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SMOOTH INTERPOLATIONWITHOUT TWIST CONSTRAINTS

by

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COMPUTER AIDED GEOMETRIC DESIGN

SMOOTH INTERPOLATION WITHOUT TWIST CONSTRAINTS

John A. Gregory

1. Introduction

Smooth or blending function interpolants, which match a given function and slopes on the boundary of a rectangle or a triangle, usually require that the cross derivative or twist terms be defined unambiguously at vertices. For example, the surfaces of Coons[3] over rectangles and the interpolation schemes of Barnhill, Birkhoff, and Gordon[1] over triangles require that certain cross derivatives be compatible at vertices. Smooth interpolation schemes which avoid such restrictions could be useful for the piecewise generation of surfaces in computer aided geometric design. This paper considers two such schemes, one over a rectangle and the other over a triangle.

The interpolation scheme for the rectangle is a modification of an interpolant of Coons, which was later developed through Boolean sum theory by Gordon [4]. The interpolant is modified by the addition of rational terms so that the compatibility constraints are removed.

The smooth interpolants over triangles of Barnhill, Birkhoff and Gordon can also be modified by the addition of rational terms, see for example, Barnhill and Gregory[2] or Mansfield[6]. This approach, together with a detailed discussion of smooth interpolation over triangles, is presented in the preceding paper by R.E. Barnhill. This present paper describes a new interpolation scheme for the triangle. This scheme has a relatively simple construction, it is symmetric in that each side of the triangle is treated in the same way, and it involves no compatibility constraints. Further details and generalizations of this scheme are given in Gregory[5].

For simplicity of presentation, the interpolation schemes in this paper are considered as bivariate surfaces which interpolate a function F and its slopes defined along the boundary of a rectangle or a triangle in cartesian (x,y) space. However, the interpolation schemes are immediately applicable to the construction of a bivariate vector-valued function

$$P\_(s,t) - (x(s,t), y(s,t), z(s,t)],$$

where s and t are parametric variables and the rectangles or triangles are defined in the parametric (s,t) space, see R.E. Barnhill's paper.

The smooth interpolants in this paper may be pieced together to give a surface which is $C^1(\Omega)$ over a rectangular and/or triangular subdivision of a polygonal region ft. Also, by defining the function and slopes along boundary interfaces in terms of data on that boundary, C^1 (Ω) finite dimensional piecewise interpolants can be derived. For example a twelve parameter interpolant for the square, and a nine parameter interpolant for the triangle, which involve the function F and its first two partial derivatives at each vertex, can be derived by suitable choice of the boundary data. Although, for incompatible boundary data, the interpolants have discontinuous cross derivatives at the vertices, they should compare favourably with other known C^1 interpolants which impose zero second order derivative conditions at each vertex, for example the Coons patch with zero twist. (See Example 4.1.)

The interpolants considered in this paper have the property that they are able to reproduce simple polynomial surfaces. The set of polynomials which are reproduced by an interpolation scheme is defined here as the *precision set* of the interpolant and these sets are derived for each of the schemes of this paper. The precision set property gives some indication that the interpolation schemes are reasonable for design purposes.

2. Smooth C^1 Interpolation on Rectangles

The unit square S with boundary ¶S is considered with vertices at (0,0), (1,0), (1,1) and (0,1). Any arbitrary rectangle can be obtained by an affine transformation of this standard square.

