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ERROR BOUNDS FOR LINEAR INTERPOLATION ON TRIANGLES,

by

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1. Introduction.

The existence of Sobolev norm bounds for interpolation remainders defined on triangles is well known [7,8], as is the relationship of such bounds to finite element analysis. A constructive method for the derivation of such bounds is proposed in a recent paper by Barnhill and the author [3]. This method is used here to derive error bounds for linear interpolation on a triangle T to functions defined in the Sobolev space $H^2(T)$. In particular, the constants which are involved in the bounds are estimated. Knowledge of such constants can lead to computable finite element error bounds for second order elliptic problems defined on convex domains [5].

Let T be the triangle with vertices at (1,0), (0,1), and (0,0). The Sobolev space $H^2(T)$ is the space of all real valued functions which, together with all their generalized derivatives of order ≤ 2, are in $L_2(T)$. A norm and semi-norm on $H^2(T)$ are respectively defined by

$$(1.1)\qquad \|u\|_{H^2(T)} = \left\{ \sum_{0\leq i+j\leq 2} \|u_{i,j}\|^2_{L_2(T)} \right\}^{1/2},$$

$$(1.2)\qquad |u|_{H^2(T)} = \left\{ \sum_{i+j=2} \|u_{i,j}\|^2_{L_2(T)} \right\}^{1/2},$$

Let linear interpolation remainders be defined by

$$(1.3)\qquad \begin{cases} R[u](s,t) = u(s,t) - su(1,0) - tu(0,1) - (1-s-t)\,u(0,0), \\ R_{1,0}[u](s,t) = \dfrac{\partial}{\partial s} R[u](s,t) = u_{1,0}(s,t) - u(1,0) + u(0,0), \\ R_{0,1}[u](s,t) = \dfrac{\partial}{\partial t} R[u](s,t) = u_{0,1}(s,t) - u(0,1) + u(0,0). \end{cases}$$

Then the bounds of interest for $u \in H^2(T)$ are those on the $L_2(T)$ norms of (1.3).

The bounds which are given in this paper are derived by means of a piecewise defined Taylor expansion of $u \in C^2(T)$. This expansion is defined in Section 2 and the bounds are given in Section 3. The use of the function space $C^2(T)$ can be justified by the use of 'smooth' or 'mollified' functions from $H^2(T)$. Finally, in Section 4, bounds are derived for an arbitrary triangle. In particular, it is shown that the bounds hold under the maximal angle condition that no angle of the triangle should approach π. This condition is derived by Synge [11] for functions in $C^2(T)$ and is also the subject of recent studies [1,2,9].

2. A Taylor expansion in $C^2(T)$.

Let $(a, b) \in T$ and let A_1, A_2, and A_3 be subsets of T defined by

$$(2.1) \qquad \begin{cases} A_1 = T - A_2 \cup A_3, \text{ where} \\ A_2 = \{(x, y)/(x, y) \in T \text{ and } x > 1 - b\}, \\ A_3 = \{(x, y)/(x, y) \in T \text{ and } y > 1 - a\}, \end{cases}$$

see Figure 2.1.

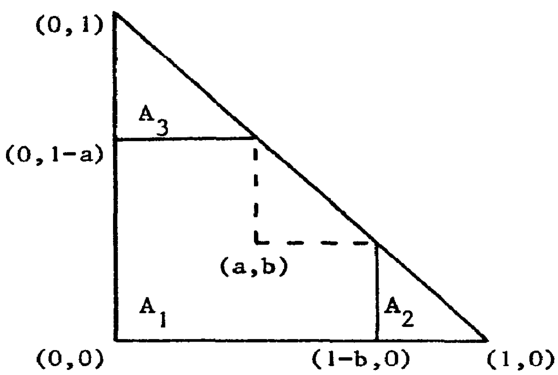

Figure 2.1

The regions A_1, A_2, and A_3

Then $u \in C^2(T)$ can be written in the piecewise defined form

(2.2) $$u(x,y) \equiv x_{A_1}(x,y)u(x,y) + x_{A_2}(x,y)u(x,y) + x_{A_3}(x,y)u(x,y), \quad (x,y) \in T,$$

