psi \end{pmatrix} - S_{\varepsilon,\mathbf{q}}\begin{pmatrix} U \\ V \end{pmatrix} - S'_{\varepsilon,\mathbf{q}}\begin{pmatrix} U \\ V \end{pmatrix}\Sigma.$$

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We are going to show that $M_{\varepsilon,\mathbf{q}}(\Sigma)$ is a contraction mapping on

$$B_{\varepsilon,\eta} = \{\Sigma \in H^2(\mathcal{M}_\varepsilon) \times H^2(\mathcal{M}_\varepsilon) | \|\Sigma\|_{H^2(\mathcal{M}_\varepsilon)} \le \eta\}.$$

We have by Lemma 3.1 and Proposition 3.2 that

$$\begin{aligned}\|M_{\varepsilon,\mathbf{q}}(\Sigma)\|_{H^2(\mathcal{M}_\varepsilon)} &\le C\Big(\|\pi_{\varepsilon,\mathbf{q}} \circ N_{\varepsilon,\mathbf{q}}(\Sigma)\|_{L^2(\mathcal{M}_\varepsilon)} + \|\pi_{\varepsilon,\mathbf{q}} \circ S_{\varepsilon,\mathbf{q}}\begin{pmatrix} U \\ V \end{pmatrix}\|_{L^2(\mathcal{M}_\varepsilon)}\Big) \\ &\le C(c(\eta)\eta + c_{\varepsilon,D})\end{aligned}$$

where $C > 0$ is independent of η, $c(\eta) \to 0$ as $\eta \to 0$ and $c_{\varepsilon,D} \to 0$ as $\max\{\frac{\varepsilon}{\sqrt{D}}, \sqrt{D}\log\frac{1}{\varepsilon^2 D \log\frac{\sqrt{D}}{\varepsilon}}\} \to 0$. Moreover, we have

$$\|M_{\varepsilon,\mathbf{q}}(\Sigma) - M_{\varepsilon,\mathbf{q}}(\Sigma')\|_{H^2(\mathcal{M}_\varepsilon)} \le Cc(\eta)\|\Sigma - \Sigma'\|_{H^2(\mathcal{M}_\varepsilon)}.$$

We choose η such that $Cc(\eta) < \frac{1}{3}$ and $Cc_{\varepsilon,D} \le \frac{1}{3}\eta$. Such a choice of η is possible if we have taken $\max\{\frac{\varepsilon}{\sqrt{D}}, \sqrt{D}\log\frac{1}{\varepsilon^2 D \log\frac{\sqrt{D}}{\varepsilon}}\}$ small enough. Then $M_{\varepsilon,\mathbf{q}}$ is a contraction mapping in $B_{\varepsilon,\eta}$. By the contraction mapping principle, there exists a solution to (3.2). Thus we have

Proposition 3.3. *There exists $\delta_0 > 0$ such that for $\max\{\frac{\varepsilon}{\sqrt{D}}, \sqrt{D}\log\frac{1}{\varepsilon^2 D \log\frac{\sqrt{D}}{\varepsilon}}\} \in (0, \delta_0)$, and $\mathbf{q} \in \mathbf{Q}_\varepsilon$, we can find a unique solution $(\phi, \psi) \in \mathcal{K}^\perp_{\varepsilon,\mathbf{q}}$ satisfying*

$$S_{\varepsilon,\mathbf{q}}\begin{pmatrix} U + \phi \\ V + \psi \end{pmatrix} \in \mathcal{C}_{\varepsilon,\mathbf{q}}$$

and

$$\|(\phi, \psi)\|_{H^2(\mathcal{M}_\varepsilon)} \le C(\xi_\varepsilon^2 + \xi_\varepsilon\varepsilon^4 R + \xi_\varepsilon^2 \sigma R_\sigma^{-\frac{1}{2}} e^{-R_\sigma}).$$

For our purpose, we need more refined estimates on ϕ. Recall that S_1 can be decomposed as $S_{11} + S_{12}$, where S_{11} in leading order is an even function in z while S_{12} in leading order is an odd function in z. So we can decompose $\phi = \phi_{\varepsilon,\mathbf{q}}$ as in the following lemma.

Lemma 3.4. *Let $\phi = \phi = \varepsilon, \mathbf{q}$ be defined in Proposition 3.3. Then for $y = p_j + z$, $|\sigma z| \le \delta_0$, we have*

$$\phi = \phi_1 + \phi_2$$

where ϕ_1 is radially symmetric in z and

$$\|\phi_2\|_{H^2(\mathcal{M}_\varepsilon)} \le C\xi_\varepsilon\Big(\xi_\varepsilon\sigma R_\sigma^{-\frac{1}{2}} e^{-R_\sigma} + \varepsilon^4 R\Big).$$

Proof. Let $S[u] = S_1(u, T(u^2))$, we first solve

$$S[U+\phi_1] - S[U] + \sum_{j=1}^{k} S_{11}(y-q_j) \in C_{\varepsilon,\mathbf{q}},$$

for $\phi_1 \in K^{\perp}_{\varepsilon,\mathbf{q}}$. Then we solve

$$S[U+\phi_1+\phi_2] - S[U+\phi_1] + \sum_{j=1}^{k} S_{12}(y-q_j) \in C_{\varepsilon,\mathbf{q}},$$

for $\phi_2 \in K^{\perp}_{\varepsilon,\mathbf{q}}$. Using the same proof as in Lemma 3.3, both the above two equations a have unique solution for $\max\{\frac{\varepsilon}{\sqrt{D}}, \sqrt{D}\log\frac{1}{\varepsilon^2 D\log\frac{\sqrt{D}}{\varepsilon}}\}$ small enough. This implies the uniqueness of $\phi = \phi_1 + \phi_2$. Moreover, it is easy to see from the estimate of S_{12} that

$$\|S_{12}\|_{L^2(\mathcal{M}_\varepsilon)} = \xi_\varepsilon\Big(\xi_\varepsilon\sigma R_\sigma^{-\frac{1}{2}} e^{-R_\sigma} + \varepsilon^4 R\Big)$$

and $S_{11} \in C^{\perp}_{\varepsilon,\mathbf{q}}$ since S_{11} is an even function. Then we conclude that ϕ_1, ϕ_2 have the required properties. □

4. The reduced problem

4.1. Deriving the reduced problem

By Proposition 3.3, for each $\mathbf{q} \in \mathbf{Q}_\varepsilon$, there exists $(u, v) = (U+\phi, V+\psi)$ such that

$$S_{\varepsilon,\mathbf{q}}\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{C}_{\varepsilon,\mathbf{q}}.$$

Now, to solve the equation exactly, we have to further choose $\mathbf{q}$ such that

$$S_{\varepsilon,\mathbf{q}}\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{C}^{\perp}_{\varepsilon,\mathbf{q}}.$$

Lemma 4.1. *Under the assumption of Proposition 3.3, the following expansion holds:*

$$\begin{aligned}&\int_{\mathcal{M}_\varepsilon} S_1(U+\phi, V+\psi) Z_{i,j}\, dv_{g_\varepsilon} \\ &= -c_2\xi_\varepsilon^2\sigma\Big[\frac{c_1\varepsilon^4}{c_2\xi_\varepsilon\sigma}\nabla\frac{\partial K(0)}{\partial x_j}\cdot q_i - \sum_{l=i+1,i-1}\tilde{G}_1'(\sigma|q_i-q_l|)\Big(\frac{q_i-q_l}{|q_i-q_l|}\Big)_j\Big] \\ &\quad + O(E_\varepsilon)\end{aligned}$$

where c_1, c_2 are given in (4.1) and (4.2), $O(E_\varepsilon) = O\Big[\xi_\varepsilon^2\varepsilon^4 R + \xi_\varepsilon^3\sigma\sum_{i\neq j}\tilde{G}_\sigma(q_i, q_j)\Big]$.

Proof. We compute

$$
\begin{aligned}
&\int_{\mathcal{M}_\varepsilon} S_1(U+\phi_{\varepsilon,\mathbf{q}}, V+\psi_{\varepsilon,\mathbf{q}}) Z_{i,j}\, dv_{g_\varepsilon} \\
&= \int_{\mathcal{M}_\varepsilon} \Big[\Delta_{g_\varepsilon}(U+\phi_{\varepsilon,\mathbf{q}}) - (U+\phi_{\varepsilon,\mathbf{q}}) + \frac{(U+\phi_{\varepsilon,\mathbf{q}})^2}{V+\psi_{\varepsilon,\mathbf{q}}}\Big] Z_{i,j}\, dv_{g_\varepsilon} \\
&= \int_{\mathcal{M}_\varepsilon} \Big[\Delta_{g_\varepsilon}(U+\phi_{\varepsilon,\mathbf{q}}) - (U+\phi_{\varepsilon,\mathbf{q}}) + \frac{(U+\phi_{\varepsilon,\mathbf{q}})^2}{V}\Big] Z_{i,j}\, dv_{g_\varepsilon} \\
&\quad + \int_{\mathcal{M}_\varepsilon} \Big[\frac{(U+\phi_{\varepsilon,\mathbf{q}})^2}{V+\psi_{\varepsilon,\mathbf{q}}} - \frac{(U+\phi_{\varepsilon,\mathbf{q}})^2}{V}\Big] Z_{i,j}\, dv_{g_\varepsilon} \\
&:= I_1 + I_2.
\end{aligned}
$$

We decompose

$$
\begin{aligned}
I_1 &= \int_{\mathcal{M}_\varepsilon} [\Delta_{g_\varepsilon}(\xi_{\varepsilon,q_i} w_i + \phi_{\varepsilon,\mathbf{q}}) - (\xi_{\varepsilon,q_i} w_i + \phi_{\varepsilon,\mathbf{q}}) + \frac{(\xi_{\varepsilon,q_i} w_i + \phi_{\varepsilon,\mathbf{q}})^2}{V(q_i)}] Z_{i,j}\, dv_{g_\varepsilon} \\
&\quad - \int_{\mathcal{M}_\varepsilon} \frac{(\xi_{\varepsilon,q_i} w_i + \phi_{\varepsilon,\mathbf{q}})^2}{V(q_i)^2} [V(q_i+z) - V(q_i)] Z_{i,j}\, dv_{g_\varepsilon} + O(\xi_\varepsilon R^{-\frac{1}{2}} e^{-2R\sin\frac{\pi}{k}}) \\
&= I_{11} + I_{12}.
\end{aligned}
$$