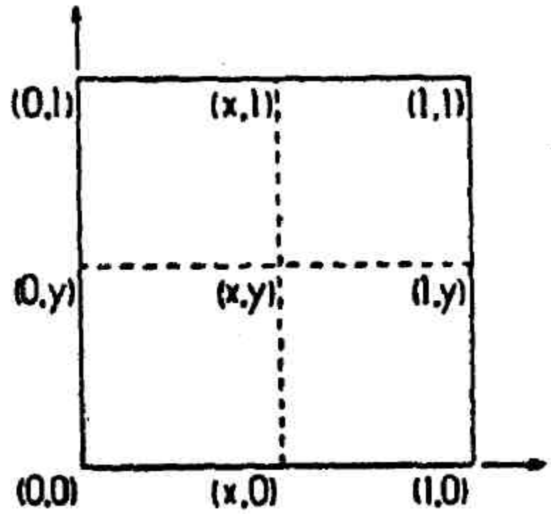

Figure 1

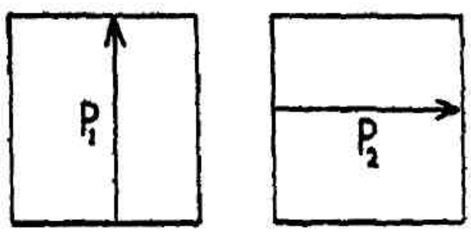

Figure 2

For $F(x,y) \in c^1(\partial S)$ and $(x,y) \in S$, cubic Hermite interpolation projectors along parallels to the sides $x = 0$ and $y = 0$ are respectively defined by

$$(2.1)\ P_1 F = \sum_{j=0}^{1} \psi_j(y)\, F_{0,j}(x,0) + \sum_{j=0}^{1} \psi_j(y) F_{0,j}(x,1),$$

$$(2.2)\ P_2 F = \sum_{i=0}^{1} \psi_i(x)\, F_{i,0}(0, y) + \sum_{i=0}^{1} \psi_i(x)\, F_{i,0}(1, y),$$

where

$$(2.3) \begin{cases} \psi_0(t) = (t-1)^2 \ (2t+1), & \psi_1(t) = (t-1)^2\, t, \\ \psi_0(t) = t^2\,(-2t+3), & \psi_1(t) = t^2\,(t-1), \end{cases}$$

are the cardinal basis functions for cubic Hermite interpolation on $0 \leq t \leq 1$, see Figures 1 and 2, The function P_1F interpolates F and its first derivatives on y - 0 and y = 1 and P_2F has dual properties on x = 0 and x = 1. The Boolean sum projector $P_1 \oplus + P_2$ is defined by

$$(2.4) \qquad (P_1 \oplus + P_2)F = (P_t + P_2 - P_1P_2)F$$

where from (2.1) and (2.2) it follows that

$$(2.5)\ P_1P_2F = \sum_{i,j\leq 1} \psi_i(x)\psi_j(y) \left[\frac{\partial^{j+i} F}{\partial y^j \partial x^i}\right](0,0)$$

$$+ \sum_{i,j\leq 1} \psi_i(x)\psi_j(y) \left[\frac{\partial^{j+i} F}{\partial y^j \partial x^i}\right](1,0)$$

$$+ \sum_{i,j\leq 1} \psi_i(x)\psi_j(y) \left[\frac{\partial^{j+i} F}{\partial y^j \partial x^i}\right](0,1)$$

$$+ \sum_{i,j\leq 1} \psi_i(x)\psi_j(y) \left[\frac{\partial^{j+i} F}{\partial y^j \partial x^i}\right](1,1)$$

If $F \in C^1(\partial S)$ and satisfies the twist compatibility condition

$$(\mathbf{2.6}) \qquad \left[\frac{\partial^2 F}{\partial x \partial y}\right](V_k) = \left[\frac{\partial^2 F}{\partial y \partial x}\right](V_k)$$

at each vertex V_k of the square, then the projectors are commutative and the Boolean sum function (2.4) interpolates F and its first derivatives on (see proof of Theorem 2.1). However, if the compatibility condition (2.6) is not satisfied at each vertex then $(P_1 \oplus P_2)F$ does not interpolate the x partial derivative on x - 0 and x = 1. The following theorem removes this discrepancy by the addition of rational terms to the Boolean sum function. It should be noted that these rational terms are zero for F satisfying (2.6), i.e. the modified interpolant reduces λ the standard Boolean sum interpolant when the boundary data is compatible.

