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VARIATIONAL INEQUALITIES AND APPROXIMATION

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ABSTRACT

The existence and uniqueness of the solution of a variational inequality is considered, and methods of approximation of the solution are given.

Some elementary theorems concerning bilinear forms and antimonotone operators are given in the appendix.

Let H be a real Hilbert Space with its dual H' , whose inner product and norm are denoted by $((\cdot))$ and $\|\cdot\|$ respectively. The pairing between $f \in H'$ and $u \in H$ is denoted by (f, u) . Let F' be the Frechet differential of a nonlinear functional F on a closed convex set M in H.

Consider also a coercive continuous bilinear form a(u,v) on H, i,e. there exists constants $\alpha > 0$, $\beta > 0$ such that

$$a(v,v) \geq \alpha \| v \|^2 \qquad \text{for all } v \in H, \tag{1}$$

$$| a(u,v)| \leq \beta \| u \| \|v\| \qquad \text{for all } u,v \in H. \tag{2}$$

Furthermore let F be a given element of H'. We now consider a functional I[v] defined by

$$I[v] = a(v,v) - 2F(v) \qquad \text{for all } v \in H.$$

Many mathematical problems either arise or can be formulated in this form. Here one seeks to minimize the functional I[v] over a whole space H or on a convex set M in H. It is well—known [1] that if F is a linear functional, then the element u which minimizes I [v] on M is given by

$$a(u,v-u) \geq (F,v-u) \qquad \text{for all } v \in M. \tag{3}$$

For a nonlinear Frechet differentiable functional F, it was shown [3] that the minimum of the functional I[v] on M is given by $u \in M$ such that

$$a(u,v-u) \geq (F'(u),v-u) \qquad \text{for all } v \in M. \tag{4}$$

Such type of inequalities are known as variational inequalities [1]. Lions-Stampacchia. [1] have studied the existence of a unique solution of (3). The motivation for this report is to show that under certain conditions there does exist a unique solution of a more general variational inequality of which (4) is a special case.

Let us consider the following problem.

PROBLEM 1

Find $u \in M$ such that

$$a(u,v-u) \geq (Au,v-u) \qquad \text{for all } v \in M, \qquad (5)$$

where A is a nonlinear operator such than $Au \in H'$.

For $M = H$, the inequality (5) is equivalent to finding $u \in H$ such that

$$a(u,v) = (Au,v) \qquad \text{for all } v \in H,$$

and thus our results include the Lax-Milgram lemma as a special case.

Definition

The operator $T : M \rightarrow H'$ is called antimonotone, if

$$(Tu-Tv, u-v) \leq 0 \qquad \text{for all } u,v \in M,$$

and is said to be hemicontinuous [4], if for all $u,v \in M$, the mapping $t \in [0,1]$ implies that $(T(u+t(v-u)),u-v)$ is continuous. Furthermore, T is Lipschitz continuous, if there exists a constant $0 < \Upsilon \leq 1$ such that

$$||Tu-Tv|| \leq \Upsilon \, ||u-v|| \qquad \text{for all } u,v \in M.$$

Theorem 1.

Let $a(u,v)$ be a coercive continuous bilinear form and M a closed convex subset in H. If A is a Lipschitz continuous antimonotone operator with $\Upsilon < \alpha$, then there exists a unique $u \in M$ such that (5) holds.

The following lemmas are needed for the proof.

Lemma 1 .

If A is an antimonotone hemicontinuous operator, then $u \in M$ is a solution of (5) if and only if u satisfies

$$a(u,v-u) \geq (Av,v-u) \qquad \text{for all } v \in M \tag{6}$$

Proof

If for a given u in M, (5) holds, then (6) follows by the antimonotonicity of A.