Note that $\phi_{\varepsilon,\mathbf{q}} = \phi_1 + \phi_2$ which implies that

$$
\begin{aligned}
&\int_{\mathcal{M}_\varepsilon} [\Delta_{g_\varepsilon}\phi_{\varepsilon,\mathbf{q}} - \phi_{\varepsilon,\mathbf{q}} + 2w_i\phi_{\varepsilon,\mathbf{q}}] Z_{i,j}\, dv_{g_\varepsilon} \\
&= \int_{\mathcal{M}_\varepsilon} (\phi_1+\phi_2)\partial_{y_j}[\Delta_{g_\varepsilon} w_i - w_i + w_i^2]\, dv_{g_\varepsilon} \\
&= O\Big(\Big(\xi_\varepsilon R_\sigma^{-\frac{1}{2}} \sigma e^{-R_\sigma} + \varepsilon^4 R\Big)\varepsilon^2 \xi_\varepsilon\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathcal{M}_\varepsilon} \frac{\phi_{\varepsilon,\mathbf{q}}^2}{\xi_{\varepsilon,q_i}} Z_{i,j}\, dv_{g_\varepsilon} \\
&= \int_{\mathcal{M}_\varepsilon} \frac{2\phi_1\phi_2 + \phi_2^2}{\xi_{\varepsilon,q_i}} Z_{i,j}\, dv_{g_\varepsilon}
\end{aligned}
$$

$$= O\Big(\xi_\varepsilon^2\Big(\xi_\varepsilon R_\sigma^{-\frac{1}{2}}\sigma e^{-R_\sigma} + \varepsilon^4 R\Big)\Big)$$

since ϕ_1 is an even function. From the expression for $\tilde{R}_1$ and using Lemma B.2 in [38], one has

$$\begin{aligned}
&\int_{\mathcal{M}_\varepsilon} \xi_{\varepsilon,q_i}[\Delta_{g_\varepsilon} w_i - w_i + w_i^2] Z_{i,j} dv_{g_\varepsilon} \\
&= \xi_{\varepsilon,q_i}\varepsilon^4 \int_{\mathbb{R}^2} \frac{1}{6}\Big[q_i \nabla^2 K(0) y^t (Q[w] - 2P[w]) + \tilde{R}_1[w]\Big]\frac{\partial w}{\partial y_j} dy + O(\xi_\varepsilon \varepsilon^5) \\
&= (-\frac{\pi}{4}\int_0^\infty (w')^2 r^3 dr)\xi_{\varepsilon,q_i}\varepsilon^4 \nabla \frac{\partial K(0)}{\partial x_j}\cdot q_i + O(\xi_\varepsilon \varepsilon^5) \\
&= -c_1 \xi_{\varepsilon,q_i}\varepsilon^4 \nabla \frac{\partial K(0)}{\partial x_j}\cdot q_i + O(\xi_\varepsilon \varepsilon^5)
\end{aligned}$$

where

$$c_1 = \frac{\pi}{4}\int_0^\infty (w')^2 r^3 dr > 0. \tag{4.1}$$

Combining the above estimates, one has

$$\begin{aligned}
I_{11} = &-c_1 \xi_{\varepsilon,q_i}\varepsilon^4 \nabla \frac{\partial K(0)}{\partial x_j}\cdot q_i \\
&+ O\Big(\xi_\varepsilon^2\Big(\xi_\varepsilon R_\sigma^{-\frac{1}{2}}\sigma e^{-R_\sigma} + \varepsilon^4 R\Big)\Big).
\end{aligned}$$

Next for I_{12}, one has

$$\begin{aligned}
I_{12} = &-\int_{\mathcal{M}_\varepsilon} \frac{(\xi_{\varepsilon,q_i} w_i + \phi_{\varepsilon,\mathbf{q}})^2}{V(q_i)^2}[V(q_i+z) - V(q_i)] Z_{i,j} dv_{g_\varepsilon} \\
= &-\Big[\int_{\mathcal{M}_\varepsilon} w_i^2 (V(q_i+z) - V(q_i)) Z_{i,j} dv_{g_\varepsilon} \\
&+ \int_{\mathcal{M}_\varepsilon} \frac{2\phi w_i}{\xi_{\varepsilon,q_i}}(V(q_i+z) - V(q_i)) Z_{i,j} dv_{g_\varepsilon} \\
&+ \int_{\mathcal{M}_\varepsilon} \frac{\phi^2}{\xi_{\varepsilon,q_i}^2}(V(q_i+z) - V(q_i)) Z_{i,j} dv_{g_\varepsilon}\Big] \\
= &-\xi_{\varepsilon,q_i}^2 \partial_{q_{i,j}} F(\mathbf{q}) \int_{\mathbb{R}^2} w^2 dy \int_{\mathbb{R}^2} w^2 \frac{\partial w}{\partial y_j} y_j dy + O\Big(\xi_\varepsilon^2\Big(\xi_\varepsilon R_\sigma^{-\frac{1}{2}}\sigma e^{-R_\sigma} + \xi_\varepsilon \varepsilon^2 \sigma^2 + \varepsilon^4 R\Big)\Big)
\end{aligned}$$

$$= c_2\xi_{\varepsilon,q_i}^2 \partial_{q_{i,j}} F(\mathbf{q}) + O\Big(\xi_\varepsilon^2\Big(\xi_\varepsilon R_\sigma^{-\frac{1}{2}}\sigma e^{-R_\sigma} + \varepsilon^4 R\Big)\Big),$$

where

$$c_2 = -\int_{\mathbb{R}^2} w^2 dy \int_{\mathbb{R}^2} w^2 \frac{\partial w}{\partial y_j} y_j dy > 0. \tag{4.2}$$

In conclusion, one has

$$I_1 = -\xi_{\varepsilon,q_i}\Big[c_1\varepsilon^4 \nabla \frac{\partial K(0)}{\partial x_j}\cdot q_i - c_2\xi_{\varepsilon,q_i}\partial_{q_{i,j}} F(\mathbf{q})\Big] + O(E_\varepsilon).$$

For I_2, recall that $\psi_{\varepsilon,\mathbf{q}}$ satisfies

$$\Delta_{g_\varepsilon}\psi_{\varepsilon,\mathbf{q}} - \sigma^2\psi_{\varepsilon,\mathbf{q}} + 2U\phi_{\varepsilon,\mathbf{q}} + \phi_{\varepsilon,\mathbf{q}}^2 = 0.$$

We can make the following decomposition

$$\Delta_{g_\varepsilon}\psi_{\varepsilon,\mathbf{q},1} - \sigma^2\psi_{\varepsilon,\mathbf{q},1} + 2U\phi_{\varepsilon,\mathbf{q},1} + \phi_{\varepsilon,\mathbf{q},1}^2 = 0$$

and

$$\Delta_{g_\varepsilon}\psi_{\varepsilon,\mathbf{q},2} - \sigma^2\psi_{\varepsilon,\mathbf{q},2} + 2U\phi_{\varepsilon,\mathbf{q},2} + \phi_{\varepsilon,\mathbf{q},2}^2 + 2\phi_{\varepsilon,\mathbf{q},1}\phi_{\varepsilon,\mathbf{q},2} = 0.$$

Then one can see that $\psi_{\varepsilon,\mathbf{q},1}$ is radially symmetric with respect to z, and

$$\|\psi_{\varepsilon,\mathbf{q},2}\|_{H^2(\mathcal{M}_\varepsilon)} = O\Big(\xi_\varepsilon\Big(\xi_\varepsilon R_\sigma^{-\frac{1}{2}}\sigma e^{-R_\sigma} + \varepsilon^4 R\Big)\Big).$$

Moreover, from the Green's representation formula,

$$\begin{aligned}\psi_{\varepsilon,\mathbf{q}}(q_i+z) - \psi_{\varepsilon,\mathbf{q}}(q_i) &= \int_{\mathcal{M}_\varepsilon}\Big[G_\sigma(q_i+z,p) - G_\sigma(q_i,p)\Big](2U\phi_{\varepsilon,\mathbf{q}} + \phi_{\varepsilon,\mathbf{q}}^2)dv_{g_\varepsilon}(p)\\ &= O(\xi_\varepsilon^3)\nabla_{q_i}F(\mathbf{q})|z| + R_e(z)\end{aligned}$$

where $R_e(z)$ is even function in z. This implies

$$\begin{aligned}I_2 &= \int_{\mathcal{M}_\varepsilon}\Big[\frac{(U+\phi_{\varepsilon,\mathbf{q}})^2}{V+\psi_{\varepsilon,\mathbf{q}}} - \frac{(U+\phi_{\varepsilon,\mathbf{q}})^2}{V}\Big]Z_{i,j}dv_{g_\varepsilon}\\ &= -\int_{\mathcal{M}_\varepsilon}\frac{(U+\phi_{\varepsilon,\mathbf{q}})^2}{V^2}\psi_{\varepsilon,\mathbf{q}}Z_{i,j}dv_{g_\varepsilon} + O(E_\varepsilon)\\ &= -\int_{\mathbb{R}^2}\frac{1}{3}\frac{\partial w_i^3}{\partial y_j}(\psi_{\varepsilon,\mathbf{q}} - \psi_{\varepsilon,\mathbf{q}}(q_i))dy + O(E_\varepsilon)\\ &= O(E_\varepsilon).\end{aligned}$$

Thus one has

$$\int_{\mathcal{M}_\varepsilon} S_1(U+\phi, V+\psi) Z_{i,j} dv_{g_\varepsilon} = -\xi_\varepsilon\Big[c_1\varepsilon^4\nabla\frac{\partial K(0)}{\partial x_j}\cdot q_i - c_2\xi_{\varepsilon,q_i}\partial_{q_{i,j}}F(\mathbf{q})\Big] + O(E_\varepsilon).$$

Recall the definition of $F(\mathbf{q})$ from (3.1):

$$F(\mathbf{q}) := \sum_{i=1}^{k}\tilde{H}_1(\sigma q_i, \sigma q_i) + \sum_{i\neq j}\tilde{G}_1(\sigma q_i, \sigma q_j)$$

and $\nabla_y \tilde{H}(y,z)|_{y=z} = 0$.

Using the asymptotic behavior of

$$\tilde{G}_1(x,y) = c|x-y|^{-\frac{1}{2}}e^{-|x-y|}(1+o(1)), \tag{4.3}$$

one has

$$\int_{\mathcal{M}_\varepsilon} S_1(U+\phi, V+\psi) Z_{i,j} dv_{g_\varepsilon}$$
$$= -c_2\xi_\varepsilon^2\sigma\Big[\frac{c_1\varepsilon^4}{c_2\xi_\varepsilon\sigma}\nabla\frac{\partial K(0)}{\partial x_j}\cdot q_i - \sum_{l=i+1,i-1}\tilde{G}'(\sigma|q_i-q_l|)\Big(\frac{q_i-q_l}{|q_i-q_l|}\Big)_j\Big] + O(E_\varepsilon). \quad \square$$

Define

$$\frac{\partial U}{\partial \mathbf{q}} = (Z_{q_1}\cdot\vec{n}_1, \cdots, Z_{q_k}\cdot\vec{n}_k, Z_{q_1}\cdot\vec{t}_1, \cdots, Z_{q_k}\cdot\vec{t}_k)^t$$

and

$$Q_i = \sigma q_i = Q_i^0 + \sigma\tilde{f}_i\vec{n}_i + \sigma\tilde{g}_i\vec{t}_i.$$

In the following, we denote

$$\tilde{\mathbf{q}} = \sigma(\tilde{f}_1, \cdots, \tilde{f}_k, \tilde{g}_1, \cdots, \tilde{g}_k)^t = (f_1, \cdots, f_k, g_1, \cdots, g_k)^t$$

and

$$R_0 = |Q_i^0| = \sigma R.$$

4.2. The reduced problem for general $k = 3, 4, \dots$

Next we analyze $\int_{\mathcal{M}_\varepsilon} S_1(U+\phi, V+\psi)\frac{\partial U}{\partial \mathbf{q}} dv_{g_\varepsilon}$. We have the following:

Lemma 4.2. $\int_{\mathcal{M}_\varepsilon} S_1(U+\phi, V+\psi)\frac{\partial U}{\partial \mathbf{q}} dv_{g_\varepsilon} = 0$ *is equivalent to the following system for the perturbation* $\tilde{\mathbf{q}}$*:*

$$\Big(\frac{\hat{d}}{d}M_1 + \frac{1}{d}M_2 + \frac{C_1}{d}M_3\Big)\tilde{\mathbf{q}} = C_2\mathbf{b}_0 + O(\mathcal{E})$$

where $d = 2R_0 \sin\frac{\pi}{k}$, *and* $\hat{d}$ *is defined in (4.5),*

$$\mathcal{E} = \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{pmatrix}$$

and $\mathcal{E}_i$ *are* k*-dimensional vectors of the form*

$$\mathcal{E}_1 = O\left([\xi_\varepsilon + \frac{1}{R_0^2} + |\tilde{\mathbf{q}}|^2]\vec{\mathbf{1}}\right), \ \mathcal{E}_2 = O\left([\frac{\xi_\varepsilon}{R_0} + \frac{1}{R_0^2} + \frac{|\tilde{\mathbf{q}}|^2}{R_0}]\vec{\mathbf{1}}\right).$$

Further, $C_1 = 4\sin^2\frac{\pi}{k}\frac{K_{22}-K_{11}}{K_{11}}$, $C_2 = -2\sin\frac{\pi}{k}\frac{K_{22}-K_{11}}{K_{11}}$ *are two constants and the matrices* M_1, M_2, M_3 *and the vector* $\mathbf{b}_0$ *are given as follows:*

$$M_1 = \begin{pmatrix} (A_1+4I)\sin^2\frac{\pi}{k} & A_2\sin\frac{\pi}{k}\cos\frac{\pi}{k} \\ -A_2\sin\frac{\pi}{k}\cos\frac{\pi}{k} & -A_1\cos^2\frac{\pi}{k} \end{pmatrix},$$

$$M_2 = \begin{pmatrix} A_1\cos^2\frac{\pi}{k} + 4\sin^2\frac{\pi}{k}I & -A_2\sin\frac{\pi}{k}\cos\frac{\pi}{k} \\ A_2\sin\frac{\pi}{k}\cos\frac{\pi}{k} & -A_1\sin^2\frac{\pi}{k} \end{pmatrix},$$

$$M_3 = \begin{pmatrix} B_1 & B_2 \\ B_2 & B_3 \end{pmatrix}, \ \mathbf{b}_0 = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}\vec{\mathbf{1}}$$

where

$$A_1 = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix},$$

$$B_1 = \mathrm{diag}\{\sin^2\theta_1, \cdots, \sin^2\theta_k\},$$

$$B_2 = \mathrm{diag}\{\sin\theta_1\cos\theta_1, \cdots, \sin\theta_k\cos\theta_k\},$$

$$B_3 = \mathrm{diag}\{\cos^2\theta_1, \cdots, \cos^2\theta_k\}.$$

Proof. Wlog, assume that

$$\nabla^2 K(0) = \begin{pmatrix} K_{11} & 0 \\ 0 & K_{22} \end{pmatrix}$$

where $K_{11}, K_{22} < 0$.

