TR/57 DECEMBER 1975.

NUMERICAL COMPUTATIONS OF THREE DIMENSIONAL THIN PANEL STRUCTURES USING REISSNER'S VARIATIONAL PRINCIPLE.

BY

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ABSTRACT

A finite element model for analysing static problems of three-dimensional thin panel constructions is presented. The potential energy of stretching of each panel is represented by the classical Reissner formulation of two-dimensional elasticity, and in the finite element discretisation piecewise constant stresses and piecewise linear u-, v- displacements are used. The potential energy of bending of each panel is represented in the finite element discretisation by Herrmann's model, i.e. piecewise constant moments and piecewise linear w - displacements. The physical appropriateness of the model is verified for different assemblages of panels, and in all cases good numerical results are obtained.

1. The description of the model.

The total potential energy of a three-dimensional thin panel structure is expressed in the form

$$\Pi = \sum_{k=1}^{N} \Pi_k \tag{1}$$

where N is the number of panels and

$$\Pi_k = \Pi_{k,s} + \Pi_{k,b} , \tag{2}$$

$\Pi_{k,s}$ and $\Pi_{k,b}$ being respectively the potential energies of the k -th panel in stretching and bending.

Using the Reissner energy formulation of two-dimensional elasticity [4], the stretching potential energy of the k -th panel can be written as

$$\Pi_{k,s} = h_k \iint_{\Omega_k} \left[-\frac{1}{2E} b(\sigma_{\bar{x}}, \sigma_{\bar{y}}, \tau_{\bar{x}\bar{y}}) + \sigma_{\bar{x}} \frac{\partial \bar{u}}{\partial \bar{x}} + \sigma_{\bar{y}} \frac{\partial \bar{v}}{\partial \bar{y}} + \tau_{\bar{x}\bar{y}} \left[\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}} \right] - (\bar{X}\bar{u} + \bar{Y}\bar{v}) \right] d\bar{x}\, d\bar{y} - h_k \int_{\Gamma_{k,s}} (p_{\bar{x}}\bar{u} + p_{\bar{y}}\bar{v})\, ds \tag{3}$$

where $\bar{x}, \bar{y}$ is a local co-ordinate system in the middle plane Ω_k of the panel, and $\Gamma_{k,s}$ is the part of the boundary Γ_k of Ω_k on which the boundary force $(p_{\bar{x}}, p_{\bar{y}})$ is prescribed. Further E is Young's modulus, h_k is the thickness of the k-th panel and the quadratic form b is defined by

$$b(\xi, \eta, \zeta) = (\xi + \eta)^2 + 2(1 + \mu)(\zeta^2 - \xi\eta) \tag{4}$$

with μ the Poisson's ratio.

In our model we assume that Ω_k has a polygonal boundary. We approximate the displacement components $\bar{u}$ and $\bar{v}$ in the directions $\bar{x}$ and $\bar{y}$, respectively, by continuous functions which are linear on the triangles T_j of a given triangulation of Ω_k while the stress components $\sigma_{\bar{x}}, \sigma_{\bar{y}}, \tau_{\bar{x}\bar{y}}$ are constant on T_j. The local components $\bar{x}$ $\bar{y}$ and $p_{\bar{x}}, p_{\bar{y}}$ of prescribed body and boundary forces, respectively, are approximated on the triangles T_j by constants.

For $\Pi_{k,b}$ we choose Herrmann's formulation [2], [5]:

$$(5) \quad \Pi_{k,b} = -\iint_{\Omega_k} \left[\frac{6}{Eh_k^3} b(M_{\bar{x}}, M_{\bar{y}}, M_{\overline{xy}}) + q_{\bar{z}}\bar{w} \right] d\bar{x}d\bar{y} - \sum_{j=1}^{n_k} \int_{\partial T_j} M_{vs} \frac{\partial \bar{w}}{\partial s} ds,$$

the quadratic form b being defined by (4). Here n_k is the number of triangles in the given triangulation of Ω_k, ∂T_j denotes the boundary of T., and the moment of M is given by

$$(6) \quad M_{vs} = -(M_{\bar{x}} - M_{\bar{y}}) \cos\alpha \sin\alpha + M_{\overline{xy}} (\cos^2\alpha - \sin^2\alpha),$$

α being the angle made by the outward normal v to a side of T_j. with the positive direction of $\bar{x}$.

The displacement component $\bar{W}$ in the direction of the axis $\bar{z}$ (which forms with the axes $\bar{x}$, $\bar{y}$ a cartesian right-handed set) is approximated by a continuous function which is linear on the triangles T_j. The moments $M_{\bar{x}}$, $M_{\bar{y}}$, $M_{\overline{xy}}$ are approximated on the triangles T_j. by constants in such a way that the normal moment M_v is continuous across all interelement boundaries. This is always possible because

$$(7) \quad M_v = M_{\bar{x}} \cos^2\alpha + M_{\bar{y}} \sin^2\alpha + 2M_{\bar{x}\bar{y}} \cos\alpha \sin\alpha.$$

Prescribing on the sides ℓ_i. of T_j. constant values M_{vi}, we get three independent equations for the three unknowns $M_{\overline{x}} M_{\overline{y}} M_{\overline{xy}}$.

The normal load $q_{\overline{z}}$ is approximated on the triangles T_j. by constants. The expressions for $\overline{X}, \overline{Y}, q_{\overline{z}}$ can be obtained by decomposing a given force $\underline{F}$ into the form

(8) $$\underline{F} = \overline{X}\, \bar{\underline{i}}_1 + \overline{Y}\, \bar{\underline{i}}_2 + q_{\overline{z}}\, \bar{\underline{i}}_3 ,$$

$\bar{\underline{i}}_1, \bar{\underline{i}}_2$ and $\bar{\underline{i}}_3$ being the unit vectors in the directions of the axes $\overline{x}$, $\overline{y}$ and $\overline{z}$ respectively.

