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BRANCH POINTS, M FRACTIONS, AND RATIONAL FUNCTIONS MATCHING BOTH DERIVATIVES AND VALUES

by

J.A.MURPHY and D.M.DREW.

Branch Points, M fractions, and Rational functions matching both derivatives and values.

INTRODUCTION

We will present and discuss three distinct ideas in this report.

In the first section we illustrate, by an example, a direct method of obtaining from the series expansions of a function approximations to the branch points of the function. This is done by matching the series with an S fraction and taking advantage of the quasi-periodicity of the coefficients in the S fraction to estimate the positions of the branch points. The essential property of periodic continued fractions is developed generally.

The heart of this report is contained in the next three sections. We develop in section 32 a neat symmetrical theory for M fractions. We then broaden the class of functions to which the theory is applicable and we indicate the important concepts of multiplying, differentiating and integrating functions expressible as M fractions. A slight shift enables us to considerably extend the usefulness of these fractions. An M fraction is developed which matches the series expansions for a function about two finite points. Subsequently this result is generalised to one that matches the Taylor expansions for a function about three more points in the complex plane. Our examples indicate the power of the approximations that can be obtained.

The third concept runs through the report and ties the various sections together. A continued fraction, and hence the rational functions obtained by truncating it, can be arranged to fit a mixture of conditions by suitably setting up the linear equations bearing in mind the error term. We can, for example, deduce rational functions which fit derivatives at more than one point and simultaneously take particular values at given points.

We generate rational function approximations by developing our continued fractions as a set of linear equations. This process often necessitates working with considerably more significant figures than is required in the final coefficients of the continued fraction.

In this report we refer directly to some basic results on continued fractions contained in section 1 of the technical report TR/25.

31. Matching Termsin a Series with an Sfraction.

Let us start by constructing rational functions which fit the derivatives of a function f at a regular point ; without losing generality we can take the point as the origin and match the Maclaurin series for f. The following solution was outlined in section 1 of TR/25.

Given $f = A_1 + A_2 x + A_3 x^2 + \ ---$ (31.1)

we successively construct the three term linear relations

$$f_1 = f - p_1 \qquad p_1 \equiv A_1$$
$$f_2 = f_1 + p_2 x f$$

$$f_j = f_{j-1} + p_j x f_{j-2}$$
----------------- (31.2)

the p_j being chosen so that the first term in the series for f_j. is of degree x^j.

Writing $f_j = A_{1,j} x^j + A_{2,j} x^{j+1} + A_{3,j} x^{j+2} + ---$ (31.3)

and equating coefficients in (31.2) we readily deduce

$$0 = A_{1,j-1} + p_j A_{1,j-2}$$
$$A_{i,j} = A_{i+1,j-1} + P_j A_{i+1,j-2} \qquad i = 1,2....$$ (31.4)

The first of these relations gives p_j,

$$p_j = -\frac{A_{1,\ j-1}}{A_{1,\ j-2}} .$$

Hence we obtain the S fraction for f

$$f = \frac{p_1}{1+}\ \frac{p_2 x}{1+}\ \frac{p_3 x}{1+}\ .\ \frac{p_n x}{1} \Big/ + \frac{p_{n+1} x}{1+} \ ......$$ (31.5)

Truncating this C.F. after the n^{th} term gives a rational function which fits f and its first n-1 derivatives at the origin as proved by (1.8) of TR/25. When one or more of the initial coefficients in the series for an f_j are zero, say the first r coefficients, the above must be modified since the p_j are of necessity non-zero. We take advantage of this situation by making the $(j+l)^{th}$ equation in (31.2)

$$f_{j+1} = f_j + p_{j+1} x^{1+r} f_{j-1}. \qquad (31.6)$$

The effect is to fit 1+r derivatives of f in one go.

In general we then obtain an S fraction whose n^{th} convergent

$$f_{0/n} = \frac{p_1}{1+} \frac{p_2 x^{1+r_2}}{1+} \frac{p_3 x^{1+r_3}}{1+} --- \frac{p_n x^{1+r_n}}{1} / \qquad (31.7)$$

matches the first $(n+r_1+r_2+..r_n)$ terms in the Maclaurin series for f.

Most rational functions which fit derivatives at just one point can be constructed by the above technique. To obtain the J fraction for f we can simply contract (31.5)

$$f = \frac{p_1}{1+p_2 x -} \quad \frac{p_2 p_3 x^2}{1+(p_3+p_4) x -} ---- \frac{p_{2n-2} p_{2n-1} x^2}{1+(p_{2n-1}+p_{2n}) x} ; \qquad (31.8)$$

while to construct Padé approximates to f we convert the series commencing with the m^{th} term of the series for f (or $\frac{1}{f}$ see (1.20)) into an S fraction. It is crucial when matching rational functions to Î series to arrange that they behave at infinity in a manner not too different from that of the function. Let us consider a simple example of (31.5). The S fraction that matches the Maclaurin series for the sum

$$(1+x)^{-\frac{1}{2}} + (1+2x)^{-\frac{1}{3}} \qquad (31.9)$$

has coefficients p_j. as follows :

j	p_j
1	2.00000000
2	0.58333333
3	0.50000000
4	0.55886243
5	0.63226183
6	0.44110785
7	0.45272735
8	0.49832286
9	0.57734169
10	0.48074673
11	0.46470721
12	0.48757704
13	0.55432919
14	0.49479358
15	0.47309508
16	0.48417838
17	0.54114572
18	0.50129380
19	0.47897225
20	0.48308332
21	0.53250687
22	0.50468423
23	0.48327576
24	0.48293517

Approximaing Branch Points

Now, suppose in the above example we knew only the Maclaurin series but not the function (31.9).From it we could deduce a number of S fractions corresponding to the Padé approximants of which the above would be one. The factor that would make this C.F stand out would be the manner in which the coefficients almost repeat every four terms and tend to settle. This quasi-periodicity of the coefficients p_j is something that we observed in TR/26 We can use it to predict the approximate positions of the branch points of the function defined by the series. In this particular example we can of course check our results as we know the function (31. 9) from which the original series came. We will develop a general analysis of periodicity and then perform this calculation as a particular case.