<u>Theorem 2.1</u>. The function

$$(2.7)\ PF = P_1F+P_2F-P_1P_2F$$

$$- \frac{x(x-\mathbf{1})^2\, y^2(y-\mathbf{1})^2}{x+y}\left[\left[\frac{\partial^2 F}{\partial x \partial y}\right](\mathbf{0},\mathbf{0}) - \left[\frac{\partial^2 F}{\partial y \partial y}\right](\mathbf{0},\mathbf{0})\right]$$

$$- \frac{x(x-\mathbf{1})^2\, y^2(y-\mathbf{1})^2}{-x+y-\mathbf{1}}\left[\left[\frac{\partial^2 F}{\partial x \partial y}\right](\mathbf{0},\mathbf{1}) - \left[\frac{\partial^2 F}{\partial y \partial y}\right](\mathbf{0},\mathbf{1})\right]$$

$$- \frac{x^2(x-\mathbf{1})\ \ y^2(y-\mathbf{1})^2}{-x+\mathbf{1}-y}\left[\left[\frac{\partial^2 F}{\partial x \partial y}\right](\mathbf{1},\mathbf{0}) - \left[\frac{\partial^2 F}{\partial y \partial y}\right](\mathbf{1},\mathbf{0})\right]$$

$$- \frac{x^2(x-\mathbf{1})\ \ y^2(y-\mathbf{1})^2}{x-y+y-\mathbf{1}}\left[\left[\frac{\partial^2 F}{\partial x \partial y}\right](\mathbf{1},\mathbf{1}) - \left[\frac{\partial^2 F}{\partial y \partial y}\right](\mathbf{1},\mathbf{1})\right]$$

where $P_1 F$, $P_2 F$, and P_1P_2F are defined by (2.1), (2.2), and (2.5), interpolates $F \in c^1(\partial S)$ and its first derivatives on the boundary ∂S of the square S.

Proof. Consider the Boolean sum function which can be written as

$$(P, \oplus P_2)F \equiv F - (I-P_1)(I-P_2)F,$$

where I is the identity operator. Then the Boolean sum function interpolates F and its first derivatives on y = 0 and y = 1 since I - P_1 and its first derivatives are null on these sides. (The condition (2.6) would imply a dual result on x = 0 and x = 1 since P_l and P_2 then commute.) Now on x = 0 the Boolean sum function interpolates F but

$$\textbf{(2.8)} \quad \left[\frac{\partial (P_1 \oplus P_2)F}{\partial x}\right]_{x=0} = F_{1,0}(0, y)$$

$$+ \psi_1(y) \left[\left[\frac{\partial^2 F}{\partial x \partial y}\right](0,0) - \left[\frac{\partial^2 F}{\partial y \partial x}\right](0,0)\right]$$

$$+ \psi_1(y) \left[\left[\frac{\partial^2 F}{\partial x \partial y}\right](0,1) - \left[\frac{\partial^2 F}{\partial y \partial x}\right](0,1)\right]$$

A dual result attains on x = 1. The discrepancy in the interpolation properties of this derivative is removed by the rational terms in (2.7). For example, the term in (2.8) which involves $\varphi_l(y) = (y-l)^2y$ is removed by the rational term in (2.7) which involves the function

$$-\frac{x(x-1)^2y^2(y-1)^2}{x+y}.$$

This function has the desired properties that it and its first derivatives are zero on ∂S except on x = 0 where

$$\left[\frac{\partial}{\partial x}\left(-\frac{x(x-1)^2y^2(y-1)^2}{x+y}\right)\right]_{x=0} - (y-1)^2y \quad \blacksquare$$

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Remark. A symmetric interpolant on. the square can be obtained by taking the average of (2. 7) and the dual modified interpolant for $(P_2 \oplus P_1)$ F.

Corollary 2.1, Let $\tilde{F} \in C^1(\partial S)$ be defined by

$$(2.9)\quad \tilde{F}(x,0) = \varphi_0(x)F(0,0) + \varphi_1,(x)F_{1\ 0}(0,0)$$
$$+ \psi_0(x)F(l,0) + \psi_1(x)F\ (1,0)$$

$$(2.10)\quad \tilde{F}_{0,\ 1}(x,0) = (l\text{-}x)F_{0,1}\ (0,0) + x\ F_{0,1}\ (1,0),$$

with dual expressions for the function and normal derivatives on (x,l), (0,y), and (l,y). Then $P\tilde{F}$ is a twelve parameter interpolant which interpolates $\tilde{F}$ and its first derivatives on the boundary ∂S of the square S.

