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LAGRANGIAN FINITE ELEMENT and
FINITE DIFFERENCE METHODS FOR
POISSON PROBLEMS.

BY

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ABSTRACT

The use of Lagrangian finite element methods for solving a Poisson problem produces systems of linear equations, the global stiffness equations. The components of the vectors which are the solutions of these systems are approximations to the exact solution of the problem at nodal points in the region of definition. There is thus associated with each nodal point an equation which can be thought of as a difference equation. Difference equations resulting from the use of polynomial trial functions of various orders on regular meshes of square and isosceles right triangular elements are derived. The rival merits of this technique of setting up a standard difference equation, as distinct from the more usual practice with finite elements of the repeated use of local stiffness matrices, are considered.

1 .Introduction and Finite Element Method

Two methods for producing numerical approximations to the solutions of elliptic boundary value problems are those of finite elements and finite differences. However, these methods are closely related. In this paper an approach is adopted whereby the linear equation which results at a mesh point from the application of the finite element method is regarded as a difference equation. Thus emphasis is placed on the idea of using the finite element method with a regular grid to produce difference molecules which are used repeatedly over the mesh and which change only when affected by the boundary.

The discussion is restricted to two dimensional Poisson problems with Dirichlet boundary conditions, Let the function $u(x, y)$ be the solution of the problem

$$\left. \begin{aligned} -\Delta | u(x, y) | &= g(x, y), & (x, y) \in \Omega, \\ u(x, y) &= 0, & (x, y) \in \partial\Omega, \end{aligned} \right] \quad (1)$$

where $\Omega \in E^2$ is an open bounded domain with polygonal boundary $\partial\Omega$ Δ is the Laplacian operator and $g \in L_2(\Omega)$. Under these assumptions the solution $u(x, y)$ of (1) minimizes the energy functional

$$I[V] = \iint_{\Omega} \left[\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 - 2gv \right] dx dy \quad (2)$$

over the class of functions $W_2^1(\Omega)$, see Zlamal (11). As

usual $W_2^1(\Omega)$ is the Sobolev space of functions which

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together with their first generalized derivatives exist

and are in L_2 , and $W_2^0(\Omega)$ is the subspace of functions in $W_2^1(\Omega)$ which are identically zero on $\partial\Omega$.

The region Ω is divided into a number of non-overlapping triangular or rectangular elements, and it is assumed that on the partition there are m mesh nodes: m in Ω and n nodes on $\partial\Omega$

An approximation $U(x, y)$ to $u(x, y)$ is constructed using the Ritz method by solving

$$\min_{S_h} (I[V]) \quad (3)$$

where S_h is an m -dimensional subspace of $W_2^1(\Omega)$, and the parameter h is a measure of the size of the elements.

The finite dimensional space S_h consists of functions which are piecewise polynomial over Ω , and satisfy the homogeneous boundary conditions on $\partial\Omega$.

In each element for Lagrangian methods the choice of the approximating function is motivated by an interpolating function which takes the values of u at k nodes in the element. Let the interpolant \tilde{u} in each element have the form

$$\tilde{u}(x, y) = \sum_{i=1}^k u_i \phi_i(x, y), \quad (4)$$

where the ϕ_i are the cardinal basis functions (shape functions) of the interpolation with respect to the values u_i of u at the k nodal points. Note that in (4) the numbering of the shape functions is local to the element. The approximating function $U^e(x, y)$ then has

in each element the form

$$U^e(x, y) = \sum_{i=1}^k U_i^e \phi_i(x, y), \quad (5)$$

where now the U_i^e are the values of U at the same nodal points of the element as those of u in (4). For the whole of Ω , the nodes have a global ordering $p = 1, 2, \dots, m$, and the collection of all shape functions associated with the p^{th} node defines a function $B_p(x, y)$. This method of construction results in the B_p being nonzero only in elements which contain the p^{th} node. Thus the shape functions Φ of (4) and (5), considered over the total number of elements in Ω , form the linearly independent set

$$\{B_p\}, \quad p = 1, 2, \dots, m,$$

members of which span the space \mathcal{S}_h A function $V \in \mathcal{S}_h$ can therefore be written as

$$V(x, y) = \sum_{p=1}^m U_p B_p(x, y). \quad (6)$$

In order that \mathcal{S}_h may be a subspace of W_2^1 the interpolating polynomials and nodes in each element

must be chosen so that $\forall \varepsilon \in \mathcal{S}_h$ satisfies certain continuity

properties across interelement boundaries. The

condition that $\mathcal{S}_h \subset W_2^1$ is the conforming condition, and

for problems of type (1) it is that $\mathcal{S}_h \subset C^0(\bar{\Omega})$ where

$$\bar{\Omega} \equiv \Omega \cup \partial\Omega.$$

Since $\mathcal{S}_h \subset W_2^1(\Omega) \forall \varepsilon \in \mathcal{S}_h$ is admissible function