By direct calculation, one has

$$\begin{aligned}
&\nabla^2 K(0)\cdot Q_i \\
&= \nabla^2 K(0)\cdot (Q_i^0 + f_i\vec{n}_i + g_i\vec{t}_i) \\
&= K_{11}(Q_i^0 + f_i\vec{n}_i + g_i\vec{t}_i) \\
&\quad +(K_{22}-K_{11})\Big[(R_0+f_i)(\sin^2\theta_i\vec{n}_i + \sin\theta_i\cos\theta_i\vec{t}_i) + g_i(\sin\theta_i\cos\theta_i\vec{n}_i + \cos^2\theta_i\vec{t}_i)\Big] \\
&= K_{11}R_0\vec{n}_i \\
&\quad +R_0(K_{22}-K_{11})(\sin^2\theta_i\vec{n}_i + \sin\theta_i\cos\theta_i\vec{t}_i) \\
&\quad +\vec{n}_i\Big[K_{11}f_i + (K_{22}-K_{11})(\sin^2\theta_i f_i + \sin\theta_i\cos\theta_i g_i)\Big] \\
&\quad +\vec{t}_i\Big[K_{11}g_i + (K_{22}-K_{11})(\sin\theta_i\cos\theta_i f_i + \cos^2\theta_i g_i\Big].
\end{aligned} \tag{4.4}$$

Next using the facts that

$$\vec{n}_{i+1} = \cos\frac{2\pi}{k}\vec{n}_i + \sin\frac{2\pi}{k}\vec{t}_i,\ \vec{t}_{i+1} = -\sin\frac{2\pi}{k}\vec{n}_i + \cos\frac{2\pi}{k}\vec{t}_i,$$
$$\vec{n}_{i-1} = \cos\frac{2\pi}{k}\vec{n}_i - \sin\frac{2\pi}{k}\vec{t}_i,\ \vec{t}_{i-1} = \sin\frac{2\pi}{k}\vec{n}_i + \cos\frac{2\pi}{k}\vec{t}_i,$$

and for $|a| >> |b|$

$$\frac{a+b}{|a+b|} = \frac{a}{|a|} + \frac{b}{|a|} - \frac{a\cdot b}{|a|^2}\frac{a}{|a|} + O(\frac{|b|}{|a|^2}),$$

one has

$$\begin{aligned}
&\frac{Q_{i+1}-Q_i}{|Q_{i+1}-Q_i|} \\
&= \frac{Q_{i+1}^0 - Q_i^0 + f_{i+1}\vec{n}_{i+1} + g_{i+1}\vec{t}_{i+1} - f_i\vec{n}_i - g_i\vec{t}_i}{|Q_{i+1}^0 - Q_i^0 + f_{i+1}\vec{n}_{i+1} + g_{i+1}\vec{t}_{i+1} - f_i\vec{n}_i - g_i\vec{t}_i|} \\
&= -\sin\frac{\pi}{k}\vec{n}_i + \cos\frac{\pi}{k}\vec{t}_i \\
&+ \frac{1}{2R_0\sin\frac{\pi}{k}}\Big[\vec{n}_i(f_{i+1}\cos\frac{2\pi}{k} - g_{i+1}\sin\frac{2\pi}{k} - f_i) + \vec{t}_i(f_{i+1}\sin\frac{2\pi}{k} + g_{i+1}\cos\frac{2\pi}{k} - g_i)\Big]
\end{aligned}$$

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$$-\frac{1}{2R_0\sin\frac{\pi}{k}}\Big[-(f_{i+1}\cos\frac{2\pi}{k}-g_{i+1}\sin\frac{2\pi}{k}-f_i)\sin\frac{\pi}{k}$$
$$+(f_{i+1}\sin\frac{2\pi}{k}+g_{i+1}\cos\frac{2\pi}{k}-g_i)\cos\frac{\pi}{k}\Big]$$
$$\times(-\vec{n}_i\sin\frac{\pi}{k}+\vec{t}_i\cos\frac{\pi}{k})+O(\frac{|\mathbf{q}|}{R_0^2}),$$

and

$$\frac{Q_{i-1}-Q_i}{|Q_{i-1}-Q_i|}$$
$$=\frac{Q^0_{i-1}-Q^0_i+f_{i-1}\vec{n}_{i-1}+g_{i-1}\vec{t}_{i-1}-f_i\vec{n}_i-g_i\vec{t}_i}{|Q^0_{i-1}-Q^0_i+f_{i-1}\vec{n}_{i-1}+g_{i-1}\vec{t}_{i-1}-f_i\vec{n}_i-g_i\vec{t}_i|}$$
$$=-\sin\frac{\pi}{k}\vec{n}_i-\cos\frac{\pi}{k}\vec{t}_i$$
$$+\frac{1}{2R_0\sin\frac{\pi}{k}}\Big[\vec{n}_i(f_{i-1}\cos\frac{2\pi}{k}+g_{i-1}\sin\frac{2\pi}{k}-f_i)+\vec{t}_i(-f_{i-1}\sin\frac{2\pi}{k}+g_{i-1}\cos\frac{2\pi}{k}-g_i)\Big]$$
$$-\frac{1}{2R_0\sin\frac{\pi}{k}}\Big[(f_{i-1}\cos\frac{2\pi}{k}+g_{i-1}\sin\frac{2\pi}{k}-f_i)\sin\frac{\pi}{k}$$
$$+(-f_{i-1}\sin\frac{2\pi}{k}+g_{i-1}\cos\frac{2\pi}{k}-g_i)\cos\frac{\pi}{k}\Big]$$
$$\times(\vec{n}_i\sin\frac{\pi}{k}+\vec{t}_i\cos\frac{\pi}{k})+O(\frac{|\mathbf{q}|}{R_0^2}).$$

Moreover, we define

$$\hat{d}=-\frac{\tilde{G}_1''(d)}{\tilde{G}_1'(d)}d=d+O(1). \tag{4.5}$$

We expand

$$\tilde{G}_1'(|Q_{i+1}-Q_i|)=\tilde{G}_1'(|Q^0_{i+1}-Q^0_i|)+\tilde{G}_1''(|Q^0_{i+1}-Q^0_i|)$$
$$\times\Big[-(f_{i+1}\cos\frac{2\pi}{k}-g_{i+1}\sin\frac{2\pi}{k}-f_i)\sin\frac{\pi}{k}+(f_{i+1}\sin\frac{2\pi}{k}+g_{i+1}\cos\frac{2\pi}{k}-g_i)\cos\frac{\pi}{k}\Big]$$

and

$$\tilde{G}_1'(|Q_{i-1}-Q_i|)=\tilde{G}_1'(|Q^0_{i-1}-Q^0_i|)+\tilde{G}_1''(|Q_{i-1}-Q_i|)$$
$$\times\Big[-(f_{i-1}\cos\frac{2\pi}{k}+g_{i-1}\sin\frac{2\pi}{k}-f_i)\sin\frac{\pi}{k}-(-f_{i-1}\sin\frac{2\pi}{k}+g_{i-1}\cos\frac{2\pi}{k}-g_i)\cos\frac{\pi}{k}\Big].$$

Combining all the above expansions, one has

$$
\begin{aligned}
&\sum_{j=i-1}^{i+1} \tilde{G}_1'(|Q_i - Q_j|)\frac{Q_j - Q_i}{|Q_j - Q_i|} \\
&= \tilde{G}_1'(d)\Big\{ -2\sin\frac{\pi}{k}\vec{n}_i \\
&-\frac{\hat{d}}{d}\Big[-\Big(f_{i+1} + f_{i-1} + 2f_i + (g_{i+1} - g_{i-1})\cot\frac{\pi}{k}\Big)\sin^2\frac{\pi}{k}\vec{n}_i \\
&+\Big((f_{i+1} - f_{i-1})\tan\frac{\pi}{k} + g_{i+1} + g_{i-1} - 2g_i\Big)\cos^2\frac{\pi}{k}\vec{t}_i\Big] \\
&+\frac{1}{d}\Big[\Big(f_{i+1} + f_{i-1} - 2f_i - (g_{i+1} - g_{i-1})\tan\frac{\pi}{k}\Big)\cos^2\frac{\pi}{k}\vec{n}_i \\
&+\Big((f_{i+1} - f_{i-1})\cot\frac{\pi}{k} - (g_{i+1} + g_{i-1} + 2g_i)\Big)\sin^2\frac{\pi}{k}\vec{t}_i\Big]\Big\} \\
&+O\left(\tilde{G}_1'(d)\left[|\tilde{\mathbf{q}}|^2\vec{n}_i + \frac{|\tilde{\mathbf{q}}|^2}{d}\vec{t}_i + \frac{|\mathbf{q}|}{R_0^2}\right]\right).
\end{aligned}
$$

Now let us define R_0 such that

$$-2\sin\frac{\pi}{k}\tilde{G}_1'\left(2R_0\sin\frac{\pi}{k}\right) + \frac{c_1\varepsilon^4 K_{11}}{c_2\xi_\varepsilon\sigma^2}R_0 = 0$$

which is possible since $\tilde{G}_1' < 0$ and $K_{11} < 0$.