Remark. The boundary function $\tilde{F}$ on a side is a cubic Hermite function interpolating data on that side, and the normal derivative is a linear function interpolating data on that side. (The tangential derivative is automatically defined by the boundary function.) Thus for piecewise interpolation, the function and slopes across a side common to two adjacent rectangles is maintained by the use of $P\tilde{F}$ over each rectangle.

Theorem 2.2. The set of polynomials for which (2.7) is exact is

$$(2.11)\qquad x^m y^n \begin{cases} 0 \le m \le 3\ , & \text{for all} \quad n \\ 0 \le n \le 3\ , & \text{for all} \quad m\ . \end{cases}$$

Proof. For polynomial F, (2.7) reduces to the commutative Boolean sum function $(P_1 \oplus P_2)$F which is precise for the union of the precision sets of P_1 and P_2, namely

$$\mathbf{P_1\, x^m y^n = x^m P_1 y^n = x^m y^n,\ 0 \le n \le 3 \text{ for all } m}$$

$$\mathbf{P_2\, x^m y^n = y^n P_2 x^m = x^m y^n,\ 0 \le m \le 3 \text{ for all } n} \ \blacksquare$$

Remark. The twelve parameter scheme defined in Corollary 2.1 has a reduced precision set.

3. Smooth C^1 Interpolation on Triangles

It is sufficient to consider the triangle T with boundary ∂T and vertices at $V_1' = (1,0)$, $V_2 = (0,1)$, and $V_3 = (0,0)$, The interpolation scheme considered below is invariant under an affine transformation which takes this "standard" triangle T onto any arbitrary triangle. The side opposite the vertex V_k is denoted by E_k and thus E_1 is the side x-0, E_2 is the side y-0, and E_3 is the side z=0, where z=1–x–y. For

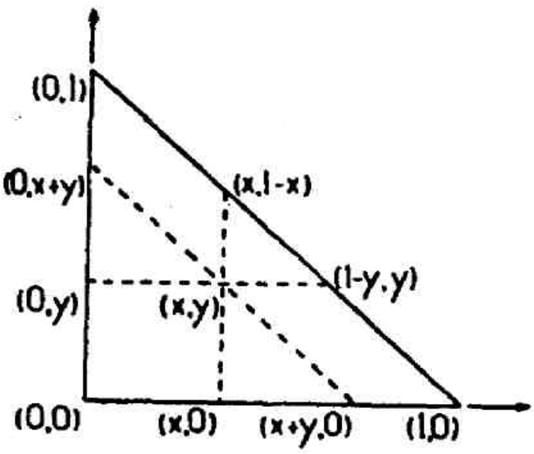

Figure 3

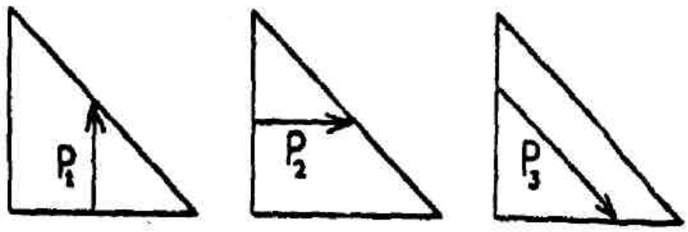

Figure 4

$F(x,y) \in C^1(\partial T)$ and $(x,y) \in T$ cubic Hermite interpolation projectors along parallels to the sides $x = 0$, $y = 0$, and $z = 0$ are respectively defined by

$$(3.1)\quad P_1F = \sum_{i=0}^{1} \psi_i\left(\frac{y}{1-x}\right)(1-x)^i F_{0,i}(x,0)$$

$$+ \sum_{i=0}^{1} \psi_i\left(\frac{y}{1-x}\right)(1-x)^i F_{0,i}(x,1-x),$$

$$(3.2)\quad P_2F = \sum_{i=0}^{1} \psi_i\left(\frac{x}{1-y}\right)(1-y)^i F_{i,0}(0,y)$$

$$+ \sum_{i=0}^{1} \psi_i\left(\frac{y}{1-y}\right)(1-y)^i F_{i,0}(1-y,y),$$

$$(3.3)\quad P_3F = \sum_{i=0}^{1} \psi_i\left(\frac{x}{x+y}\right)(x+y)^i \left(\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^i F\right)(0,x+y)$$