Conversely, suppose (6) holds, then for all $t \in [0,1]$ and $w \in M$, $v_t \equiv u + t(w-u) \in M$, since M is a convex set. Setting $v = v_t$ in (6) , we have

$$a(u,w-u) \geq (Av_t ,w-u) \qquad \text{for all } w \in M .$$

Now let $t \to o$. Since A is hemicontinuous , $Av_t \to Au$. It follows that

$$a(u,w-u) \geq (Au,w-u) \qquad \text{for all } w \in M.$$

The map $v \to a(u,v)$ is linear continuous on H, so by Reisz-Frechet theorem, there exists an element $\eta = Tu \in H'$ such that

$$a(u,v) = (Tu,v) \qquad \text{for all } v \in H \tag{7}$$

Let $\wedge$ be a canonical isomorphism from H’ onto H defined by

$$(f,v) = ((\wedge f,v)) \qquad \text{for all } v\in H,\ f\in H' \tag{8}$$

Then $\|\wedge\|_{H'} = \|\wedge^{-1}\|_H = 1$. We note first that by (1),(2) and

(7), it follows that

(i) $\|T\| \le \beta$

(ii) $\alpha \le \beta$

The next lemma is a generalization of a lemma of Lions-Stampacchia [1].

Lemma 2

Let ζ be a number such that $0 < \zeta < \dfrac{2(\alpha-\gamma)}{\beta^2-\gamma^2}$ and $\zeta < \dfrac{1}{\gamma}$

Then there exists a θ with $0 < \theta < 1$ such that

$$\|\phi(\mathbf{u_1})-\phi(\mathbf{u_2})\| \le \|\mathbf{u_1}-\mathbf{u_2}\| \ \textbf{for all } \mathbf{u_1},\mathbf{u_2} \in \mathbf{H},$$

where for $u\in H$, $\phi(u) \in H'$ is defined by

$$(\phi(u),v) = ((u,v)) - \zeta\, a(u,v) + \zeta(Au,v) \text{ for all } v\in H. \tag{9}$$

Proof:

For all $u_1,u_2\in H$.

$$\begin{aligned}(\phi(u_1) - \phi(u_2),v) &= ((u_1-u_2,v)) - \zeta\, a(u_1-u_2,v) + \zeta(Au_1-Au_2,v) \text{ for all } v\in H\\ &= ((u_1-u_2,v)) - \zeta(T(u_1-u_2),v) + \zeta(Au_1-Au_2,v), \text{ by (7)}\\ &= ((u_1-u_2,v)) - \zeta((\wedge T(u_1-u_2),v)) + \zeta((\wedge Au_1-\wedge Au_2,v)), \text{ by (8)}\\ &= ((u_1-u_2, -\zeta T(u_1-u_2),v)) + \zeta((Au_1-Au_2,v))\end{aligned}$$

Thus

$$|\phi((u_1)-\phi(u_2),v)| \le \|u_1-u_2-\zeta AT(u_1-u_2)\|\,\|v\| + \zeta\,\|Au_1-Au_2\|\,|\,|v|\,| \quad \text{for all } v\in H.$$

Now using (7) and (8) we have

$$\|u_1-u_2-\zeta\wedge T(u_1-u_2)\|^2 \le \|u_1-u_2\|^2 + \zeta^2 \|T\|^2 \|u_1-u_2\|^2 - 2\zeta\, a(u_1-u_2, u_1-u_2)$$

$$\le (1+\zeta^2 \beta^2 - 2\zeta\alpha)\|u_1-u_2\|^2, \text{ by coercivity of } a(u,v).$$

Then

$$|(\phi(u_1)-\phi(u_2),v)| \le \sqrt{(1+\zeta^2\beta^2-2\zeta\alpha)}\,\|u_1-u_2\|\,\|v\| + \zeta\|Au_1-Au_2\|\,\|v\| \text{ for all } v\in H.$$

$$\le \theta\|u_1-u_2\|\,\|v\|, \text{ by the Lipschitz coutinuity of } A,$$

and $\theta = \sqrt{1+\zeta^2\beta^2-2\zeta\alpha} + \gamma\zeta < 1$ for $0<\zeta<2\dfrac{\alpha-\gamma}{\beta^2-\gamma^2}$ and $\zeta < \dfrac{1}{\gamma}$, because $\alpha > \gamma$.

Hence for all $u_1, u_2 \in H$

$$\|\phi(u_1)-\phi(u_2)\| = \sup_{v\in H} \frac{|(\phi(u_1)-\phi(u_2),v)|}{\|v\|} \le \theta\|u_1-u_2\|.$$

The following results are proved by Mosco [2].