Then

$$\int_{\mathcal{M}_\varepsilon} S_1\frac{\partial U}{\partial \mathbf{q}}dv_{g_\varepsilon} = 0$$

is reduced to the following linear system for the perturbation $\tilde{\mathbf{q}} = (f_1, \cdots, f_k, g_1, \cdots, g_k)^t$

$$\Big(\frac{\hat{d}}{d}M_1 + \frac{1}{d}M_2 + \frac{C_1}{d}M_3\Big)\tilde{\mathbf{q}} = C_2\mathbf{b}_0 + O(\mathcal{E}) \tag{4.6}$$

where

$$\mathcal{E} = \begin{pmatrix}\mathcal{E}_1\\ \mathcal{E}_2\end{pmatrix}$$

and $\mathcal{E}_i$ are k-dimensional vectors of the form

$$\mathcal{E}_1 = O\left(\left[\xi_\varepsilon + \frac{|\mathbf{q}|}{R_0^2} + |\tilde{\mathbf{q}}|^2\right]\vec{\mathbf{1}}\right),\ \mathcal{E}_2 = O\left(\left[\frac{\xi_\varepsilon}{R_0} + \frac{|\mathbf{q}|}{R_0^2} + \frac{|\tilde{\mathbf{q}}|^2}{R_0}\right]\vec{\mathbf{1}}\right).$$

Further, we have $C_1 = 4\sin^2\frac{\pi}{k}\frac{K_{22}-K_{11}}{K_{11}}$, $C_2 = -2\sin\frac{\pi}{k}\frac{K_{22}-K_{11}}{K_{11}}$,

$$M_1 = \begin{pmatrix} (A_1 + 4I)\sin^2\frac{\pi}{k} & A_2 \sin\frac{\pi}{k}\cos\frac{\pi}{k} \\ -A_2\sin\frac{\pi}{k}\cos\frac{\pi}{k} & -A_1\cos^2\frac{\pi}{k} \end{pmatrix},$$

$$M_2 = \begin{pmatrix} A_1\cos^2\frac{\pi}{k} + 4\sin^2\frac{\pi}{k} I & -A_2\sin\frac{\pi}{k}\cos\frac{\pi}{k} \\ A_2\sin\frac{\pi}{k}\cos\frac{\pi}{k} & -A_1\sin^2\frac{\pi}{k} \end{pmatrix},$$

$$M_3 = \begin{pmatrix} B_1 & B_2 \\ B_2 & B_3 \end{pmatrix}, \ \mathbf{b}_0 = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \vec{\mathbf{1}}$$

where

$$A_1 = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix},$$

$$B_1 = \mathrm{diag}\{\sin^2\theta_1, \cdots, \sin^2\theta_k\},$$

$$B_2 = \mathrm{diag}\{\sin\theta_1\cos\theta_1, \cdots, \sin\theta_k\cos\theta_k\},$$

$$B_3 = \mathrm{diag}\{\cos^2\theta_1, \cdots, \cos^2\theta_k\}. \quad \square$$

Remark 4.3. Since for general $k \geq 2$, the linear system (4.6) is not easy to solve, we now compute $\tilde{\mathbf{q}}$ for $k = 2$. In this case, only two spikes interact with each other, and one has $|\sin\theta_1| = |\sin\theta_2|$, $|\cos\theta_1| = |\cos\theta_2|$. This will simplify our computations a lot.

4.3. The reduced problem for $k = 2$

The reduced problem for $k = 2$ is given by the following result:

Lemma 4.4. *When $k = 2$, $\int_{\mathcal{M}_\varepsilon} S_1(U + \phi, V + \psi)\frac{\partial U}{\partial \mathbf{q}} dv_{g_\varepsilon} = 0$ is equivalent to the following system for the perturbation $\tilde{\mathbf{q}}$:*

$$\mathbf{M}\tilde{\mathbf{q}} := \Big(\frac{\hat{d}}{d}M_1 + \frac{1}{R_0}M_2 + \frac{1}{R_0}M_3\Big)\tilde{\mathbf{q}} = \mathbf{b}_0 + O\begin{pmatrix} \big[\xi_\varepsilon + \frac{|\mathbf{q}|}{R_0^2} + |\tilde{\mathbf{q}}|^2\big](1,1)^t \\ \big[\frac{\xi_\varepsilon}{R_0} + \frac{|\mathbf{q}|}{R_0^2} + \frac{|\tilde{\mathbf{q}}|^2}{R_0}\big](1,1)^t \end{pmatrix} \tag{4.7}$$

where

$$M_1 = \begin{pmatrix} \beta_1 A_0 & 0 \\ 0 & 0 \end{pmatrix}, \ M_2 = \begin{pmatrix} \beta_1 I & \beta_3 A_1 \\ \beta_3 A_1 & \beta_2 I \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}\beta_1 A_0 \end{pmatrix}, \ \mathbf{b}_0 = -\beta_3 \begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix} \vec{\mathbf{1}},$$

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$$A_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\beta_1 = K_{11} + (K_{22} - K_{11}) \sin^2\theta_1, \ \beta_2 = K_{11} + (K_{22} - K_{11}) \cos^2\theta_1, \ \beta_3 = (K_{22} - K_{11}) \sin\theta_1 \cos\theta_1.$$

Proof. The proof is similar to Lemma 4.2.

First we get

$$\begin{aligned} &\tilde{G}_1'(|Q_1 - Q_2|)\frac{Q_2 - Q_1}{|Q_2 - Q_1|} \\ &= \tilde{G}_1'(d)\Big(1 - \frac{\hat{d}}{d}(f_1 + f_2)\Big)\Big(-\vec{n}_1 - \frac{1}{2R_0}(g_1 + g_2)\vec{t}_1\Big) \\ &+ O\left(\tilde{G}_1'(d)\left[|\tilde{\mathbf{q}}|^2\vec{n}_i + \frac{|\tilde{\mathbf{q}}|^2}{d}\vec{t}_i\right]\right) \end{aligned}$$

where $d = 2R_0$.

Combining with (4.4), we have

$$\begin{aligned} &\int_{\mathcal{M}_\varepsilon} S_1(U + \phi, V + \psi)\nabla w \, dv_{g_\varepsilon} \\ &= \frac{c_1\varepsilon^4}{c_2\xi_\varepsilon\sigma^2}R_0 \\ &\times\Big((K_{11} + (K_{22} - K_{11})\sin^2\theta_1)\vec{n}_1 + (K_{22} - K_{11})\sin\theta_1\cos\theta_1\vec{t}_1) \\ &+ \frac{1}{R_0}(K_{11}f_1 + (K_{22} - K_{11})(\sin^2\theta_1 f_1 + \sin\theta_1\cos\theta_1 g_1))\vec{n}_1 \\ &+ \frac{1}{R_0}(K_{11}g_1 + (K_{22} - K_{11})(\sin\theta_1\cos\theta_1 f_1 + \cos^2\theta_1 g_1))\vec{t}_1\Big) \\ &- \tilde{G}_1'(d)\left(1 - \frac{\hat{d}}{d}(f_1 + f_2)\right)\left(\vec{n}_1 + \frac{1}{d}(g_1 + g_2)\vec{t}_1\right) \\ &+ O\left(\tilde{G}_1'(d)\left[|\tilde{\mathbf{q}}|^2\vec{n}_i + \frac{|\tilde{\mathbf{q}}|^2}{d}\vec{t}_i + \frac{|\mathbf{q}|}{R_0^2}\right]\right) + O(E). \end{aligned}$$

Here by carefully checking the error estimates and using the facts that for $k = 2$, if $|\tilde{\mathbf{q}}| << 1$,

$$\frac{Q_1 - Q_2}{|Q_1 - Q_2|} = (1 + o(1))\vec{n}_1 + O\left(\frac{1}{d}\right)\vec{t}_1,$$

$$\partial_{q_{1,j}} F(\tilde{\mathbf{q}}) \cdot z = O(\sigma\tilde{G}_1'(d)z \cdot \vec{n}_i)\vec{n}_i + O\left(\frac{\sigma\tilde{G}_1'(d)}{d}z \cdot \vec{t}_i\right)\vec{t}_i,$$

and

$$Q_1 \nabla^2 K(0) z(Q[w] - 2P[w]) + \frac{1}{6}\tilde{R}_1[w] = O(R_0)\vec{n}_1 + O(1)\vec{t}_1,$$

one can have a more accurate estimate for the error term E, i.e.

$$E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = O\begin{pmatrix} \Big[\xi_\varepsilon^2 \varepsilon^4 R + \xi_\varepsilon^3 \sigma \sum_{i\neq j} \tilde{G}s_\sigma(q_i, q_j)\Big](1,1)^t \\ \frac{1}{R_0}\Big[\xi_\varepsilon^2 \varepsilon^4 R + \xi_\varepsilon^3 \sigma \sum_{i\neq j} \tilde{G}_\sigma(q_i, q_j)\Big](1,1)^t \end{pmatrix}.$$

Define R_0 as

$$\frac{c_1 \varepsilon^4}{c_2 \xi_\varepsilon \sigma^2} R_0\Big(K_{11} + (K_{22} - K_{11})\sin^2\theta_1\Big) = \tilde{G}_1'(2R_0).$$

Considering the leading order matrix M_1, the kernel in leading order is spanned by the vectors

$$\alpha(1,-1,0,0)^t,\ \beta(0,0,1,0)^t,\ \gamma(0,0,0,1)^t.$$

Since the righthand side in leading order is $\mathbf{b}_0 = -\beta_3(0,0,1,-1)^t$ we get the solvability condition $\beta_3 = 0$. Therefore we have to choose $\theta_1 = 0$ or $\frac{\pi}{2}$. By Taylor expansion,

$$R_0 = \frac{1}{2}\log\frac{1}{\varepsilon^2 D} - \frac{3}{4}\log(\log\frac{1}{\varepsilon^2 D}) - \frac{1}{2}\log\frac{c_3}{\xi_\varepsilon} + O\Big(\frac{\log(\log\frac{1}{\varepsilon^2 D})}{\log\frac{1}{\varepsilon^2 D}}\Big) \tag{4.8}$$

where

$$c_3 = -\frac{c_1\beta_1}{2cc_2} > 0$$

since $\beta_1 < 0$.

So the reduced system becomes

$$\mathbf{M}\tilde{\mathbf{q}} := \Big(\frac{\hat{d}}{d}M_1 + \frac{1}{R_0}M_2 + \frac{1}{R_0}M_3\Big)\tilde{\mathbf{q}} = \mathbf{b}_0 + O\begin{pmatrix} [\xi_\varepsilon + \frac{|\mathbf{q}|}{R_0^2} + |\tilde{\mathbf{q}}|^2](1,1)^t \\ [\frac{\xi_\varepsilon}{R_0} + \frac{|\mathbf{q}|}{R_0^2} + \frac{|\tilde{\mathbf{q}}|^2}{R_0}](1,1)^t \end{pmatrix}$$

given in (4.7), where

$$M_1 = \begin{pmatrix} \beta_1 A_0 & 0 \\ 0 & 0 \end{pmatrix},\ M_2 = \begin{pmatrix} \beta_1 I & \beta_3 A_1 \\ \beta_3 A_1 & \beta_2 I \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}\beta_1 A_0 \end{pmatrix},\ \mathbf{b}_0 = -\beta_3\begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix}\vec{\mathbf{1}},$$

$$A_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\beta_1 = K_{11} + (K_{22} - K_{11}) \sin^2 \theta_1,$$
$$\beta_2 = K_{11} + (K_{22} - K_{11}) \cos^2 \theta_1,$$
$$\beta_3 = (K_{22} - K_{11}) \sin \theta_1 \cos \theta_1.$$

This finishes the proof. □

Remark 4.5. From the definition of R_0, one can check that

$$\varepsilon |q_1| \sim \frac{\varepsilon R_0}{\sigma} \sim \sqrt{D} \log \frac{1}{\varepsilon^2 D \log \frac{\sqrt{D}}{\varepsilon}}.$$

So under assumption (1.3), one can easily see that $\varepsilon|q_1| \to 0$ as $\varepsilon \to 0$.

Finally, for $k = 2$ we solve the reduced problem and complete the proof Theorem 1.1.

Proof of Theorem 1.1. First since $\beta_3 = 0$, we have to choose $\theta_1 = 0$ or $\frac{\pi}{2}$. In this case, the reduced system becomes

$$\mathbf{M}\tilde{\mathbf{q}} = \begin{pmatrix} (\frac{\hat{d}}{d} A_0 + \frac{1}{R_0} I)\beta_1 & 0 \\ 0 & \frac{1}{R_0}(\beta_2 I - \frac{1}{2}\beta_1 A_0) \end{pmatrix} \tilde{\mathbf{q}} = O \begin{pmatrix} \big[\xi_\varepsilon + \frac{|\mathbf{q}|}{R_0^2} + |\tilde{\mathbf{q}}|^2\big](1,1)^t \\ \big[\frac{\xi_\varepsilon}{R_0} + \frac{|\mathbf{q}|}{R_0^2} + \frac{|\tilde{\mathbf{q}}|^2}{R_0}\big](1,1)^t \end{pmatrix}. \tag{4.9}$$

If $\beta_2 - \beta_1 \neq 0$, the matrix is invertible, and one can check that $\|\mathbf{M}^{-1}\| \leq C R_0$.