The whole construction is considered in terms of one global cartesian co-ordinate system x, y, z. The displacement vector

(9) $$\underline{u} = u\, \underline{i}_1 + v\, \underline{i}_2 + w\, \underline{i}_3$$

must be continuous in the construction, where $\underline{i}_1, \underline{i}_2$ and $\underline{i}_3$ are the unit vectors in the directions of the axes x, y and z, respectively. This is guaranteed by prescribing the parameters u_i, v_i, w_i at the vertices P_i of all triangles into which the construction is divided. The relations between the local components $\overline{u}_i, \overline{v}_i, \overline{w}_i$ and the global components u_i, v_i, w_i of the displacement vector at the point P_i are

(10) $$\begin{bmatrix} \overline{u}_i \\ \overline{v}_i \\ \overline{w}_i \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}$$

with $\beta_{jk} = \bar{\underline{i}}_j . \underline{i}_k$.

The parameters corresponding to stress components and normal.

moments are considered in their local context. We prescribe fifteen parameters for each triangle. These are

u_i, v_i, w_i at p_i $(i = 1,2,3)$

$\sigma_{\bar{x}}, \sigma_{\bar{y}}, \tau_{\bar{x}\bar{y}}$ at the centre of gravity

M_{v1}, M_{v2}, M_{v3} on the sides.

Let the jth triangle have the vertices P_q, P_r, P_s and let

(11) $$^{\ell}\underline{\Delta}_j = (\sigma_{\bar{x}}, \sigma_{\bar{y}}, \tau_{\bar{x}\bar{y}}, \bar{u}_q, \bar{u}_r, \bar{u}_s, \bar{v}_q, \bar{v}_r, \bar{v}_s, M_{vqr}, M_{vrs}, M_{vsq}, \bar{w}_q, \bar{w}_r, \bar{w}_s)^T$$

be the vector of local parameters of this element. The potential energy of the jth triangle can be written in the form

(12) $$\Pi_J = \frac{1}{2}\, {}^{\ell}\underline{\Delta}_J^T\, {}^{\ell}\underline{k}_j\, {}^{\ell}\underline{\Delta}_j - {}^{\ell}\underline{\Delta}_J^T\, {}^{\ell}\underline{f}_j$$

where

(13) $$^{\ell}\underline{K}_j = \begin{bmatrix} \underline{Q}_j & \underline{U}_j & \underline{V}_j & \underline{o} & \underline{o} \\ \underline{U}_j^T & \underline{o} & \underline{o} & \underline{o} & \underline{o} \\ \underline{V}_j^T & \underline{o} & \underline{o} & \underline{o} & \underline{o} \\ \underline{o} & \underline{o} & \underline{o} & \underline{G}_j & \underline{H}_j \\ \underline{o} & \underline{o} & \underline{o} & \underline{H}_j^T & \underline{o} \end{bmatrix},$$

(14) $$^{\ell}\underline{f}_j = (\underline{o}, \underline{f}_{x,j}^T, \underline{f}_{y,j}^T, \underline{o}, \underline{f}_{z,j}^T)^T .$$

The matrices of third order $\underline{Q}_j, \underline{U}_j, \underline{V}_j, \underline{G}_j, \underline{H}_j$ and the three-dimensional vectors $\underline{f}_{\bar{x},j}, \underline{f}_{\bar{y},j}, \underline{f}_{\bar{z},j}$ are computed in the local co-ordinate system $\bar{x}, \bar{y}, \bar{z}$. Details are omitted because of their simplicity.

Let

(15) $$^{g}\underline{\Delta}_j = (\sigma_{\bar{x}}, \sigma_{\bar{y}}, \tau_{\bar{x}\bar{y}}, u_q, u_r, u_s, v_q, v_r, v_s, M_{vqr}, M_{vrs}, M_{vsq}, w_q, w_r, w_s)^T$$

be the vector of global parameters of the jth triangle.

Then

(16) $$^{\ell}\underline{\Delta}_j = \underline{L}_j \, {}^{g}\underline{\Delta}_j$$

where, according to (10),

(17) $$\underline{L}_j = \begin{bmatrix} \underline{I} & \underline{o} & \underline{o} & \underline{o} & \underline{o} \\ \underline{o} & \beta_{11}\underline{I} & \beta_{12}\underline{I} & \underline{o} & \beta_{13}\underline{I} \\ \underline{o} & \beta_{21}\underline{I} & \beta_{22}\underline{I} & \underline{o} & \beta_{23}\underline{I} \\ \underline{o} & \underline{o} & \underline{o} & \underline{I} & \underline{o} \\ \underline{o} & \beta_{31}\underline{I} & \beta_{32}\underline{I} & \underline{o} & \beta_{33}\underline{I} \end{bmatrix}$$

$\underline{I}$ is the unit matrix and $\underline{o}$ is the zero matrix, both being of order 3.