Where

(2.3) $$x_{A_i}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in A_i, \\ 0 & \text{otherwise.} \end{cases}$$

For each region A, i = 1,2,3, single variable expansions of $u \in C^2(T)$ are made along parallels to the coordinate axes, see Figures 2.2 and 2.3. These expansions are such that their arguments are contained in T for all (a, b) $\in$ T and the following piecewise defined Taylor expansion of the function u(x,y) about the point (a, b) results:

(2.4) $$\begin{aligned} u(x,y) = {} & u(a,b) + (x-a)u_{1,0}(a,b) + (y-b)u_{0,1}(a,b) \\ & + x_{A_1}(x,y)\Big[\int_a^x (x-\tilde{x})u_{2,0}(\tilde{x},b)d\tilde{x} + \int_b^y \int_a^x u_{1,1}(\tilde{x},\tilde{y})d\tilde{x}\,d\tilde{y} \\ & \quad + \int_b^y (y,\tilde{y})u_{0,2}\,(a,\tilde{y})\,d\tilde{y}\Big] \\ & + x_{A_2}(x,y)\Big[\int_a^x \int_a^{\tilde{x}} u_{2,0}(x',1-\tilde{x})\,dx'd\tilde{x} + \int_a^x \int_b^{1-\tilde{x}} u_{1,1}(a,y^*)dy^*d\tilde{x} \\ & \quad + \int_a^x \int_{1-\tilde{x}}^y u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x} + \int_b^y (y-\tilde{y})\,u_{0,2}\,(a,\tilde{y})\,d\tilde{y}\Big] \\ & + x_{A_3}(x,y)\Big[\int_a^x (x-\tilde{x})\,u_{2,0}(\tilde{x},b)\,d\tilde{x} + \int_b^y \int_{1-\tilde{y}}^x u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y} \\ & \quad + \int_b^y \int_a^{1-\tilde{y}} u_{1,1}(x^*,b)\,dx^*dy + \int_b^y \int_b^{\tilde{y}} u_{0,2}(1-\tilde{y},y')dy'd\tilde{y}\Big] \end{aligned}$$

(A detailed derivation of the generalized form of this expansion for $u \in C^n(T)$

is given in Barnhill and Gregory [3].)

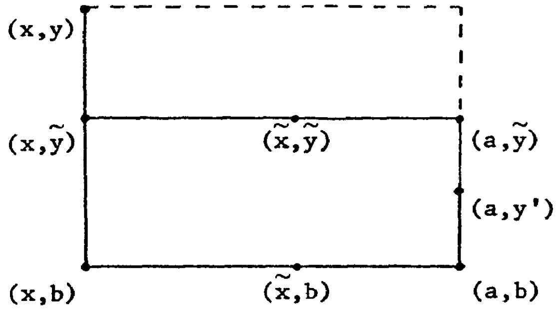

Figure 2.2

Function arguments in Taylor expansion for $(x,y) \in A_1$

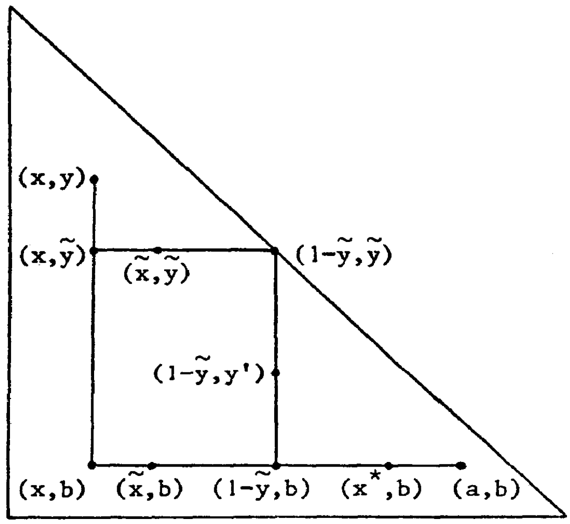

Figure 2.3

Function arguments in Taylor expansion for (x,y) A_3

($(x,y) \in A_2$ is dual)