Our idea is to first improve the top line of the right hand side of (4.9) to $O(\frac{\xi_\varepsilon}{R_0^2})$ from $O(\xi_\varepsilon)$. This is done in the following way. Since when $\theta_1 = 0$ or $\frac{\pi}{2}$, this approximate solution has some symmetry around each spike in main order, by carefully checking the calculation in Section 3 and 4, one can decompose E_ε in Lemma 4.1 as $[\delta_1 \xi_\varepsilon + \delta_2 \frac{\xi_\varepsilon}{R_0} + O(\frac{\xi_\varepsilon}{R_0^2})]\xi_\varepsilon \varepsilon^4 R$ for some $\delta_1,\ \delta_2$ which is tedious but standard. So one can decompose $f_i = f^0 + f^1 + \hat{f}_i$, where $f^0,\ f^1$ are chosen to match the $O(\xi_\varepsilon)$ and $O(\frac{\xi_\varepsilon}{R_0})$ term on the right hand side of the reduced problem. First f^0 is chosen such that $\tilde{G}_1'(2R_0 + 2f^0) = \tilde{G}_1'(2R_0)(1 + \delta_1 \xi_\varepsilon)$, which implies that $|f^0| = O(\xi_\varepsilon)$. Then we choose f^1 such that $\tilde{G}_1'(2R_0 + 2f^0 + 2f^1) - \tilde{G}_1'(2R_0 + 2f^0) = \tilde{G}_1'(2R_0)\delta_2 \frac{\xi_\varepsilon}{R_0}$ and $|f^1| = O(\frac{\xi_\varepsilon}{R_0})$. In this way we can get the reduced problem for $\{\hat{f}_i, g_i\}$ (we still denote its solution by $\tilde{\mathbf{q}}$) as follows:

$$\mathbf{M}\tilde{\mathbf{q}} = \begin{pmatrix} (\frac{\hat{d}}{d} A_0 + \frac{1}{R_0} I)\beta_1 & 0 \\ 0 & \frac{1}{R_0}(\beta_2 I - \frac{1}{2}\beta_1 A_0) \end{pmatrix} \tilde{\mathbf{q}} = O \begin{pmatrix} \big[\frac{\xi_\varepsilon}{R_0^2} + \frac{|\mathbf{q}|}{R_0^2} + |\tilde{\mathbf{q}}|^2\big](1,1)^t \\ \big[\frac{\xi_\varepsilon}{R_0} + \frac{|\mathbf{q}|}{R_0^2} + \frac{|\tilde{\mathbf{q}}|^2}{R_0}\big](1,1)^t \end{pmatrix}.$$

Since $\|\mathbf{M}^{-1}\| \leq C R_0$, one can find a solution $\tilde{\mathbf{q}}$ to by contraction mapping such that

$$|\tilde{\mathbf{q}}| \leq C \xi_\varepsilon.$$

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In conclusion, we find a solution such that $\max_i(|\hat{f}_i|+|g_i|)=O(\xi_\varepsilon)$.

It is easy to check that when $\theta_1=0$, then $\beta_2-\beta_1=K_{22}-K_{11}$; while when $\theta_1=\frac{\pi}{2}$, $\beta_2-\beta_1=K_{11}-K_{22}$. So if $\frac{K_{22}}{K_{11}}\neq 1$, one can solve the equation and get two solutions which correspond to $\theta_1=0$ and $\theta_1=\frac{\pi}{2}$, respectively. □

5. Stability study I: study of the large eigenvalues

We consider the stability of the steady-state $(u_\varepsilon, v_\varepsilon)$ constructed in Theorem 1.1.

In this section, we first study the large eigenvalues which satisfy $\lambda_\varepsilon\to\lambda_0\neq 0$ in the limit as $\max\{\frac{\varepsilon}{\sqrt{D}}, \sqrt{D}\log\frac{1}{\varepsilon^2 D\log\frac{\sqrt{D}}{\varepsilon}}\}\to 0$.

Linearizing the system around the equilibrium states $(u_\varepsilon, v_\varepsilon)$ obtained in Theorem 1.1, we obtain the following eigenvalue problem:

$$\begin{cases}\Delta_{g_\varepsilon}\phi-\phi+\frac{2u_\varepsilon}{v_\varepsilon}\phi-\frac{u_\varepsilon^2}{v_\varepsilon^2}\psi=\lambda\phi,\\ \Delta_{g_\varepsilon}\psi-\sigma^2\psi+2u_\varepsilon\phi=\tau\lambda\sigma^2\psi,\end{cases}\tag{5.1}$$

for $(\phi,\psi)\in H^2(\mathcal{M}_\varepsilon)\times H^2(\mathcal{M}_\varepsilon)$.

In this section, since we study the large eigenvalues, we may assume that $|\lambda_\varepsilon|\geq c>0$ for $\max\{\frac{\varepsilon}{\sqrt{D}}, \sqrt{D}\log\frac{1}{\varepsilon^2 D\log\frac{\sqrt{D}}{\varepsilon}}\}$ small enough. If $\mathrm{Re}(\lambda_\varepsilon)\leq -c<0$, then λ_ε is a stable large eigenvalue, we are done. Therefore, we may assume that $\mathrm{Re}(\lambda_\varepsilon)\geq -c$ and for a subsequence $\max\{\frac{\varepsilon}{\sqrt{D}}, \sqrt{D}\log\frac{1}{\varepsilon^2 D\log\frac{\sqrt{D}}{\varepsilon}}\}\to 0$, $\lambda_\varepsilon\to\lambda_0\neq 0$. We shall derive the limiting eigenvalue problem which is given by a coupled system of NLEPs.

The second equation of (5.1) is equivalent to

$$\Delta_{g_\varepsilon}\psi-\sigma^2(1+\tau\lambda_\varepsilon)\psi+2u_\varepsilon\phi=0 \text{ on } \mathcal{M}_\varepsilon.\tag{5.2}$$

We introduce the following notation:

$$\sigma_\lambda=\sigma\sqrt{1+\tau\lambda_\varepsilon},$$

where in $\sqrt{1+\tau\lambda_\varepsilon}$, we take the principal part of the square root.

Let us assume that

$$\|\phi\|_{H^2(\mathcal{M}_\varepsilon)}=1.$$

We cut off $\phi=\phi_\varepsilon$ as follows:

$$\phi_{\varepsilon,j}=\phi_\varepsilon\chi_\varepsilon(z-q_j),\ \ j=1,\cdots,k,\tag{5.3}$$

where the cutoff function χ_ε has been defined in (2.8).

From (5.1) and the exponential decay of w, it follows that

$$\phi_\varepsilon=\sum_{j=1}^k\phi_{\varepsilon,j}(1+o(1)) \text{ in } H^2(\mathcal{M}_\varepsilon).\tag{5.4}$$

Then by a standard procedure (see [15], Section 7.12), we extend $\phi_{\varepsilon,j}$ to a function defined on $\mathbb{R}^2$ such that

$$\|\phi_{\varepsilon,j}\|_{H^2(\mathbb{R}^2)} \leq C\|\phi_{\varepsilon,j}\|_{H^2(\mathcal{M}_\varepsilon)}, \ j = 1, \cdots, k.$$

Since $\|\phi_\varepsilon\|_{H^2(\mathcal{M}_\varepsilon)} = 1$, $\|\phi_{\varepsilon,j}\|_{H^2(\mathbb{R}^2)} \leq C$. By taking a subsequence, we may assume that $\phi_{\varepsilon,j} \to \phi_j$ as $\max\{\frac{\varepsilon}{\sqrt{D}}, \sqrt{D}\log\frac{1}{\varepsilon^2 D\log\frac{\sqrt{D}}{\varepsilon}}\} \to 0$ in $H^1(\mathbb{R}^2)$ for some $\phi_j \in H^1(\mathbb{R}^2)$ for $j = 1, \cdots, k$.

By (5.1), we have

$$\begin{aligned}
\psi_\varepsilon(q_j) &= \int_{\mathcal{M}_\varepsilon} G_{\sigma_\lambda}(q_j, y) 2u_\varepsilon \phi_\varepsilon(y)\, dy \\
&= \int_{\mathcal{M}_\varepsilon} G_{\sigma_\lambda}(q_j, y) 2(\sum_{i=1}^{k} \xi_{\varepsilon,q_i} w_j(y - q_i)\phi_{\varepsilon,i} + O(\xi_\varepsilon^2))\, dy \\
&= \frac{1}{2\pi}\log\frac{1}{\sigma_\lambda}\int_{\mathbb{R}^2} 2\xi_{\varepsilon,j} w_j \phi_{\varepsilon,j}(1 + o(1))\, dx.
\end{aligned}$$

Substituting the above equation into the first equation of (5.1) and using the expansion of $\xi_{\varepsilon,j}$, in the limit $\max\{\frac{\varepsilon}{\sqrt{D}}, \sqrt{D}\log\frac{1}{\varepsilon^2 D\log\frac{\sqrt{D}}{\varepsilon}}\} \to 0$ we arrive at the following nonlocal eigenvalue problem (NLEP):

$$\Delta\phi_j - \phi_j + 2w\phi_j - \frac{2}{1+\tau\lambda_0}\frac{\int_{\mathbb{R}^2_+} w\phi_j\, dx}{\int_{\mathbb{R}^2_+} w^2\, dx} w^2 = \lambda_0\phi_j, \ j = 1, \cdots, k. \tag{5.5}$$

By Theorem 3.5 in [54], (5.5) has only stable eigenvalues if τ is small enough.

In conclusion, we have shown that the large eigenvalues of the solutions given in Theorem 1.1 are all stable if τ is small enough.

6. Stability study II: study of the small eigenvalues

Now we study the eigenvalue problem (5.1) with respect to small eigenvalues. Namely, we assume that $\lambda_\varepsilon \to 0$ as $\max\{\frac{\varepsilon}{\sqrt{D}}, \sqrt{D}\log\frac{1}{\varepsilon^2 D\log\frac{\sqrt{D}}{\varepsilon}}\} \to 0$.

Our main result in the section says that if $\lambda_\varepsilon \to 0$, then

$$\lambda_\varepsilon \sim \varepsilon^4 R\sigma_0(\mathbf{M})$$

where $\sigma_0(\mathbf{M})$ is an eigenvalue of $\mathbf{M}$ defined in (4.9). So the stability of the solutions depends on the eigenvalues of $\mathbf{M}$. It turns out that it is related to the ratio $\frac{K_{22}}{K_{11}}$.

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6.1. Eigenfunctions and error estimates

Let $(u_\varepsilon, v_\varepsilon)$ be the equilibrium state constructed for equation (1.5), and define

$$u_{\varepsilon,j} = \xi_{\varepsilon,q_j} u_\varepsilon(y), \ \ j = 1, \cdots, k,$$

where ξ_{ε,q_j} is defined in (2.8) and calculated in (2.10). It is easy to see that

$$u_\varepsilon = \sum_{j=1}^{k} u_{\varepsilon,j}(1 + o(1)) \text{ in } H^2(\mathcal{M}_\varepsilon).$$

Now let us set $\lambda_0 = 0$ in (5.5), we have

$$\Delta\phi_j - \phi_j + 2w\phi_j - 2w^2 \frac{\int_{\mathbb{R}^2} w\phi_j dy}{\int_{\mathbb{R}^2} w^2 dy} = 0, \tag{6.1}$$

which is equivalent to

$$L_0\Big(\phi_j - 2\frac{\int_{\mathbb{R}^2} w\phi_j dy}{\int_{\mathbb{R}^2} w^2 dy} w\Big) = 0, \, j = 1, \cdots, k,$$

where $L_0 = \Delta - 1 + 2w$. We have

$$\phi_j - 2\frac{\int_{\mathbb{R}^2} w\phi_j dy}{\int_{\mathbb{R}^2} w^2 dy} w \in \text{span}\left\{\frac{\partial w}{\partial y_i}, i = 1, 2\right\}, \ \ j = 1, \cdots, k.$$

This implies that $\int_{\mathbb{R}^2} w\phi_j dy = 0$, and we can decompose ϕ_ε as

$$\phi_\varepsilon = \sum_{j=1}^{k}\sum_{i=1}^{2} \frac{a_{j,i}^\varepsilon}{\xi_\varepsilon} \frac{\partial u_{\varepsilon,j}}{\partial y_i} + \phi_\varepsilon^\perp$$

where

$$\phi_\varepsilon^\perp \perp \tilde{K}_{\varepsilon,\mathbf{q}} := \text{span}\left\{\frac{\partial u_{\varepsilon,j}}{\partial y_i}, \ \ j = 1, \cdots, k, \ \ i = 1, 2\right\}.$$