The potential energy Π^j of the jth triangle can be written in the form

(18) $$\Pi^{j} = \frac{1}{2}\, {}^{g}\underline{\Delta}_{j}^{T}\ {}^{g}\underline{K}_{j}\ {}^{g}\underline{\Delta}_{j} - {}^{g}\underline{\Delta}_{j}^{T}\ {}^{g}\underline{f}_{j}$$

where, according to (12) and (16),

(19) $${}^{g}\underline{K}_{j} = \underline{L}_{j}^{T}\ {}^{\ell}\underline{K}_{j}\,\underline{L}_{j}\,,$$

(20) $${}^{g}\underline{f}_{j} = \underline{L}_{j}^{T}\ {}^{\ell}\underline{f}_{j}\,.$$

The resulting system of linear equations

(21) $$\underline{K}\ \underline{\Delta} = \underline{f}$$

is constructed from the matrices ${}^{g}\underline{K}_{j}$ and vectors ${}^{g}\underline{f}_{j}$. taking into account the boundary conditions for both the global displacement components and normal moments, and the relations for normal moments on the edges of the construction (i.e., on the lines of contact of panels). Thus the vectors $\underline{\Delta}$ consist of all local stress components, independent global displacement components, and independent moments. The numbering of these parameters can be done so as to minimize the band-width of $\underline{K}$. Although $\underline{K}$ is an indefinite matrix, this will cause no trouble as long as the system (21) is solved by the Gauss elimination method.

The boundary Γ of the construction is the union of the boundaries of all panels. We shall consider the simplest case of boundary conditions only when some part Γ_1 of Γ is clamped, another part Γ_2 is simply supported and the remaining part Γ_3 is free. We assume that no external moments act on Γ_3. E.g., if Γ_2 is parallel to the axis z then the boundary conditions take the form

(22) $$u\big|_{\Gamma_1+\Gamma_2} = v\big|_{\Gamma_1+\Gamma_2} = w\big|_{\Gamma_1} = 0 \ , \ M_v\big|_{\Gamma_2+\Gamma_3} = 0.$$

As to the relations for moments on edges, they follow from the condition that the sum of moments (with respect to their orientations in the global system) must be equal to zero-
E.g., in Fig. 1a three panels have a common edge. If we orientate the local $\bar{z}$ - axes as is shown in Fig.1b we have

$$M_1 - M_2 + M_3 = 0.$$

(In Fig.1b the curved arrows denote the positive orientations of moments at the point A with respect to the local systems as it is seen from the global system).

If we use quadrilateral "macroelements" formed by four triangles and eliminate (or "condense") 19 internal parameters of each quadrilateral (i.e. 12 stress components, 4 moments and 3 displacement components) the number of unknowns in the system (21) and the band-width of $\underline{K}$ is considerably reduced In all test examples this condensation of parameters was used

In a triangular element three stress components can be condensed.

2. Some numerical examples.

2.1. Narrow Triangular Construction. A triangle consisting of three panels as shown in Fig.2 is considered. The length a of the horizontal panel is 1•6m and the length of both remaining panels is given by b = $\sqrt{2}$ a /2. The width and the thickness of all three panels is ℓ = 0.1m and h = 0.01m, respectively. The horizontal panel is subdivided into ten equal rectangles "macroelements" and each of both remaining panels into eight equal rectangles. The segment AB is fixed and the force P is equal to 2N.

According to the theory of frames consisting of one-dimensional beams we have

$$\sigma_1 = \sigma_3 = (3+8\sqrt{2})p/28\ell h = 1022\,\mathrm{Nm}^{-2},$$

$$\sigma_2 = -(3\sqrt{2}+2)\,p/28\,\ell h = 446\,\mathrm{Nm}^{-2},$$

$$M_{AB} = -(2\sqrt{2}-1)\,pb/28\ell = -1.478\ \mathrm{N},$$

$$M_{CD} = M_{EF} = (2\sqrt{2}-1)pb/14\ell = 2.955\ \mathrm{N},$$

$$M_{GH} = -(3\sqrt{2}+2)\,pb/28\ell = -5.045\ \mathrm{N},$$

where σ_i (i = 1,2,3) is the axial stress in the ith panel, i.e. the stress in the direction which is parallel to the longer side of the ith panel.

The finite element method gives the following results which show a good agreement with the exact values : $\sigma_1 = \sigma_3 = 1025\ \mathrm{Nm}^{-2}$, $\sigma_2 = -449\,\mathrm{Nm}^{-2}$, $M_{AB} = -1.458\mathrm{N}$, $M_{CD} = M_{EF} = 2.947\mathrm{N}$, $M_{GH} = -5.053\mathrm{N}$.

2.2. Wide Triangular Construction. The triangle consisting of rectangular panels with the width ℓ = 0.6m and with the same parameters a,b,h as in Example 2.1 is considered. The linear density of the given force remains the same : 20 Nm^{-1} .

Two cases are considered : (a) the u-parameters on the segments AC, CE, AE, BD, DF and BF are free; (b) the u-parameters on these six segments are equal to zero.

The results for the moments, axial stresses, v-displacements and w-displacements should he in both cases (a) and (b) the same as in Example 2.1. As to the transverse stresses they should be in case (a) equal to zero and in case (b) such that.

(23) $$\sigma_{transverse} = \mu \sigma_{axial} \cdot$$

For each panel the transverse direction is parallel to the direction of the global x-axis.

The finite element procedure gives the following results for the moments. On the segment AB;

-1.411N, -1.559N, -1.568N, -1.568N, -1.559N, -1.411N;

on the segments CD and EF :

2.748N, 3.099N, 3.153N, 3.153N, 3.099N, 2.748N

and on the segment GH:

-5.067N, -4.981N, 4.950N, -4.950N, -4,981N, -5.067N.