Periodic Continued Fractions.

Consider an S fraction with period m, it can always he written

$$R = \frac{c_1 x}{1+}\ \frac{c_2 x}{1+}\ ----\ \frac{c_m x}{1+R}\ . \tag{31.11}$$

Denoting the m[th] numerator and denominator by C_m and D_m respectively, we find

$$R = \frac{C_m + R\, C_{m-1}}{D_m + R\, D_{m-1}}$$

And hence R is a root of the quadratic equation (31.12)

$$D_{m-1} R^2 + (D_m - C_{m-1}) R - C_m = 0.$$

The discriminant Δ of this quadratic equation gives the branch points of R and is

$$\begin{aligned}\Delta &= (D_m - C_{m-1})^2 + 4D_{m-1} C_m \\ &= D_m^2 + 2D_m C_{m-1} + C_{m-1}^2 + 4(D_{m-1} C_m - D_m C_{m-1}) \\ &= (C_{m-1} + D_m)^2 + 4(-1)^{m-1}\ c_1 c_2 --- c_m x^m \end{aligned} \tag{31.13}$$

where we have used the result (1.12).

We will evaluate the cases m = 3 to 6 in detail, since they occur frequently. In doing this it is convenient to change our notation slightly and write

$$R = \frac{qx}{1+}\ \frac{rx}{1+}\ \frac{sx}{1+}\ \frac{tx}{1+}\ \frac{ux}{1+}\ \frac{vx}{1+}\ ---. \tag{31.14}$$

the first x convergents of R are

$$\frac{C_1}{D_1} = \frac{qx}{1}, \quad \frac{C_2}{D_2} = \frac{qx}{1+rx},$$

$$\frac{C_3}{D_3} = \frac{qx(1+sx)}{1+rx+sx}, \quad \frac{C_4}{D_4} = \frac{qx(1+sx+tx)}{1+(r+s+t)x+rtx^2},$$

$$\frac{C_5}{D_5} = \frac{qx[1+(s+t+u)x+sux^2]}{1+(r+s+t+u)x+(rt+ru+su)x^2},$$

$$\frac{C_6}{D_6} = \frac{qx[1+(s+t+u+v)x+(su+sv+tv)x^2]}{1+(r+s+t+u+v)x+(rt+ru+su+rv+sv+tv)x^2+rtv\,x^3}.$$

Thus when the continued fraction has:

<u>Periodicity three</u> $\quad R = \frac{qx}{1+}\;\frac{rx}{1+}\;\frac{sx}{1\;+R.}$

By (31.12) with m = 3

$$(1+rx)R^2 + [1+rx+sx = qx]R - qx(l+sx) = 0$$

and the discriminant (31.13) is

$$\Delta=[1+(q+r+s)x]^2+4qrsx^3. \qquad (31.15)$$

<u>Periodicity four</u> $\quad R = \frac{qx}{1+}\;\frac{rx}{1+}\;\frac{sx}{1+}\;\frac{tx}{1+R}$

The disoriminant, with $\sigma = q+r+s+t$, is

$$\Delta = (D_4, +C_3)^2 - 4\,qrstx^4$$

$$= [l+\sigma x+(rt+qs)x^2]^2 - 4\,qrstx^4 \qquad (31.16)$$

<u>Periodicity five</u> $\quad R = \frac{qx}{1+}\;\frac{rx}{1+}\;\frac{sx}{1+}\;\frac{tx}{1+}\;\frac{ux}{1+R}$

The discriminant, $\quad \Delta = (D_5 + C_4)^2 + 4\,qrstux^5$

$$= [1+\sigma x+\tau x^2]^2 + 4qrstux^5 \qquad (3.17)$$

where $\sigma = (q+r+s+t+u)$, $\tau = (qs+qt+rt+ru+su)$.

Periodicity six $R = \frac{qx}{1+} \frac{rx}{1+} \frac{sx}{1+} \frac{tx}{1+} \frac{ux}{1+} \frac{vx}{1+R}$

The discriminant, $\Delta = [1 + \text{o}\, x + T\, x + (qsu + rtv)x^3]^2 - 4\, qrstuvx^6$ (31.18)

where $\sigma = (q+r+s+t\text{-}m+v)$,

$\tau = q(s+t+u)+r(t+u+v) + s(u+v)+tv,$

When the continued fraction is of even periodicity the discriminant immediately factorises into two terms. We also note the number of parameters in the discriminant is less than the degree of the discriminant, e.g.with the periodicity five there are at most three parameters σ, τ and the coefficient of x^5.Consequently the five singularities (branch points) cannot be positioned independently. This suggests that some of the roots the discriminant are repeated, or in pairs.

The Branch Points.

Let us now return to our original problem which was to determine the branch points of a function defined by its Maclaurin series. The series we converted into an S fraotion whose coefficients p_j are listed in(31.10). We had observed that the coefficients p_j almost repeated themselves every four terms, we therefore write

$$R = \frac{p_n x}{1+} \frac{p_{n+1} x}{1+} \frac{p_{n+2} x}{1+} \frac{p_{n+3} x}{1+R^*} \qquad (31.19)$$

For sufficiently large n,R* is almost the same function of x as R . Thus, if we replace R* by R, the discriminant of the resulting quadratic equation must approximately give the branch points of R and hence those of the function defined by the S fraction. The periodicity is four, therefore by (31.16) the discriminant is

$$\Delta = [1 + \sigma x + \tau x^2]^2 - 4P_n P_{n+1} P_{n+3} x^4 \qquad (31.20)$$

where $\sigma = P_n + P_{n+1} + P_{n+2}\ P_{n+3}$ and $\tau = P_n P_{n+2} + P_{n+1}\ P_{n+3}$.