The decomposition of ϕ_ε implies that

$$\psi_\varepsilon = \sum_{j=1}^{k}\sum_{i=1}^{2} a_{j,i}^\varepsilon \psi_{\varepsilon,j,i} + \psi_\varepsilon^\perp$$

where $\psi_{\varepsilon,j,i}$ is the unique solution of

$$\Delta_{g_\varepsilon}\psi_{\varepsilon,j,i} - \sigma_\lambda^2 \psi_{\varepsilon,j,i} + 2\xi_\varepsilon^{-1} u_\varepsilon \frac{\partial u_{\varepsilon,j}}{\partial y_i} = 0,$$

and

$$\Delta_{g_\varepsilon}\psi_\varepsilon^\perp - \sigma_\lambda^2\psi_\varepsilon^\perp + 2\xi_\varepsilon^{-1}u_\varepsilon\phi_\varepsilon^\perp = 0.$$

Supposing $\|\phi_\varepsilon\|_{H^2(\mathcal{M}_\varepsilon)} = 1$, then we have $a_{j,i}^\varepsilon = O(1)$. Substituting the decomposition of ϕ_ε and ψ_ε into (5.1), using the fact that

$$\Delta_{g_\varepsilon}u_{\varepsilon,j} - u_{\varepsilon,j} + \frac{u_{\varepsilon,j}^2}{v_\varepsilon} = \text{h.o.t.},$$

we have

$$\begin{aligned}&\sum_{j=1}^{k}\sum_{i=1}^{2}a_{j,i}^\varepsilon\frac{u_\varepsilon^2}{v_\varepsilon^2}\Big(\frac{\partial_{y_i}v_\varepsilon}{\xi_\varepsilon} - \psi_{\varepsilon,j,i}\Big) + \sum_{j=1}^{k}\sum_{i=1}^{2}\frac{a_{j,i}^\varepsilon}{\xi_\varepsilon}\Big[\Delta_{g_\varepsilon}\frac{\partial u_{\varepsilon,j}}{\partial y_i} - \frac{\partial}{\partial y_i}\Delta_{g_\varepsilon}u_{\varepsilon,j}\Big]\\&+\Delta_{g_\varepsilon}\phi_\varepsilon^\perp - \phi_\varepsilon^\perp + \frac{2u_\varepsilon}{v_\varepsilon}\phi_\varepsilon^\perp - \frac{u_\varepsilon^2}{v_\varepsilon^2}\psi_\varepsilon^\perp - \lambda\phi_\varepsilon^\perp + \text{h.o.t.} = \lambda\sum_{j=1}^{k}\sum_{i=1}^{2}\frac{a_{j,i}^\varepsilon}{\xi_\varepsilon}\frac{\partial u_{\varepsilon,j}}{\partial y_i}.\end{aligned}\tag{6.2}$$

We set

$$\begin{aligned}\mathcal{I}_1 &= \sum_{j=1}^{k}\sum_{i=1}^{2}a_{j,i}^\varepsilon\frac{u_\varepsilon^2}{v_\varepsilon^2}\Big(\frac{\partial_{y_i}v_\varepsilon}{\xi_\varepsilon} - \psi_{\varepsilon,j,i}\Big) + \sum_{j=1}^{k}\sum_{i=1}^{2}\frac{a_{j,i}^\varepsilon}{\xi_\varepsilon}\Big[\Delta_{g_\varepsilon}\frac{\partial u_{\varepsilon,j}}{\partial y_i} - \frac{\partial}{\partial y_i}\Delta_{g_\varepsilon}u_{\varepsilon,j}\Big]\\&:= \mathcal{I}_{11} + \mathcal{I}_{12},\\ \mathcal{I}_2 &= \Delta_{g_\varepsilon}\phi_\varepsilon^\perp - \phi_\varepsilon^\perp + \frac{2u_\varepsilon}{v_\varepsilon}\phi_\varepsilon^\perp - \frac{u_\varepsilon^2}{v_\varepsilon^2}\psi_\varepsilon^\perp - \lambda\phi_\varepsilon^\perp.\end{aligned}$$

First we shall derive the estimate for $\phi_\varepsilon^\perp$. Since $\phi_\varepsilon^\perp \perp \tilde{K}_{\varepsilon,\mathbf{q}}$, we have

$$\|\phi_\varepsilon^\perp\|_{H^2} \le C\|\mathcal{I}_1\|_{L^2}.$$

By the expansion of Δ_{g_ε} in (2.1), one knows that

$$\|\mathcal{I}_{12}\| \le C\varepsilon^2\sum_{j=1}^{k}\sum_{i=1}^{2}|a_{j,i}^\varepsilon|.\tag{6.3}$$

For $\mathcal{I}_{11}$, using the equation satisfied by $\psi_{\varepsilon,j,i}$, we get

$$\begin{aligned}\psi_{\varepsilon,j,i}(y) &= \int_{\mathcal{M}_\varepsilon}\tilde{G}_{\sigma_\lambda}(y,z)\Big[2\xi_\varepsilon^{-1}u_\varepsilon\frac{\partial u_{\varepsilon,j}}{\partial z_i}\Big]dz + \text{h.o.t.}\\&= \xi_\varepsilon\int_{\mathcal{M}_\varepsilon}\tilde{G}_{\sigma_\lambda}(y,z)\frac{\partial w(z-q_j)^2}{\partial z_i}dz + \text{h.o.t.},\end{aligned}\tag{6.4}$$

and using the equation satisfied by v_ε, we have

$$\begin{aligned}\frac{\partial v_\varepsilon}{\partial y_i} &= \int_{\mathcal{M}_\varepsilon} \frac{\partial \tilde{G}_\sigma}{\partial y_i}(y,z)u_\varepsilon^2(z)dz \\ &= \xi_\varepsilon^2 \int_{\mathcal{M}_\varepsilon} \frac{\partial \tilde{G}_\sigma}{\partial y_i}(y,z)(\sum_{l=1}^k w(z-q_l)^2)dz + \text{h.o.t.}\end{aligned} \tag{6.5}$$

Combining (6.4) and (6.5), one has

$$\frac{1}{\xi_\varepsilon}\frac{\partial v_\varepsilon}{\partial y_i}(y) - \psi_{\varepsilon,j,i}(y)$$

$$= \xi_\varepsilon\Big[\int_{\mathcal{M}_\varepsilon} \frac{\partial \tilde{G}_\sigma}{\partial y_i}(y,z)(\sum_{l=1}^k w(z-q_l)^2)dz - \int_{\mathcal{M}_\varepsilon} \tilde{G}_{\sigma_\lambda}(y,z)\frac{\partial w(z-q_j)^2}{\partial z_i}dz + \text{h.o.t.}\Big] \tag{6.6}$$

$$= \xi_\varepsilon\Big[\frac{1}{2\pi}\int_{\mathcal{M}_\varepsilon} \frac{\partial}{\partial y_i}\ln\frac{1}{|y-z|}w^2(z-q_j) - \ln\frac{1}{|y-z|}\frac{\partial w^2(z-q_j)}{\partial z_i}dz \tag{6.7}$$

$$+ \int_{\mathcal{M}_\varepsilon} \frac{\partial \tilde{H}_\sigma}{\partial y_i}(y,z)w^2(y-q_j) - \tilde{H}_\sigma(y,z)\frac{\partial w^2(z-q_j)}{\partial z_i}dz \tag{6.8}$$

$$+ \sum_{l\neq j}\int_{\mathcal{M}_\varepsilon} \frac{\partial \tilde{G}_\sigma}{\partial y_i}(y,z)w^2(z-q_l)dz + \text{h.o.t.}\Big]. \tag{6.9}$$

Using the fact that $(\frac{\partial}{\partial y} + \frac{\partial}{\partial z})\log|y-z| = 0$ for $y \neq z$, we have

$$\frac{1}{\xi_\varepsilon}\frac{\partial v_\varepsilon}{\partial y_i}(y) - \psi_{\varepsilon,j,i}(y) = \xi_\varepsilon \frac{\partial F_j(y)}{\partial y_i}(\int_{\mathbb{R}^2} w^2 dz + O(\sigma))$$

where

$$F_j(y) = \tilde{H}_\sigma(y,q_j) + \sum_{\ell\neq j}\tilde{G}_\sigma(y,q_\ell).$$

From this estimate, using the fact that $\frac{\partial F_j(q_j)}{\partial y_i} = \frac{1}{2}\frac{\partial F(\mathbf{q})}{\partial q_{j,i}}$, we have

$$\begin{aligned}\mathcal{I}_{11} &= \sum_{j=1}^k\sum_{i=1}^2 \frac{a_{j,i}^\varepsilon}{\xi_\varepsilon}\frac{u_\varepsilon^2}{v_\varepsilon^2}[\frac{\partial v_\varepsilon}{\partial y_i} - \xi_\varepsilon\psi_{\varepsilon,j,i}] \\ &= \sum_{j=1}^k\sum_{i=1}^2 a_{j,i}^\varepsilon \xi_\varepsilon \frac{\partial F_j(q_j)}{\partial y_i}(\int_{\mathbb{R}^2} w^2 dz)(1+O(\sigma|y-q_j|))\frac{u_\varepsilon^2}{v_\varepsilon^2}\end{aligned}$$

$$
\begin{aligned}
&= O\left(\xi_\varepsilon \frac{\partial F(\mathbf{q})}{\partial q_{j,i}}\right) \sum_{j=1}^{k} \sum_{i=1}^{2} |a^\varepsilon_{j,i}| \\
&= O\left(\frac{\varepsilon^4 R_\sigma}{\sigma}\right).
\end{aligned} \tag{6.10}
$$

Combining (6.3) and (6.10), one has $\|\mathcal{I}_1\|_{L^2(\mathcal{M}_\varepsilon)} \le C\varepsilon^2$. So

$$
\|\phi_\varepsilon^\perp\|_{H^2(\mathcal{M}_\varepsilon)} \le C\varepsilon^2 \sum_{j=1}^{k} \sum_{i=1}^{2} |a^\varepsilon_{j,i}|. \tag{6.11}
$$

Using the equation satisfied by $\psi_\varepsilon^\perp$,

$$
\|\psi_\varepsilon^\perp\|_{H^2(\mathcal{M}_\varepsilon)} \le C\varepsilon^2 \sum_{j=1}^{k} \sum_{i=1}^{2} |a^\varepsilon_{j,i}|. \tag{6.12}
$$

6.2. Derivation of the finite-dimensional eigenvalue problem

Multiplying (6.2) by $\frac{1}{\xi_\varepsilon}\frac{\partial u_{\varepsilon,m}}{\partial y_\ell}$ and integrating over $\mathcal{M}_\varepsilon$, one has

$$
\begin{aligned}
r.h.s &= \lambda \sum_{j=1}^{k} \sum_{i=1}^{2} \frac{a^\varepsilon_{j,i}}{\xi_\varepsilon^2} \int_{\mathcal{M}_\varepsilon} \frac{\partial u_{\varepsilon,j}}{\partial y_i} \frac{\partial u_{\varepsilon,m}}{\partial y_\ell} dy \\
&= \lambda \sum_{j=1}^{k} \sum_{i=1}^{2} a^\varepsilon_{j,i} \delta_{j,m} \delta_{i,\ell} \int_{\mathbb{R}^2} \left(\frac{\partial w}{\partial y_1}\right)^2 dy + o(1) \\
&= \lambda a^\varepsilon_{m,\ell} \int_{\mathbb{R}^2} \left(\frac{\partial w}{\partial y_1}\right)^2 dy + o(1).
\end{aligned} \tag{6.13}
$$

