SYMMETRIC SMOOTH
INTERPOLATION ON TRIANGLES

by

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Rational and polynomial 'smooth' interpolation schemes are derived which interpolate a function and its derivatives along the boundary of a triangle. The interpolation schemes are symmetric and affine invariant on the triangle and can be used to construct a piecewise defined function which is $C^N(\Omega)$ over a triangular subdivision of a polygonal region $\Omega$. 
1. **INTRODUCTION**

Smooth or blending function interpolation on a triangle involves the construction of a function which matches data defined on the boundary of the triangle. Examples of such interpolants are described in Barnhill, Birkhoff, and Gordon [1], where rational functions which interpolate the boundary data are given. In particular, a symmetric rational interpolant is described which requires certain compatibility of the boundary data derivatives, as is shown in Barnhill and Mansfield [4]. The problem of smooth polynomial interpolation on triangles is considered in Barnhill and Gregory [3] but the polynomial interpolants are not symmetric on all the sides of the triangle.

This paper presents a method of deriving symmetric functions which, for any positive integer $N$, interpolate a function $F \in C^N(\bar{T})$, and its derivatives of order $N$ and less, on the boundary $\partial T$ of the triangle $T$. The interpolation functions are invariant under affine transformation and have the symmetry that each side of the triangle is treated in the same way. Two types of interpolation function are introduced. One is a rational function which, other than $F \in C^N(\bar{T})$, does not require compatibility of the boundary data derivatives. The other is a polynomial function which does require certain compatibility of the boundary data derivatives, although these conditions can be removed by the addition of rational terms.

The interpolation functions can be used to define a piecewise approximation function which is $C^N(\Omega)$ over a
triangular subdivision of a polygonal region $\Omega$. Such approximation functions have applications to finite element analysis and computer aided geometric design. The functions can be used to blend together given space curves or finite dimensional schemes can be derived by suitable choice of the triangle boundary data. Further, interpolants can be constructed on triangles adjacent to the boundary of $\Omega$ which completely satisfy boundary data on $\Omega$, whilst being compatible with finite dimensional schemes on triangles in the interior of $\Omega$. Another application of the smooth interpolation schemes is that they can be used to define transformations of a triangle $T$ onto a region with curved boundaries.

2. THE GENERAL SMOOTH INTERPOLATION SCHEME

By affine invariance, it is sufficient to consider the triangle $T$ with vertices at $V_1 = (1,0)$, $V_2 = (0,1)$, and $V_3 = (0,0)$. The side opposite the vertex $V_k$ is denoted by $E_k$, and thus $E_1$ is the side $x = 0$, $E_2$ is the side $y = 0$, and $E_3$ is the side $z = 0$, where $z = 1-x-y$.

The smooth interpolation scheme makes use of the following finite dimensional interpolant: Let $L$ be the affine invariant polynomial interpolation Projector (idempotent linear operator) over the $3(N+1)^2$ dimensional set of polynomials which are of degree $2N+1$ along parallels to the three sides of $T$, where

\[
(2.1) \quad L F = \sum_{i,j \leq N} a_{i,j}(x,y) \left( -\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F(x,y) \bigg|_{(0,0)} + \sum_{i,j \leq N} \beta_{i,j}(x,y) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) F(x,y) \bigg|_{(0,1)} + \sum_{i,j \leq N} \gamma_{i,j}(x,y) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) F(x,y) \bigg|_{(0,0)}
\]
and $\alpha_{i,j}$, $\beta_{i,j}$, and $\gamma_{i,j}$ are the cardinal basis functions.

The case $N = 1$ is the tricubic polynomial interpolant of Birkhoff [5]. An explicit representation of the interpolant (2.1) is

\[
L F = x^{N+1} \sum_{i,j \leq N} z^{(i)} y^{(j)} \left( -\frac{\partial}{\partial x} j \left[ -\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right] \frac{F(x,y)}{x^{N+1}} \right)(1,0)
\]

\[
+y^{N+1} \sum_{i,j \leq N} x^{(i)} z^{(j)} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) j \frac{F(x,y)}{y^{N+1}} \right)(1,0)
\]

\[
+z^{N+1} \sum_{i,j \leq N} x^{(i)} y^{(j)} \left( \frac{\partial^{i+j}}{\partial x^i \partial y^j} \frac{F(x,y)}{z^{N+1}} \right)(0,0)
\]

