An *a priori* error estimate for a temporally discontinuous Galerkin space-time finite element method for linear elasto- and visco-dynamics*

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Abstract

We extend the formulation and *a priori* error analysis given by Johnson (Discontinuous Galerkin finite element methods for second order hyperbolic problems, *Comp. Meth. Appl. Mech. Eng.*, 107:117—129, 1993) from the acoustic wave equation to a Voigt and Maxwell-Zener viscodynamic system incorporating Rayleigh damping. The elastic term in the Rayleigh damping introduces a multiplicative $T^{1/2}$ growth in the constant but otherwise the error bound is consistent with that obtained by Johnson, with a constant that grows *a priori* with $T^{1/2}$ and also with norms of the solution. Gronwall's inequality is not used and so we can expect that this bound is of high enough quality to afford confidence in long-time integration. The viscoelasticity is modelled by internal variables that evolve according to ordinary differential equations and so the system shares similarities with dispersive Debye and Drude metamaterial models currently being studied in electromagnetism, as well as to acoustic metamaterial systems. This appears to be the first time an *a priori* error analysis has been given for DG-in-time treatment of dispersive problems of this type.

Keywords: discontinuous Galerkin, finite element method, *a priori* error estimate, duality, viscoelasticity, dispersion.

Sub. class: 65M60, 45K05, 45D05, 74D05, 35Q74

Contents

1 Introduction

In [12] Johnson formulated a space-time finite element method for the acoustic wave equation using a continuous Galerkin (CG) discretization in space and a discontinuous Galerkin (DG) discretization in time — DGCG-FEM. Both *a priori* and *a posteriori* error estimates were derived using approximation-error estimates, error representation through a discrete or continuous dual problem, and the associated strong stability of the dual solutions.

Here we extend that formulation to the equations of linear elastodynamics with generic *Rayleigh damping*, and also with viscoelastic damping provided by either or both of a Voigt term and a Maxwell-Zener history integral with a (Prony series) kernel of decaying exponentials. This Volterra integral is not itself included in the model but is instead captured by *internal variables* that evolve according to a set of ordinary differential equations.

We note that the Prony series model of viscoelasticity allows for an efficient numerical scheme in so much as we can compute over N time levels using $O(N)$ operations. On the other hand, alternative viscoelastic kernels based on the fractional calculus, or *power laws* as in [5], require a quadrature summation over time levels $0, 1, \ldots, n$ for each time level $n = 1, \ldots, N$ and, if implemented naïvely, will require $O(N^2)$ operations. This and the associated computer memory requirements imply that long-time computations in, say, 3D over moderate to long time scales are impractical without using a method that mitigates this difficulty. For example the *sparse* method in [26] or the convolution quadrature in [24], are available for finite difference time discretizations, and in [20] McLean has proposed a fast method that is economical on storage for a DGFEM time discretization of a subdiffusion equation. These methods are of great interest because, in particular, the Prony series kernels used in viscoelastic models are sometimes felt to decay too fast to be effective in modelling 'real' materials, and may not display the correct frequency dependence (see e.g. [5]).

Nevertheless, the model described, analyzed and implemented below is of considerable importance in modelling damping and frequency dependence in dispersive 'soft' media, e.g. [10, 11], and has very close analogies in dispersive (e.g. Debye, Drude or Lorentz) electromagnetic *metamaterial* models, e.g. [16, 6, 15, 25]. Moreover, the emergence of negative dynamic mass metamaterials, e.g [29], will also involve the elastodynamic equations with the 'meta-effects' provided by companion evolution equations for, in essence, internal variables. We intend that the extension of the material in [12] offered here will provide a template for the subsequent DGCG-FEM computer modelling and numerical analysis of dispersive media as modelled by internal variable systems.

This extension is not completely trivial which is why we present it here. Some care has to be taken in how the internal variables are defined, see Remark 2.2, because this impacts on the ease with which stability estimates for the dual problem can be derived. It also affects the nature of the dual problem itself and while we do not claim that the approach below is the *only one* that can be taken, it seems clear that it is quite amenable to analysis and implementation. However, because this is an extension of [12] we have focussed more on giving details for the new terms that arise in the proofs rather than re-iterate the results in that existing work.

The plan of the paper is as follows. We outline the physical model and its main features in Section 2, and then give the DGCG-FEM approximation in Section 3. We derive an *a priori* error bound in Section 4 by following a duality argument and using a strong stability estimate for a discrete dual problem, (31). This stability estimate, Theorem 4.6, does not require Gronwall's lemma and this in turn means that the constant in the error bound does not grow exponentially in time, but only *a priori* as $T^{1/2}$ as found in [12], along with the growth stemming from norms of the exact solution. There is also an additional multiplicative temporal growth of $T^{1/2}$ of the constant that is tied to the elastic term in the Rayleigh damping — but this growth does not appear in the error estimate in Theorem 4.7 if this type of damping is not present. In either case, the absence of an e^{cT} growth means that we can expect that this bound is of high enough quality to afford confidence in long-time integration. We give some numerical results in Section 5 and finish with a discussion in Section 6.

The 1993 work by Johnson in [12] appears to have been motivated by Hughes and Hulbert's work [7, 9, 8] in elastodynamics. At around the same time French in [3] gave an alternative approach for a DG-in-time method, and French and Peterson [4] formulated a continuous-in-time approximation. Both of these were for the wave equation as a model problem. Later, Li and Wiberg in [17] gave some numerical demonstrations of how effective Johnson's scheme is for elastodynamics and those comments prompted this study. Furthermore, although we restrict attention to approximations that are piecewise linear in space and time, higher order approximations can be implemented using the decoupling approach described in [28]. A disadvantage of this is that it leads naturally to the challenge of solving complex symmetric systems, as in [14, 13], but Richter, Springer and Vexler in [22] have recently outlined an iterative approach that avoids complex arithmetic.

2 The continuum problem

To describe the problem and the constitutive relationship, let the spatial domain Ω be a time-independent open bounded polytope in $\mathbb{R}^d$ for $d = 1, 2$ or 3, and let it represent the interior of a homogeneous and isotropic linear viscoelastic compressible body with constant mass density ϱ. The boundary, $\partial\Omega$, is partitioned into $\{\Gamma_D, \Gamma_N\}$ (also time independent) with Dirichlet boundary values given on the closed set Γ_D and Neumann boundary values specified on the open (and possibly empty) set Γ_N. As usual we require that $\Gamma_D \cap \Gamma_N = \varnothing$ and $\Gamma_D \cup \Gamma_N = \partial\Omega$ and we insist that $\text{meas}_{\partial\Omega}(\Gamma_D) > 0$. The unit outward normal vector to Γ_N will be written as $\hat{\boldsymbol{n}}$. To describe the time dependence we set $I := (0, T]$ and will usually use overdots, as in $\dot{v}$, or subscripts, as in v_t, to denote partial time differentiation.

The viscoelastic body is acted upon by a system of body forces, $\boldsymbol{f} := (f_i(\boldsymbol{x}, t))_{i=1}^d$ for $\boldsymbol{x} := (x_i)_{i=1}^d \in \Omega$ and $t \in I$, and a system of surface tractions, $\boldsymbol{g} := (g_i(\boldsymbol{x}, t))_{i=1}^d$ for $\boldsymbol{x} \in \Gamma_N$ and $t \in I$, and we seek the displacement from equilibrium, $\boldsymbol{u} = (u_i(\boldsymbol{x}, t))_{i=1}^d \colon \Omega \times I \to \mathbb{R}^d$ that results from these forces.

To describe the constitutive relationship we follow the standard literature (e.g. [5, 2]), assume that $t = 0$ is a reference time such that $\boldsymbol{u} = \boldsymbol{0}$ for all $t < 0$, and introduce the (symmetric) strain tensor,

$$\varepsilon_{ij}(\boldsymbol{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1}$$

where in this and below we will usually suppress the explicit display of the $\boldsymbol{x}$ dependence. The (symmetric) stress tensor, $\underline{\boldsymbol{\sigma}} := (\sigma_{ij})_{i,j=1}^{d}$, is then given (e.g. [5]) by either of the following linear functionals of displacement,

$$\sigma_{ij}(\boldsymbol{u};t) = C_{ijkl}\varepsilon_{kl}\big(\dot{\boldsymbol{u}}(t)\big) + D_{ijkl}(0)\varepsilon_{kl}\big(\boldsymbol{u}(t)\big) - \int_0^t \frac{\partial D_{ijkl}(t-s)}{\partial s}\varepsilon_{kl}\big(\boldsymbol{u}(s)\big)\,ds, \tag{2}$$

$$= C_{ijkl}\varepsilon_{kl}\big(\dot{\boldsymbol{u}}(t)\big) + D_{ijkl}(t)\varepsilon_{kl}\big(\boldsymbol{u}(0)\big) + \int_0^t D_{ijkl}(t-s)\varepsilon_{kl}\big(\dot{\boldsymbol{u}}(s)\big)\,ds \tag{3}$$

where an integration by parts shows these to be formally equivalent. Here and below summation is implied over repeated indices.

In this $\underline{\boldsymbol{C}}$ and $\underline{\boldsymbol{D}}(t)$ are fourth order tensors with the former related to Kelvin-Voigt viscoelasticity and the latter to the Zener and Maxwell models. In fact $\underline{\boldsymbol{D}}$ is essentially a *stress relaxation* analogue of the *Hooke tensor* from linear elasticity and, with $\underline{\boldsymbol{C}} = \underline{\boldsymbol{0}}$, this is linear elasticity with memory.

In general we assume that $\underline{\boldsymbol{D}}(0)$ is positive definite so that $\gamma_{ij}\gamma_{kl}D_{ijkl}(0) > 0$ a.e. in Ω for all non-zero symmetric second order tensors $\underline{\boldsymbol{\gamma}}$ and also that (on physical grounds) $\underline{\boldsymbol{D}}$ satisfies the symmetries: $D_{ijkl}(t) = D_{jikl}(t) = D_{ijlk}(t)$. In general $D_{ijkl}(t) \neq D_{klij}(t)$ except at $t = 0$ and at the limit $t \to \infty$, but for isotropic materials this last symmetry holds for all times (see e.g. [18, equations (1.10), (2.62)]).

A much simpler formulation entails if we assume that the material is *synchronous*. This means that every component of $\underline{\boldsymbol{D}}$ has the same time dependence and means that we can replace $\underline{\boldsymbol{D}}(t)$ with the factorization $\varphi(t)\underline{\boldsymbol{D}}$. Now $\underline{\boldsymbol{D}}$ is temporally constant and φ is a *stress relaxation function* which in the material below we take as given by the Prony series

$$\varphi(t) = \varphi_0 + \sum_{q=1}^{N_\varphi} \varphi_q \exp(-t/\tau_q) \tag{4}$$

where $\varphi_q \geqslant 0$ for $q \in \{0, 1, \dots, N_\varphi\}$, $\tau_q > 0$ for $q \in \{1, \dots, N_\varphi\}$ and we normalize so that $\varphi_0 + \sum_q \varphi_q = 1$. In [5], Golden and Graham observe that $\varphi_0 = 0$ corresponds to a (very slow moving) viscoelastic fluid whereas $\varphi_0 > 0$ gives a solid. We restrict ourselves to synchronous solids below.

Moreover, due to the body being homogeneous and isotropic the tensor $\underline{\boldsymbol{D}}$ can be described by just two Lamé coefficients, $\lambda = \nu E/\big((1+\nu)(1-2\nu)\big)$ and $\mu = 2G = E/(1+\nu)$, where $E > 0$ is Young's modulus, $G > 0$ is the shear modulus and $\nu \in (-1, 1/2)$ is Poisson's ratio. The case $\nu < 0$ allows for auxetic meta-materials, but we can expect that $\nu > 0$ for most (if not all) naturally occurring materials. The action of $\underline{\boldsymbol{D}}$ is now given by $D_{ijkl}\varepsilon_{kl}(\boldsymbol{u}) = \lambda\nabla\cdot\boldsymbol{u}\delta_{ij} + \mu\varepsilon_{ij}(\boldsymbol{u})$. We assume for simplicity that λ and μ are constant in space and time.

The form of $\underline{\boldsymbol{C}}$ is not so clear cut but in *Rayleigh damping* (see e.g. Li and Wiberg [17]) we add a term proportional to $\underline{\boldsymbol{\varepsilon}}(\dot{\boldsymbol{u}})$ (a 'stiffness matrix' term) and a term proportional to $\dot{\boldsymbol{u}}$ (a 'mass matrix' term) to the momentum balance. To incorporate the stiffness part of this into our model we choose $\underline{\boldsymbol{C}} = \gamma_E \underline{\boldsymbol{D}}$ where γ_E (in units of sec) is a non-negative constant.

Introducing initial data $\breve{\boldsymbol{u}}$ and $\breve{\boldsymbol{w}}$, the resulting problem is, for each $i \in \{1, \ldots, d\}$, find $\boldsymbol{u}$ such that,

$$\varrho \dot{w}_i + \varrho\gamma_M w_i - \sigma_{ij,j} = f_i \qquad \text{in } \Omega \times I, \tag{5}$$

$$\boldsymbol{w} = \dot{\boldsymbol{u}}, \qquad \boldsymbol{u}(0) = \breve{\boldsymbol{u}}, \qquad \boldsymbol{w}(0) = \breve{\boldsymbol{w}} \tag{6}$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_D \times I \qquad \text{and} \qquad \sigma_{ij}\widehat{n}_j = g_i \text{ on } \Gamma_N \times I \tag{7}$$

where the γ_M term is the 'mass matrix' contribution to the Rayleigh damping, for γ_M (in units of sec^{-1}) a non-negative constant.