$$+ \sum_{i=0}^{1} \psi_i\left(\frac{x}{x+y}\right)(x+y)^i \left(\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^i F\right)(x+y,0),$$

where the $\varphi_i(t)$ and $\psi_1(t) = (-1)^i \varphi_i(1-t)$ are defined by (2.3), see Figures 3 and 4. The function $P_k F$ interpolates F and its first derivatives on the sides E_i, and E_j of the triangle T adjacent to the vertex V_k. The symmetric smooth interpolation scheme is defined in the following theorem.

Theorem 3.1. The function

$$(3.4)\ PF - x^2(3-2x+6yz)P_1F + y^2(3-2y+6xz)P_2F$$

$$+ z^2(3-2z+6xy)P_3F,$$

where P_1F, P_2F, and P_3F are defined by (3.1) - (3.3), interpolates $F \in C^1(\partial T)$ and its first derivatives on the boundary ∂T of the triangle T.

Proof. By symmetry it is sufficient to consider the side x = 0 where

$$(P_2F)(0,y) - (P_3F)(0,y) = F(0,y),$$

$$\left(\frac{\partial P_2F}{\partial x}\right)(0,y) = \left(\frac{\partial P_3F}{\partial x}\right)(0,y) = F_{1,0}(0,y)\ .$$

Thus from (3.4) it follows that

$$(PF)(0,y) = \left[y^2(3-2y+6xz)+z_2(3-2z+6xy)\right]_{x=0} F(0,y),$$

$$\left(\frac{\partial PF}{\partial x}\right)(0,y) = \left[\frac{\partial}{\partial x}\{y_2(3-2y+6xz)+z_2(3-2z+6xy)\}\right]_{x=0} F(0,y)$$

$$+\left[y^2(3-2y+6xz)+z_2(3-2z+6xy)\right]_{x=0} F_{1,0}(0,y).$$

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Now

(3.5) $\mathbf{x^2 (3\text{-}2x\text{+}6yz) + y^2 (3\text{-}2y\text{+}6xz) + z^2 (3\text{-}2z\text{+}6xy) = 1}$

and thus

$$\left[y^2 (3-2y+6xz) + z^2 (3-2z+6xy) \right]_{x=0} = 1,$$

$$\left\lfloor \frac{\partial}{\partial x} \left\{ y^2 (3 - 2y + 6xz) + z^2 (3 - 2z + 6xy) \right\} \right\rfloor_{x=0} = 0 .$$

Hence

$$(PF)(0,y) = F(0,y) \textbf{ and } \left(\frac{\partial PF}{\partial x} \right)$$

Also

$$\left(\frac{\partial PF}{\partial y} \right) (\mathbf{0}, y) = \frac{\partial}{\partial y} (PF)(\mathbf{0}, y) = F_{0,1} (\mathbf{0}, y),$$

or, alternatively, the dual argument to the case $\partial / \partial x$ can be applied ■

<u>Corollary 3.1</u>. Let $\tilde{F} \in C^1(\partial T)$ be defined by

(3.6) $\quad \tilde{F}(x,0)\ \varphi_0(x)F(0,0)\ \varphi_1(x)F_{1,0}(0,0)$

$\qquad\qquad + \psi_0(x)F(1,0) + \psi_1(x)F_{1,0}(1,0)$

(3.7) $\quad \tilde{F}_{0,1}(x,0) = (1-x)F_{0,1}(0,0) + xF_{0,1}(1,0)$

(3.8) $\quad \tilde{F}(0,y) = \varphi_0(y)F(0,0) + \varphi(y)F_{0,1}(0,0)$

$\qquad\qquad + \psi_0(y) F(0,1) + \psi_1(y)F_{0,1}(0,1)$

(3.9) $\tilde{F}_{1,0}(0,y) = (1-y)F_{1,0}(0,0) + yF_{1,0}(0,1)$

$$(3.10) \quad \tilde{F}(x, l\text{-}x) = \varphi_0(x)F(0,1) + \varphi_1(X)[F_{1,0}(0,1) - F_{0,1}(0.1)] + \psi_0(x)F(l,0) + \psi_1(X)[F_{1,0}(1,0) - F_{0,1}(1,0)]$$

$$(3.11) \quad \tilde{F}_{1,0}(x, l\text{-}x) + \tilde{F}_{0,1}(x, l\text{-}x) = (1\text{-}X)[F_{1,0}(0,1) + F_{0,1}(0,1)] + x[F_{1,0}(1,0) + F_{0,1}(1,0)].$$