Setting $\Delta = 0$, the approximate positions of the branch points of the function defined by the S fraction are the solutions of the equations

$$1+\sigma x+\tau x^2 \pm 2\sqrt{P_n P_{n+1} P_{n+2} P_{n+3}}\, x^2 = 0 . \qquad (31.21)$$

Using he p_j listed in (31.10) the following values are readily calculated

	σ	τ	$2\sqrt{\ }$
n = 13	2·0064	0·5018	0·5013
n = 17	2·0045	0·5014	0·5012
n = 21	2·0034	0·5011	0·5011

With the last set of values we would estimate that the branch points of the function defined by the series are at

$$x = -\ 1{\cdot}034, \quad -\ 0{\cdot}965 \text{ and } -0{\cdot}499 ,$$

the fourth root being large and negative. These are reasonable approximations to the branch points -1 and -0.5 of the original function (31.9).

The values for σ, τ and the root suggest that all we need do is increase n and (31.21) with the + sign will tend to $1 + 2x + x^2$, while with the - sign it will tend to 1 + 2x. In practice to lengthen our table for p_j significantly, with the above method, would require working to many more figures. The table was obtained using double precision arithmetic (20 figures).

The alternative is to make assumptions concerning the variation of σ, τ and the square root with n and attempt to extrapolate to their limiting values. For example a Richardson $\frac{1}{n^2}$ extrapolation on the values for σ

using the n= 13 and 21 values gives σ =2·0015,

using the n = 17 and 21 values gives σ = 2·0013.

Connecting he branch points are branch cuts. Let us simply remark that the poles and zeros of the nth convergent of the C.F. settle on to lines as n is increased, and in fact etch out these cuts, for this point is discussed in the section on quasi-periodicity in TR/26.

So far our rational functions simply match the terms in a series expansion of a function. When additional information about the function is available the approximations are often considerably improved if this information can be built into our C.F. A case of particular importance is when a series expansion, or part of one, about a second point is known. We will now construct a C.F, which matches two series, one at the origin and the other at infinity. In the subsequent section we will extend this C.F. so as to obtain a C.F. which matches two series expansions both about finite points.

32. Matching Terms in Two Series with an M fraction.

Given a function f(x) with the following expansions

$$f = L_1 x + L_2 x^2 + L_3 x^3 + \dots \qquad \text{for} \quad |x| \quad \text{small} \qquad (32.1)$$

$$\sim H_1 + \frac{H_2}{x} + \frac{H_3}{x^2} + \dots\dots \qquad \text{for} \quad |x| \quad \text{large,}$$

we can successively construct the linear relations

$$f_1 = (L_1 x + H_1)f - L_1 H_1 x$$

$$f_2 = (\alpha_2 x + \beta_2) f_1 - \alpha_2 \beta_2 x f$$

$$f_j = (a_j x + \beta_j) f_{j-1} - \alpha_j \beta_j x f_{j-2} \qquad (32.2)$$

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if we choose the α_j and β_j so that the ratio $\frac{f_j}{f_{j-1}}$ maintains the form of f. To do this write

$$fj = L_{1,j} x^{j+1} + L_{2,j} x^{j+2} + L_{3,} x^{j+3} + \dots \quad \text{for} |x| \ \text{small}$$

$$\sim H_{1,j} + \frac{H_{2,j}}{x} + \frac{H_{3,j}}{x^2} + \ \dots\dots \quad \text{for} |x| \ \text{large.} \qquad (32.3)$$

Then substituting these series for f_j in (32.2) and equating coefficients, we readily deduce for $j \geq 2$ and $i \geq 1$,

$$\alpha_j = \frac{L_{1,j-1}}{L_{1,j-2}}, \qquad \beta_j = \frac{H_{1,j-1}}{H_{1,j-2}}$$

$$L_{i,j} = \alpha_j L_{i,j-1} + \beta_j L_{i+1,j-1} - \alpha_j \beta_j L_{i+1,j-2} \qquad (32.4)$$

$$P_{i,j} = \beta_j H_{i,j-1} + \alpha j H_{i+1,j-1} - \alpha_j \beta_j H_{i+1,j-2} \quad ,$$

provided that the $L_{1,j}$ and $H_{1,j}$ are non-zero for all j < **n** when constructing n of the equations (32.2).

The set of equations 32.2) are equivalent to Murphy's M fraction for f(x)

$$f = \frac{\alpha_1 \beta_1 x}{\alpha_1 x+\beta_1 -} \quad \frac{\alpha_2 \beta_2 x}{\alpha_2 x+\beta_2 -} \quad \frac{\alpha_3 \beta_3 x}{\alpha_3 x+\beta_3 -} \quad .... \quad \frac{\alpha_n \beta_n x}{\alpha_n x + \beta_n - \frac{f_n}{f_{n-1}}} \tag{32.5}$$

where $\alpha_1 = L_1$ and $\beta_1 = H_1$.

This M fraction possesses the important property that its n^{th} convergent matches n terms in each of the series(32.1),a fact readily established by differentiating the error term for the n^{th} convergent

$$f - \frac{c_n}{D_n} = \frac{f_n}{D_n} = \begin{cases} \frac{L_{1,n}}{H_{1,n-1}} \quad x^{n+1} + \ldots \\ \frac{H_{1,n}}{L_{1,n-1}} \quad \frac{1}{x^n} + \ldots \end{cases} \tag{32.6}$$

$$\text{for } D_n = (\alpha_1 \alpha_2 \ldots \alpha_n x_n + \ldots + \beta_1 \beta_2 .. \beta_n) L_{1,n-1} x^n + \ldots + H_{1,n-1}$$

The M fraction (32.5) terminates when f is a rational function. Otherwise in the above derivation none of the first terms, $L_{1,j-1}$ and H_{1j+1} , in the series can be zero. When one is zero there is some choice as to how to rectify the situation. We will simply construct the j^{th} partial quotient to match two terms from each series bearing in mind the ratio $\frac{f_j}{f_{j-1}}$ must be in a suitable form for continuing the M fraction.

Suppose $L_{1,j-1} = 0$, but $H_{1,j-1} \neq 0$ and $L_{2,j-1} \neq 0$.