We could work with the memory integrals, (2) or (3), for the constitutive time dependence, but when the stress relaxation function is given by (4) we can capture the history with *internal variables*. For this we set $\beta_q := (\varphi_q\tau_q)^{1/2}$ and note that

$$\begin{aligned}
&\varphi(t)\underline{\boldsymbol{\varepsilon}}\big(\breve{\boldsymbol{u}}\big) + \int_0^t \varphi(t-s)\underline{\boldsymbol{\varepsilon}}\big(\dot{\boldsymbol{u}}(s)\big)\,ds \\
&\qquad = \underline{\boldsymbol{\varepsilon}}\left(\varphi(t)\breve{\boldsymbol{u}} + \int_0^t \varphi_0\dot{\boldsymbol{u}}(s)\,ds + \sum_{q=1}^{N_\varphi}\int_0^t \varphi_q\dot{\boldsymbol{u}}(s)e^{-(t-s)/\tau_q}\,ds\right), \\
&\qquad = \underline{\boldsymbol{\varepsilon}}\left((\varphi(t)-\varphi_0)\breve{\boldsymbol{u}} + \varphi_0\boldsymbol{u}(t) + \sum_{q=1}^{N_\varphi}\beta_q\int_0^t \left(\frac{\varphi_q}{\tau_q}\right)^{1/2}\dot{\boldsymbol{u}}(s)e^{-(t-s)/\tau_q}\,ds\right), \\
&\qquad = (\varphi(t)-\varphi_0)\underline{\boldsymbol{\varepsilon}}(\breve{\boldsymbol{u}}) + \varphi_0\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}(t)) + \sum_{q=1}^{N_\varphi}\beta_q\underline{\boldsymbol{\varepsilon}}(\boldsymbol{z}_q(t)),
\end{aligned}$$

[¶[1]] where the internal variables are defined as,

$$\boldsymbol{z}_q(t) := \int_0^t \left(\frac{\varphi_q}{\tau_q}\right)^{1/2}\dot{\boldsymbol{u}}(s)e^{-(t-s)/\tau_q}\,ds \tag{8}$$

or, equivalently, recalling that $\boldsymbol{w} := \dot{\boldsymbol{u}}$,

$$\boldsymbol{z}_q(t) + \tau_q\dot{\boldsymbol{z}}_q(t) = \beta_q\boldsymbol{w}(t), \quad \text{with } \boldsymbol{z}_q(0) = \boldsymbol{0} \tag{9}$$

for $q = 1, 2, \ldots, N_\varphi$. With this the constitutive law (3) can be written as,

$$\begin{aligned}
\underline{\boldsymbol{\sigma}}(\boldsymbol{u};t) &= \underline{\boldsymbol{C}}\underline{\boldsymbol{\varepsilon}}\big(\boldsymbol{w}(t)\big) + \underline{\boldsymbol{D}}\left(\varphi(t)\underline{\boldsymbol{\varepsilon}}\big(\breve{\boldsymbol{u}}\big) + \int_0^t \varphi(t-s)\underline{\boldsymbol{\varepsilon}}\big(\boldsymbol{w}(s)\big)\,ds\right), \\
&= \gamma_E\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}\big(\boldsymbol{w}(t)\big) + \underline{\boldsymbol{D}}\left((\varphi(t)-\varphi_0)\underline{\boldsymbol{\varepsilon}}(\breve{\boldsymbol{u}}) + \varphi_0\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}(t)) + \sum_{q=1}^{N_\varphi}\beta_q\underline{\boldsymbol{\varepsilon}}(\boldsymbol{z}_q(t))\right).
\end{aligned} \tag{10}$$

[1]Short Version:

$$\varphi(t)\underline{\boldsymbol{\varepsilon}}\big(\breve{\boldsymbol{u}}\big) + \int_0^t \varphi(t-s)\underline{\boldsymbol{\varepsilon}}\big(\dot{\boldsymbol{u}}(s)\big)\,ds = (\varphi(t)-\varphi_0)\underline{\boldsymbol{\varepsilon}}(\breve{\boldsymbol{u}}) + \varphi_0\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}(t)) + \sum_{q=1}^{N_\varphi}\beta_q\underline{\boldsymbol{\varepsilon}}(\boldsymbol{z}_q(t)),$$

[¶[2]]

To give a weak formulation of (5) with (10)[¶[3]] we first recall the product Hilbert spaces, $\boldsymbol{H}^s(\Omega) := H^s(\Omega)^d$, for $s = 0, 1, 2, \ldots$, with inner products given for all $\boldsymbol{w}$, $\boldsymbol{v} \in \boldsymbol{H}^s(\Omega)$ by $(\boldsymbol{w}, \boldsymbol{v})_s := \sum_{i=1}^{d} (w_i, v_i)_{H^s(\Omega)}$. These spaces have the natural norms $\| \cdot \|_s := \sqrt{(\cdot, \cdot)_s}$ and, of course, $\boldsymbol{L}_2(\Omega) \equiv \boldsymbol{H}^0(\Omega)$. We use $(\cdot, \cdot)$ to denote the inner product on both $L_2(\Omega)$ and $\boldsymbol{L}_2(\Omega)$ and will introduce additional notation as and when necessary below. In particular, the natural energy space for this problem is given by

$$\boldsymbol{X} := \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \colon \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D \right\}, \tag{12}$$

and we also define the symmetric bilinear forms a, $b\colon \boldsymbol{X} \times \boldsymbol{X} \to \mathbb{R}$ by

$$a(\boldsymbol{\vartheta}, \boldsymbol{v}) := \int_\Omega D_{ijkl} \varepsilon_{kl}(\boldsymbol{\vartheta}) \varepsilon_{ij}(\boldsymbol{v}) \, d\Omega, \tag{13}$$

$$b(\boldsymbol{\vartheta}, \boldsymbol{v}) := \gamma_M (\varrho \boldsymbol{\vartheta}, \boldsymbol{v}) + \gamma_E a(\boldsymbol{\vartheta}, \boldsymbol{v}) \tag{14}$$

for all $\boldsymbol{\vartheta}$, $\boldsymbol{v} \in \boldsymbol{X}$.

It is easy to see that $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are continuous on $\boldsymbol{H}^1(\Omega)$, but not so easy to see that for a positive constant c we also have $a(\boldsymbol{v}, \boldsymbol{v}) \geqslant c \|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)}$ for all $\boldsymbol{v} \in \boldsymbol{X}$. This coercivity of $a(\cdot, \cdot)$ follows from our requirement that $\text{meas}_{\partial\Omega}(\Gamma_D) > 0$ in (7), and is a consequence of Korn-type inequalities. If $\Gamma_D = \partial\Omega$ this coercivity is easily established but in the more general case a non-trivial technical argument is needed to show that the coercivity results from excluding the possibility of rigid-body translations and rotations. The details of both of these coercivity results are given in, for example, [21, Thm. 3.1; Def. 3.1 and Thm. 3.5], and from them it follows that $(\boldsymbol{X}, a(\cdot, \cdot))$ is a Hilbert space equivalent to $(\boldsymbol{H}^1(\Omega), (\cdot, \cdot)_1)$ and with topological dual $\boldsymbol{X}'$. We will use the induced *energy norm* $\|\boldsymbol{v}\|_{\boldsymbol{X}} := \sqrt{a(\boldsymbol{v}, \boldsymbol{v})}$ extensively below.

Testing (5), integrating by parts, using (10)[¶[4]], and imposing $\boldsymbol{w} = \dot{\boldsymbol{u}}$ and each of the internal variable evolution equations, (9), individually in the energy inner product $a(\cdot, \cdot)$ we arrive at the weak problem: find $\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{N_\varphi} \colon I \to \boldsymbol{X}$ such that,

$$(\varrho \dot{\boldsymbol{w}}(t), \boldsymbol{v}) + a(\boldsymbol{u}(t), \varphi_0 \boldsymbol{v}) + b(\boldsymbol{w}(t), \boldsymbol{v}) + \sum_{q=1}^{N_\varphi} a(\boldsymbol{z}_q(t), \beta_q \boldsymbol{v}) = \langle L(t), \boldsymbol{v} \rangle, \tag{15}$$

$$a(\boldsymbol{z}_q(t) + \tau_q \dot{\boldsymbol{z}}_q(t) - \beta_q \boldsymbol{w}(t), \boldsymbol{v}) = 0 \qquad \text{for each } q = 1, \ldots, N_\varphi, \tag{16}$$

$$a(\dot{\boldsymbol{u}}(t), \varphi_0 \boldsymbol{v}) = a(\boldsymbol{w}(t), \varphi_0 \boldsymbol{v}) \tag{17}$$

where each in turn holds for all $\boldsymbol{v} \in \boldsymbol{X}$, with $\boldsymbol{u}(0) = \check{\boldsymbol{u}}$, $\dot{\boldsymbol{u}}(0) = \check{\boldsymbol{w}}$, $\boldsymbol{z}_q(0) = \boldsymbol{0}$ for each q, and where $L\colon I \to \boldsymbol{X}'$ is the time dependent linear form defined by,

$$\langle L(t), \boldsymbol{v} \rangle := \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f}(t) \, d\Omega + \oint_{\Gamma_N} \boldsymbol{v} \cdot \boldsymbol{g}(t) \, d\Gamma + (\varphi_0 - \varphi(t)) a(\check{\boldsymbol{u}}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{X}.$$

[2] Short Version:

$$\underline{\boldsymbol{\sigma}}(\boldsymbol{u}; t) = \gamma_E \underline{\boldsymbol{D}} \underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t)) + \underline{\boldsymbol{D}} \left((\varphi(t) - \varphi_0) \underline{\boldsymbol{\varepsilon}}(\check{\boldsymbol{u}}) + \varphi_0 \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}(t)) + \sum_{q=1}^{N_\varphi} \beta_q \underline{\boldsymbol{\varepsilon}}(\boldsymbol{z}_q(t)) \right). \tag{11}$$

[3] Short Version: (11)

[4] Short Version: (11)

Our first (unsurprising) result confirms the dissipativity introduced by the viscoelastic damping terms.

Theorem 2.1 (energy balance, dissipation) *We have*

$$\|\varrho^{1/2}\boldsymbol{w}(t)\|_0^2 + \|\varphi_0^{1/2}\boldsymbol{u}(t)\|_{\boldsymbol{X}}^2 + \sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\boldsymbol{z}_q(t)\|_{\boldsymbol{X}}^2 + 2\int_0^t b(\boldsymbol{w}(s),\boldsymbol{w}(s))\,ds$$
$$+ 2\sum_{q=1}^{N_\varphi}\|\boldsymbol{z}_q\|_{L_2(0,t;\boldsymbol{X})}^2 = 2\int_0^t \langle L(s),\boldsymbol{w}(s)\rangle\,ds + \|\varrho^{1/2}\breve{\boldsymbol{w}}\|_0^2 + \|\varphi_0^{1/2}\breve{\boldsymbol{u}}\|_{\boldsymbol{X}}^2$$

for every $t \in I$. Moreover,

$$\|\varrho^{1/2}\boldsymbol{w}(t)\|_0^2 + \frac{1}{2}\|\varphi_0^{1/2}\boldsymbol{u}(t)\|_{\boldsymbol{X}}^2 + \sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\boldsymbol{z}_q(t)\|_{\boldsymbol{X}}^2$$
$$+ 2\int_0^t b(\boldsymbol{w}(s),\boldsymbol{w}(s))\,ds + 2\sum_{q=1}^{N_\varphi}\|\boldsymbol{z}_q\|_{L_2(0,t;\boldsymbol{X})}^2$$
$$\leqslant 2\|\varrho^{1/2}\breve{\boldsymbol{w}}\|_0^2 + \frac{9}{4}\|\varphi_0^{1/2}\breve{\boldsymbol{u}}\|_{\boldsymbol{X}}^2 + 32\|\varphi_0^{-1/2}L\|_{L_\infty(0,t;\boldsymbol{X}')}^2 + 16\|\varphi_0^{-1/2}\dot{L}\|_{L_1(0,t;\boldsymbol{X}')}^2$$

also for every $t \in I$.

Proof. Choose $\boldsymbol{v} = 2\boldsymbol{w}$ in (15), $\boldsymbol{v} = 2\boldsymbol{z}_q$ for each q in (16), $\boldsymbol{v} = 2\boldsymbol{u}$ in (17), and then add the results together and note that the terms involving $\sum_q a(\boldsymbol{w},\beta_q\boldsymbol{z}_q)$ cancel out. This gives,

$$\frac{d}{dt}\|\varrho^{1/2}\boldsymbol{w}(t)\|_0^2 + \frac{d}{dt}\|\varphi_0^{1/2}\boldsymbol{u}(t)\|_{\boldsymbol{X}}^2 + \sum_{q=1}^{N_\varphi}\frac{d}{dt}\|\tau_q^{1/2}\boldsymbol{z}_q(t)\|_{\boldsymbol{X}}^2$$
$$+ 2b(\boldsymbol{w}(t),\boldsymbol{w}(t)) + 2\sum_{q=1}^{N_\varphi}\|\boldsymbol{z}_q(t)\|_{\boldsymbol{X}}^2 = 2\langle L(t),\boldsymbol{w}(t)\rangle.$$

Next, integrating by parts and using three Young inequalities with $\epsilon = 8$ in each gives,

$$2\int_0^t \langle L(s),\boldsymbol{w}(s)\rangle\,ds = 2\langle L(t),\boldsymbol{u}(t)\rangle - 2\langle L(0),\breve{\boldsymbol{u}}\rangle - 2\int_0^t\langle \dot{L}(s),\boldsymbol{u}(s)\rangle\,ds$$
$$\leqslant 2\|\varphi_0^{-1/2}L\|_{L_\infty(0,t;\boldsymbol{X}')}\|\varphi_0^{1/2}\boldsymbol{u}\|_{L_\infty(0,t;\boldsymbol{X})} + 2\|\varphi_0^{-1/2}L\|_{L_\infty(0,t;\boldsymbol{X}')}\|\varphi_0^{1/2}\breve{\boldsymbol{u}}\|_{\boldsymbol{X}}$$
$$+ 2\|\varphi_0^{-1/2}\dot{L}\|_{L_1(0,t;\boldsymbol{X}')}\|\varphi_0^{1/2}\boldsymbol{u}\|_{L_\infty(0,t;\boldsymbol{X})},$$
$$\leqslant 8\|\varphi_0^{-1/2}L\|_{L_\infty(0,t;\boldsymbol{X}')}^2 + \frac{1}{8}\|\varphi_0^{1/2}\boldsymbol{u}\|_{L_\infty(0,t;\boldsymbol{X})}^2 + 8\|\varphi_0^{-1/2}L\|_{L_\infty(0,t;\boldsymbol{X}')}^2 + \frac{1}{8}\|\varphi_0^{1/2}\breve{\boldsymbol{u}}\|_{\boldsymbol{X}}^2$$
$$+ 8\|\varphi_0^{-1/2}\dot{L}\|_{L_1(0,t;\boldsymbol{X}')}^2 + \frac{1}{8}\|\varphi_0^{1/2}\boldsymbol{u}\|_{L_\infty(0,t;\boldsymbol{X})}^2,$$
$$= \frac{1}{8}\|\varphi_0^{1/2}\breve{\boldsymbol{u}}\|_{\boldsymbol{X}}^2 + 16\|\varphi_0^{-1/2}L\|_{L_\infty(0,t;\boldsymbol{X}')}^2 + 8\|\varphi_0^{-1/2}\dot{L}\|_{L_1(0,t;\boldsymbol{X}')}^2 + \frac{1}{4}\|\varphi_0^{1/2}\boldsymbol{u}\|_{L_\infty(0,t;\boldsymbol{X})}^2$$

and, therefore, noting that

$$\|\varrho^{1/2}\boldsymbol{w}(t)\|_0^2 + \|\varphi_0^{1/2}\boldsymbol{u}\|^2_{L_\infty(0,t;\boldsymbol{X})} + \sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\boldsymbol{z}_q(t)\|^2_{\boldsymbol{X}} + 2\int_0^t b(\boldsymbol{w}(s),\boldsymbol{w}(s))\,ds$$
$$+ 2\sum_{q=1}^{N_\varphi}\|\boldsymbol{z}_q\|^2_{L_2(0,t;\boldsymbol{X})} \leqslant 2\|\varrho^{1/2}\breve{\boldsymbol{w}}\|_0^2 + 2\|\varphi_0^{1/2}\breve{\boldsymbol{u}}\|^2_{\boldsymbol{X}}$$
$$+ 2\Big(\frac{1}{8}\|\varphi_0^{1/2}\breve{\boldsymbol{u}}\|^2_{\boldsymbol{X}} + 16\|\varphi_0^{-1/2}L\|^2_{L_\infty(0,t;\boldsymbol{X}')} + 8\|\varphi_0^{-1/2}\dot{L}\|^2_{L_1(0,t;\boldsymbol{X}')} + \frac{1}{4}\|\varphi_0^{1/2}\boldsymbol{u}\|^2_{L_\infty(0,t;\boldsymbol{X})}\Big)$$

then completes the proof.[¶[5]]