In each panel the axial stresses differ from element to element contrary to Example 2.1. They are constant in the mean only, i.e. in each panel the expression

$$\bar{\sigma}_{ax} = \frac{1}{6} \sum_{strip} \sigma_{ax}$$

is invariant, where the sum is taken over the six elements lying in the same strip which is parallel to the segment AB

In the case (a) the transverse stresses are equal to zero in the strips close to the segments AB and GH. The transverse stresses in the strips which are very close to the edges CD and EF are almost of the same order as the axial stresses. This can be explained by the fact that according to the results of the finite element method the edges CD and EF do not remain straight after the deflection, whilst the segment AB is fixed, and thus straight.

In the case (b) the computed stresses σ_{trans} and σ_{ax} satisfy (23) "in the mean", i.e.

$$\sum_{strip} \sigma_{trans} = \mu \sum_{strip} \sigma_{ax} ,$$

where again the sum is taken over six elements lying in the same strip which is parallel to the edge AB. Moreover, (23) is exactly satisfied in these elements which are very close either to the segment AB or to the segment GH.

2.3. <u>Rectangular Panel Construction</u>. As the third example the construction of Fig.4a is considered. Because of the computational technique each node is numbered the same number of times as the number of panels to which it belongs. Such a numbering seems to be necessary in the case of edges in which three or more panels meet. The thickness h and the width ℓ of each panel is given by h = 0.01 m and ℓ = 0.1m, respectively. The edges AB and CD are simply supported and on the edge GH a force P with linear density 20 Nm^{-1} acts. The lengths a, b, c are given by a = 1.6m, b = 1m, c = 0.6m.

According to the theory of frames consisting of one dimensional beams, we have for the absolute values of the vertical and. horizontal reactions that

$$V_1 = cP/b = 1.2N, \quad V_2 = (1 + c/b)\, P = 3.2N,$$

$$H_1 = H_2 = 3c(3a+4b)\, P/8a(2a + 3b) = 0.399N.$$

Thus we get the axial stresses

$$\sigma_1 = V_1 / h\ell = 1.2\ 10^3\ Nm^{-2}\ ,$$

$$\sigma_2 = H_2 / h\ell = 3.99\ 10^2\ Nm^{-2}\ ,$$

$$\sigma_3 = (p - V_2) / h\ell = -1.2\ 10^3\ Nm^{-2}\ ,$$

$$\sigma_4 = -V_2 / h\ell = -3.2\ 10^3\ Nm^{-2}\ ,$$

The moments can also be derived easily, e.g.,

$$M_{EF,3} = (V_1\ b - H_1\ a/2)/\ell = 8.808N,$$

$$M_{EF,4} = -\ H_2\ a\ /\ 2\ell = -\ 3.192N,$$

$$M_{EF,5} = -\ Pc/\ell = 12N.$$

The symbol $M_{EF,i}$ $(i = 3,4,5)$ denotes the normal moment on the edge EF in the ith panel.

The computed values are in very good agreement with the exact values :

$$\sigma_1 = 1.2000\ 10^3\ NM^{-2}, \quad \sigma_2 = 3.986\ 10^2\ NM^{-2}\ ,$$

$$\sigma_3 = -1.2000\ 10^3\ NM^{-2}, \quad \sigma_4 = -3.2000,\ 10^3\ NM^{-2}\ ,$$

$$M_{EF,3} = 8.811N, \quad M_{EF,4} = -3.189N, \quad M_{EF,5} = -12.000N$$

2.4. <u>L - shaped Wide Construction</u>. A construction consisting of two square panels as shown in Fig.5 is considered. The edge with the end points 1 and 9 is clamped and at the corners 145 and 153 respectively two opposing forces parallel to the y-axis but with the same magnitude P = 10N act.

In the horizontal panel, according to the technical theory of bending of plates, at a sufficiently large distance from the edge with the end points 73 and 81 we have

$$\left|\sigma_y\right| = Pa\rho \;/\; J = 12P\rho/ha^2,$$

where σ_y is the axial stress, a is the length of the side of the panel, h the thickness of the panel, and ρ the distance of a node from the line determined by the points 5 and 77.

In our case a = 0.8m, h = 0.01m. Thus

$$\left|\sigma_y\right| = 3\,10^5\,\rho/16 \;.$$

The results are introduced in Table 1. (In this table, the values are expressed in 10^3 Nm^{-2}.)

2.5. Uniformly Loaded Cube. As the last example we calculate the deflections and moments of a cube, the faces of which are uniformly loaded.

The length of edges of the cube, and the thickness of the panels were a = 3m, and h = 0.1m, respectively. The Young's modulus, the Poisson's ratio and the intensity of a uniform load were $E = 2.1\,10^{11}$ Nm^{-2}, μ = 0.3, and p = 10^4 Nm^{-2}, respectively.

The results for maximum deflections and moments are given in Table 2. The exact values of deflections, and moments at centres of edges are taken to be the values computed by

Conway [1]. As they are almost the same as the values derived for a clamped plate by Timoshenko [4], we consider the moment at the centre of a clamped plate as the exact value for our case.

The results for moments are very satisfactory, the results for deflections comparatively good. It should be noted that the results for a clamped and uniformly loaded square plate are practically the same ; they differ from the results in Table 2 at most by about 2%.