Where $x^{(i)} = x^i / i!$ etc., see Gregory [6]. (Barnhill and Mansfield [4] show that $L$ is the product operator of the three Hermite projectors defined in Section 3 of this paper.) The first three cardinal functions of (2.1) have the properties that

\[
\alpha_{0,0}(x,y) + \beta_{0,0}(x,y) + \gamma_{0,0}(x,y) = 1
\]

\[
(D^\nu \alpha_{0,0})(E_1) = (D^\nu \beta_{0,0})(E_2) = (D^\nu \gamma_{0,0})(E_3) \text{ C , } |\nu| \leq N,
\]

where $\nu = (m,n)$, $D^\nu = \partial^{m+n}/\partial x^m \partial y^n$, and $(D^\nu \alpha_{0,0})(E_1)$ denotes

$D^\nu \alpha_{0,0}(x,y)$ evaluated on the side $E_1$, etc. The property (2.3) follows from the precision of (2.1) for $F = 1$ and (2.4) follows since the basis functions have factors $x^{N+1}$, $y^{N+1}$, and $z^{N+1}$ respectively, see (2.2).
Examples

(i) For $N = 0$

\[
\begin{align*}
\alpha_{0,0}(x, y) &= x, \\
\beta_{0,0}(x, y) &= y, \\
\gamma_{0,0}(x, y) &= z,
\end{align*}
\]

(2.5)

(ii) For $N = 1$

\[
\begin{align*}
\alpha_{0,0}(x, y) &= x^2 (3 - 2x + 6yz), \\
\beta_{0,0}(x, y) &= y^2 (3 - 2z + 6xy), \\
\gamma_{0,0}(x, y) &= z^2 (3 - 2z + 6xy),
\end{align*}
\]

(2.6)

The smooth interpolation scheme is defined in the following theorem.

**Theorem 2.1.** Let $P_k$, $k = 1, 2, 3$, be linear operators such that $P_k F$ is a function which interpolates $F \in C^N(T)$, and its derivatives of order $N$ and less, on the sides $E_i$ and $E_j$ of the triangle $T$ adjacent to the vertex $V_k$. Then the function

\[
P F = \alpha_{0,0}(x,y) P_1 F + \beta_{0,0}(x,y) P_2 F + \gamma_{0,0}(x,y) P_3 F
\]

interpolates $F$, and its derivatives of order $N$ and less, on the boundary $\partial T$ of the triangle $T$, where $P$ is a linear operator.

**Proof** It is sufficient to consider the side $E_1$, where, from (2.4),

\[
(D^\nu P F)(E_1) = (D^\nu [\beta_{0,0} P_2 F + \gamma_{0,0} P_3 F])(E_1), |\nu| \leq N.
\]
Now
\[(\nu^0 P^3 F)(E_1) = (\nu^0 P_3 F)(E_1) = (\nu^0 F)(E_1), \quad |\nu| \leq N,\]
and (2.3) and (2.4) give that
\[(\beta_{0,0} + \gamma_{0,0})(E_1) = 1 \quad \text{and} \quad D^\nu[\beta_{0,0} + \gamma_{0,0}](E_1) = 0, \quad |\nu| \leq N\]
Thus application of Leibnitt's rule in (2.8) gives the required result that
\[(\nu^0 PF)(E_1) = (\nu^0 F)(E_1), \quad |\nu| \leq N.\]

Remark 1. Let \(\tau\) be the intersection of the sets of polynomials for which each of the operators \(P_k\) \(k = 1, 2, 3\), ia exact. Then \(P\) is exact for at least the set \(\tau\) (the precision set). The proof of this result follows immediately from (2.3) and (2.7). The precision set is important in that it indicates the accuracy of the interpolation function.

Remark 2. If the operators \(P_k\), \(k = 1, 2, 3\), are affine invariant and symmetric on the sides of the triangle, then the operator \(P\) is affine invariant and symmetric.

Remark 3. If the \(P\) are linear operators such that \(P_k(PF) = P_kF\), \(k = 1, 2, 3\), then it follows that \(P\) is a projector, i.e. \(P\) is a linear operator and \(P(PF) = PF\). This property holds for the schemes discussed in Sections 3 and 4.

3. **A RATIONAL SMOOTH INTERPOLANT**

The Hermite two point Taylor projectors \(P_k\), \(k = 1, 2, 3\), along parallels to the sides \(E_k\) of the triangle \(T\), are appropriate operators for the interpolation scheme (2.7). Explicitly these
The projectors are defined by

(3.1) \[ P_1 F = \sum_{i \leq N} \phi_i \left( \frac{y}{1-x} \right) (1-x)^i F_{o,i} (x, o) + \sum_{i \leq N} \psi_i \left( \frac{y}{1-x} \right)(1-x)^i F_{o,i} (x, 1-x), \]

(3.2) \[ P_2 F = \sum_{i \leq N} \phi_i \left( \frac{x}{1-y} \right) (1-y)^i F_{o,i} (o, y) + \sum_{i \leq N} \psi_i \left( \frac{x}{1-y} \right)(1-y)^i F_{o,i} (1-y, y), \]

(3.3) \[ P_3 F = \sum_{i \leq N} \phi_i \left( \frac{x+y}{x+y} \right) (x+y)^i \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right)^i F(o, x+y) + \sum_{i \leq N} \phi_i \left( \frac{x+y}{x+y} \right) (x+y)^i \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right)^i F(x+y, o), \]

where the \( \Phi_i(t) \) and \( \Psi_i(t) = (-1)^i \Phi_i(1-t) \) are the cardinal basis functions for Hermite two-point Taylor interpolation on \([0,1]\).