We take in our M fraction for the j[th] partial quotient

$$\frac{-\alpha_j \beta_j x^2}{\alpha_j x^2 + y_j x + \beta_j} \qquad (32.7)$$

and choose the y_j to fit a second terra of the H series. In the linear relations (32.2), we then have

$$f_j = (\alpha_j x^2 + y_j x + \beta_j) f_{j-1} - \alpha_j \beta_{j x 2} f_{j-2} . \qquad (32.8)$$

For | x | small

$$f_j = \beta_j (L_{2,j-1} - \alpha_j L_{1,j-2}) x^{j+1} + 0(x^{j+2}) ,$$

while for |x| large

$$f_j \sim \alpha_j (H_{1,j-1} - \beta_j H_{1,j-2} | x^2 + (y_j H_{1,j-1} + \alpha_j H_{2,j-1} - \alpha_j \beta_j H_{2,j-2}) x + 0(1),$$

consequently if we let

$$\alpha_j = \frac{L_{2,j-1}}{L_{1,j-2}} , \quad \beta_j = \frac{H_{1,j-1}}{H_{1,j-2}} \text{ and } y_j = \frac{\alpha_j (\beta_j H_{2,j-2} ,-H_{2,j-1}}{H_{1,j-2}} \qquad (32.9)$$

the ratio $\frac{f_j}{f_{j-1}}$ regains the form of f and we can continue to spin out subsequent terms of the M fraction.

When $H_{1,j-1} = 0$ and $L_{1,j-1} \neq 0$ and $H_{2,j-1} \neq 0$.

The appropriate form for the j[th] partial quotient is

$$\frac{-\alpha_j \beta_j}{\alpha_j x + y_j + \frac{\beta_j}{x}} \qquad (32.10)$$

where the y_j is to be chosen so that a second term of the L series is matched. From the linear relation

$$f_j = (\alpha_j x + \gamma_j + \frac{\beta_j}{x}) f_{j-1} - \alpha_j \beta_j f_{j-2} \qquad (32.11)$$

we then have, for $|x|$ small, that

$$f_j = \beta_j \, (L_{1,j-1} - \alpha_j \, L_{1,j-2}) x_{j-1} + (\gamma_j \, L_{1,j-1} + \beta_j \, L_{2,j-1} - \alpha_j \, \beta_j \, L_{2,j-2}) x^j + 0(x^{j+1})$$

while for $|x|$ large,

$$f_j \sim \alpha_j \, (H_{2,i-1} - \beta_j \, H_{1,j-2}) + 0(\frac{1}{x}) \, .$$

Thus taking

$$\alpha j = \frac{L_{1,\,j-1}}{L_{1,\,j-2}}, \quad \beta_j \, \frac{H_{2,\,j-1}}{H_{1,\,j-2}} \quad \text{and} \quad y_j = \beta_j \, \frac{(\alpha_j L_{2,\,j-2} - L_{2,\,j-1})}{L_{1,\,j-1}}, \qquad (32.12)$$

again the series for $\frac{f_j}{f_{j-1}}$ are in the correct form for producing the subsequent partial quotients in the M fraction.

With (32.7), or (32.10), as the j^{th} partial quotient in Murphy's M fraction a simple consideration of the error term immediately verifies that the j^{th} convergent matches $j + 1$ terms in each of the series (32.1) for f. In general (32.5) may include a number of terms like (32.7) and (32.10), but even when it does we will still refer to it as an M fraction.

The special case (32.7) and (32.10) can be extended to allow for two or more leading L and/or H to be zero by simply increasing the number of unknowns in the partial denominator. The partial numerator is the product of the first and last terms in the denominator. Only one further case is worth mentioning. When both $L_{1,j-1}$ and $H_{1,j-1}$ are zero, the form for the j^{th} partial quotient is

$$\frac{-\alpha_j \beta_j x}{\alpha_j x^2 + \gamma_j x + \sigma_j + \frac{\beta_j}{x}} \, . \qquad (32.13)$$

Our special cases are in fact of considerable interest. The product of two functions with expansions of the form (32.1) give a function g(x) which behaves as

$$g = Lx^2 + 0(x^3) \quad \text{for } |x| \text{ small,}$$
$$\sim H + 0(1/x) \quad \text{for } |x| \text{ large.}$$

By (32.7) the M fraction for such a function has as first partial quotient

$$\frac{LHx^2}{Lx^2 + yx + H} \qquad (32.14)$$

when γ is chosen so that a second term of the H series is matched.

Similarly (32.10) gives the form of the first partial quotient of the M fraction for the derivative of (32.1)

$$xf' = L1x + 2L_2x^2 + \ldots. \quad \text{for } |x| \text{ small}$$

and so

$$\sim \frac{-H_2}{x} - \frac{2H_3}{x} - \ldots. \quad \text{for } |x| \text{ large},$$

$$xf' = \frac{-L_1L_2}{L_1x + y - \frac{H_2}{x} - R}$$

where R has an M fraction representation.

Hence the derivative

$$f' = \frac{L_1H_2}{H_2 - yx - L_1x^2 + xR} \qquad (32.15)$$

and we can deduce its M fraction.

Only a few common functions can be expressed in the form (32.1). It would therefore be useful to generalise the concept of fitting a C.F. simultaneously to an expansion at the origin and one at infinity, to a function g(x) that can be written

$$g(x) = L_1 x + L_2 x^2 + L_3 x^3 + \; .... \text{ for } |x| \text{ small} \qquad (32.16)$$
$$\sim h(x) + H_1 + \frac{H_2}{x} + \frac{H_3}{x^2} + \; .... \text{ for } |x| \text{ large}$$

where h(x) is an irrational function which cannot be expanded in integer powers of x. A neat general solution is not known. Particular examples can, however, be dealt with numerically by subtracting from f (x) a simple function with a Maclaurin series expansion and which tends to h(x) as $x \to \infty$, then expressing the difference as an M fraction.

As an example we construct such a solution for the integral of the function,

$$f(x) = L_1 + L_2 X + L_3 x^2 + \; ... \text{ for } |x| \text{ small,} \qquad (32.17)$$
$$\sim \frac{H_1}{x} + \frac{H_2}{x^2} + \frac{H_3}{x^3} + \; .... \text{ for } |x| \text{ large ,}$$

which can itself be written as an M fraction.

Integrating

$$g = \int_o^x f(x)dx = L_1 x + L_2 \frac{x^2}{2} + L_3 \frac{x^3}{3} + \; ....$$
$$(32.18)$$
$$\sim H_1 \log x + c - \frac{H_2}{x} - \frac{H_3}{2x^2} - ....$$

where C is the constant of integration and must be evaluated.