Remark 1. The fact that (a,b) can be an arbitrary point of T is significant since (a,b) = (s,t) is permissible, where (s,t) is the point of evaluation of the interpolation remainders (1.3). This choice of (a,b) enables bounds to be derived in Sobolev function spaces. This is observed by Birkhoff, Schultz, and Varga |6|, who use the Kernel Theory of Sard |10| to derive error bounds for Hermite interpolation on rectangles. However, the theory of Sard is based on Taylor expansions which correspond to the case of $(x,y) \in A_1$ in (2.4), and such expansions have a rectangular domain of influence, see Figure 2.2. This precludes the choice of $(a,b) = (s,t)$ in the application of Sard's theory on triangles (although this theory can be applied with $(a,b) = (0,0)$, [2,4]).

Remark 2. The remainder terms which involve $u_{2,0}$ in the Taylor expansion (2.4) are constants in the variable y and, dually, the terms which involve $u_{0,2}$ are constants in the variable x. This property, which is a result of the development of the Taylor expansion along parallels to the coordinate axes, has an important consequence in the following section and leads to the maximal angle condition of Section 4.

3. Error bounds for linear interpolation on T .

The application of the linear interpolation remainders defined by (1.3) to the Taylor expansion (2.4), with $(a,b) = (s,t)$, gives the following representations:

$$(3.1)\qquad \begin{aligned} R[u](s,t) = &-s\int_s^1\int_s^{\tilde{x}} u_{2,0}(x',1-\tilde{x})\,dx'\,d\tilde{x} - s\int_s^1\int_t^{1-\tilde{x}} u_{1,1}(s,y^*)\,dy^*\,d\tilde{x} \\ &- s\int_s^1\int_{1-\tilde{x}}^0 u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x} - t\int_t^1\int_{1-\tilde{y}}^0 u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y} \\ &- t\int_t^1\int_s^{1-\tilde{y}} u_{1,1}(x^*,t)\,dx^*\,dy - t\int_t^1\int_t^{\tilde{y}} u_{0,2}(1-\tilde{y},y')\,dy'\,d\tilde{y} \\ &+ (1-s)\int_s^0 \tilde{x}u_{2,0}(\tilde{x},t)\,d\tilde{x} - (1-s-t)\int_t^0\int_s^0 u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y} \\ &+ (1-t)\int_t^0 \tilde{y}u_{0,2}(s,\tilde{y})\,d\tilde{y}, \end{aligned}$$

$$(3.2)\quad R_{1,0}[u](s,t) = -\int_s^1 \int_s^{\tilde{x}} u_{2,0}(x',1-\tilde{x})\,dx'\,d\tilde{x} - \int_s^1 \int_t^{1-\tilde{x}} u_{1,1}(s,y^*)\,dy^*d\tilde{x}$$

$$-\int_s^1 \int_{1-\tilde{x}}^0 u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x} - \int_s^0 \tilde{x}\,u_{2,0}(\tilde{x},t)\,d\tilde{x}$$

$$+\int_t^0 \int_s^0 u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y}$$

($R_{0,1}$ [u](s,t) has a dual representation to (3.2)). Equations (3.1) and (3.2) contain terms which involve only second derivatives since R and $R_{1,0}$ annihilate the linear terms in (2.4). Also, a consequence of the form of the Taylor expansion (2.4) and the form of the interpolation remainders (1.3) is that (3.2) contains no term in $u_{0,2}$. This is because R[g(y)](s,t) is a constant in the variable s for all univariate functions g(y), and hence $R_{1,0}$ [g(y)] (s,t) = 0 (see Remark 2, Section 2). A generalization of this property forms the basis of the Zero Kernel Theorem in Barnhill and Gregory [2]