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(see | 8 |) so that we may write

$$\begin{aligned} \|V\| &= \sum_e \iint_{\Omega_e} \left[\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 - 2gV \right] dx dy \\ &\cong \sum_e \|U^e\|, \end{aligned} \tag{7}$$

where u^e is the local trial function in the e^{th} element and Ω_e is the area of that element. From (5) and (7) it follows that for the e^{th} element:

$$\begin{aligned} \|U^e\| &= \left\| \sum_{i=1}^k U_i^e \phi_i(x, y) \right\|, \\ &= \|U^e\|_{lxk}^T K_{kxk}^e U_{kx1}^e - 2 \|U^e\|_{lxk}^T z_{kx1}, \end{aligned} \tag{8}$$

where, see Zlamal [11], K_e is the elements stiffness

matrix, $U^e = \text{vec}(u_1, u_2^e, \dots, u_k^e)$, z is a column vector the components of which depend on $g(x, y)$. The summing of $\|U^e\|$ to form $\|V\|$ as in (7) produces a quadratic function of the variables U_1, U_2, \dots, U_m , where these U 's are simply the total number over all the elements of the U^e associated with the m nodes in Ω . The vector $U = \text{vec}(U_1, U_2, \dots, U_m)$ can be found by solving, the linear system

$$\frac{\partial \|V\|}{\partial U_p} = 0, \quad p = 1, 2, \dots, m. \tag{9}$$

which, after the boundary conditions have been taken into account, is written

$$KU = Z. \tag{10}$$

The mxm matrix K is the global stiffness matrix. Substitution of u in (6) gives the approximation U(x,y) to the solution u(x,y) of (1).

2. Difference Stars Using Langrangian Interpolation with C⁰ Approximating Functions

The above procedure may be termed a standard finite element approach for setting up the global stillness equations (10), and requires that the local stiffness matrix K^e be calculated for each element. For a regular mesh, using co-ordinates local to each element, the K^e will all be the same. However, their repeated use will be necessary for the construction of (10).

An alternative method, which is useful for regular meshes, is suggested by thinking of the equations (10) as difference equations, so that the equation $\partial I |V| / \partial U_p = 0$ is the difference equation at the pth node. Nonzero contributions to the function $\partial I[V] / \partial U_p$ come only from the set of elements, {N_p}, which contain the pth node. Thus the summation in (7) need only be performed over these elements. The technique adopted here is for every node in Ω to form $I | u^e |$, I, and hence $\partial I | u^e | / \partial U_p$, for all elements of {N_p} and then to form

$$\sum_{N_p} \frac{\partial I [U^e]}{\partial U_p} = 0 \quad (11)$$

Equation (11) is exactly the pth equation of (9). As a difference equation it is derived only once. However, the region Ω is covered with a regular mesh, and so equation (11) is used for all nodal points at

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which the element configuration is as in $\{ N_p \}$. As before the boundary conditions have to be taken into account

In Sections 2.1. - 2.3 various interpolating functions are used to produce difference molecules at nodes on regular triangular and square meshes. As the main interest is the form of the difference molecules at general interior mesh nodes, no account is taken of the boundary or boundary conditions.

2.1 Standard Triangular Elements

Consider a mesh made up entirely of isosceles right angled triangles as in Figure 1.