Now

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \; ....$$
$$= \log x + \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \; .... \; ,$$

hence the difference

$$\int_0^x f(x)dx - H_1 \log(1+x) = (L_1 - H_1)x + (L_2 + H_1)\frac{x^2}{2} + (L_3 - H_1)\frac{x^3}{3} + \;.....$$

$$\sim c - (H_2 + H_1)\frac{1}{x} - (H_3 - H_1)\frac{1}{2x^2} - \;......... \qquad (32.19)$$

is in the form of (32.1) and can therefore be expressed as an M fraction.

Finally let us make two remarks concerning the numerical determination of the coefficients in an M fraction. The numerical stability of the equations (32.4) is such that it is usually necessary to work Double length in order to evaluate sufficient coefficients α_j,$ß_j$ to single length accuracy.

Since $ß_j$ is never zero we can rewrite (32.5) in the alternative form

$$f = \frac{P_1 x}{1 + q_1 x \;\; +} \;\; \frac{P_2 x}{1 + q_2 x +} \;\; .... \;\; \frac{P_j x}{1 + q_j x +} \;\; .... \qquad (32.20)$$

We will often tabulate p_j and q_j rather than α_j and β_j. Two examples of M fractions are given in TR/26.

The matching of a C.F. to the asymptotic behaviour of a function at infinity, as well as its Maclaurin expansion, is often not practical. In the next section we will modify our M fraction and construct a C.F. which matches the series expansions of a function about two points in the finite complex plane. This will considerably extend our use of M fractions.

33. M fraction matching Series at Two Finite Points.

Suppose the function F(t) is regular at two finite points in the complex plane and we are given its series expansions about these points. Without loss of generality we can adjust the points so that they are at the origin and at $t = 1$. Further we can also arrange that F(t) is zero at both points, so that

$$F(t) = A_1 t + A_2 t^2 + A_3 t^3 + \ldots \quad \text{for } |t| \text{ small}$$

$$= B_1 \tau + B_2 \tau^2 + B_3 \tau^3 + \ldots \quad \text{for } |\tau| \text{ small} \tag{33.1}$$

where $\tau = 1 - t$.

Proceeding along lines parallel to those of the previous section, as a first approximation to F(t) we have

$$\frac{A_1 B_1 \, t\tau}{A_1 t + B_1 \tau}$$

and in general

$$F(t) \quad \frac{\alpha_1 \beta_1 \, t\tau}{\alpha_1 t + \beta_1 \tau -} \quad \frac{\alpha_2 \beta_2 t\tau}{\alpha_2 t + \beta_2 \tau -} \ldots \frac{\alpha_n \beta_n t\tau}{\alpha_n t + \beta_n \tau -} \ldots \tag{33.2}$$

where $\alpha_1 \equiv A_1$ and $ß_1 \equiv B_1$. When necessary we incorporate terms equivalent to (32.7) and (32.10). This we refer to as an M fraction for two finite points.

To deduce (33.2) we can define

$$F_j(t) = A_{1,j} t^{j+1} + A_{2,j} t^{j+2} + \ldots$$
$$= B_{1,j} \tau^{j+1} + B_{2,j} \tau^{j+2} + \ldots \tag{33.3}$$

and construct the linear relations for $j \geq 2$,

$$F_j = (\alpha_j t + \beta_j \tau) F_{j-1} - \alpha_j \beta_j t\tau F_{j-2} \tag{33.4}$$

From which we find

$$\alpha_j = \frac{A_{1,j-1}}{A_{1,j-2}} \quad , \quad \beta_j = \frac{B_{1,j-1}}{B_{1,j-2}} \tag{33.5}$$

and $\quad A_{1,j} = (\alpha_j - \beta_j)\ A_{i,j-1} + \beta_j A_{i+1,j-1} + \alpha_j \beta_j\ (A_{i,j-2} - A_{i+1,j-2})$

with a similar expression for $B_{i,j}$

Further,since the denominator polynomials,

$$D_n = (\alpha_1\alpha_2..\alpha_n t^n + ... \quad + \beta_1\beta_2 \ ..\beta_n \tau^n) = A_{1,n-1} t^n + ... + \beta_{1,n-1} \tau^n ,$$

the error in approximating F(t) by the n^{th} covergent of (33.2) is

$$F(t) - \frac{c_n(t)}{D_n(t)} = \frac{F_n(t)}{D_n(t)} = \begin{cases} \dfrac{A_{1,n}}{B_{1,n-1}} t^{n+1} + \ldots , \\ \dfrac{B_{1,n}}{A_{1,n-1}} \tau^{n+1} + \ldots . \end{cases} \tag{33.6}$$

As a corollary we can then verify that the n^{th} convergent

$\frac{c_n(t)}{D_n(t)}$ fits n terms in each of the series(33.1) of F(t).

Alternatively we could derive the continued fraction (33.2) directly from (32.5) by putting $x = \frac{t}{1-t}$ In fact this was how we first obtained (33.2), Setting

$$F(t) = (1-t)f(x) \tag{33.7}$$

$$F(t) = L_1 t + L_2 t^2 (1-t)^{-1} + L_3 t^3 (1-t)^{-2} + ... \text{ for } |t| \text{ small},$$

$$\sim H_1 \tau + H_2 \tau^2 (1-\tau)^{-1} + H_3 \tau^3 (1-\tau)^{-2} + ... \text{ for } |\tau| \text{ small},$$

It follows that the A_j are related to the L_j, and similarly the ß$_j$ to the H_j, by linear relations of a Pascal triangle type.

$$A_2 = L_2 \qquad L_2 = A_2$$
$$A_3 = L_2 + L_3 \qquad L_3 = A_3 - A_2$$
$$A_4 = L_2 + 2L_3 + L_4 \qquad L_4 = A_4 - 2A_3 + A_2 \tag{33.8}$$
$$A_5 = L_2 + 3L_3 + 3L_4 + L_5 \qquad L_5 = A_5 - 3A_4 + 3A_3 - A_2$$

We will now demonstrate the power of the convergents of the M fraction (33.2) in approximating functions. The coefficients in the M fraction in our two examples were evaluated by changing the series (33.1) to the series (32.1) and determining the M fraction (32.20). We therefore tabulate p_j and q_j rather than a_j and $ß_j$

A function with the correct form (33.1) at t = o and t = 1 is $\sin \pi t$,

$$\sin \pi t = \pi t - \frac{\pi^3}{3!} t^3 + \frac{\pi^5}{5!} t^5 - \ldots. ,$$
$$= \pi\tau - \frac{\pi^3}{3!}\tau^3 + \frac{\pi^5}{5!}\tau^5 - \ldots\ldots$$

where $\tau = 1 - t$.