Then $P\tilde{F}$ is a nine parameter interpolant which interpolates $\tilde{F}$ and its first derivatives on the boundary ∂T of the triangle T.

Remark. For piecewise interpolation, the nine parameter interpolant $P\tilde{F}$ is restricted to a regular mesh of right angled triangles. This is because the transformation of $P\tilde{F}$ onto an arbitrary triangle will not in general take normals into normals. However, the smooth interpolant (3.4) is invariant under affine transformation since the $P_k F$ are defined along the invariant parallels to the sides E_k.

Theorem 3.2. The set of polynomials for which (3.4) is exact is at least T_3, the set of polynomials of degree three or less along parallels to the three sides of T, i.e.

$$(3.12) \quad T_3, = \{l,x,y,x^2,xy,y^2,x^3,x^2y,xy^2,y^3,(x+y)x^2y, (x+y)xy^2\}.$$

Proof. The intersection of the precision . set of P_1, P_2, and P_3 is T_3. Thus from (3 . 5) it Follows that for $F \in T_3$

$$PF = x^2(3-2x+6yz)P_1F + y^2(3-2y+6xz)P_2F$$

$$+ z^2(3-2z+6xy)P_3 F$$

$$= [x^2(3-2x+6yz) + y^2(3-2y+6xz) + z^2(3-2z+6xy)]F$$

$= F$ ■

4. Examples.

The examples discussed in this section have been implemented by R.J. McDermott, see the following paper of these Proceedings. The examples illustrate the twelve parameter interpolation scheme for the rectangle described in Corollary 2.1. Examples of interpolation schemes for triangles are discussed in R.E. Barnhill's paper. The data for each example is supplied by some given primitive function F(x,y).

Example 4.1. $F(x,y) = xy$. The twelve parameter interpolant exactly reproduces thefunction $F(xy) = xy$ since it is contained in the precision set of the interpolant (see Figure 5). Figure 6 illustrates the effect of defining zero twist parameters for the Coons patch (16 parameter bicubic tensor product with zero twist conditions). Close examination reveals the "flat spot" effect at the vertices which would be more apparent in a three dimensional model.

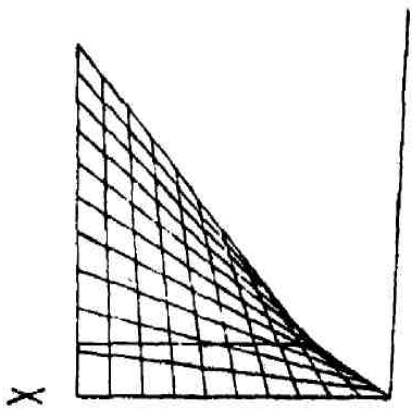

Figure 5

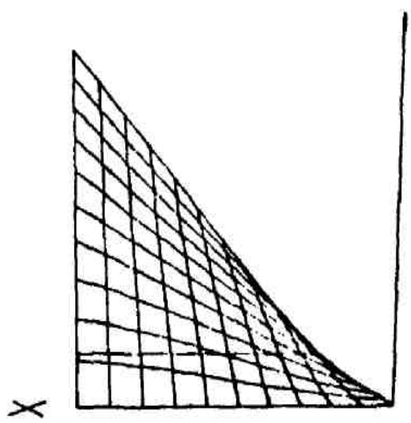

Figure 6

Example 4.2. $F(x,y) = (1-x)^2(1-y)^2$. This function (see Figure 7) is not contained in the precision set of the interpolant. However, the interpolant (see Figure 8) is a good approximation to this function.