Remark 2.2 (the choice of internal variable definition) *The result just given in Theorem 2.1 did not require Gronwall's lemma and so is in some sense* sharp. *In fact the cancellation of the* $\sum_q a(\boldsymbol{w},\beta_q\boldsymbol{z}_q)$ *terms rendered the proof almost trivial, and this is why we used* (3) *rather than* (2) *to define the internal variables in* (8)*. In fact we could define internal variables using* (2)*, as in [23], and arrive at ODE's similar to those in* (9)*. On a physical level the approaches are equivalent but, in the latter case, the analogue to* (16) *will contain* $\boldsymbol{u}$ *and not* $\boldsymbol{w}$ *and the cancellation used above will not occur. Similarly high quality stability estimates can still be derived in that case but with considerably more effort, and in the space-time Galerkin framework set forth below, this additional effort seems not to bring additional rewards. On the contrary, it will make the definition of a discrete dual problem, as later in* (31)*, more obscure and impede the duality argument used in the derivation of a priori error bounds.*

In the next section we give a space-time finite element approximation of this problem using a continuous Galerkin scheme in space and a discontinuous Galerkin scheme in time (DGCG-FEM).

3 The discrete scheme

The finite element spatial discretization is performed in a standard way by generating a family of boundary conforming quasi-uniform meshes indexed by an element-size parameter h, and then constructing a corresponding family of standard conforming nodal (Lagrange) finite element spaces, $\boldsymbol{X}^h \subset \boldsymbol{X}$, of piecewise polynomials of degree $p \geqslant 1$. We assume that these spaces have the usual approximation property,

$$\inf_{\boldsymbol{v}^h\in\boldsymbol{X}^h}\Big\{\|\boldsymbol{v}-\boldsymbol{v}^h\|_0 + h\|\boldsymbol{v}-\boldsymbol{v}^h\|_1\Big\} \leqslant Ch^{p+1}\|\boldsymbol{v}\|_{H^{p+1}(\Omega)} \tag{18}$$

[5]Short Version: We then integrate by parts and use three Young inequalities with $\epsilon = 8$ in each to get,

$$2\int_0^t \langle L(s),\boldsymbol{w}(s)\rangle\,ds = 2\langle L(t),\boldsymbol{u}(t)\rangle - 2\langle L(0),\breve{\boldsymbol{u}}\rangle - 2\int_0^t\langle \dot{L}(s),\boldsymbol{u}(s)\rangle\,ds$$
$$\leqslant 2\|\varphi_0^{-1/2}L\|_{L_\infty(0,t;\boldsymbol{X}')}\|\varphi_0^{1/2}\boldsymbol{u}\|_{L_\infty(0,t;\boldsymbol{X})} + 2\|\varphi_0^{-1/2}L\|_{L_\infty(0,t;\boldsymbol{X}')}\|\varphi_0^{1/2}\breve{\boldsymbol{u}}\|_{\boldsymbol{X}}$$
$$+ 2\|\varphi_0^{-1/2}\dot{L}\|_{L_1(0,t;\boldsymbol{X}')}\|\varphi_0^{1/2}\boldsymbol{u}\|_{L_\infty(0,t;\boldsymbol{X})},$$
$$\leqslant \frac{1}{8}\|\varphi_0^{1/2}\breve{\boldsymbol{u}}\|^2_{\boldsymbol{X}} + 16\|\varphi_0^{-1/2}L\|^2_{L_\infty(0,t;\boldsymbol{X}')} + 8\|\varphi_0^{-1/2}\dot{L}\|^2_{L_1(0,t;\boldsymbol{X}')} + \frac{1}{4}\|\varphi_0^{1/2}\boldsymbol{u}\|^2_{L_\infty(0,t;\boldsymbol{X})}.$$

The proof is then completed by using a standard kick-back argument.

for all $\boldsymbol{v} \in \boldsymbol{H}^{p+1}(\Omega)$. For the time discretization we choose $N \in \mathbb{N}$, define the time step $k = T/N$ and set $I_n = (t_{n-1}, t_n)$ with $t_n = nk$. Note that although we could anticipate an adaptive solver and allow the time steps and $\boldsymbol{X}^h$ to vary with time by using the same approach as in [12], we don't because we are concerned only with an *a priori* error analysis and we want to keep the exposition simple.

We recall the $\boldsymbol{L}_2(\Omega)$ and elliptic projections, P_0 and $P_{\boldsymbol{X}}$, defined by

$$(P_0\boldsymbol{v} - \boldsymbol{v}, \boldsymbol{\chi}) = 0 \qquad \text{and} \qquad a(P_{\boldsymbol{X}}\boldsymbol{v} - \boldsymbol{v}, \boldsymbol{\chi}) = 0 \tag{19}$$

each for all $\boldsymbol{\chi} \in \boldsymbol{X}^h$, and we note from (18) that

$$\|\boldsymbol{v} - P_0\boldsymbol{v}\|_0 \leqslant Ch^{p+1}\|\boldsymbol{v}\|_{H^{p+1}(\Omega)} \qquad \text{and} \qquad \|\boldsymbol{v} - P_{\boldsymbol{X}}\boldsymbol{v}\|_{\boldsymbol{X}} \leqslant Ch^{p}\|\boldsymbol{v}\|_{H^{p+1}(\Omega)}. \tag{20}$$

Our other notation is either standard and/or well known in this context. We define the limits,

$$v_n^{\pm} = \lim_{\epsilon \downarrow 0} v(t_n \pm \epsilon), \qquad \text{the jumps}, \qquad [\![v]\!]_n := v_n^+ - v_n^-,$$

and the temporally local and global space-time forms

$$(\!(\cdot,\cdot)\!)_n := \int_{I_n} (\cdot,\cdot)\,dt \qquad \text{and} \qquad (\!(\cdot,\cdot)\!) := \sum_{n=1}^{N} (\!(\cdot,\cdot)\!)_n$$

with the obvious extensions to $a(\!(\cdot,\cdot)\!)_n$ and $\langle\!\langle\cdot,\cdot\rangle\!\rangle_n$ locally, and to $a(\!(\cdot,\cdot)\!)$ and $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ globally. The fully discrete finite element space is built from the space of temporally discontinuous piecewise polynomials of degree $r \geqslant 0$ which have target space $\boldsymbol{X}^h$:

$$\mathbb{V}_n := \mathbb{P}_r(I_n; \boldsymbol{X}^h) \qquad \text{and} \qquad \mathbb{V} = \Big\{\boldsymbol{v} \in L_\infty(0,T;\boldsymbol{X})\colon \boldsymbol{v}|_{I_n} \in \mathbb{V}_n\Big\}.$$

For convenience below we set $\mathbb{V}_n^{\times} := \mathbb{V}_n^{2+N_\varphi}$ and $\mathbb{V}^{\times} := \mathbb{V}^{2+N_\varphi}$.

The fully discrete approximation of the problem (15), with (16) and (17), is then: for $n = 1, \ldots, N$ in turn, find $(\boldsymbol{U}, \boldsymbol{W}, \boldsymbol{Z}_1, \ldots)|_{I_n} \in \mathbb{V}_n^{\times}$ such that,

$$\begin{aligned}
(\!(\varrho\dot{\boldsymbol{W}}, \boldsymbol{\vartheta})\!)_n + (\varrho\,[\![\boldsymbol{W}]\!]_{n-1}, \boldsymbol{\vartheta}_{n-1}^+) + a(\!(\boldsymbol{U}, \varphi_0\boldsymbol{\vartheta})\!)_n + b(\!(\boldsymbol{W}, \boldsymbol{\vartheta})\!)_n + \sum_{q=1}^{N_\varphi} a(\!(\boldsymbol{Z}_q, \beta_q\boldsymbol{\vartheta})\!)_n \\
+ \sum_{q=1}^{N_\varphi} a(\!(\boldsymbol{Z}_q + \tau_q\dot{\boldsymbol{Z}}_q - \beta_q\boldsymbol{W}, \boldsymbol{\xi}_q)\!)_n + \sum_{q=1}^{N_\varphi} a(\tau_q\,[\![\boldsymbol{Z}_q]\!]_{n-1}, \boldsymbol{\xi}_{q,n-1}^+) \\
+ a(\!(\dot{\boldsymbol{U}} - \boldsymbol{W}, \varphi_0\boldsymbol{\zeta})\!)_n + a([\![\boldsymbol{U}]\!]_{n-1}, \varphi_0\boldsymbol{\zeta}_{n-1}^+) = \langle\!\langle L, \boldsymbol{\vartheta}\rangle\!\rangle_n
\end{aligned} \tag{21}$$

for all $(\boldsymbol{\theta}, \boldsymbol{\zeta}, \boldsymbol{\xi}_1, \ldots) \in \mathbb{V}_n^{\times}$ and where we define

$$\boldsymbol{U}_0^- := P_{\boldsymbol{X}}\breve{\boldsymbol{u}} \qquad \text{and} \qquad \boldsymbol{W}_0^- := P_0\breve{\boldsymbol{w}}, \tag{22}$$

from (19), and $\boldsymbol{Z}_{q,0}^- = \boldsymbol{0}$ for each q. The discrete analogue of the first part of Theorem 2.1 now follows, for which Remark 2.2 remains relevant. The stability estimate is deferred to later (in Theorem 4.6) where we need it for a discrete dual problem.

Theorem 3.1 (dissipation) *There exists a unique solution to* (21) *such that,*

$$\|\varrho^{1/2}\boldsymbol{W}_n^-\|_0^2+\|\varphi_0^{1/2}\boldsymbol{U}_n^-\|_{\boldsymbol{X}}^2+\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\boldsymbol{Z}_{q,n}^-\|_{\boldsymbol{X}}^2+2\int_0^{t_n} b(\boldsymbol{W},\boldsymbol{W})\,dt+2\sum_{q=1}^{N_\varphi}\|\boldsymbol{Z}_q\|_{L_2(0,t_n;\boldsymbol{X})}^2$$
$$+\sum_{m=1}^{n}\left(\|\varrho^{1/2}\,[\![\boldsymbol{W}]\!]_{m-1}\,\|_0^2+\|\varphi_0^{1/2}\,[\![\boldsymbol{U}]\!]_{m-1}\,\|_{\boldsymbol{X}}^2+\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\,[\![\boldsymbol{Z}_q]\!]_{m-1}\,\|_{\boldsymbol{X}}^2\right)$$
$$=\|\varrho^{1/2}\boldsymbol{W}_0^-\|_0^2+\|\varphi_0^{1/2}\boldsymbol{U}_0^-\|_{\boldsymbol{X}}^2+\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\boldsymbol{Z}_{q,0}^-\|_{\boldsymbol{X}}^2+2\int_0^{t_n}\langle L,\boldsymbol{W}\rangle\,dt$$

for every $n\in\{1,2,\dots,N\}$.

Proof. Given this result we see that zero initial data, $\boldsymbol{W}_0^-=\boldsymbol{U}_0^-=\boldsymbol{Z}_{q,0}^-=\boldsymbol{0}$ (for each q), and load, $L=0$, would imply only a trivial discrete solution. It follows that a discrete solution exists and is unique for any given set of these data. It remains only to prove the stated equality and for this we first note the identity

$$(\!(\varrho\dot{\boldsymbol{W}},\boldsymbol{W})\!)_m+([\![\boldsymbol{W}]\!]_{m-1}\,,\varrho\boldsymbol{W}_{m-1}^+)=\frac{1}{2}\|\varrho^{1/2}\boldsymbol{W}_m^-\|_0^2-\frac{1}{2}\|\varrho^{1/2}\boldsymbol{W}_{m-1}^-\|_0^2+\frac{1}{2}\|\varrho^{1/2}\,[\![\boldsymbol{W}]\!]_{m-1}\,\|_0^2$$

along with the analogues for $a(\dot{\boldsymbol{U}},\varphi_0\boldsymbol{U})$ and $a(\dot{\boldsymbol{Z}}_q,\tau_q\boldsymbol{Z}_q)$. Choosing $(\boldsymbol{\vartheta},\boldsymbol{\zeta},\boldsymbol{\xi}_q,\dots)=2(\boldsymbol{W},\boldsymbol{U},\boldsymbol{Z}_q,\dots)$ in (21) and noting that the terms involving $a(\!(\boldsymbol{Z}_q,\beta_q\boldsymbol{W})\!)_m$ cancel out results in

$$\|\varrho^{1/2}\boldsymbol{W}_m^-\|_0^2-\|\varrho^{1/2}\boldsymbol{W}_{m-1}^-\|_0^2+\|\varphi_0^{1/2}\boldsymbol{U}_m^-\|_{\boldsymbol{X}}^2-\|\varphi_0^{1/2}\boldsymbol{U}_{m-1}^-\|_{\boldsymbol{X}}^2$$
$$+\sum_{q=1}^{N_\varphi}\left(\|\tau_q^{1/2}\boldsymbol{Z}_{q,m}^-\|_{\boldsymbol{X}}^2-\|\tau_q^{1/2}\boldsymbol{Z}_{q,m-1}^-\|_{\boldsymbol{X}}^2\right)+2b(\!(\boldsymbol{W},\boldsymbol{W})\!)_m+2\sum_{q=1}^{N_\varphi}a(\!(\boldsymbol{Z}_q,\boldsymbol{Z}_q)\!)_m$$
$$+\|\varrho^{1/2}\,[\![\boldsymbol{W}]\!]_{m-1}\,\|_0^2+\|\varphi_0^{1/2}\,[\![\boldsymbol{U}]\!]_{m-1}\,\|_{\boldsymbol{X}}^2+\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\,[\![\boldsymbol{Z}_q]\!]_{m-1}\,\|_{\boldsymbol{X}}^2=2\langle\!\langle L,\boldsymbol{W}\rangle\!\rangle_m,$$

and the proof is completed by summing over $m=1,\dots,n$.[¶[6]]

This discrete energy balance is consistent with that given in Theorem 2.1 for the exact solution, and we also see clearly the numerical dissipation introduced by the jump terms.