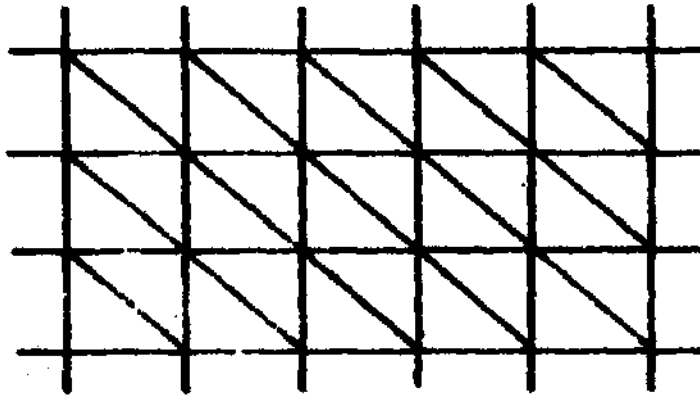


Figure 1

The complete M^{th} order polynomial

$$I_M(x) = \sum_{i+j=0}^M a_{ij} x^i y^j$$

can be used to interpolate a function at $k = \frac{1}{2}(M+1)(M+2)$

symmetrically placed nodes in a triangle. This k is the same as the upper limit of summation of the series in (5). Interpolation using different orders of polynomial is now illustrated.

Linear Interpolation ; $M = 1$

Here $k = 3$ and in any triangle of the mesh the nodes are taken at the vertices. Linear interpolation to the values of U at these nodes produces over Ω a global approximating function which is continuous, so that the conforming condition is satisfied.

Use is now made of the standard triangle T , see Birkhoff and Mansfield [1]. In terms of local co-ordinates (X, Y) this has vertices

$(0, 0)$, $2 (h, 0)$ and $3 \equiv (0, h)$, as in Figure 2.

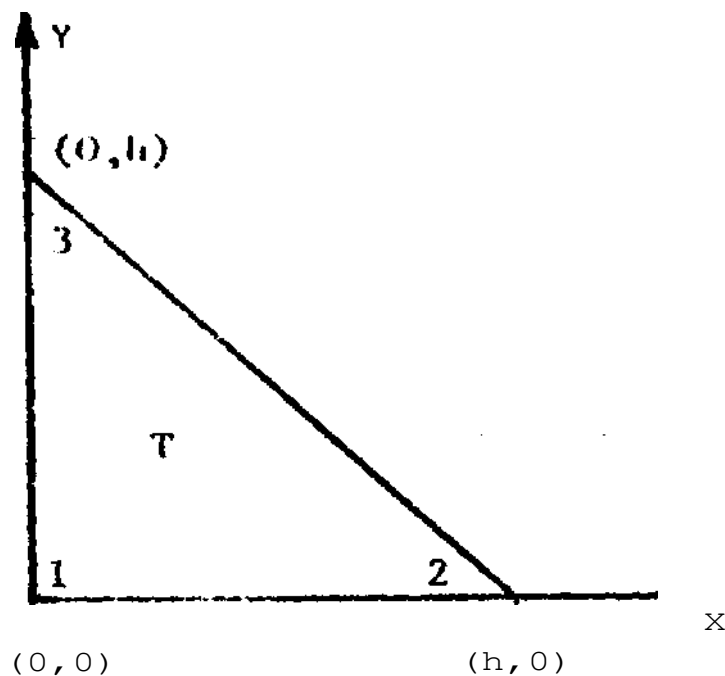


Figure 2

The Function which is linear in X and Y and which interpolates to the values $U_1^e \equiv U^e(0,0)$, $U_2^e \equiv U^e(h,0)$, $U_3^e \equiv U^e(0,h)$ can be written in cardinal basis form as

$$U^e(X, Y) = U_1^e(1 - X/h - Y/h) + U_2^e X/h + U_3^e Y/h . \quad (12)$$

Thus for the standard triangle, if $G(X, Y) = g(x, y)$,

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$$\begin{aligned}
 I[U^e] &= \iint_I \left[\frac{1}{h^2} \{ (U_2^e - U_1^e)^2 + (U_3^e - U_1^e)^2 \} \right. \\
 &\quad \left. - 2G(x, y) \{ U_1^e (1 - \frac{x}{h} - \frac{y}{h}) + U_2^e \frac{x}{h} + U_3^e \frac{y}{h} \} dx dy \right] \\
 &= \frac{1}{2} \{ (U_2^e - U_1^e)^2 + (U_3^e - U_1^e)^2 \} - 2F^e(U_1^e, U_2^e, U_3^e), \tag{13}
 \end{aligned}$$

where F^e is the appropriate linear function of the variables U_1^e, U_2^e, U_3^e , so that

$$\frac{\partial I[U^e]}{\partial U_1^e} = 2U_1^e - U_2^e - U_3^e - 2 \frac{\partial F^e}{\partial U_1^e}, \tag{14}$$

$$\frac{\partial I[U^e]}{\partial U_2^e} = U_2^e - U_1^e - 2 \frac{\partial F^e}{\partial U_2^e} \tag{15}$$

and

$$\frac{\partial I[U^e]}{\partial U_3^e} = U_3^e - U_1^e - 2 \frac{\partial F^e}{\partial U_3^e}. \tag{16}$$

