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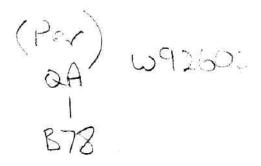
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A NEW CLASS OF MATRICES WITH POSITIVE INVERSES.

BY

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## ABSTRACT

It is well known that irreducibly diagonallydominant matrices with positive diagonal and non-positive off-diagonal elements have positive inverses. A whole class of symmetric circulant and symmetric quindiagonal Toeplitz matrices with positive inverses which do not satisfy the above conditions is found.

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## Introduction

A condition that the zeros of a certain polynomial are positive implies that the corresponding quindiagonal matrix has a positive inverse. A similar condition means that the related circulant with five entries per row has a positive inverse. The quindiagonal matrix is analysed by expressing the first row of its inverse as the solution of a finite difference equation and showing that this solution is positive. A result of Trench (1964) may then "be used to prove that for symmetric quindiagonal Toeplitz matrices, the complete inverse is positive if the first row of the inverse is positive. The behaviour of the first row of the inverse for very large orders is considered and examples of this new class for matrices are given.

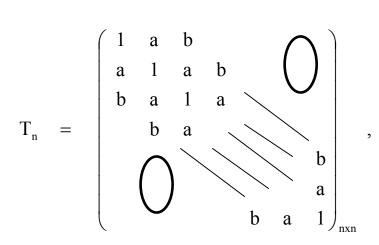
## Main Theorems

Theorem 1. (Proof in section 3)

If the real numbers a and b, b non-zero, are chosen so that the symmetric polynomial

$$bx^4 + ax^3 + x^2 + ax + b$$
 (1)

has real positive zeros, then the inverse of the symmetric quindiagonal Toeplitz matrix,



consists of positive elements for all order n.

The values of a and b for which the polynomial (1) has real positive zeros may be described as follows. Suppose the real zeror. of polynomial (1) are  $r_1$ ,  $l/r_1$ ,  $r_2$  and  $l/r_2$  where  $r_1$  and  $r_2$  are positive, then the values a and b may be expressed as

$$a = -\frac{(r_1 + 1/r_1) + (r_2 + 1/r_2)}{(r_1 + 1/r_1) + (r_2 + 1/r_2) + 2} ,$$

and

$$b = \frac{1}{(r_1 + 1/r_1) + (r_2 + 1/r_2) + 2} .$$

It is easy to verify that -2/3 < a < 0 and  $0 < b \le 1/6$ . The relation  $a + b \ge \frac{1}{2}$  follows from the inequality  $(r_1 + l/r_1 - 2)(r_2 + l/r_2 - 2) \ge 0$  and  $a^2 + 8b^2 - 4b \ge 0$  is the condition that  $r_1$  and  $r_2$  are real. The above inequalities describe a region, R, in the a - b plane bounded by the three curves b = 0,  $a + b = -\frac{1}{2}$  and  $a^2 + 8b^2 - 4b = 0$  with intersection points (0,0), (-1/2,0) and (-2/3, 1/6).

Theorem 2.

If the real numbers a and b, b non-zero, are chosen such that the symmetric polynomial

$$bx^4 + ax^3 + x^2 + ax + b$$
 (1)

has real positive zeros, not equal to one, then the inverse of the symmetric circulant matrix

$$C_n = \text{circ} (l, a, b, 0 \dots, 0, b, a)_{nxn}$$
 (3)

consists of positive entries for all orders n.

Proof

The matrix  $C_n\ ,\ in\ (3),\ can be factored into the product of two matrices,$ 

 $C_n = [(r_1 + 1/r_1) I - (P + P^{-1})] [(r_2 + 1/r_2) I - (P + P^{-1})]$ , wh ere P is the permutation matrix circ  $(0,1,0,...,0)_{nxn}$  and  $r_1$  and  $r_2$  are zeros of (1) not equal to 1. However, both factors are irreducibly diagonally dominant and have positive inverses, Varga (1962), and consequently, the inverse of the matrix Cn is positive.

The above method of proof may be used for a more general theorem concerning circulant matrices, Meek (1973).

Theorem 3. (Trench(1964)).

If  $T_n$  is the n<sup>th</sup> order,  $n \ge 3$ , symmetric Toeplitz matrix in (2)

with  $T_n^{-1} > 0$  and the first row of the matrix  $T_{n+1}^{-1}$  is positive, then the matrix  $T_{n+1}^{-1} > 0$ .