Summing over all time levels, we see that the global formulation of (21) is to find $(\boldsymbol{U},\boldsymbol{W},\boldsymbol{Z}_1,\dots)\in\mathbb{V}^\times$ such that,

$$\mathscr{A}((\boldsymbol{U},\boldsymbol{W},\boldsymbol{Z}_1,\dots),(\boldsymbol{\vartheta},\boldsymbol{\zeta},\boldsymbol{\xi}_1,\dots))=\mathscr{L}((\boldsymbol{\vartheta},\boldsymbol{\zeta},\boldsymbol{\xi}_1,\dots))\qquad\forall(\boldsymbol{\theta},\boldsymbol{\zeta},\boldsymbol{\xi}_1,\dots)\in\mathbb{V}^\times\tag{23}$$

where the linear form is defined by

$$\mathscr{L}((\boldsymbol{\vartheta},\boldsymbol{\zeta},\boldsymbol{\xi}_1,\dots))=(\boldsymbol{W}_0^-,\varrho\boldsymbol{\vartheta}_0^+)+a(\boldsymbol{U}_0^-,\varphi_0\boldsymbol{\zeta}_0^+)+\langle\!\langle L,\boldsymbol{\vartheta}\rangle\!\rangle\tag{24}$$

[6]Short Version: Next choose $(\boldsymbol{\vartheta},\boldsymbol{\zeta},\boldsymbol{\xi}_q,\dots)=2(\boldsymbol{W},\boldsymbol{U},\boldsymbol{Z}_q,\dots)$ in (21), note that the terms involving $a(\!(\boldsymbol{Z}_q,\beta_q\boldsymbol{W})\!)_m$ cancel out and then sum over $m=1,\dots,n$.

and the bilinear form by,

$$\mathscr{A}((\boldsymbol{U},\boldsymbol{W},\boldsymbol{Z}_1,\ldots),(\boldsymbol{\vartheta},\boldsymbol{\zeta},\boldsymbol{\xi}_1,\ldots)) = (\!(\varrho\dot{\boldsymbol{W}},\boldsymbol{\vartheta})\!) + a(\!(\boldsymbol{U},\varphi_0\boldsymbol{\vartheta})\!) + b(\!(\boldsymbol{W},\boldsymbol{\vartheta})\!)$$
$$+ a(\!(\dot{\boldsymbol{U}}-\boldsymbol{W},\varphi_0\boldsymbol{\zeta})\!) + \sum_{q=1}^{N_\varphi} a(\!(\boldsymbol{Z}_q,\beta_q\boldsymbol{\vartheta})\!) + \sum_{q=1}^{N_\varphi} a(\!(\boldsymbol{Z}_q+\tau_q\dot{\boldsymbol{Z}}_q-\beta_q\boldsymbol{W},\boldsymbol{\xi}_q)\!)$$
$$+ \sum_{n=1}^{N-1}\left(([\![\boldsymbol{W}]\!]_n, \varrho\boldsymbol{\vartheta}_n^+) + a([\![\boldsymbol{U}]\!]_n, \varphi_0\boldsymbol{\zeta}_n^+) + \sum_{q=1}^{N_\varphi} a([\![\boldsymbol{Z}_q]\!]_n, \tau_q\boldsymbol{\xi}_{q,n}^+)\right)$$
$$+ (\boldsymbol{W}_0^+,\varrho\boldsymbol{\vartheta}_0^+) + a(\boldsymbol{U}_0^+,\varphi_0\boldsymbol{\zeta}_0^+) + \sum_{q=1}^{N_\varphi} a(\boldsymbol{Z}_{q,0}^+,\tau_q\boldsymbol{\xi}_{q,0}^+). \tag{25}$$

Noting that $(\boldsymbol{U},\boldsymbol{W},\boldsymbol{Z}_1,\ldots)$ can, on recalling (22), be replaced by $(\boldsymbol{u},\boldsymbol{w},\boldsymbol{z}_1,\ldots)$ in (23) we obtain the following Galerkin orthogonality relationship

$$\mathscr{A}((\boldsymbol{u},\boldsymbol{w},\boldsymbol{z}_1,\ldots)-(\boldsymbol{U},\boldsymbol{W},\boldsymbol{Z}_1,\ldots),(\boldsymbol{\vartheta},\boldsymbol{\zeta},\boldsymbol{\xi}_1,\ldots)) = 0 \qquad \forall(\boldsymbol{\theta},\boldsymbol{\zeta},\boldsymbol{\xi}_1,\ldots)\in\mathbb{V}^\times. \tag{26}$$

In the next section we address the convergence of this scheme.

4 *A priori* error estimates

To give *a priori* error bounds for the discrete approximation, (21), or (23), of (15), (16) and (17) we make some mostly-standard assumptions regarding regularity and data. The important ones are captured in the following block.

Assumptions 4.1 (technical assumptions) *For the error analysis in this section we restrict to the specific case of piecewise linear polynomial approximation in space and time. As already mentioned we assume that the material coefficients are constant in space and time, that the body is a synchronous linear viscoelastic solid with* $0<\varphi_0\leqslant 1$, *that the domain* Ω *is a convex polytope that is exactly represented by the finite element mesh, and also that* $\Gamma_N=\varnothing$ *so that* $\boldsymbol{X}=\boldsymbol{H}_0^1(\Omega)$. *We further assume regularity of data and domain sufficient to guarantee that the system* (15), (16), (17) *has a unique solution* $\boldsymbol{u}\in W_\infty^3(I;\boldsymbol{X}\cap\boldsymbol{H}^3(\Omega))$ *and we assume* elliptic regularity *such that for every* $\boldsymbol{\ell}\in\boldsymbol{L}_2(\Omega)$ *the solution,* $\boldsymbol{q}\in\boldsymbol{X}$ *to the elasticity problem* $a(\boldsymbol{q},\boldsymbol{v})=(\boldsymbol{\ell},\boldsymbol{v})$ *for all* $\boldsymbol{v}\in\boldsymbol{X}$ *satisfies* $\|\boldsymbol{q}\|_{\boldsymbol{H}^2(\Omega)}\leqslant C_e\|\boldsymbol{\ell}\|_0$.

As a consequence of these assumptions and the Riesz representation theorem we may define a linear elasticity analogue of the *inverse Laplacian* as $\mathcal{G}\colon \boldsymbol{L}_2(\Omega)\to\boldsymbol{X}$ by the relationship $a(\mathcal{G}\boldsymbol{\ell},\boldsymbol{v})=(\boldsymbol{\ell},\boldsymbol{v})$ for all $\boldsymbol{v}\in\boldsymbol{X}$ as well as its discrete analogue $\mathcal{G}_h\colon \boldsymbol{L}_2(\Omega)\to\boldsymbol{X}^h$ given by $a(\mathcal{G}_h\boldsymbol{\ell},\boldsymbol{v})=(\boldsymbol{\ell},\boldsymbol{v})$ for all $\boldsymbol{v}\in\boldsymbol{X}^h$.

Theorem 4.2 (e.g. [27, Chap. 2]) *The map* $\mathcal{G}\colon \boldsymbol{L}_2(\Omega)\to\boldsymbol{X}$ *defined above is self-adjoint and positive definite on* $\boldsymbol{L}_2(\Omega)$. *Also,* $\mathcal{G}_h\colon \boldsymbol{L}_2(\Omega)\to\boldsymbol{X}^h$ *is self-adjoint and positive semi-definite on* $\boldsymbol{L}_2(\Omega)$. *Furthermore, there are positive constants,* C, $C_\star$, *such that*

$$\|(\mathcal{G}-\mathcal{G}_h)\boldsymbol{\ell}\|_0 \leqslant Ch^2\|\boldsymbol{\ell}\|_0 \tag{27}$$
$$|(\mathcal{G}_h\boldsymbol{\varkappa},\boldsymbol{\ell})| \leqslant C\|\boldsymbol{\varkappa}\|_{\boldsymbol{X}'}\|\boldsymbol{\ell}\|_0, \tag{28}$$
$$\|\ell\|_{\boldsymbol{X}'}^2 \leqslant |(\boldsymbol{\ell},\mathcal{G}_h\boldsymbol{\ell})| + C_\star h^2\|\boldsymbol{\ell}\|_0^2 \tag{29}$$

for all $\boldsymbol{\ell}, \boldsymbol{\varkappa} \in \boldsymbol{L}_2(\Omega)$.

Proof. For arbitrary $\boldsymbol{\ell} \in \boldsymbol{L}_2(\Omega)$ we have $(\mathcal{G}\boldsymbol{\ell}, \boldsymbol{\varkappa}) = a(\mathcal{G}\boldsymbol{\ell}, \mathcal{G}\boldsymbol{\varkappa}) = (\boldsymbol{\ell}, \mathcal{G}\boldsymbol{\varkappa})$, as well as $(\mathcal{G}\boldsymbol{\ell}, \boldsymbol{\ell}) = \|\mathcal{G}\boldsymbol{\ell}\|_{\boldsymbol{X}}^2 \geqslant 0$ with $\mathcal{G}\boldsymbol{\ell} = \mathbf{0}$ if and only if $\boldsymbol{\ell} = \mathbf{0}$. Furthermore, by the same reasoning $(\mathcal{G}_h\boldsymbol{\ell}, \boldsymbol{\varkappa}) = a(\mathcal{G}_h\boldsymbol{\ell}, \mathcal{G}_h\boldsymbol{\varkappa}) = (\boldsymbol{\ell}, \mathcal{G}_h\boldsymbol{\varkappa})$ and $(\mathcal{G}_h\boldsymbol{\ell}, \boldsymbol{\ell}) = \|\mathcal{G}_h\boldsymbol{\ell}\|_{\boldsymbol{X}}^2 \geqslant 0$ for all $\boldsymbol{\ell} \in \boldsymbol{L}_2(\Omega)$.

Next, by standard energy and approximation error estimates, followed by the Aubin-Nitsche duality technique we get $\|(\mathcal{G} - \mathcal{G}_h)\boldsymbol{\ell}\|_0 \leqslant Ch^2\|\mathcal{G}\boldsymbol{\ell}\|_{\boldsymbol{H}^2(\Omega)}$ and (27) then follows from elliptic regularity.

Notice now that $\|\mathcal{G}_h\boldsymbol{\ell}\|_{\boldsymbol{X}}^2 = (\mathcal{G}_h\boldsymbol{\ell}, \boldsymbol{\ell}) \leqslant \|\mathcal{G}_h\boldsymbol{\ell}\|_{\boldsymbol{X}}\|\boldsymbol{\ell}\|_{\boldsymbol{X}'}$ which gives $\|\mathcal{G}_h\boldsymbol{\ell}\|_{\boldsymbol{X}} \leqslant C\|\boldsymbol{\ell}\|_0$ because

$$\|\boldsymbol{\ell}\|_{\boldsymbol{X}'} = \sup_{\boldsymbol{v}\in\boldsymbol{X}\setminus\{0\}} \frac{(\boldsymbol{\ell}, \boldsymbol{v})}{\|\boldsymbol{v}\|_{\boldsymbol{X}}} \leqslant \|\boldsymbol{\ell}\|_0 \sup_{\boldsymbol{v}\in\boldsymbol{X}\setminus\{0\}} \frac{\|\boldsymbol{v}\|_0}{\|\boldsymbol{v}\|_{\boldsymbol{X}}} \leqslant C\|\boldsymbol{\ell}\|_0$$

and therefore $|(\mathcal{G}_h\boldsymbol{\varkappa}, \boldsymbol{\ell})| \leqslant \|\boldsymbol{\varkappa}\|_{\boldsymbol{X}'}\|\mathcal{G}_h\boldsymbol{\ell}\|_{\boldsymbol{X}} \leqslant C\|\boldsymbol{\varkappa}\|_{\boldsymbol{X}'}\|\boldsymbol{\ell}\|_0$ as claimed in (28). Lastly, for (29) we notice the isometry $\|\mathcal{G}\boldsymbol{\ell}\|_{\boldsymbol{X}} = \|\boldsymbol{\ell}\|_{\boldsymbol{X}'}$ for all $\ell \in \boldsymbol{L}_2(\Omega)$ from the Riesz theorem. Therefore, for every $\boldsymbol{\ell} \in \boldsymbol{L}_2(\Omega)$

$$\|\boldsymbol{\ell}\|_{\boldsymbol{X}'}^2 = \|\mathcal{G}\boldsymbol{\ell}\|_{\boldsymbol{X}}^2 = (\boldsymbol{\ell}, \mathcal{G}\boldsymbol{\ell}) = (\boldsymbol{\ell}, \mathcal{G}_h\boldsymbol{\ell}) + (\boldsymbol{\ell}, \mathcal{G}\boldsymbol{\ell} - \mathcal{G}_h\boldsymbol{\ell})$$

and, from (27), $\|\boldsymbol{\ell}\|_{\boldsymbol{X}'}^2 \leqslant |(\boldsymbol{\ell}, \mathcal{G}_h\boldsymbol{\ell})| + Ch^2\|\boldsymbol{\ell}\|_0^2$ which is (29).