<u>Lemma 3</u>

Let M be a convex subset of H. Then, given z H we have

$$x = P_M z,$$

if and only if

$$x\ M;\quad ((x-z,y-x)) \ge 0 \quad \text{for all } y\in M.$$

where P_M is the projection of H in M.

Lemma 4.

P_M is non-expansive, i.e.,

$$\|P_M Z_1 - P_M Z_2\| \le \|Z_1-Z_2\| \qquad \text{for all } z_1,z_2 \in H.$$

Using the technique of Lions-Stampacchia [1] , we now prove theorem 1,

Proof of theorem 1,

(a) Uniqueness

Let u_i , i=1,2 be solutions in M of

$$a(u_i, v-u_i.) \geq (Au_i-v-u_i) \qquad \text{for all } v \in M.$$

Setting $v = u_{3-i}$, i = 1 , 2 in the above inequality, by addition we have

$$a(u_1-u_2,u_1-u_2) \leq (Au_1-Au_2,u_1-u_2) .$$

Since a(u,v) is a coercive bilinear form, there exists a constant

a > 0 such that

$$\alpha \| u_1-u_2 \|^2 \leq (Au_1-Au_2,u_1-u_2) \leq 0,$$

by the antimonotonicity of A. From which the uniqueness of the solution $u \in M$ follows.

(b) Existence

For a fixed ζ as in Lemma 2, and u H, define $\phi(u) \in H'$ by (9). By lemma 3, there exists a unique $w \in M$ such that

$$((w,v-w)) \geq (\phi (u),v-w) \qquad \text{for all } v \in M,$$

and w is given by

$$w = P_M \wedge \phi (u) = Tu,$$

which defines a map from H into M.

Now for all u.,u H,

$$\| Tu_1 - Tu_2 \| = \| P_M \wedge \phi(u_1) - P_M \wedge \phi(u_2) \|,$$

$$\leq \| \wedge \phi(u_1) - \wedge \phi(u_2) \|, \text{ by lemma 4,}$$

$$\leq \| \phi(\upsilon_1) - \phi(u_2) \|,$$

$$\leq \theta \| u_1 - u_2 \|, \text{ by lemma 2.}$$

Since $\theta < 1$. Tu is a contraction and has a fixed point u = Tu, which belongs to M, a closed convex set and satisfies

$$((u,v-u)) > (\phi(u),v-u) = ((u,v-u)) - \zeta [a(u,v-u) - (Au,v-u)]$$

Thus for $\zeta > 0$,

$$a(u,v-u) \geq (Au,v-u) \qquad \text{for all } v \in M$$

showing that u is a unique solution of problem 1.

<u>Remarks</u>

1; It is obvious that for Au = F' (u), the existence of a unique solution of a variational inequality (4) follows under the assumptions of theorem. 1 .

2: If A is independent of u, that is Au = A' (say), then the Lipschitz constant γ y is zero, and lemma 2 reduces to a lemma of

Lions-Stampacchia [1] and ζ is a number such that $0 < \zeta < \frac{2\alpha}{\beta^2}$.

Consequently theorem 1 is exactly the same as one proved by

Lions-Stampacchia for the linear case. It is obvious that our result

not only generalizes their result, but also includes it as a special case.

Method of Approximation

Suppose that the bilinear form is non-negative, i.e.

$$a(v,v) \geq 0 \qquad \text{for all } v\in H. \tag{10}$$

Assume that there exists at least one solution $u \in M$ of

$$a(u,v-u) \geq (Au,v-u) \qquad \text{for all } v\in M \tag{11}$$

and X is the set of all solutions of (11). Let, finally, $b(u,v)$ be a coercive bilinear form on H, that is there exists a constant $\alpha > 0$ such that

$$b(v,v) \geq \alpha \, ||v|| \qquad \text{for all } v\in H \tag{12}$$

First of all we prove some elementary but important lemmas.