Clearly $\alpha_j = ß_j$ and there fore $q_j = 1$. Hence

$$F(t) = \frac{P_1 t\tau}{1 +} \frac{P_2 t\tau}{1 +} \ldots \frac{P_n t\tau}{1 +} \ldots \tag{33.9}$$

where the first eight p_j are

j	P_j
1	3·1415926536
2	-1.0000000000
3	0.6449340668
4	-0.0943879701
5	0.1896765053
6	-0.0359889578
7	0.0936058123
8	-0.0186821135

Some values of the fifth and eighth convergents of (33.9) for a few values of t are listed in the following table :

x	$t = \frac{x}{x+1}$	$F_{0/5}$	$F_{0/8}$	sinπt
$\frac{1}{4}$	$\frac{1}{5}$	0·5877852288	as	0·5877852523
$\frac{1}{2}$	$\frac{1}{3}$	0.8660252297	sinπt	0.8660254038
$\frac{3}{4}$	$\frac{3}{7}$	0.9749275967	column	0.9749279122
1	$\frac{1}{2}$	0.9999996420		1.0000000000

The Theconvergents of F(t) are symmetrical about t = ½ .

The continued fraction (33.9) can be recommended for its simplicity. Nevertheless more effective M fractions for calculating the sine (and cosine) function can be constructed. The fifth convergent of (33.10) yields sin $^{\pi}/_{2}$ t to nine decimal places.

The function $\sin(\frac{\pi}{2}t)$ -t has the series expansions

$$\sin\frac{\pi}{2}t - t = (\frac{\pi}{2}-1)t - \frac{(\frac{\pi}{2})^3}{3!}t^3 \quad \frac{(\frac{\pi}{2})^5}{5!}t^5 - ....$$

$$= \tau - \frac{(\frac{\pi}{2})^2}{2!}\tau^2 + \frac{(\frac{\pi}{2})^4}{4!}\tau^4 - ....$$

where τ = 1 - t,

and these series can be matched simultaneously by the M fraction

$$F(t) = \frac{P_1 t\tau}{\tau + q_1 t -} \quad \frac{P_2 t\tau}{\tau + q_2 t -} \quad .... \quad \frac{P_n t\tau}{\tau + q_n t -} \quad .... \qquad (33.10)$$

The first eight values pf p_j and q_j are

j	p_j	q_j
1	0·5707963268	0·5707963268
2	-0·5707963268	1·9296159323
3	0·0530340085	0·7213280285
4	-0·0727047212	1·3386167399
5	0·0154574880	0·8114789728
6	-0·0282468746	1·2123789943
7	0·0074338105	0·8575377249
8	-0·0149091582	1·1550923321

We give just a We give just a short table of t plus the fourth convergent and t plus the fifth convergent of (33.10).

x	$t=\frac{x}{x+1}$	$t+F_{0/4}$	$t+F_{0/5}$	$\sin\pi/2t$
$\frac{1}{4}$	$\frac{1}{5}$	0.3090169922	0.3090169944	0.3090169944
$\frac{1}{2}$	$\frac{1}{3}$	0.4999999887	0.4999999999	0.5000000000
1	$\frac{1}{2}$	0.7071067605	0.7071067810	0.7071067812
2	$\frac{2}{3}$	0.8660253922	0.8660254038	0.8660254038
4	$\frac{4}{5}$	0.9510565141	0.9510565163	0.9510565163

34. Matching Series at three or more Points.

Our concept of matching the derivatives of a function at two points by rational functions readily extends to matching the derivatives of a function at three or more points in the complex plane. The analysis in section 33 generalises and for three finite points is as follows.

Suppose F(t) is regular at the origin, t = b and t = c and has the following Taylor expansions at these points.

$$\begin{aligned} F(t) &= A_1 t + A_2 t^2 + A_3 t^3 + \ldots && \text{for } |t| \text{ small,} \\ &= B_1 \tau + B_2 \tau^2 + B_3 \tau^3 + \ldots && \text{for } |\tau| \text{ small,} \\ &= c_1 T + c_2 T^2 + c_3 T^3 + \ldots && \text{for } |T| \text{ small,} \end{aligned} \qquad (34.1)$$

where $\tau = 1 - \frac{t}{b}$ and $T = 1 - \frac{t}{c}$ . ...

Now the rational function

$$\frac{A_1 B_1 C_1 \quad t\tau T}{B_1 C_1 \, \tau T + A_1 C_1 tT + A_1 B_1 t\tau}$$

correctly matches the first terms of the above series at the three points. This suggests that we construct the linear relations

$$\begin{aligned} F_1 &= (B_1 C_1 \tau T + A_1 C_1 \, tT + A_1 B_1 \, t\tau) F - A_1 B_1 C_1 \; t\tau T \\ F_j &= (\beta_j \gamma_j \tau T + \alpha_j \gamma_j tT + \alpha_j \beta_j t\tau) F_{j-1} - \alpha_j \beta_j \gamma_j t\tau T F_{j-2} \text{ for } j \geq 2. \end{aligned} \qquad (34.2)$$

in which we successively choose the α_j ß$_j$ γ_j so that the form of F_j is

$$\begin{aligned} F_j(t) &= A_{1,j} \, t^{j+1} + A_{2,j} \, t^{j+2} + \ldots && \text{for } |t| \text{ small,} \\ &= B_{1,j} \, \tau^{j+1} + B_{2,j} \, \tau^{j+2} + \ldots && \text{for } |\tau| \text{ small,} \\ &= C_{1,j} \, T^{j+1} + C_{2,j} \, T^{j+2} + \ldots && \text{for } |T| \text{ small.} \end{aligned} \qquad (34.3)$$