To handle the time discretization errors we introduce, piecewise for each n, the projection $P_I|_{I_n} : C(\bar{I}_n) \to \mathbb{P}_1(I_n)$ defined by

$$(P_I v)_n^- = v_n^- \qquad \text{and} \qquad \int_{t_{n-1}}^{t_n} v(t) - P_I v(t)\, dt = 0. \tag{30}$$

We will need the following estimates of the approximation error associated with P_I as well as the error associated with the piecewise constant approximation of a function $v \in L_1(I_n)$ by its average value $\bar{v} := \frac{1}{k}\int_{t_{n-1}}^{t_n} v(s)\, ds$. [¶[7]]

Lemma 4.3 *For $t \in \bar{I}_n$, for each n, we have that,*

$$(P_I v)(t) = v(t_n) - \frac{2(t_n - t)}{k^2}\int_{t_{n-1}}^{t_n} (s - t_{n-1})\dot{v}(s)\, ds,$$

$$(I - P_I)v(t) = \int_t^{t_n} (s - t)\ddot{v}(s)\, ds - \frac{(t_n - t)}{k^2}\int_{t_{n-1}}^{t_n} (s - t_{n-1})^2\ddot{v}(s)\, ds.$$

Moreover, $|(I - P_I)v(t)| \leqslant C(q)k^{2-1/p}\|\ddot{v}\|_{L_p(I_n)}$ *and* $\|(I - P_I)v\|_{L_p(I_n)} \leqslant C(q)k^2\|\ddot{v}\|_{L_p(I_n)}$ *for Hölder conjugates* $p, q \in [1, \infty]$.

Proof. We first write $(P_I v)(t) = v(t_n) + (t_n - t)\psi$ on $\bar{I}_n$ and determine $\psi \in \mathbb{R}$. Noting

[7] Short Version: The proofs are standard and are omitted.

that $v(t) = v(t_n) + \int_{t_n}^{t} \dot{v}(s)\, ds$ we get,

$$
\begin{aligned}
0 &= \int_{I_n} v(t) - P_I v(t)\, dt, \\
&= \int_{I_n} \Big[v(t_n) + \int_{t_n}^{t} \dot{v}(s)\, ds\Big] - \Big[v(t_n) + (t_n - t)\psi\Big]\, dt, \\
&= -\int_{t_{n-1}}^{t_n} \int_{t}^{t_n} \dot{v}(s)\, ds dt - \psi \int_{t_{n-1}}^{t_n} t_n - t\, dt, \\
&= -\int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{s} dt\ \dot{v}(s)\, ds - \psi \frac{(t_n - t)^2}{2}\Big|_{t_n}^{t_{n-1}}, \\
&= -\int_{t_{n-1}}^{t_n} (s - t_{n-1})\dot{v}(s)\, ds - \frac{\psi k^2}{2}
\end{aligned}
$$

from which $\psi = -\frac{2}{k^2}\int_{t_{n-1}}^{t_n} (s - t_{n-1})\dot{v}(s)\, ds$ which establishes the first claim. Next we see that

$$
\begin{aligned}
v(t) - P_I v(t) &= \Big[v(t_n) + \int_{t_n}^{t} \dot{v}(s)\, ds\Big] - \Big[v(t_n) - \frac{2(t_n - t)}{k^2} \int_{t_{n-1}}^{t_n} (s - t_{n-1})\dot{v}(s)\, ds\Big], \\
&= \int_{t_n}^{t} \dot{v}(s)\, ds + \frac{2(t_n - t)}{k^2} \int_{t_{n-1}}^{t_n} (s - t_{n-1})\dot{v}(s)\, ds, \\
&= (s - t)\dot{v}(s)\Big|_{t_n}^{t} - \int_{t_n}^{t} (s - t)\ddot{v}(s)\, ds \\
&\quad + \frac{(t_n - t)}{k^2}\Big[(s - t_{n-1})^2 \dot{v}(s)\Big|_{t_{n-1}}^{t_n} - \int_{t_{n-1}}^{t_n} (s - t_{n-1})^2 \ddot{v}(s)\, ds\Big], \\
&= \int_{t}^{t_n} (s - t)\ddot{v}(s)\, ds - \frac{(t_n - t)}{k^2} \int_{t_{n-1}}^{t_n} (s - t_{n-1})^2 \ddot{v}(s)\, ds
\end{aligned}
$$

which is the second claim. Therefore, for $1 < p, q < \infty$ with $p^{-1} + q^{-1} = 1$,

$$
\begin{aligned}
|v(t) - P_I v(t)| &\leqslant \left| \int_{t}^{t_n} (s - t)\ddot{v}(s)\, ds - \frac{(t_n - t)}{k^2} \int_{t_{n-1}}^{t_n} (s - t_{n-1})^2 \ddot{v}(s)\, ds \right|, \\
&\leqslant \left(\int_{t}^{t_n} (s - t)^q\, ds\right)^{1/q} \left(\int_{t}^{t_n} |\ddot{v}(s)|^p\, ds\right)^{1/p} \\
&\quad + \frac{(t_n - t)}{k^2} \left(\int_{t_{n-1}}^{t_n} (s - t_{n-1})^{2q}\, ds\right)^{1/q} \left(\int_{t_{n-1}}^{t_n} |\ddot{v}(s)|^p\, ds\right)^{1/p}, \\
&\leqslant \left[\left(\frac{k^{q+1}}{q+1}\right)^{1/q} + \frac{t_n - t}{k^2}\left(\frac{k^{2q+1}}{2q+1}\right)^{1/q}\right] \|\ddot{v}\|_{L_p(I_n)}, \\
&\leqslant C(q)\Big(k^{(q+1)/q} + k^{-1+(2q+1)/q}\Big)\|\ddot{v}\|_{L_p(I_n)}
\end{aligned}
$$

which, after noting that $(q+1)/q = 2 - 1/p$ and $(2q+1)/q = 3 - 1/p$ proves the third claim for $p, q \in (1, \infty)$. The cases $p = 1, \infty$ follow from the first line of the above argument and completes the proof of the third claim. The fourth claim follows easily by using the third claim to estimate the $L_p(I_n)$ norm of $(I - P_I)v$.

Lemma 4.4 *For the average value of* $v \in L_1(I_n)$ *defined by*

$$\bar{v} := \frac{1}{k}\int_{t_{n-1}}^{t_n} v(s)\,ds \qquad \text{we have} \qquad v(t) - \bar{v} = \frac{1}{k}\int_{t_{n-1}}^{t_n}\int_s^t \dot{v}(\eta)\,d\eta ds$$

and $|v(t) - \bar{v}| \leqslant \|\dot{v}\|_{L_1(I_n)}$.

Proof. We have

$$\begin{aligned} v(t) - \bar{v} &= v(t) - \frac{1}{k}\int_{t_{n-1}}^{t_n}\left[v(t) + \int_t^s \dot{v}(\eta)\,d\eta\right] ds, \\ &= v(t) - \frac{1}{k}\int_{t_{n-1}}^{t_n} ds\; v(t) - \frac{1}{k}\int_{t_{n-1}}^{t_n}\int_t^s \dot{v}(\eta)\,d\eta ds, \\ &= \frac{1}{k}\int_{t_{n-1}}^{t_n}\int_s^t \dot{v}(\eta)\,d\eta ds \end{aligned}$$

as required. The estimate then follows immediately. ꝏꝏꝏ

[¶[8]]

Lemma 4.5 *If* $(\boldsymbol{Y}, (\cdot,\cdot))$, *with induced norm* $\|\cdot\|_{\boldsymbol{Y}}$, *is either* $\boldsymbol{L}_2(\Omega)$ *or one of its Hilbert subspaces, then for any* $p \in [1,\infty]$ *we have* $\|\boldsymbol{v} - \bar{\boldsymbol{v}}\|_{L_p(I_n;\boldsymbol{Y})} \leqslant k\|\dot{\boldsymbol{v}}\|_{L_p(I_n;\boldsymbol{Y})}$ *and* $\|(I - P_I)\boldsymbol{v}\|_{L_p(I_n;\boldsymbol{Y})} \leqslant 2k^2\|\ddot{\boldsymbol{v}}\|_{L_p(I_n;\boldsymbol{Y})}$.

Proof. We begin with the second claim because it is a less tedious argument. For $K(t,s) := (s-t)_+ - k^{-2}(t_n - t)(s - t_{n-1})^2$ we have from Lemma 4.3 that

$$\boldsymbol{v}(t) - (P_I\boldsymbol{v})(t) = \int_{t_{n-1}}^{t_n} K(t,s)\ddot{\boldsymbol{v}}(s)\,ds$$

for $t \in \bar{I}_n$. Therefore, for such t,

$$\begin{aligned} \|(I - P_I)\boldsymbol{v}(t)\|_{\boldsymbol{Y}}^2 &= \left(\int_{t_{n-1}}^{t_n} K(t,\eta)\ddot{\boldsymbol{v}}(\eta)\,d\eta, \int_{t_{n-1}}^{t_n} K(t,\xi)\ddot{\boldsymbol{v}}(\xi)\,d\xi\right), \\ &= \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} K(t,\eta)K(t,\xi)\big(\ddot{\boldsymbol{v}}(\eta), \ddot{\boldsymbol{v}}(\xi)\big)\,d\eta d\xi, \\ &\leqslant \left(\int_{t_{n-1}}^{t_n} |K(t,s)|\,\|\ddot{\boldsymbol{v}}(s)\|_{\boldsymbol{Y}}\,ds,\right)^2, \\ \Longrightarrow \qquad \|(I - P_I)\boldsymbol{v}(t)\|_{\boldsymbol{Y}} &\leqslant 2k^{1+1/q}\|\ddot{\boldsymbol{v}}(s)\|_{L_p(I_n;\boldsymbol{Y})} \end{aligned}$$

for $1 \leqslant p, q \leqslant \infty$ because, clearly, $\|K\|_{L_\infty(I_n\times I_n)} \leqslant 2k$. Taking the $L_p(I_n)$ norm on both sides then proves the second claim of the lemma.

For the first claim we use Lemma 4.4 to write

$$\begin{aligned} \|\boldsymbol{v}(t) - \bar{\boldsymbol{v}}\|_{\boldsymbol{Y}}^2 &= \frac{1}{k^2}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n}\int_{s_1}^t\int_{s_2}^t \big(\dot{\boldsymbol{v}}(\eta), \dot{\boldsymbol{v}}(\xi)\big)\,d\eta d\xi\,ds_1 ds_2, \\ &\leqslant \frac{1}{k^2}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n}\int_{\min\{s_1,t\}}^{\max\{s_1,t\}}\int_{\min\{s_2,t\}}^{\max\{s_2,t\}} \|\dot{\boldsymbol{v}}(\eta)\|_{\boldsymbol{Y}}\,\|\dot{\boldsymbol{v}}(\xi)\|_{\boldsymbol{Y}}\,d\eta d\xi\,ds_1 ds_2, \\ \Longrightarrow \qquad \|\boldsymbol{v}(t) - \bar{\boldsymbol{v}}\|_{\boldsymbol{Y}} &\leqslant \int_{t_{n-1}}^{t_n} \|\dot{\boldsymbol{v}}(\xi)\|_{\boldsymbol{Y}}\,d\xi. \end{aligned}$$

[8] Short Version:

The first claim then follows from this.

[¶[9]]

The next step is to introduce a discrete dual backward problem and establish strong stability estimates for its solution. We then use this dual problem to obtain an error representation formula and the error bound will follow from that, the dual stability estimates and approximation results. The discrete dual backward problem is: find $(\mathcal{U}, \mathcal{W}, \mathcal{Z}_1, \ldots) \in \mathbb{V}^\times$ such that,

$$\mathscr{A}^*((\mathcal{W}, \mathcal{U}, \mathcal{Z}_1, \ldots), (\boldsymbol{\vartheta}, \boldsymbol{\zeta}, \boldsymbol{\xi}_1, \ldots)) = \mathscr{G}((\boldsymbol{\vartheta}, \boldsymbol{\zeta}, \boldsymbol{\xi}_1, \ldots)) \qquad \forall (\boldsymbol{\theta}, \boldsymbol{\zeta}, \boldsymbol{\xi}_1, \ldots) \in \mathbb{V}^\times \tag{31}$$

where the linear form (with data $\mathcal{W}_N^+$, $\mathcal{U}_N^+$ and $\mathfrak{g}$ to be chosen later) is defined by

$$\mathscr{G}((\boldsymbol{\vartheta}, \boldsymbol{\zeta}, \boldsymbol{\xi}_1, \ldots)) = (\mathcal{W}_N^+, \varrho\boldsymbol{\zeta}_N^-) + a(\mathcal{U}_N^+, \varphi_0\boldsymbol{\vartheta}_N^-) + \langle\!\langle \mathfrak{g}, \boldsymbol{\zeta} \rangle\!\rangle \tag{32}$$

and the bilinear form is defined by,

$$\begin{aligned}
\mathscr{A}^*((\mathcal{W}, \mathcal{U}, \mathcal{Z}_1, \ldots), (\boldsymbol{\vartheta}, \boldsymbol{\zeta}, \boldsymbol{\xi}_1, \ldots)) &= -(\!(\varrho\dot{\mathcal{W}}, \boldsymbol{\zeta})\!) - a(\!(\mathcal{U}, \varphi_0\boldsymbol{\zeta})\!) + b(\!(\boldsymbol{\zeta}, \mathcal{W})\!) \\
+ a(\!(\mathcal{W} - \dot{\mathcal{U}}, \varphi_0\boldsymbol{\vartheta})\!) &- \sum_{q=1}^{N_\varphi} a(\!(\mathcal{Z}_q, \beta_q\boldsymbol{\zeta})\!) + \sum_{q=1}^{N_\varphi} a(\!(\mathcal{Z}_q - \tau_q\dot{\mathcal{Z}}_q + \beta_q\mathcal{W}, \boldsymbol{\xi}_q)\!) \\
- \sum_{n=1}^{N-1} \Bigg(([\![\mathcal{W}]\!]_n, \varrho\boldsymbol{\zeta}_n^-) &+ a([\![\mathcal{U}]\!]_n, \varphi_0\boldsymbol{\vartheta}_n^-) + \sum_{q=1}^{N_\varphi} a([\![\mathcal{Z}_q]\!]_n, \tau_q\boldsymbol{\xi}_{q,n}^-) \Bigg) \\
+ (\mathcal{W}_N^-, \varrho\boldsymbol{\zeta}_N^-) &+ a(\mathcal{U}_N^-, \varphi_0\boldsymbol{\vartheta}_N^-) + \sum_{q=1}^{N_\varphi} a(\mathcal{Z}_{q,N}^-, \tau_q\boldsymbol{\xi}_{q,N}^-).
\end{aligned} \tag{33}$$

If we define $\boldsymbol{X} \to \boldsymbol{X}'$ maps A and B^* using the bilinear forms so that $\langle A\boldsymbol{\chi}, \boldsymbol{\theta} \rangle = a(\boldsymbol{\chi}, \boldsymbol{\theta})$ and $\langle B^*\boldsymbol{\chi}, \boldsymbol{\theta} \rangle = b(\boldsymbol{\theta}, \boldsymbol{\chi})$ each for all $\boldsymbol{\theta}, \boldsymbol{\chi} \in \boldsymbol{X}$ then this corresponds to a discrete approximation to a backward problem which in 'strong form', and with $\mathcal{W} = \dot{\mathcal{U}}$, looks like $\varrho\dot{\mathcal{W}} + \varphi_0 A\mathcal{U} - B^*\mathcal{W} + \sum_q \beta_q A\mathcal{Z}_q = -\mathfrak{g}$ and $\tau_q\dot{\mathcal{Z}}_q - \mathcal{Z}_q = \beta_q\mathcal{W}$ for each q.