Lemma 5

If $a(u,v)$ is a non-negative bilinear form and $u\in M$, then the inequality (5) is equivalent to the inequality

$$a(v,v-u) \geq (A(u),v-u) \qquad \text{for all } v\in M. \tag{13}$$

Proof

Let (5) hold, then

$$a(v,v-u) \geq (A(u),v-u) + a(v-u,v-u) \geq (A(u),v-u), \text{ by (10).}$$

Thus (13) holds;

Conversely let (13) hold, then for all $t \in [0,1]$ and $w\in M$, $v_t = u+ t(w-u) \in M$. Setting $v=v_t$ in (13) it follows that

$$a(u,w-u) + t\, a(w-u,w-u) \geq (A(u),w-u), \text{ for all } w\in M.$$

Letting $t \to 0$, (5) follows.

As a consequence of lemma 1 and lemma 5 we have the following result.

Lemma 6.

If $a(u,v)$ is non-negative bilinear form and A is hemicontinuous antimonotone operator, then the inequality (5) is equivalent to

$$a(v,v-u) \geq (A(v),v-u) \quad \text{for all } v \in M.$$

Theorem 2

If $b(u,v)$ is a coercive continuous bilinear form and B is a Lipschitz continuous antimonotone operator with $\Upsilon < \alpha$ then there exists a unique solution $u_o \in X$ such that

$$b(u_o, v-u_o) \geq (Bu_o, v-u_o) \text{ for all } v \in x. \tag{14}$$

Proof:

Obviously X is closed. In order to prove theorem (2), it is enough to show that X is convex. Since $a(u,v)$ is non-negative, so (11) is equivalent to

$$a(v,v-u) \geq (Av,v-u) \text{ , by lemma 6 .}$$

Now for all $t \in [0,1]$, u_1 , $u_2 \in X$,

$$\begin{aligned} a(v,v-u_2-t(u_1-u_2)) &= a(v,v-u_2) - t\, a(v,u_1-u_2) \\ &= a(v,v-u_2) - t\, a(v,u_1 - v + v - u_2) \\ &= a(v,v-u_2) + t\, a(v,v-u_1) - t\, a(v,v-u_2) \\ &= (1-t)\, a(v,v-u_2) + t\, a(v,v-u_1) \\ &\geq (1-t)\, (Av,v-u_2) + t\, (Av,v-u_1), \end{aligned}$$

by lemma 6 .

Thus for all $t \in [0,1]$, $U_1, U_2 \in x$, $tu_1 + (l-t)u_2 \in x$, which implies that X is a convex set. Hence by theorem (1), there does exist a unique solution $u_o \in X$ satisfying (14).

Theorem 3

Assume that (10) and (12) hold. If $a(u,v) + \in b(u,v)$ is a continuous bilinear form and A, B are both antimonotone Lipschitz continuous with Y < α , then there exists a unique solution $u_\varepsilon \in M$ such that

$$a(u_\varepsilon, v-u_\varepsilon) + \varepsilon\, b(u_\varepsilon, v-u_\varepsilon) \geq (Au_\varepsilon - \varepsilon Bu_\varepsilon, v-u_\varepsilon) \qquad \text{for all } v \in M. \qquad (15)$$

Proof:

Since for $\in > 0$ and by (10), (12), the continuous bilinear form $a(u,v) + \varepsilon\, b(u,v)$ is coercive on H, then by theorem 1, there exists a unique $u_\varepsilon \in M$ satisfying (15).

Using lemma 1 and the methods of Sibony [4] and Lions-Stampacchia [1], we prove that the elements of X can be approximated.

Theorem 4

Suppose A,B:M $\rightarrow$ H' are both hemicontinuous operators and the assumptions of theorems (2) and (3) hold. If u_o is the element of X defined by (14) satisfying

$$a(u_o, v-u_o) \geq (Au_o, v-u_o) \qquad \text{for all } v \in X. \qquad (16)$$

and u_ε is the element of M defined by (15), then

$u_\varepsilon \rightarrow u_o$ strongly in H as $\varepsilon \rightarrow 0$.

Proof:

Ihis is proved in three steps.

(i) $u_£$ is bounded in H.