Proof

The matrix  $T_{n+1}^{-1}$  may be partitioned in the form

$$T_{n+1}^{-1} = \begin{bmatrix} t & W_n^T \\ W_n & T_n^{-1} + W_n W_n^T / t \end{bmatrix}_{n+1 \times n+1}$$

If the first row of  $T_{n+1}^{-1} > 0$ . is positive, then both t and the lxn Vector  $W_n^T$  are positive. However, thus  $T_n^{-1} > 0$ , all of the elements of the matrix  $T_{n+1}^{-1}$  are positive. 3. Proof of Theorem 1.

The first row of the matrix  $T_n^{-1}$  is shown positive by solving a difference equation so that theorem 3 is an induction step in the proof of theorem 1. It is easy to verify that  $T_3^{-1} > 0$  when a and b are chosen so that (a,b) is in region R.

The difference equation for the elements of the first row of the matrix  $T - \frac{1}{n}$  is

 $bD_{r-2} + aD_{r-1}$ ,  $+D_r + aD_{r+1} + bD_{r+2} = e_r$ , r = 1, 2, ..., n, (4) with end conditions  $D_{-1} = D_0 = D_{n+1} = D_{n+2} = 0$ , where

$$e_r = \begin{cases} 1 & r = 1 \\ 0 & r = 2,3,...,n. \end{cases}$$

A particular solution to equation (4) is the function

$$H_{r} = \begin{cases} 1/b & r = -1 \\ 0 & r = 0, 1, ..., n + 2, \end{cases}$$

while the general solution falls into four cases, depending upon the zeros of the polynomial (1). The more general case of positive real distinct zeros not equal to 1 will be discussed first.

The zeros of the polynomial (1) are positive, distinct and not equal to 1, if and only if a and b are such that the point (a,b) lies in the interior of the region R. For convenience take  $r_1 > r_2 > 1$  to be zeros of (1), then the general solution to the difference equation (4) is of the form

$$D_r = Ar \frac{r}{1} + Br \frac{r}{1} + Cr \frac{r}{2} + Dr \frac{-r}{2} H_r, r = -1,0,...,n+2$$

If the function f(r,n) is defined

$$f(\mathbf{r},\mathbf{n}) = \mathbf{s}_{\mathbf{r}}\mathbf{t}_{1} + \mathbf{s}_{1}\mathbf{t}_{\mathbf{r}} + \mathbf{s}_{n+2}\mathbf{t}_{n-\mathbf{r}+1} - \mathbf{s}_{n-\mathbf{r}+2}\mathbf{t}_{n+1} - \mathbf{s}_{n+1}\mathbf{t}_{n-\mathbf{r}+2} + \mathbf{s}_{n-\mathbf{r}+1}\mathbf{t}_{n+2} , \quad (5)$$
  
where  $\mathbf{s}_{\omega} \equiv \mathbf{r}_{1}^{\omega} - \mathbf{r}_{1}^{-\omega}$  and  $\mathbf{t}_{\omega} \equiv \mathbf{r}_{2}^{\omega} - \mathbf{r}_{2}^{-\omega}$ , then the solution satisfy-

ing the end condition is

$$D_r = f(r,n)/(b f(l,n+l)) + H_r, r = -1,0,...,n+2.$$
 (6)

As n becomes large, Dr approaches the function

$$\lim_{n \to \infty} D_r = \begin{cases} 1/(br_1, r_2) & t = 0\\ 0 & 0 < t \le 1 \end{cases},$$

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where  $r \equiv (n-l)t+l$ .

The substitution  $r_1 = e^{(\theta^+ | \Psi^+)}$ ,  $r_2 = e^{(\theta^+ | \Psi^+)}$ , where 0, transforms the function f(r, n) in (5) into

 $\theta \ge \psi \ge 0$ , transforms the function f(r,n) in (5) into

 $f(r,n) = 16\sinh(n+1)\theta \sinh(n-r+2)\theta \sinh\psi$ 

-  $16\sinh(n+1)\psi \sinh(n-r+2)\psi \sinh\theta$ 

which is positive for r = 1, 2, ..., n since both

$$\sinh(n+1)\theta \sinh\psi - \sinh(n+1)\psi \sinh\theta$$

and

$$\sinh(n-r+2)\theta \sinh\psi - \sinh(n-r+2)\psi \sinh\theta$$

are positive r = 1,2,...,n, (see the appendix) - Now tooth b and f(r,n) are positive, thus  $D_r$ , r=1,2,...,n in equation (6) is positive.