We find for $j \geq 2$,

$$\alpha_j = \frac{A_{1,j-1}}{A_{1,j-2}} , \qquad \beta_j = \frac{B_{1,j-1}}{B_{1,j-2}} , \qquad \gamma_j = \frac{C_{1,j-1}}{C_{1,j-2}} \tag{34.4}$$

and from the coefficient of t^{J+1}, the general expression for the $A_{i,j}$, $i \geq 1$, is

$$A_{i,j} = \beta_j\gamma_j A_{i+1,j-1} - [(\frac{1}{b}+\frac{1}{c})\beta_j\gamma_j - \alpha_j (\beta_j + \gamma_j)] A_{i,j-1} +$$

$$+ [\frac{\beta_j \gamma_j}{bc} - \alpha j (\frac{\beta_j}{b} + \frac{\gamma_j}{c})] A_{i-1,j-1} -$$

$$-\alpha_j \beta_j \gamma_j [A_{i+1,j-2} -(\frac{1}{b}+\frac{1}{c}) A_{i,j-2} + \frac{1}{bc} A_{i-1,j-2}] , \tag{34.5}$$

with similar expressions for the $B_{i,j}$ and the $C_{i,j}$

The linear relations (34.2) can be spun out to give for F(t) the continued fraction form

$$F(t) = \frac{\alpha_1\beta_1\gamma_1 \ t\tau T}{(\beta_1\gamma_1\tau T+\alpha_1\gamma_1 tT+\alpha_1\beta_1 t\tau)-} \ ..... \ \frac{\alpha_n\beta_n\gamma_n t\tau T}{-(\beta_n\gamma_n\tau T+\alpha_n\gamma_n tT+\alpha_n\beta_n t\tau)-\frac{F_n}{F_{n-1}}} \tag{34.6}$$

where $\alpha_1 = A_1$,$ß_1 = B_1$ and $y_1 = C_1$.

We see that the denominator polynomial $D_n(t)$ of the n^{th} Convergent is dominated, when either t or τ or T is small, by the product

$$(\beta_1\gamma_1\tau T + \alpha_1\gamma_1 tT + \alpha_1\beta_1 t\tau) (\beta_2\gamma_2\tau T + \alpha_2\gamma_2 tT + \alpha_2\gamma_2 t\tau) \ .. \\ .. (\beta_n\gamma_n\tau T + \alpha_n\gamma_n tT + \alpha_n\beta_n t\tau) .$$

In particular when t is small the constant term is

$$\beta_1\gamma_1\beta_2\gamma_2 \ ..... \ \beta_n\gamma_n = B_{1,n-1} C_{1,n-1} ;$$

the corresponding result when r is small is $A_{1,n-1}\ C_{1,n-1}$, and

when T is small is $A_{1,n-1}\ B_{1,n-1}$

Consequently the error in approximating F(t) by the n^{th} convergent of (34.6) is

$$F(t) - \frac{c_n(t)}{D_n(t)} = \frac{F_n(t)}{D_n(t)} = \begin{cases} \dfrac{A_{1,n}}{S_{1,n-1}C_{1,n-1}}\, t^{n+1} + \ldots\ldots \\ \dfrac{B_{1,n}}{A_{1,n-1}C_{1,n-1}}\, \tau^{n+1} + \ldots\ldots \\ \dfrac{C_{1,n}}{A_{1,n-1}B_{1,n-1}}\, T^{n+1} + \ldots\ldots \end{cases} \qquad (34.7)$$

for $|t|$, $|\tau|$ and $|T|$ small respectively. From this result we can immediately establish that the rational function $\dfrac{C_n(t)}{D_n(t)}$ matches n terms in each of the series (34.1) .

35. Fitting Values at Given Points

Just as we have generated a C.F, to fit derivatives of a function at one or more points, so we can design a C.F. to take on particular values at given points. Let us start with an example.

A rearrangement of the S fraction for $(1+t)^x$ gives

$$\frac{(1+t)^x-1}{(1+t)^x+1} = \frac{xt}{2+t+} \; \frac{(x^2-1)t^2}{3(2+t)+} \; \frac{(x^2-4)t^2}{5(2+t)+} \; \cdots\cdots \; \frac{(x^2-n^2)t^2}{(2n+1)\,(2+t)+} \; \cdots$$

Substituting, $e^{12Ø} = 1+t$, we find.

$$\tan x\Phi = \frac{x\tan\phi}{1-} \; \frac{(x^2-1)\tan^2\phi}{3-} \; \frac{(x^2-4)\tan^2\phi}{5-} \; .. \; \frac{(x^2-n^2)\tan^2\phi}{2n+1-} \; .. \qquad (35.1)$$

and in particular

$$\tan x\frac{\pi}{4} = \frac{x}{1-} \; \frac{(x^2-1)}{3-} \; \frac{(x^2-4)}{5-} \; \cdots \; \frac{(x^2-n^2)}{2n+1-} \; \cdots \qquad (35.2)$$

Clearly this C.F. is fitting values of the function $\tan x\frac{\pi}{4}$ at $x = 0, \pm 1, \pm 2, \ldots$

Its n^{th} convergent is a rational function which as n is increased progressively approximates more of the function. (35.2) is a member of a class of C.F's which fit values of even and odd functions; as few appear in the literature we give some examples.

a) Fitting the values of $\cos\frac{\pi}{2}x$ at $x = 0, \pm 1, \pm 2 \ldots \pm (n+1)$, we obtain

$$1 - \frac{x^2}{1+} \; \frac{(x^2-1)}{3\;-} \; \frac{2(x^2-4)}{5\;+} \; \frac{3(x^2-9)}{7\;-} \; \frac{(-1)^{n+1}(x^2-n^2)}{2n+1} \qquad (35.3)$$

b) Fitting the values of $\sin\frac{\pi}{2}x$ at $x = 0, \pm 1, \pm 2 \ldots \pm(n+1)$ gives

$$x[\,1 - \frac{(x^2-1)}{3\;+} \; \frac{3(x^2-4)}{5\;-} \; \frac{2(x^2-9)}{7\;+} \; \cdots \; \frac{(-1)^n(n+(-1)^n)(x^2-n^2)}{2n+1}\,].$$