Excluding the apparent singularity at the vertex \( V_k \), the functions \( P_k F, \ k = 1,2,3 \), are \( N \) times continuously differentiable on \( T \), provided that the boundary data in \( P F \) is \( I \) times continuously differentiable with respect to its single variables on the sides of \( T \). Also, Barnhill and Gregory [2] show that a sufficient condition that \( P_k F \) is \( N \) times continuously differentiable as the limit to the vertex \( V_k \) is approached from \( T \), is that the functions \( F \) in \( P_k F \) be \( N+1 \) times continuously differentiable. These are sufficient conditions that (2.7) can define a piecewise interpolation function which is \( C^N(\Omega) \) over a triangular subdivision of a polygonal region \( \Omega \). These conditions are satisfied if the boundary data on the triangle is polynomial.

From Remark 1, Section 2, and the definition of the projectors \( P_k \), it follows that the interpolation scheme (2.7) is exact for \( \tau_{2N+1} \) the set of polynomials which are of degree \( 2N+1 \) along parallels to the three sides of \( T \).
Example

(i) For \( N = 0 \),

\[
\text{PF} = x\left[\frac{Z}{1-x} F(x,0) + \frac{Y}{1-x} F(x,i-x) \right] \\
+ y\left[\frac{Z}{1-y} F(o,y) + \frac{X}{1-y} F(1-y,y) \right] \\
+ z\left[\frac{Z}{1-z} F(o,x+y) + \frac{X}{1-z} F(x+y,0) \right]
\]

with precision set \( J_i \).

(ii) For \( N = 1 \)

\[
\text{PF} = z^2(3 - 2z + 6xy) \left[ \varphi_0 \left( \frac{x}{x+y} \right) F(o,x+y) + \varphi_1 \left( \frac{x}{x+y} \right)(x+y)(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})F(o,x,y) \\
+ \psi_0 \left( \frac{x}{x+y} \right) F(x+y,0) + \psi_1 \left( \frac{x}{x+y} \right)(x+y)(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})F(x+y,0) \right] \\
+ \text{dual terms in } P_1F \text{ and } P_2F,
\]

where

\[
\begin{align*}
\varphi_0 \ (t) &= (t=1)^2 (2t+1) , \\
\varphi_1 \ (t) &= (t-1)^2 t , \\
\psi_0 \ (t) &= t^2 (-2t+3) , \\
\psi_1 \ (t) &= t^2 (t-1) .
\end{align*}
\]

The precision set is \( \tau_3 \).

4. **A POLYNOMIAL SMOOTH INTERPOLANT**

The polynomial Taylor projectors \( T_i^j \) on the sides \( E_j \) of the triangle, along parallels to the sides \( E_i \), are defined by
\[
\begin{align*}
T_1^2 F &= \sum_{i \leq N} y^{(i)} F_{0,i} (x,0) , \quad T_1^1 F = \sum_{i \leq N} (x+y-1)^{(i)} F_{0,1}(x,1-x) \\
T_2^1 F &= \sum_{i \leq N} x^{(i)} F_{i,0} (0,y) , \quad T_2^2 F = \sum_{i \leq N} (x+y-1)^{(i)} F_{0,1}(1-y-y) \\
T_3^1 F &= \sum_{i \leq N} x^{(i)} \left( \left[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right]^i F \right) (0,x,y) , \quad T_3^2 F = \sum_{i \leq N} y^{(i)} \left( \left[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right]^i F \right) (x+y,0)
\end{align*}
\]

Where \( x^{(i)} = x^i / i! \) etc. Polynomial projectors \( P_k, \ k = 1,2,3, \) appropriate for the interpolation scheme (2.7), are defined in the following theorem.