Integrating by parts in time and using,

$$\begin{aligned}
\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\varrho\dot{\boldsymbol{\zeta}}, \mathcal{W})\,dt + \sum_{n=1}^{N-1} (\varrho\,[\![\boldsymbol{\zeta}]\!]_n, \mathcal{W}_n^+) + (\varrho\boldsymbol{\zeta}_0^+, \mathcal{W}_0^+) \\
= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (-\varrho\dot{\mathcal{W}}, \boldsymbol{\zeta})\,dt + \sum_{n=1}^{N-1} (-\varrho\,[\![\mathcal{W}]\!]_n, \boldsymbol{\zeta}_n^-) + (\varrho\mathcal{W}_N^-, \boldsymbol{\zeta}_N^-),
\end{aligned}$$

for all $\mathcal{W}$ and $\boldsymbol{\zeta}$ such that $\mathcal{W}|_{I_n} \in W_1^1(I_n; \boldsymbol{X})$ and $\boldsymbol{\zeta}|_{I_n} \in W_1^1(I_n; \boldsymbol{X})$ for each $n \in \{1, \ldots, N\}$, with similar results for the terms involving $a(\dot{\mathcal{U}}, \varphi_0\boldsymbol{\vartheta})$ and $a(\dot{\mathcal{Z}}_q, \tau_q\boldsymbol{\xi}_q)$, gives that

$$\mathscr{A}^*((\mathcal{W}, \mathcal{U}, \mathcal{Z}_1, \ldots), (\boldsymbol{\vartheta}, \boldsymbol{\zeta}, \boldsymbol{\xi}_1, \ldots)) = \mathscr{A}((\boldsymbol{\vartheta}, \boldsymbol{\zeta}, \boldsymbol{\xi}_1, \ldots), (\mathcal{W}, \mathcal{U}, \mathcal{Z}_1, \ldots)). \tag{34}$$

Let $\Pi_u, \Pi_w, \Pi_1, \ldots, \Pi_{N_\varphi} \colon H^1(I; \boldsymbol{X}) \to \mathbb{V}$ be projections, as yet unspecified. Then, on choosing $(\boldsymbol{\theta}, \boldsymbol{\zeta}, \boldsymbol{\xi}_1, \ldots) = (\boldsymbol{U}, \boldsymbol{W}, \boldsymbol{Z}_1, \ldots) - (\Pi_u\boldsymbol{u}, \Pi_w\boldsymbol{w}, \Pi_1\boldsymbol{z}_1, \ldots)$ in (31), and using

[9] Short Version:

(34) and the Galerkin orthogonality in (26), we obtain the error representation formula,

$$\begin{aligned}\mathscr{G}((\boldsymbol{\vartheta},\boldsymbol{\zeta},\boldsymbol{\xi}_1,\ldots)) &= \mathscr{G}((\boldsymbol{U}-\Pi_u\boldsymbol{u},\boldsymbol{W}-\Pi_w\boldsymbol{w},\boldsymbol{Z}_1-\Pi_1\boldsymbol{z}_1,\ldots)),\\ &= \mathscr{A}^*((\mathcal{W},\mathcal{U},\mathcal{Z}_1,\ldots),(\boldsymbol{U}-\Pi_u\boldsymbol{u},\boldsymbol{W}-\Pi_w\boldsymbol{w},\boldsymbol{Z}_1-\Pi_1\boldsymbol{z}_1,\ldots)),\\ &= \mathscr{A}((\boldsymbol{U}-\Pi_u\boldsymbol{u},\boldsymbol{W}-\Pi_w\boldsymbol{w},\boldsymbol{Z}_1-\Pi_1\boldsymbol{z}_1,\ldots),(\mathcal{W},\mathcal{U},\mathcal{Z}_1,\ldots)),\\ &= \mathscr{A}((\boldsymbol{u}-\Pi_u\boldsymbol{u},\boldsymbol{w}-\Pi_w\boldsymbol{w},\boldsymbol{z}_1-\Pi_1\boldsymbol{z}_1,\ldots),(\mathcal{W},\mathcal{U},\mathcal{Z}_1,\ldots)). \end{aligned}\tag{35}$$

The terms $(\boldsymbol{u}-\Pi_u\boldsymbol{u},\boldsymbol{w}-\Pi_w\boldsymbol{w},\boldsymbol{z}_1-\Pi_1\boldsymbol{z}_1,\ldots)$ on the right can be bounded by approximation results and then once the terms involving $(\mathcal{W},\mathcal{U},\mathcal{Z}_1,\ldots)$ are bounded by the data in $\mathscr{G}$, and suitable choices for those data are made, we will obtain an *a priori* estimate for $\boldsymbol{U}-\Pi_u\boldsymbol{u}$ and $\boldsymbol{W}-\Pi_w\boldsymbol{w}$. The estimates for $\boldsymbol{u}-\boldsymbol{U}$ and $\boldsymbol{w}-\boldsymbol{W}$ then follow from more approximation estimates and the triangle inequality.

We begin by determining an analogue of Theorem 3.1, and derive stability estimates for the discrete dual problem where the final values of the dual internal variables are zero. In this and below it is to be understood that the temporal norms of time derivatives are 'broken' so that $\|\dot{\mathcal{W}}\|_{L_p(I;\cdot)} = (\sum_n \|\dot{\mathcal{W}}\|^p_{L_p(I_n;\cdot)})^{1/p}$ with the 'max$\{\ldots\}$' modification for $p=\infty$.

Theorem 4.6 (discrete dual stability) *Let Assumptions 4.1 hold and then, with $\mathcal{Z}_q(T)=\mathbf{0}$ for each q, there exists a unique solution to* (31) *that satisfies*

$$\begin{aligned}\|\varrho^{1/2}\mathcal{W}_n^+\|_0^2 + \|\varphi_0^{1/2}\mathcal{U}_n^+\|_{\boldsymbol{X}}^2 + \sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\mathcal{Z}_{q,n}^+\|_{\boldsymbol{X}}^2 + 2\int_{t_n}^T b(\mathcal{W},\mathcal{W})\,dt + 2\sum_{q=1}^{N_\varphi}\|\mathcal{Z}_q\|^2_{L_2(t_n,T;\boldsymbol{X})}\\ + \sum_{m=n+1}^{N}\left(\|\varrho^{1/2}\,[\![\mathcal{W}]\!]_m\,\|_0^2 + \|\varphi_0^{1/2}\,[\![\mathcal{U}]\!]_m\,\|_{\boldsymbol{X}}^2 + \sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\,[\![\mathcal{Z}_q]\!]_m\,\|_{\boldsymbol{X}}^2\right)\\ = \|\varrho^{1/2}\mathcal{W}_N^+\|_0^2 + \|\varphi_0^{1/2}\mathcal{U}_N^+\|_{\boldsymbol{X}}^2 + 2\int_{t_n}^T\langle\mathfrak{g},\mathcal{W}\rangle\,dt\end{aligned}$$

for every $n=0,1,\ldots,N-1$. If in addition $\mathfrak{g}=0$ and $h\leqslant c_Tk$ for a positive constant c_T then

$$\begin{aligned}\|\varrho^{1/2}\mathcal{W}\|^2_{L_\infty(I;\boldsymbol{L}_2(\Omega))} + \|\varrho^{1/2}\dot{\mathcal{W}}\|^2_{L_p(I;\boldsymbol{X}')} + \|\varphi_0^{1/2}\mathcal{U}\|^2_{L_\infty(I_n;\boldsymbol{X})} + \|\varphi_0^{1/2}\dot{\mathcal{U}}\|^2_{L_\infty(I;\boldsymbol{L}_2(\Omega))}\\ + \sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|^2_{L_\infty(I;\boldsymbol{X})} \leqslant CT^{2/p}(1+c_T^2)\left(\|\varrho^{1/2}\mathcal{W}_N^+\|_0^2 + \|\varphi_0^{1/2}\mathcal{U}_N^+\|_{\boldsymbol{X}}^2\right)\end{aligned}$$

for a constant C independent of T, h and k and where we can choose $p=\infty$ if $\gamma_E=0$ in (14) *and $p=2$ if $\gamma_E>0$.*

Proof. Notice that if the data, $\mathcal{W}_N^+$, $\mathcal{U}_N^+$ and $\mathfrak{g}$, are zero in the discrete dual problem then the first claim of the theorem provides uniqueness of solution, and existence then follows. To prove this equality, in analogy to the proof of Theorem 3.1, we choose in (31) $(\boldsymbol{\vartheta},\boldsymbol{\zeta},\boldsymbol{\xi}_1,\ldots)=(\mathcal{U},\mathcal{W},\mathcal{Z}_1,\ldots)$ on (t_n,T), and zero on $(0,t_n)$, for an arbitrary

$n = 0, 1, \ldots, N-1$, to get (with the notational convenience that $\int_{t_n}^T = \sum_m \int_{I_m}$),

$$\int_{t_n}^T \langle \mathfrak{g}, \mathcal{W} \rangle \, dt + (\mathcal{W}_N^+, \varrho \mathcal{W}_N^-) + a(\mathcal{U}_N^+, \varphi_0 \mathcal{U}_N^-) = -\int_{t_n}^T (\varrho \dot{\mathcal{W}}, \mathcal{W}) \, dt - \int_{t_n}^T a(\mathcal{U}, \varphi_0 \mathcal{W}) \, dt$$
$$+ \int_{t_n}^T b(\mathcal{W}, \mathcal{W}) \, dt + \int_{t_n}^T a(\mathcal{W}, \varphi_0 \mathcal{U}) \, dt - \int_{t_n}^T a(\dot{\mathcal{U}}, \varphi_0 \mathcal{U}) \, dt$$
$$- \sum_{q=1}^{N_\varphi} \int_{t_n}^T a(\mathcal{Z}_q, \beta_q \mathcal{W}) \, dt + \sum_{q=1}^{N_\varphi} \int_{t_n}^T a(\mathcal{Z}_q, \mathcal{Z}_q) - a(\tau_q \dot{\mathcal{Z}}_q, \mathcal{Z}_q) + a(\beta_q \mathcal{W}, \mathcal{Z}_q) \, dt$$
$$- \sum_{m=n+1}^{N-1} \left((\llbracket \mathcal{W} \rrbracket_m, \varrho \mathcal{W}_m^-) + a(\llbracket \mathcal{U} \rrbracket_m, \varphi_0 \mathcal{U}_m^-) + \sum_{q=1}^{N_\varphi} a(\llbracket \mathcal{Z}_q \rrbracket_m, \tau_q \mathcal{Z}_{q,m}^-) \right)$$
$$+ (\mathcal{W}_N^-, \varrho \mathcal{W}_N^-) + a(\mathcal{U}_N^-, \varphi_0 \mathcal{U}_N^-) + \sum_{q=1}^{N_\varphi} a(\mathcal{Z}_{q,N}^-, \tau_q \mathcal{Z}_{q,N}^-).$$

Noting that [¶[10]]

$$- \sum_{m=n+1}^{N} \int_{t_{m-1}}^{t_m} \frac{1}{2} \frac{d}{dt} \|\varrho^{1/2} \mathcal{W}\|_0^2 \, dt - \sum_{m=n+1}^{N-1} \left(\mathcal{W}_m^+ - \mathcal{W}_m^-, \varrho \mathcal{W}_m^- \right) + \|\varrho^{1/2} \mathcal{W}_N^-\|_0^2$$
$$= \frac{1}{2} \sum_{m=n+1}^{N} \|\varrho^{1/2} \mathcal{W}_m^-\|_0^2 + \frac{1}{2} \sum_{m=n+1}^{N} \|\varrho^{1/2} \mathcal{W}_{m-1}^+\|_0^2 - \sum_{m=n+1}^{N-1} (\mathcal{W}_m^+, \varrho \mathcal{W}_m^-),$$
$$= \frac{1}{2} \sum_{m=n}^{N-1} \|\varrho^{1/2} \mathcal{W}_m^+\|_0^2 - \frac{1}{2} \sum_{m=n+1}^{N} \|\varrho^{1/2} \mathcal{W}_m^+\|_0^2 + (\mathcal{W}_N^+, \varrho \mathcal{W}_N^-)$$
$$+ \frac{1}{2} \sum_{m=n+1}^{N} \Big[(\mathcal{W}_m^-, \varrho \mathcal{W}_m^-) + (\mathcal{W}_m^+, \varrho \mathcal{W}_m^+) - 2(\mathcal{W}_m^+, \varrho \mathcal{W}_m^-) \Big],$$

which leads eventually to

$$- \sum_{m=n+1}^{N} \int_{t_{m-1}}^{t_m} (\varrho \dot{\mathcal{W}}, \mathcal{W}) \, dt - \sum_{m=n+1}^{N-1} (\llbracket \mathcal{W} \rrbracket_m, \varrho \mathcal{W}_m^-) + (\varrho \mathcal{W}_N^-, \mathcal{W}_N^-)$$
$$= \frac{1}{2} \|\varrho^{1/2} \mathcal{W}_n^+\|_0^2 - \frac{1}{2} \|\varrho^{1/2} \mathcal{W}_N^+\|_0^2 + \frac{1}{2} \sum_{m=n+1}^{N} \|\varrho^{1/2} \llbracket \mathcal{W} \rrbracket_m \|_0^2 + (\varrho \mathcal{W}_N^+, \mathcal{W}_N^-),$$