The results for the remaining three cases may now be summarized. The polynomial (l) may have one repeated zero, that is the zeros are  $r_1$ ,  $1/r_1$ , l,  $r_1 > 1$ , and a+b = -1/2, or it may have two pairs of repeated zeros, that is the zeros are  $r_1$ ,  $1/r_1$ ,  $r_1$ ,  $1/r_1$ ,  $r_1 > 1$  and  $a^2 + 8b^2 - 4b = 0$ , of it. may have all. Four zeros equal to 1, whence  $a = -\frac{2}{3}$   $b = \frac{1}{6}$ . The solution of the difference equation (4) is of the same form as in equation (6) with the function f(r,n) being defined

$$\begin{split} f(r,n) &= s_{r+} r s_{1+} \left( \ n{-}r{+}1 \right) s_{n+2} \mbox{-}( \ n{+}1) \ s_{n-r+2} \mbox{-}(n{-}r{+}2) s_{n+1} \mbox{+}^{(n+2)} s_{n-r+1}, \\ f(r,n) &= r s_{n+1} s_{n-r+2} \mbox{-}(n{+}1) (n{-}r{+}2) \ s_r s_1 \quad , \end{split}$$

and

$$f(r,n) = r(n-r+1)(n-r+2)$$

respectively. As n becomes large,  $D_{\rm r}$  , in these three cases approaches the functions

$$\begin{split} &\lim_{n \to \infty} \ D_r = \begin{cases} 1/(br_1) & t = 0\\ (1-t)/(b(r_1-1)) & 0 < t \le 1 \end{cases}, \\ &\lim_{n \to \infty} \ D_r = \begin{cases} 1/(br^2_1) & t = 0\\ 0 & 0 < t \le 1 \end{cases} \end{split}$$

and

$$\lim_{n \to \infty} D_r = \begin{cases} \mathbf{6} & t = \mathbf{0} \\ \mathbf{6}nt(\mathbf{1} - t)^2 & \mathbf{0} < t \le \mathbf{1} \end{cases},$$

,

where r = (n-1)t+1.

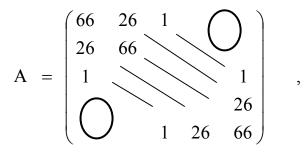
The substitution  $r_1 = e^{2\theta}$ ,  $\theta > 0$ , in the first case yields  $f(r,n) = 8\sinh(n-r+2)\theta \sinh\theta[(n+1)\cosh(n+1)\theta \sinh r\theta - r\cosh r\theta \sinh(n+1)\theta]$ 

+  $8\sinh(n+1)\theta$  sinhr9 [  $(n-r+2)\cosh(n-r+2)\theta$  sinh $\theta$ -cosh $\theta$  sinh $(n-r+2)\theta$ ] in which both of the terms are positive for each n and r=1,2,...,n (see the appendix). The substitution  $r_1 = e^{\theta}$ ,  $\theta > 0$ , in the second case gives

$$\begin{split} f(r,n) &= 4[rsinh(n+1)\theta]:sinh(n-r+2)\theta] - U[(n+1)sinhre3L(n-p+2)sinhe] \\ \text{which is also positive for each n and } r=1,2,--n \quad (\text{see the appendix}). \\ \text{In the third case, } f(r,n) \text{ is obviously positive for each n and} \\ r &= 1,2,..., n \;. \end{split}$$

4. Examples

The matrix arising from quintic polynomial spline interpolation on a uniform partition with non-periodic boundary conditions is the quindiagonal

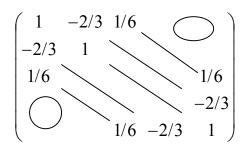


Ahlberg, Nilson and Walsh, p. 124 (1967). The related matrix  $DAD^{T}$ , where D is the diagonal matrix diag  $(1,-1,...,(-1)^{n-1})_{nxn}$ , has a positive inverse since the zeros of

$$1/66 z^4$$
 - 26/66 z3 +  $z^2$  - 26/66 z + 1/66

are real and positive

Hoskins and Ponzo (1972) have found another class of symmetric Toeplitz matrices with positive inverses which intersects with the class described here in the nxn matrix



Lemma 1.

The function  $(\sinh m\theta)m$  is greater than  $\theta$  and monotone increasing

in m when  $\theta > 0$ , m > 0.

Proof

The result follows from a Maclaurin expansion of sinhm $\theta$ . Lemma 2.

The function m cothm $\theta$  is greater than  $\theta$  and is monotone increasing in m,  $\theta > 0$ , m > 0.

Proof

The derivative of m cothm9 with respect to m can be shown to be positive using lemma 1. for  $\theta > 0$ , m > 0 and  $\lim_{m \to 0} (m \text{ cothm}\theta) = \theta$ .

Lemma 3.

The function  $(sinhm\theta\,)/(sinhm\,\psi\,)$  is greater than  $\theta/\psi\,$  and increases

with m,  $\theta > 0$ ,  $\psi > 0$ , m > 0.

The derivative of  $(\sinh m\theta)/(\sinh m\psi)$  with respect to m can be shown to be positive using lemma 2. and  $\lim_{m\to 0} (\sinh m\theta)/(\sinh my) = \theta/\psi$ .

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