Any interior point of the mesh is a node for six elements, the assemblage of these being as in Figure 3. In particular the p^{th} node of the global ordering may be taken as the node 1 of the standard triangle T. Considering only the part of $I[U]$ which is associated with the Laplacian $-\Delta U$, it follows from (14) that the contribution to $\frac{\partial I[U]}{\partial U_p}$ from the triangle T at the point $(x, y) \in \Omega$ is

$$2 U(x, y) - U(x+h, y) - U(x, y+h) .$$

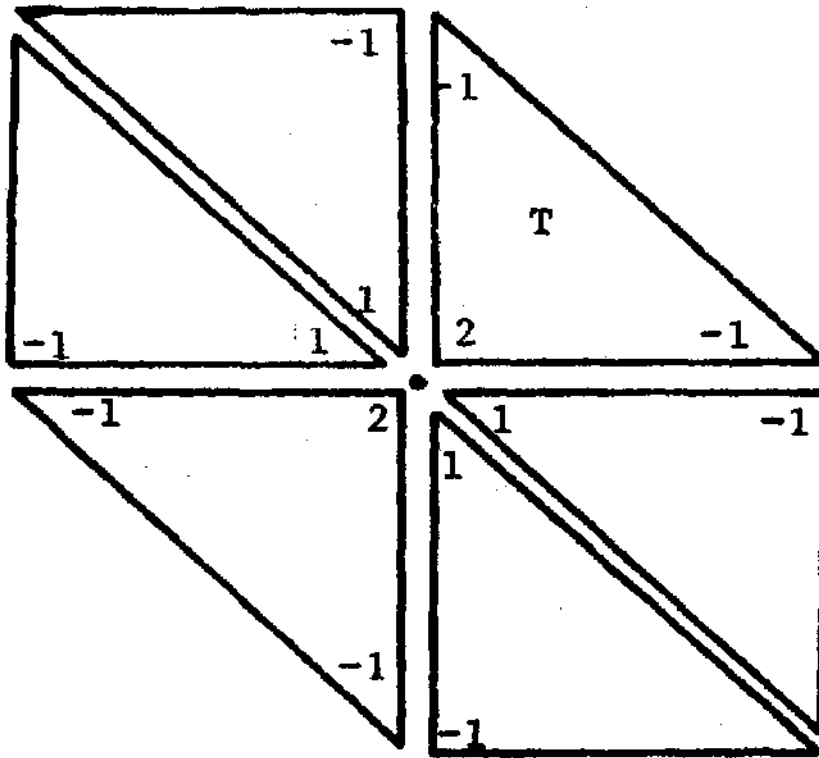


Figure 3

The contributions from the other five elements of N_p follow immediately from (14)-(16) using symmetry, and on summation produce at the point (x,y) the familiar five-point finite difference replacement

$$4 U(x,y) - U(x+h,y) - U(x,y+h) - U(x-h,y) - U(x,y-h) \quad (17)$$

for the Laplacian, which has $O(h^2)$ local discretization error.

The fact that this five point formula can be obtained using linear interpolants is well known, and has been pointed out by Fix and Strang [6], Pian [7] and Whiteman [10].

10.

Quadratic Interpolation; M = 2

Here $k = 6$ and in any triangle the nodes are taken as the vertices and the centre points of the sides. Interpolation with the complete quadratic in x and y to values at these nodes again produces a continuous global approximating function. For the standard triangle the quadratic interpolant to the function values U^e_i $i = 1, 2, \dots, 6$ at the six nodes $1 \equiv (0, 0)$,