For the l.h.s., we get

$$
\begin{aligned}
\int_{\mathcal{M}_\varepsilon} \mathcal{I}_2 \xi_\varepsilon^{-1} \frac{\partial u_{\varepsilon,m}}{\partial y_\ell} dy &= \int_{\mathcal{M}_\varepsilon} \xi_\varepsilon^{-1} [\Delta\phi_\varepsilon^\perp - \phi_\varepsilon^\perp + \frac{2u_\varepsilon}{v_\varepsilon}\phi_\varepsilon^\perp - \lambda\phi_\varepsilon^\perp - \frac{u_\varepsilon^2}{v_\varepsilon^2}\psi_\varepsilon^\perp] \frac{\partial u_{\varepsilon,m}}{\partial y_\ell} dy \\
&= -\lambda \int_{\mathcal{M}_\varepsilon} \xi_\varepsilon^{-1} \phi_\varepsilon^\perp \frac{\partial u_{\varepsilon,m}}{\partial y_\ell} dy \\
&\quad + \int_{\mathcal{M}_\varepsilon} \frac{u_{\varepsilon,m}^2}{v_\varepsilon^2} \xi_\varepsilon^{-1} \Big(\frac{\partial v_\varepsilon}{\partial y_\ell}\phi_\varepsilon^\perp - \frac{\partial u_{\varepsilon,m}}{\partial y_\ell}\psi_\varepsilon^\perp\Big) dy + O(\frac{\varepsilon^6 R_\sigma}{\sigma}) \\
&= -\lambda \int_{\mathcal{M}_\varepsilon} \xi_\varepsilon^{-1} \phi_\varepsilon^\perp \frac{\partial u_{\varepsilon,m}}{\partial y_\ell} dy
\end{aligned}
$$

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$$
\begin{aligned}
&+\int_{\mathcal{M}_\varepsilon} \frac{u_{\varepsilon,m}^2}{v_\varepsilon^2}\xi_\varepsilon^{-1}\phi_\varepsilon^\perp\Big(\frac{\partial v_\varepsilon}{\partial y_\ell}(q_m+y)-\frac{\partial v_\varepsilon}{\partial y_\ell}(q_m)\Big)dy \\
&+\int_{\mathcal{M}_\varepsilon} \frac{u_{\varepsilon,m}^2}{v_\varepsilon^2}\xi_\varepsilon^{-1}\phi_\varepsilon^\perp\frac{\partial v_\varepsilon}{\partial y_\ell}(q_m)dy \\
&-\int_{\mathcal{M}_\varepsilon} \frac{u_{\varepsilon,m}^2}{v_\varepsilon^2}\xi_\varepsilon^{-1}\frac{\partial u_{\varepsilon,m}}{\partial y_l}(\psi_\varepsilon^\perp(q_m+y)-\psi_\varepsilon^\perp(q_m))dy \\
&-\int_{\mathcal{M}_\varepsilon} \frac{u_{\varepsilon,m}^2}{v_\varepsilon^2}\xi_\varepsilon^{-1}\frac{\partial u_{\varepsilon,m}}{\partial y_\ell}\psi_\varepsilon^\perp(q_m)dy+o(\varepsilon^4) \\
&= J_1+J_2+J_3+J_4+J_5+o(\varepsilon^4). \qquad (6.14)
\end{aligned}
$$

Using the equation for $\psi_\varepsilon^\perp$, one has

$$
\begin{aligned}
&\psi_\varepsilon^\perp(q_m)=\int_{\mathcal{M}_\varepsilon} G_{\sigma_\lambda}(y,z)(2\xi_\varepsilon^{-1}u_\varepsilon\phi_\varepsilon^\perp)(z)dz=O(\|\phi_\varepsilon^\perp\|_{H^2})=O(\varepsilon^2), \\
&\psi_\varepsilon^\perp(q_m+y)-\psi_\varepsilon^\perp(q_m)=\int_{\mathcal{M}_\varepsilon}[\tilde G_{\sigma_\lambda}(y+q_m,z)-\tilde G_{\sigma_\lambda}(q_m,z)]2\xi_\varepsilon^{-1}u_\varepsilon\phi_\varepsilon^\perp(z)dz \\
&=2\int_{\mathcal{M}_\varepsilon}\nabla_{q_m}\tilde G_{\sigma_\lambda}(q_m,z)\cdot y\xi_\varepsilon^{-1}u_\varepsilon\phi_\varepsilon^\perp dz \\
&=O(\varepsilon^2\frac{\varepsilon^4R_\sigma}{\sigma\xi_\varepsilon}|y|)=o(\varepsilon^4|y|).
\end{aligned} \qquad (6.15)
$$

Similarly, using the equation satisfied by v_ε, one has

$$
\begin{aligned}
&\frac{\partial v_\varepsilon}{\partial y_\ell}(q_m)=O\left(\xi_\varepsilon^2\frac{\partial F(\mathbf{q})}{\partial q_{m,\ell}}\right), \\
&\frac{\partial v_\varepsilon}{\partial y_\ell}(q_m+y)-\frac{\partial v_\varepsilon}{\partial y_\ell}(q_m)=O\left(\xi_\varepsilon^2\frac{\partial^2F(\mathbf{q})}{\partial q_{m,\ell}\partial q_{j,i}}|y|\right).
\end{aligned} \qquad (6.16)
$$

So using the definition of $\phi_\varepsilon^\perp$, one has $J_1=0$. Using (6.16),

$$
J_2+J_3=O\left(\varepsilon^2\frac{\varepsilon^4R_\sigma}{\sigma}\right)=o(\varepsilon^4),
$$

while using (6.15), one has

$$
J_4+J_5=o(\varepsilon^4).
$$

Combining all the above estimates, one has

$$\int_{\mathcal{M}_\varepsilon} \mathcal{I}_2 \xi_\varepsilon^{-1} \frac{\partial u_{\varepsilon,m}}{\partial y_{ell}} dy = o(\varepsilon^4). \tag{6.17}$$

Next recall the estimate for $\frac{1}{\xi_\varepsilon}\frac{\partial v_\varepsilon}{\partial y_i}(y) - \psi_{\varepsilon,j,i}$ in (6.9), we have

$$\begin{aligned}
&\int_{\mathcal{M}_\varepsilon} \mathcal{I}_{11} \xi_\varepsilon^{-1} \frac{\partial u_{\varepsilon,m}}{\partial y_\ell} dy \\
&= \sum_{j=1}^{k}\sum_{i=1}^{2} a_{j,i}^{\varepsilon} \xi_\varepsilon \left(\int_{\mathcal{M}_\varepsilon} \frac{\partial F_j(y)}{\partial y_i} \frac{u_\varepsilon^2}{v_\varepsilon^2} \frac{\partial w_j(y-q_m)}{\partial y_\ell} dy + \text{h.o.t.} \right) \int_{\mathbb{R}^2} w^2 dy \\
&= \sum_{j=1}^{k}\sum_{i=1}^{2} a_{j,i}^{\varepsilon} \xi_\varepsilon \frac{\partial F(\mathbf{q})}{\partial q_{j,i} \partial q_{m,\ell}} \delta_{i,\ell} \left(\int_{\mathbb{R}^2} w^2(y) y_i \frac{\partial w}{\partial y_i} dy + o(1) \right) \left(\int_{\mathbb{R}^2} w^2 dy + o(1) \right) \\
&= -c_2 \sum_{j=1}^{k}\sum_{i=1}^{2} a_{j,i}^{\varepsilon} \xi_\varepsilon \frac{\partial^2 F(\mathbf{q})}{\partial q_{j,i} \partial q_{m,\ell}} (\delta_{i,\ell} + o(1))
\end{aligned} \tag{6.18}$$

where c_2 is defined in (4.2).

For $\mathcal{I}_{12}$, we get

$$\begin{aligned}
&\int_{\mathcal{M}_\varepsilon} I_{12} \xi_\varepsilon^{-1} \frac{\partial u_{\varepsilon,m}}{\partial y_\ell} dy \\
&= \sum_{j=1}^{k}\sum_{i=1}^{2} a_{j,i}^{\varepsilon} \int_{\mathcal{M}_\varepsilon} [\Delta_{g_\varepsilon} \frac{\partial w}{\partial y_i}(y-q_j) - \frac{\partial}{\partial y_i} \Delta_{g_\varepsilon} w(y-q_j)] \frac{\partial w(y-q_m)}{\partial y_\ell} dy \delta_{m,j} \delta_{i,\ell} + o(1).
\end{aligned} \tag{6.19}$$

Consider the expansion of Δ_{g_ε} around each point q_j, i.e. replacing 0 by εq_j in (2.1), we have

$$\begin{aligned}
\frac{\partial}{\partial y_i} \Delta_{g_\varepsilon} w - \Delta_{g_\varepsilon} \frac{\partial w}{\partial y_i} &= \frac{1}{3} K(\varepsilon q_j) \varepsilon^2 \Big[\frac{\partial}{\partial y_i}(Q[w] - 2P[w]) - (Q[\partial_i w] - 2P[\partial_i w]) \Big] \\
&+ \frac{1}{6} (\nabla K(\varepsilon q_j) \cdot y) \varepsilon^3 \Big[\frac{\partial}{\partial y_i}(Q[w] - 2P[w]) - (Q[\partial_i w] - 2P[\partial_i w]) \Big] \\
&+ \frac{1}{6} \frac{\partial}{\partial y_i} (\nabla K(\varepsilon q_j) \cdot y) \varepsilon^3 (Q[w] - 2P[w]) \\
&+ \frac{1}{20} (y \nabla^2 K(\varepsilon q_j) y^t) \varepsilon^4 \Big[\frac{\partial}{\partial y_i}(Q[w] - 2P[w]) - (Q[\partial_i w] - 2P[\partial_i w]) \Big] \\
&+ \frac{1}{20} \frac{\partial}{\partial y_i} (y \nabla^2 K(\varepsilon q_j) y^t) \varepsilon^4 (Q[w] - 2P[w])
\end{aligned}$$

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$$+\frac{1}{45}K^2(\varepsilon q_j)|y|^2\varepsilon^4\Big[\frac{\partial}{\partial y_i}(3Q[w]-4P[w])-(3Q[\partial_i w]-4P[\partial_i w])\Big]$$
$$+\frac{2}{45}K^2(\varepsilon q_j)y_i\varepsilon^4(3Q[w]-4P[w])$$
$$+\frac{1}{6}\varepsilon^3\Big[\frac{\partial}{\partial y_i}R_1[w]-R_1[\partial_i w]\Big]$$
$$+\frac{1}{10}\varepsilon^4[\frac{\partial}{\partial y_i}R_2[w]-R_2[\partial_i w]]+o(\varepsilon^4).$$

Using Lemma A.1, one has

$$\begin{aligned}\int_{\mathbb{R}^2}[\frac{\partial}{\partial y_i}\Delta_{g_\varepsilon}w-\Delta_{g_\varepsilon}\frac{\partial w}{\partial y_i}]\frac{\partial w}{\partial y_i}dy&=-\frac{1}{4}\frac{\partial^2 K}{\partial x_i^2}(\varepsilon q_j)\int_{\mathbb{R}^2}(w')^2y_i^2dy\\&=-\frac{\varepsilon^4}{4}\frac{\partial^2 K}{\partial x_i^2}(0)\int_{\mathbb{R}^2}(w')^2y_i^2dy(1+o(1))\\&=-c_1\varepsilon^4\frac{\partial^2 K}{\partial x_i^2}(0)(1+o(1)).\end{aligned}\tag{6.20}$$

Combining (6.18) and (6.20),

$$\int_{\mathcal{M}_\varepsilon}\mathcal{I}_1\xi_\varepsilon^{-1}\frac{\partial u_{\varepsilon,m}}{\partial y_\ell}dy=\sum_{j=1}^{k}\sum_{i=1}^{2}a_{j,i}^\varepsilon[-c_2\xi_\varepsilon\frac{\partial^2F(\mathbf{q})}{\partial q_{j,i}\partial q_{m,\ell}}+c_1\varepsilon^4\frac{\partial^2K}{\partial x_i^2}(0)\delta_{j,m}](\delta_{i,\ell}+o(1)).\tag{6.21}$$

So one has

$$l.h.s=\sum_{j=1}^{k}\sum_{i=1}^{2}a_{j,i}^\varepsilon[-c_2\xi_\varepsilon\frac{\partial F(\mathbf{q})}{\partial q_{j,i}\partial q_{m,\ell}}\delta_{i,\ell}+c_1\varepsilon^4\frac{\partial^2K}{\partial x_i^2}(0)\delta_{i,\ell}\delta_{j,m}+o(1)].\tag{6.22}$$

Combining the l.h.s. and r.h.s.,

$$\begin{aligned}&\sum_{j=1}^{k}\sum_{i=1}^{2}a_{j,i}^\varepsilon[-c_2\xi_\varepsilon\frac{\partial^2F(\mathbf{q})}{\partial q_{j,i}\partial q_{m,\ell}}\delta_{i,\ell}+c_1\varepsilon^4\frac{\partial^2K}{\partial x_i^2}(0)\delta_{j,m}\delta_{i,\ell}]+o(\varepsilon^4)\\&=\lambda a_{m,\ell}^\varepsilon(\int_{\mathbb{R}^2}\Big(\frac{\partial w}{\partial y_1}\Big)^2dy+o(1))\end{aligned}\tag{6.23}$$

Finally, for $k=2$ we solve the finite-dimensional eigenvalue problem and complete the proof of Theorem 1.4.