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	i	i + 1	i + 2	i + 3	i + 4	i + 5	i + 6	i + 7	i + 8
i = 1	-8.10	-6.44	-4.28	-2.16	0.00	2.16	4.28	6.44	8.10
i = 10	-8.04	-6.39	-4.21	-2.03	0.00	2.03	4.21	6.39	8.04
i = 19	-7.90	-6.29	-4.12	-1.99	0.00	1.99	4.12	6.29	7.90
i = 28	-7.73	-6.17	-4.04	-1.97	0.00	1.97	4.04	6.17	7.73
i = 37	-7.63	-6.07	-3.91	-1.88	0.00	1.88	3.91	6.07	7.63
i = 46	-7.71	-5.98	-3.66	-1.68	0.00	1.68	3.66	5.98	7.71
i = 55	-8.20	-5.84	-3.12	-1.31	0.00	1.31	3.12	5.84	8.20
i = 64	-9.67	-5.26	-2.24	-0.86	0.00	0.86	2.24	5.26	9.67
Exact	-7.50	-5.63	-3.75	-1.88	0.00	1.88	3.75	5.63	7.50

Table 1 Axial stresses at the nodal points of the horizontal panel

(Example 2.4)

Division of plates into squares	4 × 4	6 × 6	8 × 8	Exact values	
Deflections at centres of plates	0.871	0.692	0.627	0.535	10^{-4} m
Moments at centres of plates	0.1874	0.1997	0.2032	0.2079	10^{4} N
Moments at centres of edges	-0.3621	-0.4092	-0.4305	-0.4653	10^{4} N

Table 2. Extremal displacements and moments of the uniformly loaded box (Example 2.5)