**Theorem 4.1.** Let \( F \in C^N(\bar{T}) \) satisfy the compatibility condition

\[
\left( \frac{\partial^m+n}{\partial^n_i \partial^m_j} \right) (V_k) = \left( \frac{\partial_{n+m}}{\partial^n_i \partial^m_j} \right) (V_k) , \ m,n \leq N ; \ m+n > N ,
\]

at the vertex \( V_k \), with adjacent sides \( E_i \) and \( E_j \) where \( \partial / \partial \ell \)
denotes differentiation along the side \( E_i \). Then the polynomial

Boolean sum function

\[
P_{k,F} = (T_1^1 \oplus T_2^2) F
\]

interpolates \( F \in C^N(\bar{T}) \), and its derivatives of order \( N \) and
less, on the sides \( E_i \) and \( E_j \) of the triangle \( T \).

**Proof** By affine transformation it is sufficient to consider
the case

\[
P_{3,F} = (T_1^1 \oplus T_2^2) F
\]

\[
= \sum_{i \leq N} x^{(i)} F_{i,0} (0,y) + \sum_{j \leq N} y^{(j)} F_{0,j} (x,0)
\]

\[
+ \sum_{i,j \leq N} x^{(i)} y^{(j)} \left( \frac{\partial^{i+j}}{\partial x^i \partial y^j} \right) (0,0)
\]
Since $D^{\nu}(I - T_{12}^1)$ is null on $E_1$ for all $|\nu| \leq N$, it follows that

$$D^{\nu}F - D^{\nu}P_3F = D^{\nu}(I - T_{12}^1)(I - T_{21}^2)F$$

is zero on $E_{1i}$. Also, the compatibility condition (4.2), and $F \in C^N(\overline{T})$ imply that $T_{12}^1$ and $T_{21}^2$ are commutative and hence the dual result holds on $E_2$.

The compatibility condition can be removed by the addition of a suitable rational term to the projector, as is shown in [3]. Let $T_k$, $k = 1,2,3$, be the rational Hermits projectors defined by equations (3.1) - (3.3) respectively. Then

$$T_k[F - (T_{ij}^i \oplus T_{ij}^j)F]$$

is a rational function which interpolates the remainder function $F - (T_{ij}^i \oplus T_{ij}^j)F$ on $E_i$ and $E_j$. Hence

(4.5) $$P_kF = (T_{ij}^i \oplus T_{ij}^j)F + T_k[F - (T_{ij}^i \oplus T_{ij}^j)F]$$

Interpolates $F \in C^N(\overline{T})$ on the sides $E_i$ and $E_j$ of the triangle $T$ adjacent to the vertex $V_k$. Further, the symmetry of the interpolation scheme (2.7) is retained if the projectors $P_k$ are defined by the average

(4.6) $$P_kF = \frac{1}{2}(T_{ij}^i \oplus T_{ij}^j)F + \frac{1}{2}(T_{ij}^i \oplus T_{ij}^j)F + T_k[F - \frac{1}{2}(T_{ij}^i \oplus T_{ij}^j)F - \frac{1}{2}(T_{ij}^i \oplus T_{ij}^j)F].$$

If $F$ satisfies (4.2) at each vertex, the rational terms are zero and the projector reduces to (4.3).

The projector $P_k$ is precise for the union of the precision sets of $T_{ij}^i$ and $T_{ij}^j$, namely
\[(4.7) \quad \begin{align*}
\xi^m_i \xi^n_j & \quad \begin{cases}
0 \leq m \leq N \text{ for all } n, \\
0 \leq n \leq N \text{ for all } m,
\end{cases}
\end{align*}\]

where \( \xi_1 = x \), \( \xi_2 = y \), and \( \xi_3 = z \). Thus, from Remark 1, Section 2, the interpolation scheme (2.7) is exact for \( P_{2N+1} \), the set of polynomials which are of degree \( 2N+1 \) or less.

Examples

(i) For \( N = 0 \),

\[(4.8) \quad PF = x[F(1-y,y) + F(x+y,0) - F(1,0)] + y[F(x,1-x) + F(0,x+y) - F(0,1)] + z[F(0,y) + F(x,0) - F(0,0)] \]

with precision set \( P_1 \).

(ii) For \( N = 1 \),

\[(4.9) \quad PF = \begin{aligned}
z^2(3-2z+6xy)[F(0,y) + xf_{1,0}(0,y) + F(x,0) - F(0,0)]
- xF_{1,0}(0,0) - yF_{0,1}(0,0) - \frac{xy}{2} \left\{ \frac{\partial^2 F}{\partial x \partial y}(0,0) + (\frac{\partial^2 F}{\partial y e^x})(0,0) \right\} \\
+ \frac{xy(x-y)}{2(x+y)} \left\{ (\frac{\partial^2 F}{\partial x \partial y})(0,0) - (\frac{\partial^2 F}{\partial y e^x})(0,0) \right\}
\end{aligned} + \text{ dual terms in } P_1 F \text{ and } P_2 F ,
\]

where the rational term is the compatibility correction.

The precision set is \( P_3 \).
REFERENCES


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