Setting $v = u_o$ in (15) and $v = u_\varepsilon$ in (16), we get

$$a(u_\varepsilon, u_o - u_\varepsilon) + \varepsilon b(u_\varepsilon, u_o - u_\varepsilon) \geq (Au_\varepsilon + \varepsilon Bu_\varepsilon, u_o - u_\varepsilon)$$

and

$$a(u_o, u-u_o) \geq (Au_o, u_\varepsilon - u_o)$$

By addition of these inequalities, it follows from (10) and the antimonotonicity of A that

$$b(u_\varepsilon, u_o - u_\varepsilon) \geq (Bu_\varepsilon, u_o - u_\varepsilon) \tag{17}$$

Since $b(u_\varepsilon, u_\varepsilon)$ is a coercive bilinear form, there exists a constant $a > 0$ such that

$$\alpha \, ||u_\varepsilon||^2 \leq b(u_\varepsilon, u_o) + (Bu_\varepsilon, u_\varepsilon - u_o) .$$

It follows that $|| u_\varepsilon || \leq$ constant, independent of ε. Hence there exists a subsequence u_ε which converges to ξ , say.

(ii) ξ belongs to X.

Since A and B are antimonotone operators, by (15) and the application of lemma 1, we get

$$a(u_\varepsilon, v - u_\varepsilon) + \varepsilon b(u_\varepsilon, v - u_\varepsilon) \geq (Av + \varepsilon Bv, v - u\varepsilon) \qquad \text{for all } v \in M.$$

Now let $\varepsilon \to 0$, then $u_\varepsilon \to \xi$ and $\liminf a(u_\varepsilon, u_\varepsilon) \geq a(\xi,\xi)$,[1] We have

$$a(\xi, v-\xi) \geq (Av, v-\xi) \qquad \text{for all } v \in X ,$$

which is by lemma 1 equivalent to

$$a(\xi, v-\xi) \geq (A\xi, v-\xi) \qquad \text{for all } v \in X.$$

Thus $\xi \in X$.

(iii) Finally $|| u_\varepsilon || \to || c||$ when $\varepsilon \to 0$,

Setting $v=u \in X$ in (15) and $v-u_\varepsilon \in X$ in (11)

We obtain

$$a(u_\varepsilon, u-u_\varepsilon) + \varepsilon b(u_\varepsilon, u-u_\varepsilon) \geq (Au_\varepsilon + \varepsilon Bu_\varepsilon, u-u_\varepsilon),$$

which is, by lemma 1, equivalent to

$$a(u_\varepsilon , u-u_\varepsilon) +\varepsilon b(u_\varepsilon , u-u_\varepsilon) \geq (Au+\varepsilon Bu,u-u_\varepsilon)$$

Also,

$$a(u,u_\varepsilon -u) \geq (Au,u_\varepsilon -u)$$

By addition one has

$$a(u_\varepsilon -u,u—u_\varepsilon) + \varepsilon b(u_\varepsilon ,u-u_\varepsilon) \geq \varepsilon(Bu,u-u_\varepsilon)$$

Using (10) , and for $\varepsilon > 0$, we get

$$b(u_\varepsilon ,u-u_\varepsilon) \geq (Bu,u-u_\varepsilon) \qquad \text{for all } u\in X.$$

Letting $\varepsilon \to 0$, $u_\varepsilon \to \xi$, we have

$$\begin{aligned} b(\xi ,u-\xi) &\geq (Bu.u-\xi) \\ &\geq (B\xi ,u-\xi), \qquad \text{by lemma 1.} \end{aligned}$$

Thus $\xi \in X$ is a solution of (14) and since the solution is unique, it follows that $\xi - u_o$.

Also from (17), by the coercivity of $b(u_\varepsilon ,u_\varepsilon)$, it follows that there exists a constant $\alpha >0$ such that

$$\begin{aligned} \alpha \| u_\varepsilon -u_o \| &\leq b(u_\varepsilon -u_o ,u_\varepsilon -u_o) \\ &\leq (Bu_\varepsilon ,u_\varepsilon -u_o) - b(u_o ,u_\varepsilon -u_o) \\ &\leq (Bu_o ,u_\varepsilon -u_o)-b(u_o , u_\varepsilon — u_o), \text{ by lennna 1,} \end{aligned}$$

which $\to 0$, as, $\varepsilon \to 0$. Hence it follows that $u_\varepsilon \to u$ strongly in H.