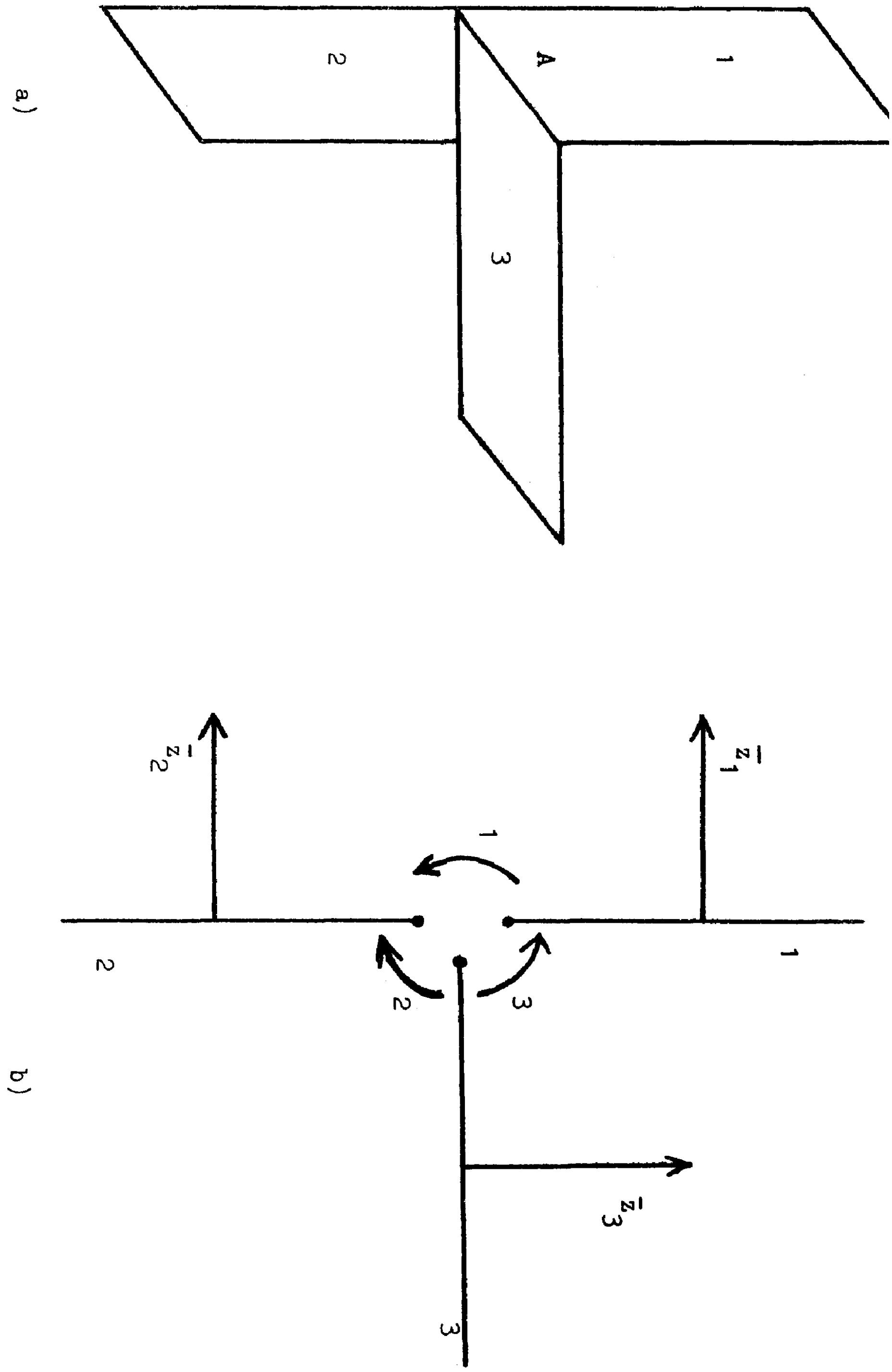


Fig.1

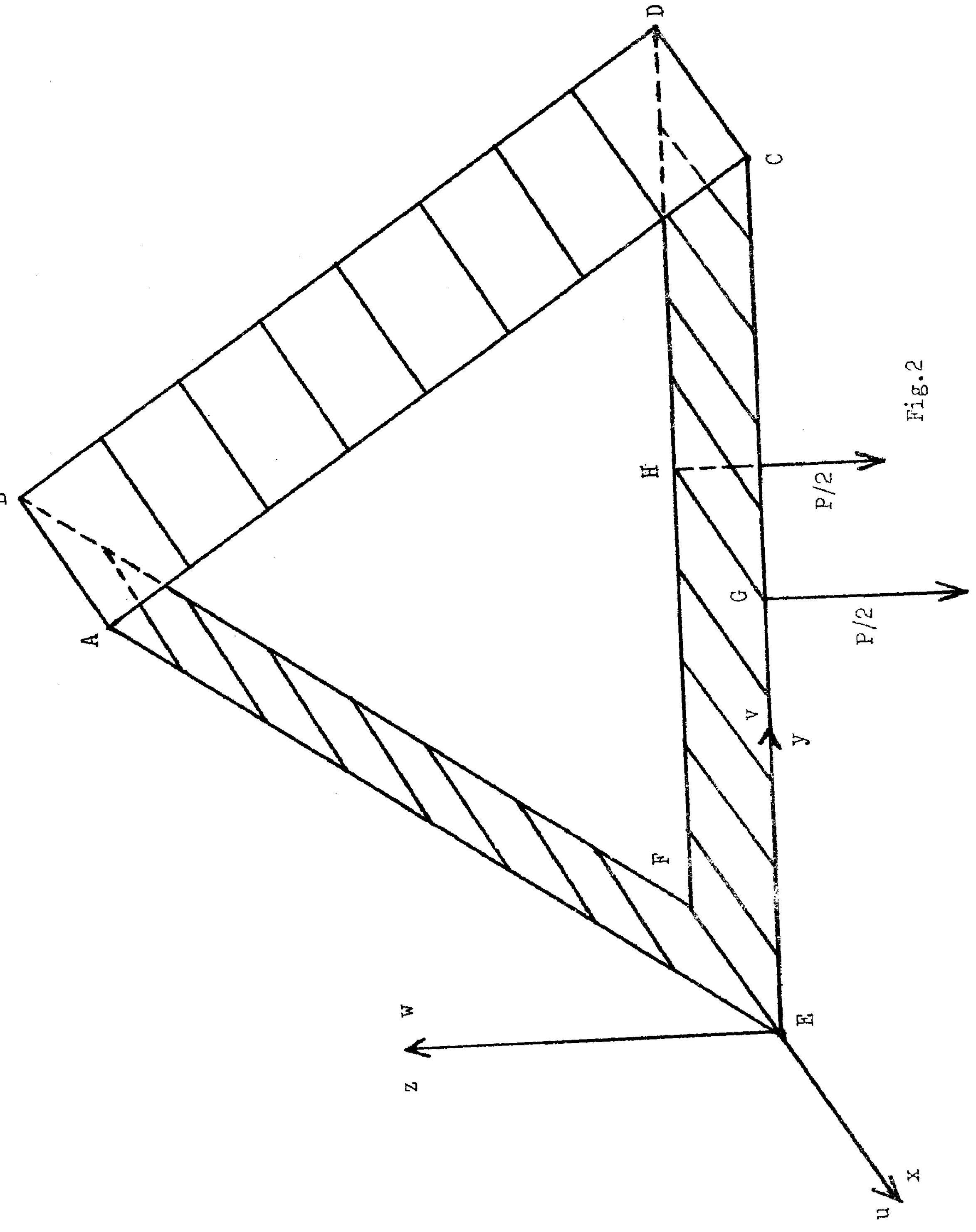


Fig.2

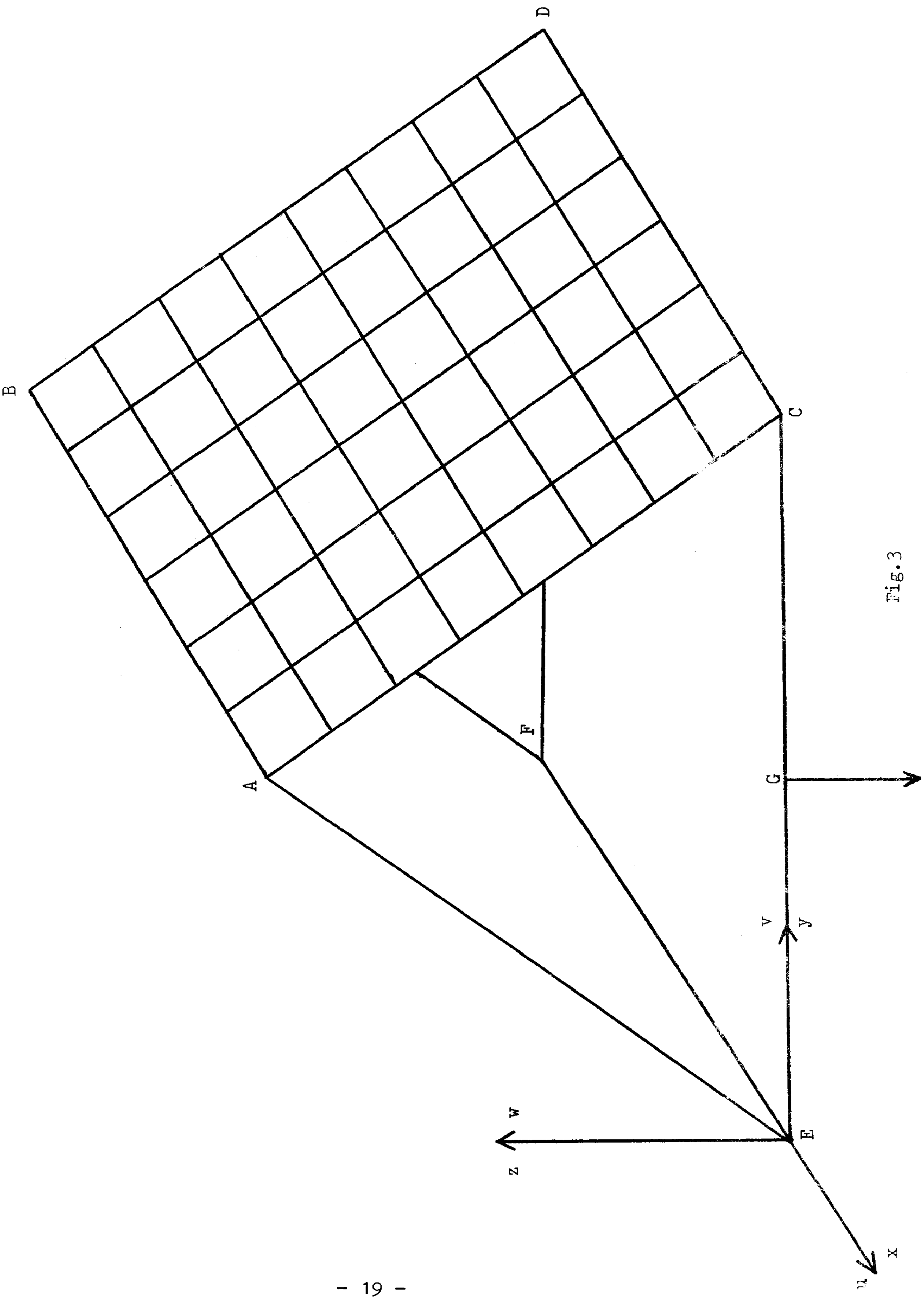