The $L_2(T)$ norm of (3.1) and (3.2) is now taken with respect to (s,t) and the triangle inequality for norms is applied to the sum of terms of the right hand sides. For $u \in H^2$ (T), the norms of the terms which result can be bounded appropiately. For example, for one such term (cf. (3.2)) the following inequalities hold:

$$(3.3)\quad \left\| \int_s^1 \int_t^{1-\tilde{x}} u_{1,1}(s,y^*)\,dy^*\,d\tilde{x} \right\|_{L_2(T)}^2$$

$$= \left\| \int_t^{1-s} \int_s^{1-y^*} u_{1,1}(s,y^*)\,d\tilde{x}\,dy^* - \int_0^t \int_{1-y^*}^1 u_{1,1}(s,y^*)d\tilde{x}\,dy^* \right\|_{L_2(T)}^2$$

$$\leq \left\| \left\{ \int_t^{1-s} (1-y^*-s)^2 dy^* \int_t^{1-s} |u_{1,1}(s,y^*)|^2\,dy^* \right\}^{1/2}\right.$$

$$\left. + \left\{ \int_0^t (y^*)^2 dy^* \int_0^t \left|u_{1,1}(s,y^*)\right|^2 dy^* \right\}^{1/2} \right\|_{L_2(T)}^2$$

$$= \int_0^1 \int_0^{1-s} \left[\frac{(1-s-t)^3}{3} \int_t^{1-s} |u_{1,1}(s,y^*)|^2\, dy^* + \frac{t^3}{3} \int_0^t |u_{1,1}(s,y^*)|^2\, dy^* \right.$$

$$\left. + \left\{ \frac{(1-s-t)^{3/2}\, t^{3/2}}{3} \int_t^{1-s} |u_{1,1}(s,y^*)|^2\, dy^* \int_0^t |(u_{1,1}s,y^*)|^2\, dy^* \right\}^{1/2} \right] dt\, ds$$

$$\leq \int_0^1 (1-s)^4 (1/6 + \pi/32) \int_0^{1-s} |u_{1,1}(s,y^*)|^2\, dy^* ds$$

$$\leq (1/6 + \pi/32)\, \| u_{1,1} \|^2_{L_2(T)}$$

The details of the calculations which involve the other terms are omitted for brevity, and the final bounds obtained are

(3.4) $\| R[u] \|_{L_2(T)} \leq 0.17 \| u_{2,0} \|_{L_2(T)} + 0.39 \| u_{1,1} \|_{L_2(T)} + 0.17 \| u_{0,2} \|_{L_2(T)}$,

(3.5) $\| R_{1,0}[u] \|_{L_2(T)} \leq 0.65 \| u_{2,0} \|_{L_2(T)} + 0.93 \| u_{1,1} \|_{L_2(T)}$,

(3.6) $\| R_{0,1}[u] \|_{L_2(T)} \leq 0.93 \| u_{1,1} \|_{L_2(T)} + 0.65 \| u_{0,2} \|_{L_2(T)}$.

4. <u>Error bounds for linear interpolation on a general triangle</u>.

Consider the linear transformation defined by

(4.1) $$\hat{u}(\xi, \eta) = u[\xi(x,y), \eta(x,y)],$$

where

(4.2) $$\begin{cases} \xi = \xi(x,y) \equiv x + ay \\ \eta = \eta(x,y) \equiv by, \end{cases}$$

and $1 \geq a \geq 0$, $1 \geq b \geq 0$.. Equations (4.2) define a transformation of the triangle T with vertices at (1,0), (0,0), and (0,1) in the (x,y) plane, to the triangle $\hat{T}$ with vertices at (1,0), (0,0), and (a,b) respectively in the (ξ,η) plane. Further, let the angles at the vertices of $\hat{T}$ be denoted by α, β, and γ respectively, where $\alpha \le \beta \le \gamma$, see Figure 4.1. Thus, the length of the greatest side of $\hat{T}$ is 1. (Results for a triangle of greatest side h follow immediately from the analysis in this section.)