$2 = (h, 0)$, $3 \equiv (0, h)$, $4 \equiv (\frac{h}{2}, 0)$, $5 \equiv (\frac{h}{2}, \frac{h}{2})$, $6 \equiv (0, \frac{h}{2})$ can be written in cardinal basis form as

$$U^e(X, Y) = U_1^e \left(1 - \frac{X}{h} - \frac{Y}{h}\right) \left(1 - \frac{2X}{h} - \frac{2Y}{h}\right) + U_2^e \frac{X}{h} \left(\frac{2X}{h} - 1\right) + U_3^e \frac{Y}{h} \left(\frac{2Y}{h} - 1\right) + U_4^e \frac{4X}{h} \left(1 - \frac{X}{h} - \frac{Y}{h}\right) + U_5^e \frac{4XY}{hh} + U_6^e \frac{4Y}{h} \left(1 - \frac{X}{h} - \frac{Y}{h}\right). \quad (18)$$

In a completely analogous manner to that used with linear interpolation, the interpolant (18) leads at a mesh point with co-ordinates (x, y) , which is a vertex of a triangle, to a non-standard nine-point difference replacement for the Laplacian of the form

$$12U(x, y) - 4\left\{U\left(x + \frac{h}{2}, y\right) + U\left(x, y + \frac{h}{2}\right) + U\left(x - \frac{h}{2}, y\right) + U\left(x, y - \frac{h}{2}\right)\right\} + U(x + h, y) + U(x, y + h) + U(x - h, y) + U(x, y - h). \quad (19)$$

Formula (19) is clearly a combination of five-point replacements at (x, y) based on mesh lengths h and $\frac{h}{2}$.

However, at a mid-side node (e.g. nodes 4, 5 or 6 of the standard triangle) contributions to $\partial I[U] / \partial U_p$

come only from two elements, and in this case the difference replacement is again the familiar five-point formula (17), except that now it is based on a mesh of length $h/2$.

It is thus clear that the use of this form of quadratic interpolation leads to a double system of difference equations the form of which differs between vertex and mid-side nodes.

Cubic and Higher Order Interpolation

In the previous two cases the difference replacements produced have been either of a familiar form or closely related to this. There is with the increase in order of interpolating polynomial an increase in the number of nodes in each triangle ($M = 3, k = 10; M = 4, k = 15; \dots$). There are also more classes of nodes and for each class the replacement is so complicated as to be worthless in its own right as a difference replacement.

As an illustration, we give in Figure 4 in tabular form the replacement at a mesh point which is an element vertex obtained using cubic interpolants to function values at the vertices, the points of trisection of the sides and the centre of the triangle of the mesh of isosceles right angled triangles.

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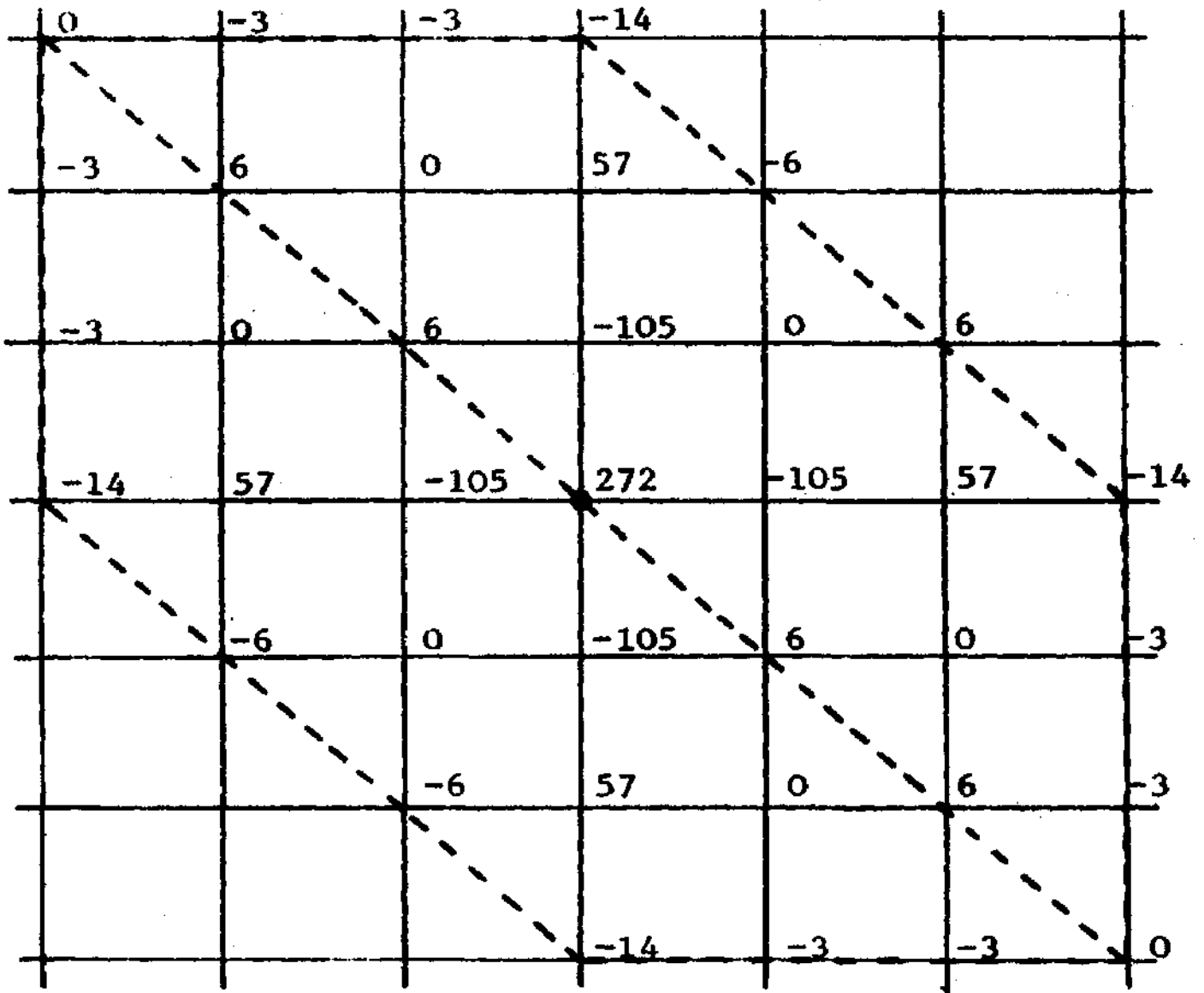