Fig.3

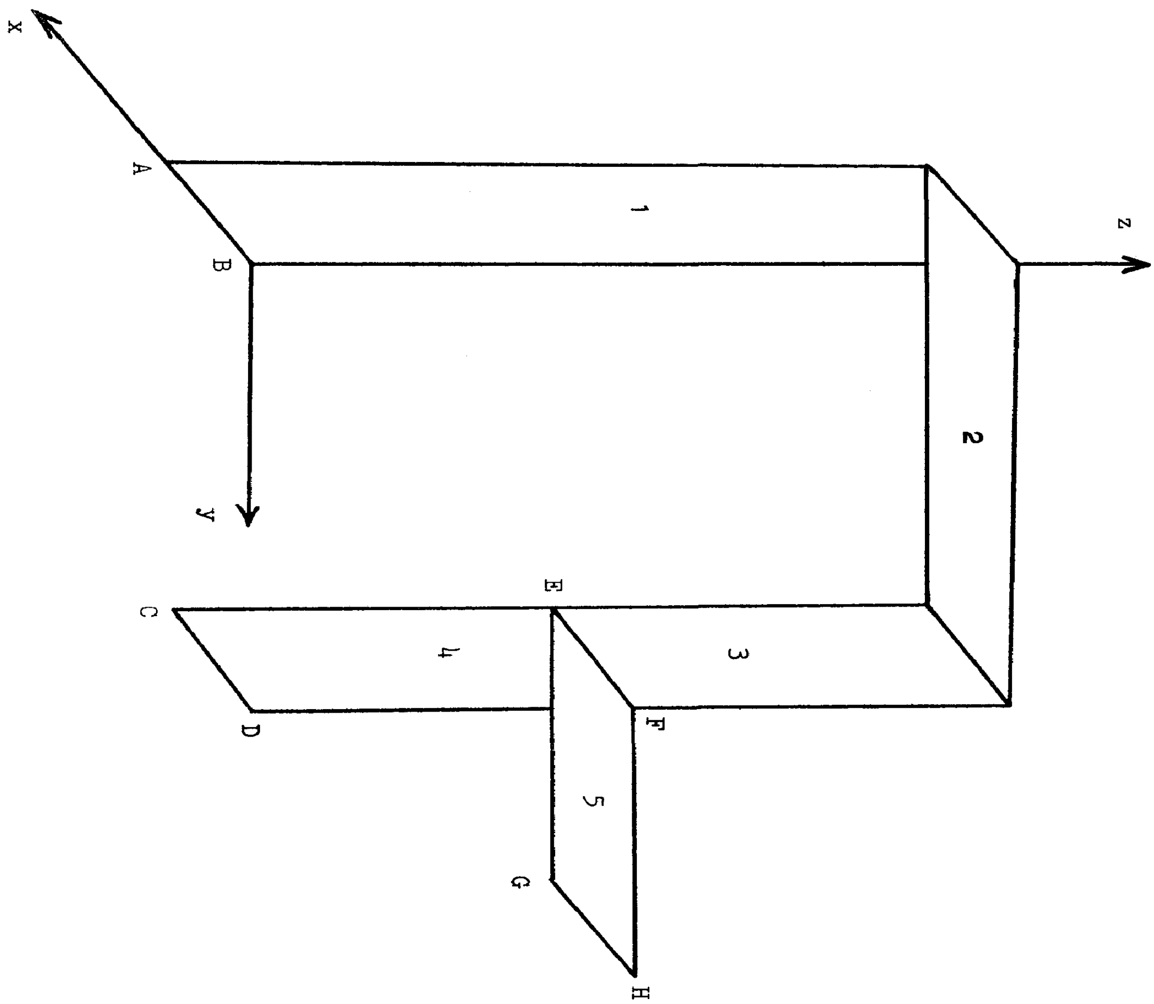


Fig.4a

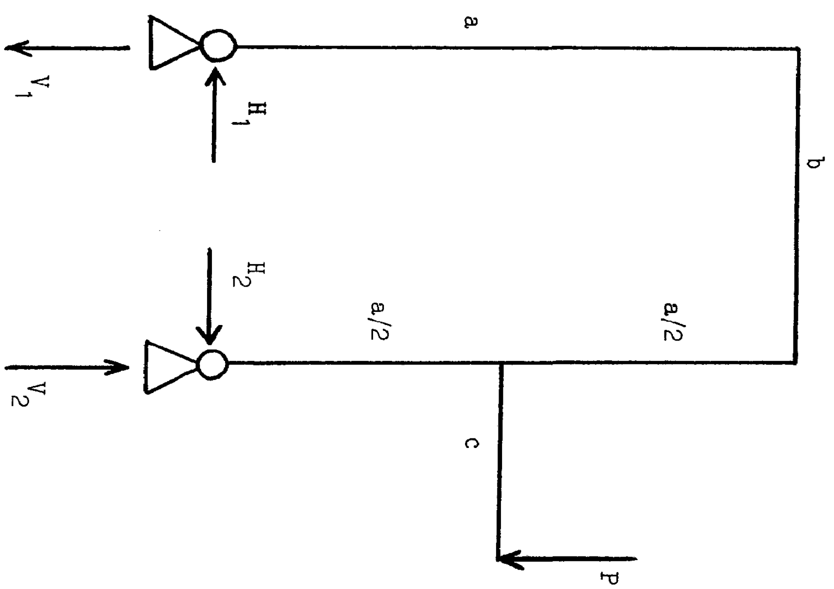