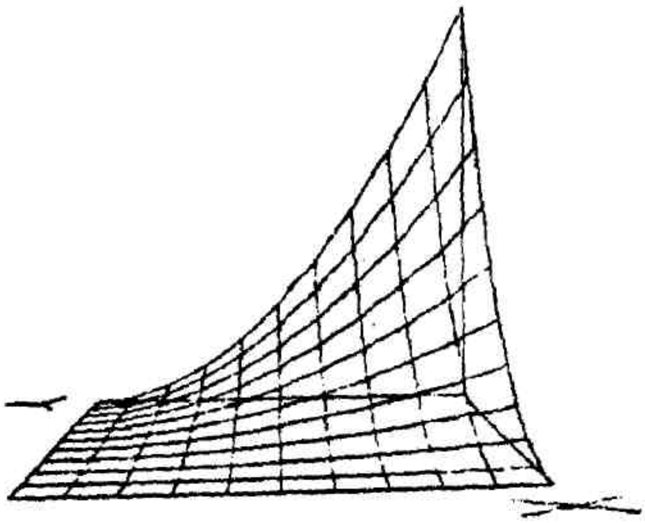

Figure 7

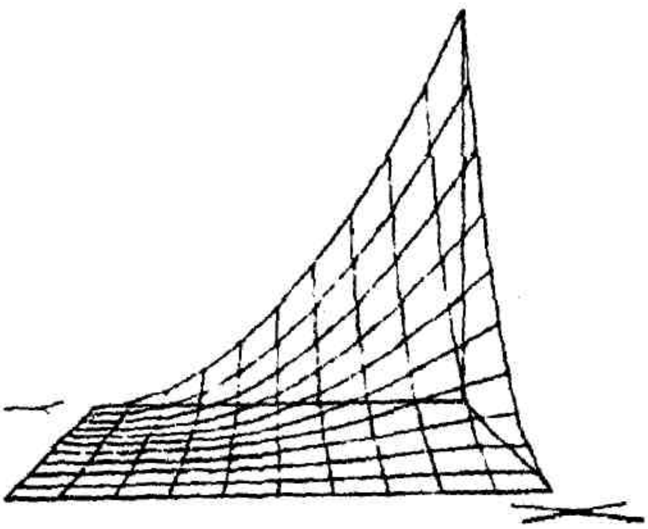

Figure 8

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Example 4.3. $F(x,y) = 0.1/(x-0.5)$. The vertex data supplied by this function is well behaved although the function itself has a line singularity at $x = 0.5$ (see Figure 9). The interpolant (see Figure 10) smooths out the singularity.

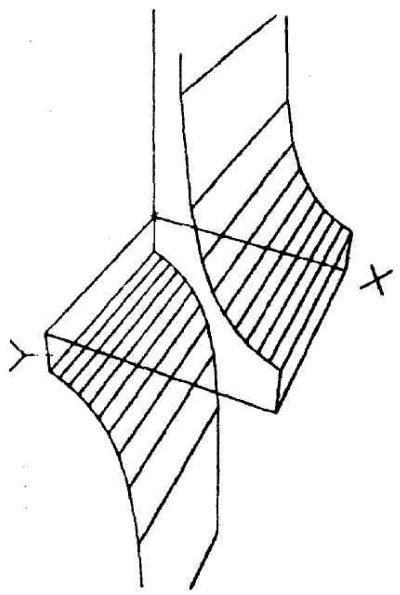

Figure 9

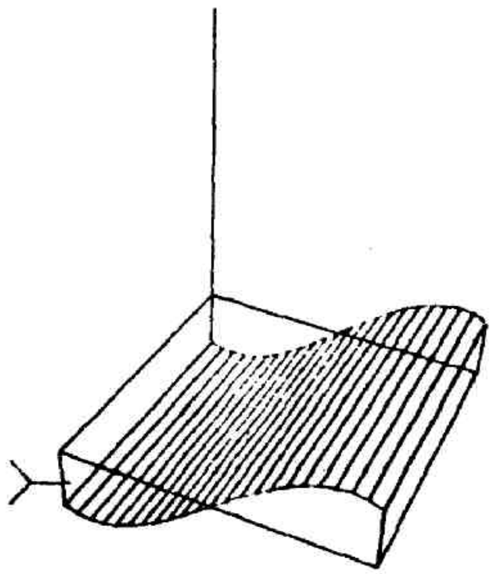

Figure 10

Example 4.4. $F(x,y) = \sin(2x)/(y+1)$. This example illustrates the join of four separate patches (see Figure 11). The function is shown in Figure 12 and the four patch interpolant is shown in Figure 13, this being a good approximation to the function.