[10] Short Version: and noting that

along with the analogues for $a(\dot{\mathcal{U}},\varphi_0\mathcal{U})$ and $a(\dot{\mathcal{Z}}_q,\tau_q\mathcal{Z}_q)$, then gives,

$$\int_{t_n}^{T}\langle\mathfrak{g},\mathcal{W}\rangle\,dt=\int_{t_n}^{T}b(\mathcal{W},\mathcal{W})\,dt+\sum_{q=1}^{N_\varphi}\|\mathcal{Z}_q\|^2_{L_2(t_n,T;\boldsymbol{X})}$$
$$+\frac{1}{2}\|\varrho^{1/2}\mathcal{W}_n^+\|_0^2+\frac{1}{2}\|\varphi_0^{1/2}\mathcal{U}_n^+\|_{\boldsymbol{X}}^2+\frac{1}{2}\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\mathcal{Z}_{q,n}^+\|_{\boldsymbol{X}}^2$$
$$-\frac{1}{2}\|\varrho^{1/2}\mathcal{W}_N^+\|_0^2-\frac{1}{2}\|\varphi_0^{1/2}\mathcal{U}_N^+\|_{\boldsymbol{X}}^2-\frac{1}{2}\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\mathcal{Z}_{q,N}^+\|_{\boldsymbol{X}}^2$$
$$+\frac{1}{2}\sum_{m=n+1}^{N}\left(\|\varrho^{1/2}\,[\![\mathcal{W}]\!]_m\,\|_0^2+\|\varphi_0^{1/2}\,[\![\mathcal{U}]\!]_m\,\|_{\boldsymbol{X}}^2+\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\,[\![\mathcal{Z}_q]\!]_m\,\|_{\boldsymbol{X}}^2\right)$$

and the proof — of the first part — is completed by setting each $\mathcal{Z}_{q,N}^+=\mathcal{Z}_{q,N}^-=\mathbf{0}$. [¶[11]]

Next, in (31) we choose $\boldsymbol{\zeta}|_{I_n}=(t_n-t)\mathcal{G}_h\dot{\mathcal{W}}$ to obtain

$$(\varrho\dot{\mathcal{W}},\mathcal{G}_h\dot{\mathcal{W}})=\frac{2}{k^2}\int_{t_{n-1}}^{t_n}(t_n-t)b(\mathcal{G}_h\dot{\mathcal{W}},\mathcal{W})\,dt$$
$$-\frac{2}{k^2}\int_{t_{n-1}}^{t_n}(t_n-t)(\varphi_0\mathcal{U},\dot{\mathcal{W}})\,dt-\sum_{q=1}^{N_\varphi}\frac{2}{k^2}\int_{t_{n-1}}^{t_n}(t_n-t)(\beta_q\mathcal{Z}_q,\dot{\mathcal{W}})\,dt.$$

[11] Short Version: along with the analogues for $a(\dot{\mathcal{U}},\varphi_0\mathcal{U})$ and $a(\dot{\mathcal{Z}}_q,\tau_q\mathcal{Z}_q)$, then gives the first part of the theorem once we set $\mathcal{Z}_{q,N}^+=\mathcal{Z}_{q,N}^-=\mathbf{0}$.

Recalling that $\beta_q^2 = \varphi_q\tau_q$ and using (29) and then (28) now gives,

$$
\begin{aligned}
\|\varrho^{1/2}\mathcal{W}\|^2_{\boldsymbol{X}'} &\leqslant C_* h^2\|\varrho^{1/2}\dot{\mathcal{W}}\|_0^2 + (\varrho\dot{\mathcal{W}}, \mathcal{G}_h\dot{\mathcal{W}}),\\
&\leqslant C_* h^2\|\varrho^{1/2}\dot{\mathcal{W}}\|_0^2 + \frac{2}{k^2}\int_{t_{n-1}}^{t_n}(t_n-t)b(\mathcal{G}_h\dot{\mathcal{W}},\mathcal{W})\,dt\\
&\quad - \frac{2}{k^2}\int_{t_{n-1}}^{t_n}(t_n-t)(\varphi_0\mathcal{U},\dot{\mathcal{W}})\,dt - \sum_{q=1}^{N_\varphi}\frac{2}{k^2}\int_{t_{n-1}}^{t_n}(t_n-t)(\beta_q\mathcal{Z}_q,\dot{\mathcal{W}})\,dt,\\
&\leqslant C_* h^2\|\varrho^{1/2}\dot{\mathcal{W}}\|_0^2 + \frac{2}{k^2}\int_{t_{n-1}}^{t_n}(t_n-t)\Big(\gamma_M(\varrho\mathcal{G}_h\dot{\mathcal{W}},\mathcal{W}) + \gamma_E(\dot{\mathcal{W}},\mathcal{W})\Big)\,dt\\
&\quad - \frac{2}{k^2}\int_{t_{n-1}}^{t_n}(t_n-t)(\varphi_0\mathcal{U},\dot{\mathcal{W}})\,dt - \sum_{q=1}^{N_\varphi}\frac{2}{k^2}\int_{t_{n-1}}^{t_n}(t_n-t)(\beta_q\mathcal{Z}_q,\dot{\mathcal{W}})\,dt,\\
&\leqslant C_* h^2\|\varrho^{1/2}\dot{\mathcal{W}}\|_0^2 + \frac{2C\gamma_M}{k^2}\int_{t_{n-1}}^{t_n}(t_n-t)\|\varrho^{1/2}\mathcal{W}\|_0\,dt\;\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'}\\
&\quad + \frac{2\gamma_E}{\varrho^{1/2}k^2}\int_{t_{n-1}}^{t_n}(t_n-t)\|\mathcal{W}\|_{\boldsymbol{X}}\,dt\;\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'}\\
&\quad + \frac{2\varphi_0^{1/2}}{\varrho^{1/2}k^2}\int_{t_{n-1}}^{t_n}(t_n-t)\|\varphi_0^{1/2}\mathcal{U}\|_{\boldsymbol{X}}\,dt\;\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'}\\
&\quad + \sum_{q=1}^{N_\varphi}\frac{2\varphi_q^{1/2}}{\varrho^{1/2}k^2}\int_{t_{n-1}}^{t_n}(t_n-t)\|\tau_q^{1/2}\mathcal{Z}_q\|_{\boldsymbol{X}}\,dt\;\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'},\\
&\leqslant C_* h^2\|\varrho^{1/2}\dot{\mathcal{W}}\|_0^2 + C\gamma_M\|\varrho^{1/2}\mathcal{W}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))}\;\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'}\\
&\quad + \frac{C(q)\gamma_E k^{-1/p}}{\varrho^{1/2}}\|\mathcal{W}\|_{L_p(I_n;\boldsymbol{X})}\;\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'}\\
&\quad + \left(\frac{\varphi_0}{\varrho}\right)^{1/2}\|\varphi_0^{1/2}\mathcal{U}\|_{L_\infty(I_n;\boldsymbol{X})}\;\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'}\\
&\quad + \sum_{q=1}^{N_\varphi}\left(\frac{\varphi_q}{\varrho}\right)^{1/2}\|\tau_q^{1/2}\mathcal{Z}_q\|_{L_\infty(I_n;\boldsymbol{X})}\;\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'},
\end{aligned}
$$

because, from Hölder's inequality

$$
\begin{aligned}
\frac{2\gamma_E}{\varrho^{1/2}k^2}\int_{t_{n-1}}^{t_n}(t_n-t)\|\mathcal{W}\|_{\boldsymbol{X}}\,dt &\leqslant \frac{2\gamma_E}{\varrho^{1/2}k^2}\left(\int_{t_{n-1}}^{t_n}|t_n-t|^r\,dt\right)^{1/r}\|\mathcal{W}\|_{L_p(I_n;\boldsymbol{X})},\\
&\leqslant \frac{2\gamma_E}{\varrho^{1/2}k^2}\left(\frac{k^{r+1}}{r+1}\right)^{1/r}\|\mathcal{W}\|_{L_p(I_n;\boldsymbol{X})},\\
&\leqslant \frac{2\gamma_E k^{1+1/r-2}}{\varrho^{1/2}(r+1)^{1/r}}\|\mathcal{W}\|_{L_p(I_n;\boldsymbol{X})},\\
&\leqslant \frac{C(p)\gamma_E k^{-1/p}}{\varrho^{1/2}}\|\mathcal{W}\|_{L_p(I_n;\boldsymbol{X})}
\end{aligned}
$$

for $1 \leqslant p \leqslant \infty$. Using Young's inequality four times in the form $ab \leqslant a^2/8 + 2b^2$ then gives,

$$
\begin{aligned}
\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'}^2 \leqslant{}& C_* h^2 \|\varrho^{1/2}\dot{\mathcal{W}}\|_0^2 + C\gamma_M \|\varrho^{1/2}\mathcal{W}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))}\,\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'} \\
&+ \frac{C(q)\gamma_E k^{-1/p}}{\varrho^{1/2}}\|\mathcal{W}\|_{L_p(I_n;\boldsymbol{X})}\;\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'} \\
&+ \left(\frac{\varphi_0}{\varrho}\right)^{1/2}\|\varphi_0^{1/2}\mathcal{U}\|_{L_\infty(I_n;\boldsymbol{X})}\;\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'} \\
&+ \left(\sum_{q=1}^{N_\varphi}\frac{\varphi_q}{\varrho}\right)^{1/2}\left(\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\mathcal{Z}_q\|_{L_\infty(I_n;\boldsymbol{X})}^2\right)^{1/2}\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'}, \\
\leqslant{}& C_* h^2\|\varrho^{1/2}\dot{\mathcal{W}}\|_0^2 + C\|\varrho^{1/2}\mathcal{W}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))}^2 + \frac{1}{8}\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'}^2 \\
&+ C\gamma_E^2 k^{-2/p}\|\mathcal{W}\|_{L_p(I_n;\boldsymbol{X})}^2 + \frac{1}{8}\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'}^2 \\
&+ C\|\varphi_0^{1/2}\mathcal{U}\|_{L_\infty(I_n;\boldsymbol{X})}^2 + \frac{1}{8}\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'}^2 \\
&+ C\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\mathcal{Z}_q\|_{L_\infty(I_n;\boldsymbol{X})}^2 + \frac{1}{8}\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'}^2,
\end{aligned}
$$

which is, [¶[12]]

$$
\begin{aligned}
\|\varrho^{1/2}\dot{\mathcal{W}}\|_{\boldsymbol{X}'}^2 \leqslant{}& C\gamma_E^2 k^{-2/p}\|\mathcal{W}\|_{L_p(I_n;\boldsymbol{X})}^2 + Ch^2\|\varrho^{1/2}\dot{\mathcal{W}}\|_0^2 + C\|\varrho^{1/2}\mathcal{W}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))}^2 \\
&+ C\|\varphi_0^{1/2}\mathcal{U}\|_{L_\infty(I_n;\boldsymbol{X})}^2 + C\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\mathcal{Z}_q\|_{L_\infty(I_n;\boldsymbol{X})}^2
\end{aligned}
$$

for every $p \in [1,\infty]$. If there is no stiffness term in the Rayleigh damping then $\gamma_E = 0$ and this estimate is sufficient for our needs, but if $\gamma_E \neq 0$ then we need to eliminate the $k^{-2/p}$ term on the right. To do this we take $p = 2$ and obtain,

$$
\begin{aligned}
\|\varrho^{1/2}\dot{\mathcal{W}}\|_{L_2(I;\boldsymbol{X}')}^2 \leqslant{}& C\gamma_E^2 k^{1-2/2}\|\mathcal{W}\|_{L_2(I;\boldsymbol{X})}^2 + Ck\sum_{n=1}^{N} h^2\|\varrho^{1/2}\dot{\mathcal{W}}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))}^2 \\
+ Ck\sum_{n=1}^{N}\|\varrho^{1/2}\mathcal{W}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))}^2 &+ Ck\sum_{n=1}^{N}\|\varphi_0^{1/2}\mathcal{U}\|_{L_\infty(I_n;\boldsymbol{X})}^2 + Ck\sum_{n=1}^{N}\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\mathcal{Z}_q\|_{L_\infty(I_n;\boldsymbol{X})}^2
\end{aligned}
$$

because $\int_{I_n} 1^2\,dt = k$. From this we have,

$$
\begin{aligned}
\|\varrho^{1/2}\dot{\mathcal{W}}\|_{L_2(I;\boldsymbol{X}')}^2 \leqslant{}& CT\max_{1\leqslant n\leqslant N}\Big\{h^2\|\varrho^{1/2}\dot{\mathcal{W}}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))}^2 + \|\varrho^{1/2}\mathcal{W}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))}^2 \\
&+ \|\varphi_0^{1/2}\mathcal{U}\|_{L_\infty(I_n;\boldsymbol{X})}^2 + \sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\mathcal{Z}_q\|_{L_\infty(I_n;\boldsymbol{X})}^2\Big\} + C\gamma_E\int_0^T b(\mathcal{W},\mathcal{W})\,dt
\end{aligned}
$$

[12]Short Version: after recalling that $\beta_q = (\varphi_q\tau_q)^{1/2}$, using (29) and then (28) with several applications of Hölder's and Young's inequalities,

or, more compactly, [¶13] in the general case,

$$\|\varrho^{1/2}\dot{\mathcal{W}}\|^2_{L_p(I;\boldsymbol{X}')} \leqslant CT^{2/p} \max_{1\leqslant n\leqslant N} \Big\{ h^2\|\varrho^{1/2}\dot{\mathcal{W}}\|^2_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))} + \|\varrho^{1/2}\mathcal{W}\|^2_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))} + \|\varphi_0^{1/2}\mathcal{U}\|^2_{L_\infty(I_n;\boldsymbol{X})} + \sum_{q=1}^{N_\varphi} \|\tau_q^{1/2}\mathcal{Z}_q\|^2_{L_\infty(I_n;\boldsymbol{X})} \Big\} + C\gamma_E \int_0^T b(\mathcal{W},\mathcal{W})\,dt$$

with $p=2$ when $\gamma_E>0$ and $p=\infty$ when $\gamma_E=0$. Noting now that on each I_n we have $\mathcal{W}|_{I_n} = k^{-1}(t_n-t)\mathcal{W}^+_{n-1} + k^{-1}(t-t_{n-1})\mathcal{W}^-_n$ and so on, we can obtain,

$$k\|\varrho^{1/2}\dot{\mathcal{W}}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))} + \|\varrho^{1/2}\mathcal{W}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))} \leqslant \|\varrho^{1/2}\mathcal{W}^+_n\|_0 + \|\varrho^{1/2}\mathcal{W}^+_{n-1}\|_0 + \|\varrho^{1/2}\,[\![\mathcal{W}]\!]_n\,\|_0,$$

with similar results for $\mathcal{U}$ and $\mathcal{W}$ in the $\boldsymbol{X}$ norm, and these imply,

$$\begin{aligned}\|\varphi_0^{1/2}\mathcal{U}\|^2_{L_\infty(I_n;\boldsymbol{X})} + \|\varrho^{1/2}\mathcal{W}\|^2_{L_\infty(I;\boldsymbol{L}_2(\Omega))} + \|\varrho^{1/2}\dot{\mathcal{W}}\|^2_{L_p(I;\boldsymbol{X}')} \leqslant C\gamma_E \int_0^T b(\mathcal{W},\mathcal{W}) \\ + CT^{2/p} \max_{1\leqslant n\leqslant N} \Big\{ \Big(1+k^{-2}h^2\Big)\Big(\|\varrho^{1/2}\mathcal{W}^+_n\|^2_0 + \|\varrho^{1/2}\mathcal{W}^+_{n-1}\|^2_0 + \|\varrho^{1/2}\,[\![\mathcal{W}]\!]_n\,\|^2_0 \Big) \\ + \|\varphi_0^{1/2}\mathcal{U}^+_n\|^2_{\boldsymbol{X}} + \|\varphi_0^{1/2}\mathcal{U}^+_{n-1}\|^2_{\boldsymbol{X}} + \|\varphi_0^{1/2}\,[\![\mathcal{U}]\!]_n\,\|^2_{\boldsymbol{X}} \\ + \sum_{q=1}^{N_\varphi} \Big(\|\tau_q^{1/2}\mathcal{Z}^+_{q,n}\|^2_{\boldsymbol{X}} + \|\tau_q^{1/2}\mathcal{Z}^+_{q,n-1}\|^2_{\boldsymbol{X}} + \|\tau_q^{1/2}\,[\![\mathcal{Z}_q]\!]_n\,\|^2_{\boldsymbol{X}} \Big) \Big\}.\end{aligned}$$