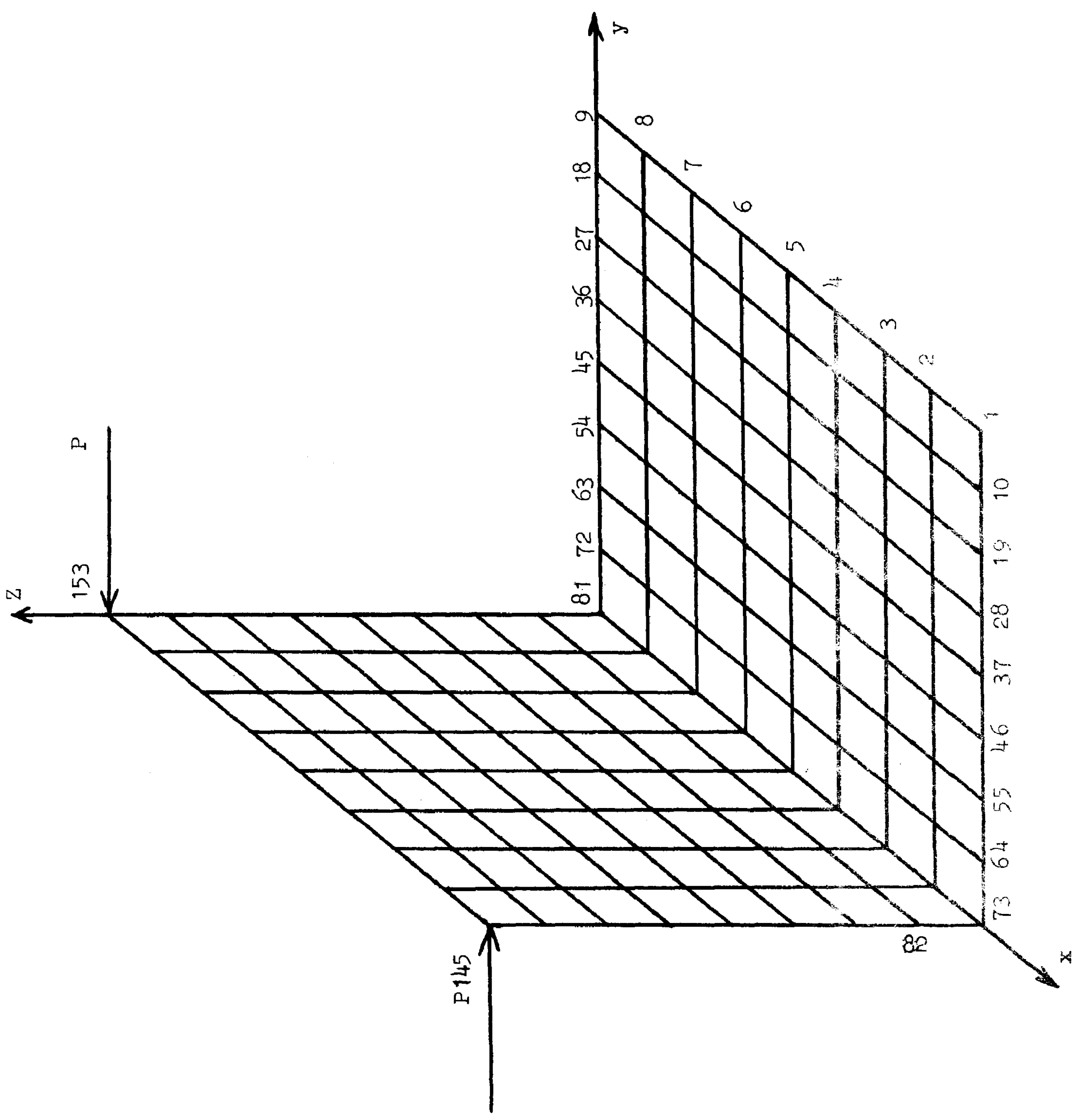


Fig.4b



z
153
P
y
9
18
27
36
45
54
63
72
81
8
7
6
5
4
3
2
1
10
19
28
37
46
55
64
73
82
x
P145

Fig.5