Figure 4.

2.2 Standard Square Elements

Consider now a mesh made up entirely of squares of side h . The interpolation is performed with polynomials of the form

$$Q_M(x, y) = \sum_{i=0}^M \sum_{j=0}^M \alpha_{ij} x^i y^j,$$

where successively $M = 1$ (bilinear), $M = 2$ (biquadratic) etc.

Bilinear Interpolation; $M = 1$

In this case there are four terms in $Q(x, y)$ so that, using square elements, a continuous global approximating function can be obtained if interpolation to function values at the corners of the squares is performed.

The bilinear function which interpolates to the values $U_1^e \equiv U^e(0, 0)$, $U_2^e \equiv U^e(h, 0)$, $U_3^e \equiv U^e(h, h)$, $U_4^e \equiv U^e(0, h)$ at the corners of the standard square is, in cardinal basis form

$$U^e(x, y) = U_1^e \left(1 - \frac{x}{h}\right) \left(1 - \frac{y}{h}\right) + U_2^e \frac{x}{h} \left(1 - \frac{y}{h}\right) + U_3^e \frac{x}{h} \frac{y}{h} + U_4^e \left(1 - \frac{x}{h}\right) \frac{y}{h}. \quad (20)$$

Use of (20) leads at a point (x, y) of the mesh to another non-standard nine-point finite difference replacement for the Laplacian of the form

$$\begin{aligned} 8U(x, y) - U(x+h, y) - U(x, y+h) - U(x-h, y) - U(x, y-h) \\ - U(x+h, y+h) - U(x-h, y+h) - U(x-h, y-h) - U(x+h, y-h) \end{aligned} \quad (21)$$

14.

This replacement has $O(h^2)$ local discretization error, and is given by Birkhoff, Schultz and Varga in [2].

Biquadratic Interpolation; $M = 2$

For biquadratic interpolation nine nodes on the standard square are used, and these are taken at the

Points $1 \equiv (0, 0)$, $2 \equiv (\frac{h}{2}, 0)$, $3 \equiv (h, 0)$, $4 \equiv (0, \frac{h}{2})$,
 $5 \equiv (\frac{h}{2}, \frac{h}{2})$, $6 \equiv (h, \frac{h}{2})$, $7 \equiv (0, h)$, $8 \equiv (\frac{h}{2}, h)$, $9 \equiv (h, h)$.