<u>Theorem 5</u>

If a(u,v), b(u,v) are coercive continuous bilinear forms, M is a closed convex set in H, and A,B are heniicontinuous antimonotone Lipschitz continuous operators with $\alpha > \gamma$, then problem 1 has a unique solution if and only if there exists a constant L, independent of ε, such that the solution of (15) satisfies

$$\| u_\varepsilon \| \leq L \tag{18}$$

Proof:

If there exists a solution, then from theorem 4, it follows that (18) holds. Conversely suppose that (18) holds , then there exists a subsequence u_η of u_ε which converges to

w weakly in H. Since M is a closed convex set, $w \in M$, Further writing (15) in. the form

$$a(u ,u -v) + \varepsilon b(u_\varepsilon ,u_\varepsilon - v) \leq (Av+\varepsilon Bv,u_\varepsilon - v) \qquad \text{for all } v \in M$$

and taking $\varepsilon = \eta = 0$, we find that

$$a(w,w) \leq a(w,v) + (Av,w - v) \qquad \text{for all } v \in M,$$

which is by lemma 1 , equivalent to

$$a(w,w - v) \leq (Aw,w - v) \qquad \text{for all } v \in M.$$

Thus w is the solution satisfying (11).

Existence of Solutions

In this section, the existence of the solution satisfying (10) for the cases, when M is bounded or an unbounded convex subset of H is considered.

Theorem 6

If M s a bounded closed convex subset, and A is a hemicontinuous Lipschiltz antimono tone operation, then there exists a unique solution of problem (1).

Proof:

Let $u_\varepsilon \in M$ be the element defined by (15). Since M is bounded, then $\|u_\varepsilon\|$ is bounded, and theorem (6) follows from theorem (5).

Consider now the case when the set M is bounded. Let $M_R = \{k \,;\, k \in M, \|k\| \leq R\}$ with R large enough so that $M_R \neq \phi$. Assume that A is hemicontinuous antimonotone operator, then by theorem (6), there exists a non-empty set,

$X_R \equiv$ set of all solution of $w \in M_R$ with (19)

$a(w,v-w) \geq (Aw,v-w)$ for all $v \in M_R$

Theorem 7

Suppose a(u,v) is a continuous bilinear form and A is a hemicontinuous antimonotone operator. If $u \in X_R$ with $\|u\| < R$, then u satisfies (11).

Proof

In fact, let w be any solution in M. Then for $0 < \varepsilon < 1$, $u+\varepsilon(w-u) \in M$ and $\| u+\in(w-u)\| \leq \| u| + \varepsilon \|w-u\| < R$ for sufficiently small ε. Thus for $0 < \varepsilon < \varepsilon_1$, $v=u + \varepsilon(w-u) \in M_R$.

Consequently such a v is allowed in (19) with w =u

and it follows that

$$a(u,w-u) \geq (Au,w-u) \qquad \text{for all. } w \in M.$$

This proves theorem 7.

Let a(u,v) be a coercive continuous bilinear form on H. The Cauchy-Schwarz inequality holds for a(u,v) and is given by

$$|a(u,v)|^2 \leq a(u,u)a(v,v) \qquad \text{for all } u,v \in H.$$

Theorem 8

A bounded bilinear form is continuous with respect to the norm convergence.

Proof:

Let $u_n \rightarrow u$ and $v_n \rightarrow v$, these sequences are bounded. We let Y be their bound, and then $\| u_n \| \leq Y$.

Now

$$\begin{aligned} [a(u_n ,v_n) - a(u,v) | &= | a(u_n ,v_n) - a(u_n ,v) + a(u_n ,v) - a(u,v) | \\ &\leq | a(u_n ,v_n -v)| + |a(u_n -u,v) | \\ &\leq C \gamma \| v_n-v \| + C_1 \| u_n-u \| \|v\|, \end{aligned}$$

by the Cauchy-Schwarz inequality. But $\|u_n — u \| \rightarrow 0$ and $\| [v_n -v \| \rightarrow 0$ as $n \rightarrow \infty$, and therefore

$| a(u_n ,v_n) - a(u,v) | \rightarrow 0$, i.e.,

$a(u_n ,v_n) \rightarrow a(u,v)$.