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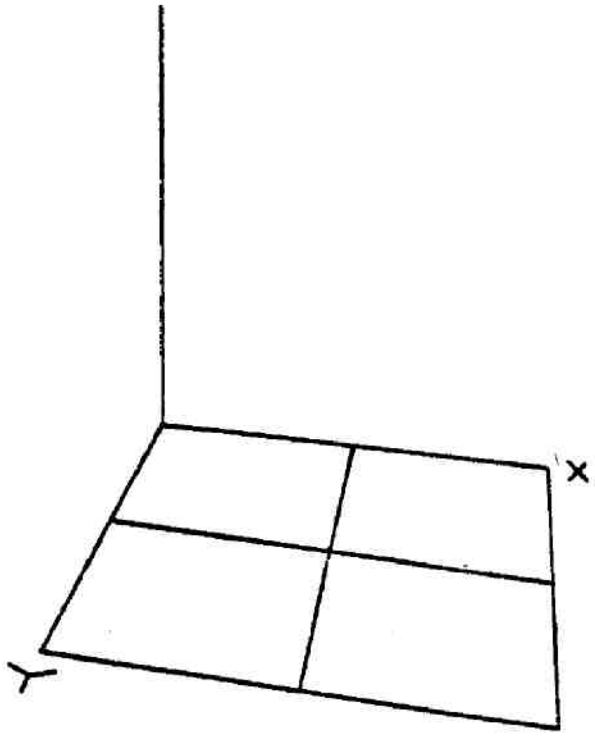

Figure 11

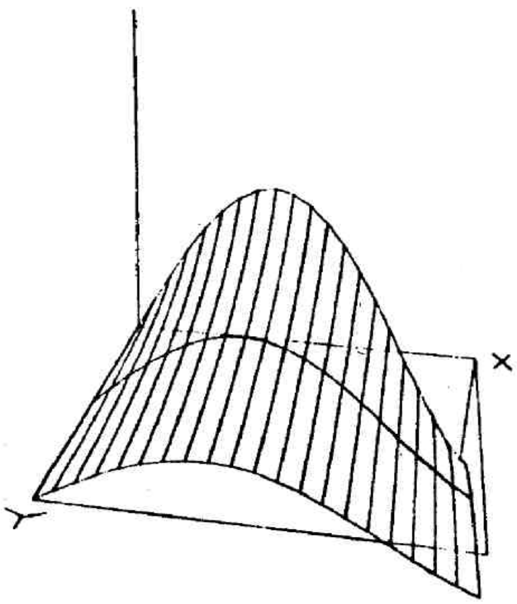

Figure 12

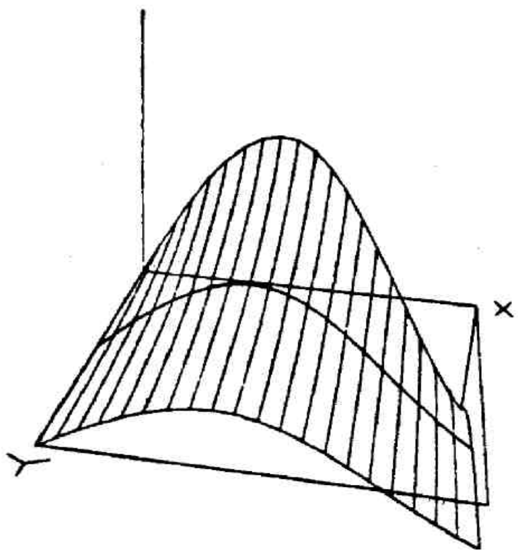

Figure 13

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Acknowledgement. This paper represents a continuation and extension of Joint work with R.E. Barnhill. The author also wishes to thank R.J. McDermott for the computer-generated pictures.

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