The biquadratic interpolant to function values at these nodes is, in cardinal basis form,

$$\begin{aligned} U^e(X, Y) = & \{ U_1^e p(X) + U_2^e q(X) + U_3^e r(X) \} p(Y) \\ & + \{ U_4^e p(X) + U_5^e q(X) + U_6^e r(X) \} q(Y) \\ & + \{ U_7^e p(X) + U_8^e q(X) + U_9^e r(X) \} r(Y) , \end{aligned} \quad (22)$$

where $p(t) = (1-t)(1-2t)$,

$$q(t) = 4t(1-t) ,$$

$$r(t) = t(2t-1) .$$

Use of (22) leads at a node (x, y) which is a corner of a mesh square to the twenty-five-point replacement of Figure 5.

	1	5	3	5	1
5		-16	-18	-16	5
-3	-18	112	-18	-3	
5	-16	-18	-16	5	
-1	5	-3	5	-1	

Figure 5

At nodes which are respectively centre points of the squares and mid-points of sides of squares the replacements of Figures 6 and 7 are obtained. The difference replacements of Figures 5, 6 and 7 all have $O(h^2)$ local discretization errors.

16.

-1	-3	-1
-3	16	-3
-1	-3	-1

Figure 6 (Mid-point of square)

5	0	5
-16	-48	-16
-18	176	-18
-16	-48	-16
5	0	5

Figure 7 (Mid-point of side)

2.3 A. Nonconforming Element

If in each element of a triangular mesh a Linear trial function interpolates to function values at the mid-points of the sides of the triangle, then the global approximating function so produced is not in general continuous and the conforming condition is violated.

Suppose that for the standard triangle of Section 2.1 the nodes are now taken at the points $(\frac{h}{2}, 0)$, $(\frac{h}{2}, \frac{h}{2})$ and $(0, \frac{h}{2})$. In cardinal basis form the linear interpolant to the function values at these points can be written as

$$U^e(X, Y) = U(\frac{h}{2}, 0) (1 - 2\frac{Y}{h}) + U(\frac{h}{2}, \frac{h}{2}) (2\frac{X}{h} + 2\frac{Y}{h} - 1) + U(0, \frac{h}{2}) (1 - 2\frac{X}{h}). \quad (23)$$

Use of (23) leads at a point of type $(\frac{h}{2}, \frac{h}{2})$ that is the mid-point of the hypotenuse of a triangle, to the difference replacement

$$4U(x, y) - U(x + \frac{h}{2}, y) - U(x, y + \frac{h}{2}) - U(x - \frac{h}{2}, y) - U(x, y - \frac{h}{2}). \quad (24)$$

However, at the mid-point of a vertical side ($x = \text{constant}$) of a triangle, the replacement is

$$2U(x, y) - U(x + \frac{h}{2}, y) - U(x - \frac{h}{2}, y) \quad (25)$$

and at the mid-point of a horizontal side ($y = \text{constant}$) it is

$$2U(x, y) - U(x, y + \frac{h}{2}) - U(x, y - \frac{h}{2}).$$

Both of the difference replacements (25) and (26) are inconsistent.

3. Discussion and Application to Model Problem

It has been shown in Sections 2.1 - 2.3 that finite element techniques, based on Lagrangian interpolants with triangular and square elements, produce difference replacements for the Laplacian, and that the systems of equations so derived are exactly those of (9) and (10). Most of these replacements are not those with which the users of finite differences are familiar; see for example Collatz [4], pp.542-543. However, the methods of Sections 2.1 and 2.2 produce consistent replacements for which, with the usual techniques based on Taylor's series, it can be shown that the local discretization errors are $O(h^2)$. Some of these schemes are unconventional in that with a single trial function the finite element technique produces several difference replacements; the particular replacement which is appropriate to a mesh point being determined by the position of this (nodal) point in an element. The mesh points thus fall into classes with each of which is associated a particular type of replacement. This property suggests that it may be possible, by permuting the rows and columns, to rearrange the global stiffness matrix K so that it can be partitioned in a form which facilitates the solution of the linear system (10).

Many theoretical bounds have been derived for the error in the finite element approximations to the solutions of problems of the type (1); see for example Bramble and Zlamal [3] for triangular elements and

Birkhoff, Schultz and Varga [2] for rectangular elements. Provided that the conforming condition

$(S^0, W_{1/2}^0(\Omega))$ is satisfied, and that the solution u of

(1) has specific derivatives bounded throughout $\bar{\Omega}$, these bounds hold. The difference methods of Sections 2.1 and 2.2, being exactly "conforming finite element methods", are subject to these same error bounds and so, under the same conditions on u , convergence with decreasing mesh size of the "finite difference" solution to the true solution is assured.