(35.4)

c) While for the tangent function $\tan\frac{\pi}{2}x$, we have

$$\frac{\pi x}{2}[\,\frac{1}{1-} \; \frac{x^2}{1-} \; \frac{(x^2-1)}{3\;-} \; \frac{4(x^2-4)}{5\;-} \; \cdots \; \frac{-n^2(x^2-n^2)}{2n+1}\,]\,. \qquad (35.5)$$

Successive convergents of this last expression lie on opposite sides of tan $^{\pi}/_{2}$ x and provide reasonable approximations to it. But in general the addition of a partial quotient only has a large influence on the approximations near the points being fitted; for a given x the convergents tend to oscillate. These examples are of analytic interest but we will not, discuss them further here.

In principle to construct a rational function which fits a set of values of a function f(x) at given points x_1 x_2 .... x_n , we simply set up the linear equations

$$P_o = f + (x - x_1,)f_1$$
$$p_l f = f_l + (x - x_2)f_2$$
$$\text{---------------} \qquad (35.6)$$
$$P_{n-1} f_{n-2} = f_{n-1}, + (x - x_n)f_n$$

and successively compute p_{i-1} at x_i and f_i at x_{i+1}. . .... x_n

These equations can be written as the C. F.

$$f = \frac{P_o}{1+} \quad \frac{P_1(x-x_1)}{1\ +} \quad \frac{P_2(x-x_2)}{1\ \ +} \cdots \frac{P_{n-1}(x-x_{n-1})}{1 \quad +(x-x_n)\ \frac{f_n}{f_{n-1}}} \qquad (35.7)$$

and hence we obtain a rational function which fits the set of values as shown by the error term for this C.P.

$$f - \frac{P_n}{Q_n} = (-1)^n \ (x-x_1)\ (x-x_2)\ ..\ (x-x_n)\ \frac{f_n}{Q_n}. \qquad (35.8)$$

P_n (x) and Q_n(x) are the n^{th} numerator and denominator polynomials.

However there is little to ensure that the rational function is a good approximation to f. There is not a unique rational function which fits the data. Different rational approximations would be obtained, for example, by fitting some values to a polynomial, and then the remainder to a C.F. in the above manner The position is rather similar to that of S fractions, we require further information on the general form that the rational approximations should take.

36. General Fitting of Conditions

The coefficients in all the continued fractions that we have discussed have been generated by setting up three term relations, thus in principle, by propagating the necessary information through these linear relations we can switch from producing one type of fraction to developing another type. When the Maclaurin series for a function is known, hut only a limited number of asymptotic terms in the form $\frac{H_{n+1}}{x^n}$ are available, it is possible to incorporate the latter by initially forming M fraction type partial quotients and later switching to the S fraction type. Similarly the rational function approximations obtained by truncating a C.F. designed to match derivatives can be made to take a particular value at a given point (or points) by appending to the truncated C.F. a suitable additional partial quotient.

To illustrate this point let us modify the convergents of the M fraction (33.9) for sin πt so that the rational functions take the value one when t = ½, as this is the point where the maximum error occurs. Computing the linear relations successively with t = ½, we find the term to be added to the truncated C.F to produce this value of one.

The values of the fifth convergent of (33.9) $F_{\circ/5}$ are given in the table on page 20. The values of the sixth convergent of

$$\frac{P_1 t\tau}{1+} \quad \frac{P_2 t\tau}{1\ +} \quad .... \quad \frac{P_5 t\tau}{1\ +} \quad \frac{P_6^* t\tau}{1} \,/ \; + \; 0(t-\frac{1}{2}) \qquad (36.1)$$

with P^*_6 = - 0·0351623 instead of the value of p_6 of (3.3.9) are given for the same values of t below

x	t	sixth convergent of (36·1)
$\frac{1}{4}$	$\frac{1}{5}$	0·58773 5252
$\frac{1}{2}$	$\frac{1}{3}$	0· 86602 5404
$\frac{3}{4}$	$\frac{3}{7}$	0·97492 7912
1	$\frac{1}{2}$	0.974927912

These figures, which are the correct values of sin π t to nine decimal places for the four values of t, indicate a significant improvement in the numerical accuracy of the approximations. We have, of course, lost some of the simplicity in our method of generating the approximations to sin πt.

Conclusion

The approach to rational functions through linear equations that we advocate in this report is both powerful and flexible. It is clear that rational functions, like polynomials, can incorporate much of the available information on a given function, and produce a satisfactory representation of it. Many of our approximations fit the function smoothly and hence, unlike Chebyshev series, can give reasonable approximations to the derivatives of the function as well as the function itself.

The most significant result is contained in Sections 33 and 34 where we design rational functions that can fit the derivatives of a function at two or more points in the complex plane. There will be difficulties in obtaining the initial series, but the method has considerable potential for approximating the solutions of differential equations and for approximating indefinite integrals etc. The technique is simple and the approximations often efficient, further the maximum error can often be roughly located and troublesome branch cuts will frequently be squeezed away fro the region of interest in the complex plane. The error terms are given only in terms of series, and, except in a few special cases, it will not be easy to obtain asymptotic estimates for these errors.

Determining the branch points from a Taylor series for the function is not an easy problem, and indeed the position of the branch cuts is dependent on the method of approximating the function. Our approach in the first section is simple but calls for a deeper and more thorough analysis especially of the nature of the branch points. Nevertheless some useful expressions for the discriminant Δ for quasi-periodic continued fractions are recorded.

References

Murphy, J.A. and Drew, D.M. Continued fractions with error estimates for some special functions (1973). Tech. Report TR/25 Brunel University.

Murphy, J.A. and Drew, D.M continued fraction solutions of linear differential fractions (1973). Tech Report TR/26 Brunel University

Wall, H.S. Analytical theory of continued fractions