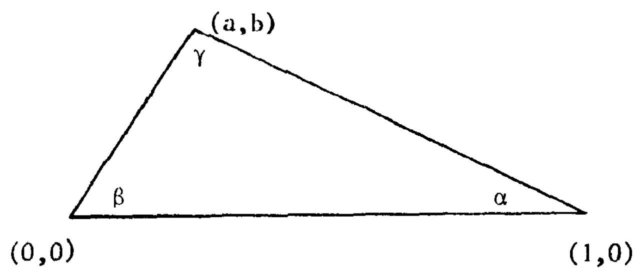

Figure 4.1

The triangle $\hat{T}$

Let R, $\hat{R}$, $\hat{R}_{1,0}$, and $\hat{R}_{0,1}$, denote the linear interpolation remainder operators on $\hat{T}$. Then, from (3.4)-(3.6) and the transformation (4.1), the following results can be obtained:

$$(4.3)\quad \|\hat{R}[\hat{u}]\|_{L_2(\hat{T})} = \sqrt{b}\,\|R[u]\|_{L_2(T)}$$
$$\le (0.17 + 0.39\,a + 0.17\,a^2)\,\|\hat{u}_{2,0}\|_{L_2(\hat{T})}$$
$$+ (0.39\,b + 0.34\,a\,b)\,\|\hat{u}_{1,1}\|_{L_2(\hat{T})} + 0.17\,b^2\,\|\hat{u}_{0,2}\|_{L_2(\hat{T})},$$

$$(4.4)\quad \|\hat{R}_{1,0}[u]\|_{L_2(\hat{T})} = \sqrt{b}\,\|R_{1,0}[u]\|_{L_2(T)}$$
$$\le (0.65 + 0.93\,a)\,\|\hat{u}_{2,0}\|_{L_2(\hat{T})} + 0.93\,b\,\|\hat{u}_{1,1}\|_{L_2(\hat{T})},$$

$$(4.5)\quad \|\hat{R}_{0,1}[u]\|_{L_2(\hat{T})} = \sqrt{b}\,\|-\frac{a}{b}R_{1,0}[u] + \frac{1}{b}R_{0,1}[u]\|_{L_2(T)}$$
$$\le \frac{a}{b}1.58(1+a)\,\|\hat{u}_{2,0}\|_{L_2(\hat{T})}$$
$$+ (0.93 + 1.3\,a)\,\|\hat{u}\| + 0.65\,b\,\|\hat{u}\|$$

From (4.4) and (4.5), the following semi-norm result is obtained:

(4.6) $$|\hat{R}[\hat{u}]|_{H^1(\hat{T})} = \left\{\| \hat{R}_{1,0}[\hat{u}] \|^2_{L_2(\hat{T})} + \| \hat{R}_{0,1}[\hat{u}] \|^2_{L_2(\hat{T})}\right\}^{1/2}$$
$$\leq (c_1 + c_2 \tan \beta) \, | \hat{u} |_{H^2(\hat{T})},$$

where C_1 and C_2 are constants independent of $\hat{T}$ and $\hat{u}$, and $\tan \beta = a/b$. Moreover, since the semi-norms in (4.6) are invariant under translation and rotation, (4.6) is true for any triangle of greatest side 1 and with angles $\alpha \leq \beta \leq \gamma$ From (4.6) there follows the condition that $\tan \beta \leq K < \infty$, that is $\beta \geq \beta_o > 0$, where $\alpha \leq \beta \leq \gamma$. This is equivalent to the maximal angle condition that $\gamma \leq \gamma_o < \pi$. Finally, it should be noted that had (3.6) contained a term in $u_{2,0}$, then (4.5) would have contained the additional from $1/b \| \hat{u}_{2,0} \|_{L_2(\hat{T})}$. This term would give the more usually quoted minimum angle condition that $\alpha \geq \alpha_o > 0$.

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