The discussion in this paper has been limited to finite element methods based on Lagrangian interpolation. In higher order problems it is usual for trial functions to interpolate to values of derivatives at nodal points as well as to function values. These are finite element methods based on Hermite interpolation, such as are discussed in (2). In biharmonic problems, for example, the integrand in the energy functional which corresponds to (2) contains second derivatives, and conforming global trial functions possess continuous first derivatives. The final system of global stiffness equations in this case involves derivative values. Clearly such linear equations involving function and derivative values as unknowns cannot be thought of as conventional difference equations. However, in the difference context such Hermitean methods (mehrstellenverfahren) have been considered by Collatz [5].

The difference approach based on finite element techniques does not allow for the full exploitation of

the versatility of the standard finite element method; namely that the form of local trial function in each element may be varied over Ω so as to produce better numerical approximations. Further, the difference approach is only really useful when regular meshes are involved. However, as has been indicated, when the difference approach is viable, it produces equations which are identical with the global stiffness equations.

The criterion by which to judge the two methods is the respective computation times taken to produce the same numerical solutions. The approach adopted here is to use the standard finite element method to solve a model problem and to analyse the times taken for the various parts of the computation. This computation involves the repeated call of local stiffness matrices for the assembly of the global stiffness matrix; a process which will largely be absent in the difference approach. In the model problem $\bar{\Omega}$ is the square

$$\bar{\Omega} \{ (x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1 \}$$

with interior Ω and boundary $\partial\Omega$, and the function $u(x,y)$ satisfies

$$-\Delta[u(x,y)] = x-x^2+y-y^2, \quad (x,y) \in \Omega,$$

$$u(x,y) = 0, \quad (x,y) \in \partial\Omega.$$

This problem has exact solution $u(x, y) = \frac{1}{2} xy (1-x) (1-y)$.

The region Ω is partitioned into right triangular elements by subdivision first into equal squares, and then by further subdivision of each square into two triangles along the diagonal parallel to $x+y = 1$. Numerical approximations U to u are calculated using respectively piecewise linear, quadratic and cubic approximating functions on meshes of the above type. As accuracy is not the main aim here, the partitions for the three cases are chosen so that the total number of nodes is approximately constant. Results at two points in are given in Table 1. Details of the partitions and the global stiffness matrices are given

Trial functions in Each Element	Total Number of Nodes	Values of U at $(\frac{1}{2}, \frac{1}{2})$	Values of U at $(\frac{3}{4}, \frac{1}{4})$
Linear	289	.031017	.017496
Quadratic	289	.031254	.017578
Cubic	256	.031243	.017581
Exact solution		.031250	.017578

Table 1

in Table 2, together with a break down of the computation times. It is seen that the generation of the local stiffness matrix, and of the global stiffness equations through its repeated call, in the three cases takes respectively 45, 30 and 35 seconds. When the difference approach is adopted, the generation of the

TABLE 2

PARTITIONS			
Trial Functions	Linear	Quadratic	Cubic
Length of short sides of each triangle	1/16	1/8	1/5
No. of nodes in $\bar{\Omega}$	289	289	256
No. of boundary nodes	64	64	60
No. of elements	512	128	50
GLOBAL STIFFNESS MATRICES			
Bandwidth	42	82	96
No. of non-zero elements (i) in half of g.s.m.	709	793	1197
(ii) with fill in using Gaussian elimination	3363	4653	6737
COMPUTATION TIMES IN SECS ON CDC 6500			
Generation of mesh, co-ordinates, data	30	25	20
Generation of standard element stiffness matrix	7	14	24
Generation of global stiffness matrix	38	16	11
Gaussian elimination	8	15	35
Back substitution	3	6	6
TOTAL TIMES	86	76	96

global stiffness equations in each case takes approximately 5 seconds. As can be seen from Table 2, this results in savings of the order of 45%, 40% and 30% in the respective computation times.

Clearly this model problem is ideally suited for the difference approach, so that this is not a totally fair comparison. However, it does illustrate the potential of this alternative way of viewing the global stiffness equations.

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