Theorem 9

Let v be in H and M be a closed convex subset of H. If u is a minimizing vector and a(x,v) is any continuous bilinear form such that $a(x,y) = ((x,y))$, for all $x,y \in H$, then

$$a(u-v,w—u) \geq 0 \qquad \text{for all } w \in M. \qquad (20)$$

Conversely if (19) holds and $a(u,v)$ Is also coercive, then

$$||u-v|| \leq \alpha^{-1} c \,||w-v|| \text{ for all } w \in M.$$

Proof:

If u is the unique minimizing vector, then we have to show that $a(u-v,w—u) \geq 0$ for all $w \in M$.

Suppose to the contrary that there is a vector $v_1 \in M$ such that $a(u-v,u-v_1) = \varepsilon > 0$. For all $t \in [0,l]$ and $v_1 \in M$, $v_t \equiv u + t(v_1-u) \in M$,

Now

$$\begin{aligned} || v_t -v||^2 &= || u+t\,(v_1-u) -v||^2 \\ &= ||u-v||^2 + t^2 \,|| v_1-u ||^2 + 2t(u-v,v_1-u) \\ &< || u-v ||^2, \end{aligned}$$

for, small positive t , which contradicts the minimizing property of u. Hence no such v_1 can exist.

Conversely let ueM such that (20) holds, then for any $w \neq u$, $w \in M$,

$$0 \leq a(\,u—v,w-u) = a(u-v,w-v +v— u)$$

implies that

$$a(u—v,u-v) \leq a(u—v,w—v).$$

Since a (u, v) is a continuous coercive bilinear form, there exist constants $c > 0$, $\alpha > 0$ such that

$$\alpha\, ||u-v||^2 \leq c|| u-v||\, || w-v || \qquad \text{for all } w \in M,$$

i.e.,

$$|| u- v || \geq a^{-1} c\, ||w-v || \qquad \text{for all } w \in M.$$

The following representation of the differentiable functions is needed

$$F(u) - F(v) = {}_0\int^1 (u - v, F'(v + s(u - v)))\, ds.$$

Theorem 10.

If F' is antimonotone, then the real-valued functional F is weakly upper semicontinuous and concave.

Proof:

Consider

$$F(u_n) - F(u) = {}_0\int^1 (u_n - u, F'(u + s(u_n - u)))\, ds$$

$$= {}_0\int^1 (u_n - u, F'(u) ds + {}_0\int^1 (u_n - u, F'(u_n + s(u_n - u)) - F'(u))ds$$

If $u_n \to u$ weakly, as $n \to \infty$, then the first term on R.H.S. tends to zero. The second term is always non-positive. In fact, by antimonotonicity $(u_n - u, F'(u+s(u_n - u)) - F'(u)) \leq 0$ for all $0 \leq s \leq 1$, and therefore the integrand ≤ 0 for all $0 \leq s \leq 1$. Hence for a sufficiently large n, there exists $\varepsilon_n \to 0$ as $n \to \infty$ such that $F(u\ ) - F(u) \leq \varepsilon_n$, i.e.,

$$\lim_{n\to\infty} \sup F(u_n) \leq F(u)\ .$$

Thus F is a weakly upper semicontinuous functional. Using a similar argument, it can be seen that the antimcnotonicity of F' guarantees concavity of F.

Theorem 11.

If a functional F is concave on a convex set M, then the

Frechet differential F' of F is antimonotone .

Proof:

For all $t \in [0,1]$ and $u,v \in M$, $tu+(1-t)v = v + t(u-v) \in M$.

By definition

$$F(v+t(u-v) \geq t\,F(u) + (1-t)F(v)$$

Dividing both sides by t, and letting $t \to 0$, we get

$$F'(v),u-v) \geq F(u) - F(v)$$

Similarly

$$(F'(u),v-u) \geq F(v) - F(u)$$

By addition, it follows that

$$(F'(v) - F'(u), u-v) \geq 0 \qquad \text{for all } u,v \in M.$$

Thus from theorem 10 and theorem 11 one concludes that "A real—valued functional on a convex set in a Hilbert space is <u>concave</u> if and only if its Frechet differential is an antimonotone operator".

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