Proof of Theorem 1.4. Equation (6.23) shows that the small eigenvalues λ_ε of (5.1) are given by

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$$\lambda_\varepsilon \sim \sigma_0\Big(-c_2\xi_\varepsilon \frac{\partial^2 F(\mathbf{q})}{\partial q_{j,i}\partial q_{m,\ell}}\delta_{i,\ell} + c_1\varepsilon^4 \frac{\partial^2 K}{\partial x_i^2}(0)\delta_{j,m}\delta_{i,\ell}\Big)_{j,m=1,\cdots,k,i,\ell=1,2} \sim c_1\varepsilon^4 R\sigma_0(\mathbf{M}) \tag{6.24}$$

where $\mathbf{M}$ is given in (4.7). From the expression of $\mathbf{M}$, we know that if $\theta_1 = 0$, the eigenvalues are given by $\lambda_1 \sim \frac{K_{11}}{R_0}$, $\lambda_2 \sim (\frac{2\hat{d}}{d} + \frac{1}{R_0})K_{11}$, $\lambda_3 \sim \frac{K_{22}}{R_0}$, $\lambda_4 \sim \frac{1}{R_0}(K_{22} - K_{11})$; while when $\theta_1 = \frac{\pi}{2}$, the eigenvalues are given by $\lambda_1 \sim \frac{K_{22}}{R_0}$, $\lambda_2 \sim (\frac{2\hat{d}}{d} + \frac{1}{R_0})K_{22}$, $\lambda_3 \sim \frac{K_{11}}{R_0}$, $\lambda_4 \sim \frac{1}{R_0}(K_{11} - K_{22})$. So since $K_{11} \neq K_{22}$, it follows that one of the solutions is stable and the other one is unstable. □

7. Discussion

In this section we discuss the main results given in Theorems 1.1 and 1.2. We consider specific two-dimensional Riemannian manifolds without boundary. In particular let us choose the surface of a three-dimensional ellipsoid.

First we study the surface of a tri-axial ellipsoid with semi-axes $a_1 < a_2 < a_3$. There are two maximum points of the Gaussian curvature near each of which two different two-spike cluster solutions exist. The orientation of the stable cluster is towards the smaller principal curvature and the orientation of the unstable cluster is towards the larger principal curvature. There are also two saddle points of the Gaussian curvature for which a single two-spike cluster exists whose spikes are orientated in the direction in which the saddle point is a local maximum of the Gaussian curvature. These spike clusters are unstable. Finally, there are two minimum points of the Gaussian curvature near which no two-spike cluster exists.

Second we consider an American football for which the semi-axes are $a_1 = a_2 < a_3$. This surface has two maximum points of the Gaussian curvature. Near each of them multiple two-spike clusters exist. Since the manifold is invariant under rotation around the maximum points any orientation is possible. All of these two-spike clusters are stable. This result is not proved in the current paper but it will follow by adapting our analysis to the case of rotationally symmetric manifolds (which is simpler than the more general non-rotationally symmetric setting considered here), then the finite-dimensional problems for existence and stability can be handled as in [56]. Further, for the American football case there is also a minimum point of the Gaussian curvature near which no spike cluster exists.

The degenerate case of a point for which the two principal curvatures are the same but the manifold is not rotationally symmetric is more difficult to handle. Further expansions are required which will determine the existence and stability of two-spike cluster solutions near this point.

Spike clusters of more than two spikes have not been considered in this paper since higher-order expansions of the contributions from the local geometry of the manifold are required to determine the orientation of the cluster. We are currently investigating this problem.

Appendix A

In this appendix, we will give some useful identities and we will compute the eigenvalues of the matrix M.

A.1. Some identities

By direct calculation (following Appendix B of [38]), one has the following lemma:

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Lemma A.1. *If w is a radial function, then the following identities hold:*

$$\int_{\mathbb{R}^2} (Q[w]-2P[w])y_j \frac{\partial w}{\partial y_j} dy = -\int_{\mathbb{R}^2} (w')^2 y_i^2 dy = -\pi \int_0^\infty (w')^2 r^3 dr,$$

$$\int_{\mathbb{R}^2} (3Q[w]-4P[w])y_i \frac{\partial w}{\partial y_i} dy = -\int_{\mathbb{R}^2} (w'(r))^2 y_i^2 dy,$$

$$\int_{\mathbb{R}^2} [\frac{\partial}{\partial y_i} R_1[w] - R_1[\partial_i w]] \frac{\partial w}{\partial y_j} dy = 0,$$

$$\int_{\mathbb{R}^2} [\frac{\partial}{\partial y_i} R_2[w] - R_2[\partial_i w]] \frac{\partial w}{\partial y_i} dy = -\frac{3}{2}\frac{\partial^2 K}{\partial x_i^2}(\varepsilon q_j) \int_{\mathbb{R}^2} (w')^2 y_i^2 dy,$$

$$\frac{\partial}{\partial y_i}(Q[w]-2P[w]) - (Q[\partial_i w] - 2P[\partial_i w]) = 0.$$

A.2. Eigenvalues of the matrix M

Next we will compute the eigenvalues of the matrix $M = M_1 + \frac{1}{\hat{d}}(M_2 + C_1 M_3)$ given in Lemma 4.2. By direct calculation, the eigenvalues of A_1 are given by

$$\lambda_{1,l} = -2 + \varepsilon^{l-1} + \varepsilon^{(k-1)(l-1)} = -4\sin^2 \frac{(l-1)\pi}{k}$$

and the eigenvalues of A_2 by

$$\lambda_{2,l} = \varepsilon^{l-1} - \varepsilon^{(k-1)(l-1)} = 2i \sin \frac{2(l-1)\pi}{k}$$

for $l = 1, \cdots, k$. Denote the diagonal matrices of A_1 and A_2 by

$$D_1 = \text{diag}(\lambda_{1,1}, \cdots, \lambda_{1,k}) \text{ and } D_2 = \text{diag}(\lambda_{2,1}, \cdots, \lambda_{2,k}), \text{ respectively.}$$

Using the matrix P_k of eigenvectors for a $k \times k$ circulant matrix, we have

$$P^{-1}\left(M_1 + \frac{1}{\hat{d}}(M_2 + C_1 M_3)\right) P = \begin{pmatrix} P_k^{-1} & 0 \\ 0 & P_k^{-1} \end{pmatrix} \left(M_1 + \frac{1}{\hat{d}}(M_2 + C_1 M_3)\right) \begin{pmatrix} P_k & 0 \\ 0 & P_k \end{pmatrix}$$
$$= \begin{pmatrix} (D_1+4I)\sin^2\frac{\pi}{k} + \frac{1}{\hat{d}}(D_1\cos^2\frac{\pi}{k} + 4\sin^2\frac{\pi}{k} I + C_1 B_1) & D_2 \sin\frac{\pi}{k}\cos\frac{\pi}{k}(1-\frac{1}{\hat{d}}) + \frac{1}{\hat{d}} C_1 B_2 \\ -D_2 \sin\frac{\pi}{k}\cos\frac{\pi}{k}(1-\frac{1}{\hat{d}}) + \frac{1}{\hat{d}} C_1 B_2 & -D_1(\cos^2\frac{\pi}{k} + \frac{1}{\hat{d}}\sin^2\frac{\pi}{k}) + \frac{1}{\hat{d}} C_1 B_3 \end{pmatrix}.$$

Since the matrix $M_1 + \frac{1}{\hat{d}}(M_2 + C_1 M_3)$ is symmetric and its entries are all real numbers, its eigenvalues are also real and satisfy the equations

$$\Lambda_l^2 + b_l \Lambda_l + c_l = 0,$$

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where

$$b_l = \lambda_{1,l}\cos\frac{2\pi}{k}\left(1-\frac{1}{\hat d}\right) - 4\sin^2\frac{\pi}{k}\left(1+\frac{1}{\hat d}\right) - \frac{1}{\hat d}C_1$$
$$= -4\left(\sin^2\frac{(l-1)\pi}{k}\cos\frac{2\pi}{k} + \sin^2\frac{\pi}{k}\right) + \frac{4}{\hat d}\left(\sin^2\frac{(l-1)\pi}{k}\cos\frac{2\pi}{k} - \sin^2\frac{\pi}{k} - \frac{C_1}{4}\right)$$

and

$$\begin{aligned} c_l &= \lambda_{2,l}^2\left(1-\frac{1}{\hat d}\right)^2\sin^2\frac{\pi}{k}\cos^2\frac{\pi}{k} \\ &\quad -\lambda_{1,l}^2\left(\cos^2\frac{\pi}{k}+\frac{1}{\hat d}\sin^2\frac{\pi}{k}\right)\left(\sin^2\frac{\pi}{k}+\frac{1}{\hat d}\cos^2\frac{\pi}{k}\right) \\ &\quad -4\lambda_{1,l}\left(1+\frac{1}{\hat d}\right)\sin^2\frac{\pi}{k}\left(\cos^2\frac{\pi}{k}+\frac{1}{\hat d}\sin^2\frac{\pi}{k}\right) \\ &\quad +\frac{4}{\hat d}C_1\sin^2\frac{\pi}{k}\cos^2\frac{(l-1)\pi}{k} + \frac{4}{\hat d^2}C_1\left[-\sin^2\frac{(l-1)\pi}{k} + \sin^2\frac{\pi}{k}\left(1+\sin^2\frac{(l-1)\pi}{k}\right)\right] \\ &= \frac{16}{\hat d}\sin^2\frac{(l-1)\pi}{k}\left[\sin^2\frac{\pi}{k}\left(1+\cos^2\frac{\pi}{k}\right) - \sin^2\frac{(l-1)\pi}{k}\right] + \frac{4}{\hat d}C_1\sin^2\frac{\pi}{k}\cos^2\frac{(l-1)\pi}{k} \\ &\quad -\frac{16}{\hat d^2}\sin^2\frac{(l-1)\pi}{k}\sin^2\frac{\pi}{k}\cos\frac{2\pi}{k} \\ &\quad +\frac{4}{\hat d^2}C_1\left[-\sin^2\frac{(l-1)\pi}{k} + \sin^2\frac{\pi}{k}\left(1+\sin^2\frac{(l-1)\pi}{k}\right)\right]. \end{aligned}$$

For $k \geq 3$, we get $b_l \leq -\frac{8}{\hat d} < 0$. Denote the solutions by

$$\Lambda_{1,l} = -\frac{b_l}{2}\left(1-\sqrt{1-\frac{4c_l}{b_l^2}}\right) \text{ and } \Lambda_{2,l} = -\frac{b_l}{2}\left(1+\sqrt{1-\frac{4c_l}{b_l^2}}\right), \text{ respectively.}$$

For $k = 3, 5, 6, 7, \ldots$, we have

$$\Lambda_{1,1} = 0, \quad \Lambda_{2,1} = 4\left(1+\frac{1}{\hat d}\right)\sin^2\frac{\pi}{k} > 0$$

and for $l = 2, \ldots, k,\ i = 1, 2$ it follows that

$$|\Lambda_{l,i}| > \frac{c_4}{\hat d^2}$$

for some $c_4 > 0$.

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