Returning to (31) with, this time, $\boldsymbol{\vartheta}|_{I_n} = \varphi_0^{-1}(t_n-t)\mathcal{G}_h\dot{\mathcal{U}}$ and $\boldsymbol{\vartheta}=\mathbf{0}$ on $I\setminus I_n$ we get,

$$\frac{k^2}{2}\|\dot{\mathcal{U}}\|^2_0 = \int_{t_{n-1}}^{t_n} (t_n-t)(\mathcal{W},\dot{\mathcal{U}})\,dt$$

which gives $\|\dot{\mathcal{U}}\|_0 \leqslant \|\mathcal{W}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))}$ on I_n and, therefore, [¶14]

$$\|\varphi_0^{1/2}\dot{\mathcal{U}}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))} \leqslant C\Big(\|\varrho^{1/2}\mathcal{W}^+_n\|_0 + \|\varrho^{1/2}\mathcal{W}^+_{n-1}\|_0 + \|\varrho^{1/2}\,[\![\mathcal{W}]\!]_n\,\|_0 \Big).$$

In a similar way, with $\boldsymbol{\xi}_q|_{I_n} = (t_n-t)\mathcal{G}_h\dot{\mathcal{Z}}_q$ in (31) and zero elsewhere we get,

$$\sum_{q=1}^{N_\varphi}\int_{t_{n-1}}^{t_n} (t_n-t)a(\tau_q\dot{\mathcal{Z}}_q,\mathcal{G}_h\dot{\mathcal{Z}}_q)\,dt = \sum_{q=1}^{N_\varphi}\int_{t_{n-1}}^{t_n} (t_n-t)a(\mathcal{Z}_q+\beta_q\mathcal{W},\mathcal{G}_h\dot{\mathcal{Z}}_q)\,dt$$

and therefore, for each q,

$$\begin{aligned}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|^2_0 &= \frac{2}{k^2}\int_{t_{n-1}}^{t_n} (t_n-t)a(\tau_q^{-1/2}\mathcal{Z}_q + \tau_q^{-1/2}\beta_q\mathcal{W}, \tau_q^{1/2}\mathcal{G}_h\dot{\mathcal{Z}}_q)\,dt, \\ &= \frac{2}{k^2}\int_{t_{n-1}}^{t_n} (t_n-t)(\tau_q^{-1/2}\mathcal{Z}_q + \tau_q^{-1/2}\beta_q\mathcal{W}, \tau_q^{1/2}\dot{\mathcal{Z}}_q)\,dt, \\ &\leqslant \|\tau_q^{-1/2}\mathcal{Z}_q + \tau_q^{-1/2}\beta_q\mathcal{W}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|_0.\end{aligned}$$

[13] Short Version:

[14] Short Version:

[¶[15]] In an obvious way, and recalling that $\beta_q = (\varphi_q\tau_q)^{1/2}$, this implies,

$$\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|_0 \leqslant \|\tau_q^{-1/2}\mathcal{Z}_q\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))} + \left(\frac{\varphi_q}{\varrho}\right)^{1/2}\|\varrho^{1/2}\mathcal{W}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))}$$

and so,

$$\begin{aligned}\frac{1}{2}\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|_0^2 &\leqslant \sum_{q=1}^{N_\varphi}\|\tau_q^{-1/2}\mathcal{Z}_q\|^2_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))} + \sum_{q=1}^{N_\varphi}\left(\frac{\varphi_q}{\varrho}\right)\|\varrho^{1/2}\mathcal{W}\|^2_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))},\\ &= \sum_{q=1}^{N_\varphi}\|\tau_q^{-1/2}\mathcal{Z}_q\|^2_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))} + \frac{\varphi(0)-\varphi_0}{\varrho}\|\varrho^{1/2}\mathcal{W}\|^2_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))}.\end{aligned}$$

[¶[16]] and, therefore, on I_n (using $\frac{1}{3}(a+b+c)^2 \leqslant a^2+b^2+c^2$),

$$\begin{aligned}\frac{1}{6}\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|_0^2 &\leqslant \sum_{q=1}^{N_\varphi}\Big(\|\tau_q^{-1/2}\mathcal{Z}_{q,n}^+\|_0^2 + \|\tau_q^{-1/2}\mathcal{Z}_{q,n-1}^+\|_0^2 + \|\tau_q^{-1/2}\left[\!\left[\mathcal{Z}_q\right]\!\right]_n\|_0^2\Big)\\ &\quad + \left(\frac{\varphi(0)-\varphi_0}{\varrho}\right)\Big(\|\varrho^{1/2}\mathcal{W}_n^+\|_0^2 + \|\varrho^{1/2}\mathcal{W}_{n-1}^+\|_0^2 + \|\varrho^{1/2}\left[\!\left[\mathcal{W}\right]\!\right]_n\|_0^2\Big).\end{aligned}$$

Assembling these recent estimates gives,

$$\begin{aligned}&\|\varrho^{1/2}\mathcal{W}\|^2_{L_\infty(I;\boldsymbol{L}_2(\Omega))} + \|\varrho^{1/2}\dot{\mathcal{W}}\|^2_{L_p(I;\boldsymbol{X}')} + \|\varphi_0^{1/2}\mathcal{U}\|^2_{L_\infty(I_n;\boldsymbol{X})} + \|\varphi_0^{1/2}\dot{\mathcal{U}}\|^2_{L_\infty(I;\boldsymbol{L}_2(\Omega))}\\ &\quad + \sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|^2_{L_\infty(I;\boldsymbol{X})} \leqslant C\gamma_E\int_0^T b(\mathcal{W},\mathcal{W})\,dt\\ &+ CT^{2/p}\max_{1\leqslant n\leqslant N}\bigg\{\Big(1+k^{-2}h^2\Big)\Big(\|\varrho^{1/2}\mathcal{W}_n^+\|_0^2 + \|\varrho^{1/2}\mathcal{W}_{n-1}^+\|_0^2 + \|\varrho^{1/2}\left[\!\left[\mathcal{W}\right]\!\right]_n\|_0^2\Big)\\ &\quad + \|\varphi_0^{1/2}\mathcal{U}_n^+\|^2_{\boldsymbol{X}} + \|\varphi_0^{1/2}\mathcal{U}_{n-1}^+\|^2_{\boldsymbol{X}} + \|\varphi_0^{1/2}\left[\!\left[\mathcal{U}\right]\!\right]_n\|^2_{\boldsymbol{X}}\\ &\quad + \sum_{q=1}^{N_\varphi}\Big(\|\tau_q^{1/2}\mathcal{Z}_{q,n}^+\|^2_{\boldsymbol{X}} + \|\tau_q^{1/2}\mathcal{Z}_{q,n-1}^+\|^2_{\boldsymbol{X}} + \|\tau_q^{1/2}\left[\!\left[\mathcal{Z}_q\right]\!\right]_n\|^2_{\boldsymbol{X}}\Big)\bigg\}\end{aligned}$$

and on recalling the first claim of the theorem now we obtain,

$$\begin{aligned}&\|\varrho^{1/2}\mathcal{W}\|^2_{L_\infty(I;\boldsymbol{L}_2(\Omega))} + \|\varrho^{1/2}\dot{\mathcal{W}}\|^2_{L_p(I;\boldsymbol{X}')} + \|\varphi_0^{1/2}\mathcal{U}\|^2_{L_\infty(I_n;\boldsymbol{X})} + \|\varphi_0^{1/2}\dot{\mathcal{U}}\|^2_{L_\infty(I;\boldsymbol{L}_2(\Omega))}\\ &\quad + \sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|^2_{L_\infty(I;\boldsymbol{X})} \leqslant CT^{2/p}(1+c_T^2)\Big(\|\varrho^{1/2}\mathcal{W}_N^+\|_0^2 + \|\varphi_0^{1/2}\mathcal{U}_N^+\|^2_{\boldsymbol{X}}\Big)\end{aligned}$$

[15] Short Version:

$$\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|_0^2 \leqslant \|\tau_q^{-1/2}\mathcal{Z}_q + \tau_q^{-1/2}\beta_q\mathcal{W}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|_0.$$

[16] Short Version:

$$\frac{1}{2}\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|_0^2 \leqslant \sum_{q=1}^{N_\varphi}\|\tau_q^{-1/2}\mathcal{Z}_q\|^2_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))} + \frac{\varphi(0)-\varphi_0}{\varrho}\|\varrho^{1/2}\mathcal{W}\|^2_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))}$$

for a constant C independent of T, h and k. This concludes the proof. [¶[17]]

For the linear elasticity operator we can introduce the map $\Delta\colon \boldsymbol{X} \to \boldsymbol{L}_2(\Omega)$ which is well defined for every $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega)$ by $(\Delta\boldsymbol{v}, \boldsymbol{\vartheta}) = -a(\boldsymbol{v}, \boldsymbol{\vartheta})$ for all $\boldsymbol{\vartheta} \in \boldsymbol{X}$. The discrete version of this is then $\Delta_h\colon \boldsymbol{X}^h \to \boldsymbol{X}^h$ and defined by $(\Delta_h\boldsymbol{v}, \boldsymbol{\vartheta}) = -a(\boldsymbol{v}, \boldsymbol{\vartheta})$ for all $\boldsymbol{\vartheta} \in \boldsymbol{X}^h$. We note that for any $\boldsymbol{\vartheta} \in \boldsymbol{X}^h$ and $\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) \cap \boldsymbol{H}^2(\Omega)$,

$$(\Delta_h P_{\boldsymbol{X}}\boldsymbol{v}, \boldsymbol{\vartheta}) = -a(P_{\boldsymbol{X}}\boldsymbol{v}, \boldsymbol{\vartheta}) = -a(\boldsymbol{v}, \boldsymbol{\vartheta}) = (\Delta\boldsymbol{v}, \boldsymbol{\vartheta})$$

and therefore $\|\Delta_h P_{\boldsymbol{X}}\boldsymbol{v}\|_0 \leqslant \|\Delta\boldsymbol{v}\|_0$. [¶[18]]We can now give the error bound.

Theorem 4.7 (*a priori* error bound) *Let Assumptions 4.1 hold and in addition assume that $h \leqslant c_T k$ for a positive constant c_T. Then*

$$\|\varrho^{1/2}(\boldsymbol{w} - \boldsymbol{W})^-_N\|_0 + \|\varphi_0^{1/2}(\boldsymbol{u} - \boldsymbol{U})^-_N\|_{\boldsymbol{X}} \leqslant C T^{1/2+1/p} \mathscr{R}(\boldsymbol{u}) \left(h + k^3 + k^{-1/2}h^2\right)$$

for a constant C, dependent on data, but independent of T, h and k and where

$$\mathscr{R}(\boldsymbol{u}) = \|\boldsymbol{u}\|_{W^1_r(I;\boldsymbol{H}^2(\Omega))} + \|\boldsymbol{u}\|_{W^1_1(I;\boldsymbol{H}^2(\Omega))} + \|\boldsymbol{u}\|_{W^1_\infty(I;\boldsymbol{H}^2(\Omega))} + \|\boldsymbol{u}\|_{W^3_1(I;\boldsymbol{H}^2(\Omega))} + \|\boldsymbol{u}\|_{W^3_r(I;\boldsymbol{H}^3(\Omega))}.$$

In this bound we can take $p = \infty$ if $\gamma_E = 0$ in (14) *and $p = 2$ if $\gamma_E > 0$.*

Proof. From the error representation formula, (35), (34) and (33) we have,

$$\begin{aligned}(\varrho\mathcal{W}^+_N, (\boldsymbol{W} - \Pi_w\boldsymbol{w})^-_N) &+ a(\varphi_0\mathcal{U}^+_N, (\boldsymbol{U} - \Pi_u\boldsymbol{u})^-_N) + \langle\!\langle \mathfrak{g}, \boldsymbol{W} - \Pi_w\boldsymbol{w}\rangle\!\rangle \\ &= \mathscr{G}((\boldsymbol{U} - \Pi_u\boldsymbol{u}, \boldsymbol{W} - \Pi_w\boldsymbol{w}, \boldsymbol{Z}_1 - \Pi_1\boldsymbol{z}_1, \ldots)) \\ &= \mathscr{A}((\boldsymbol{u} - \Pi_u\boldsymbol{u}, \boldsymbol{w} - \Pi_w\boldsymbol{w}, \boldsymbol{z}_1 - \Pi_1\boldsymbol{z}_1, \ldots), (\mathcal{W}, \mathcal{U}, \mathcal{Z}_1, \ldots)), \\ &= \mathscr{A}^*((\mathcal{W}, \mathcal{U}, \mathcal{Z}_1, \ldots), (\boldsymbol{u} - \Pi_u\boldsymbol{u}, \boldsymbol{w} - \Pi_w\boldsymbol{w}, \boldsymbol{z}_1 - \Pi_1\boldsymbol{z}_1, \ldots)),\end{aligned}$$

[17] Short Version: Assembling these estimates and recalling the first claim of the theorem then completes the proof.

[18] Short Version:

and so

$$
\begin{aligned}
(\varrho \mathcal{W}_N^+, &(\boldsymbol{W} - \Pi_w \boldsymbol{w})_N^-) + a(\varphi_0 \mathcal{U}_N^+, (\boldsymbol{U} - \Pi_u \boldsymbol{u})_N^-) + \langle\!\langle \mathfrak{g}, \boldsymbol{W} - \Pi_w \boldsymbol{w} \rangle\!\rangle \\
&= -(\!(\varrho \dot{\mathcal{W}}, \boldsymbol{w} - \Pi_w \boldsymbol{w})\!) - a(\!(\mathcal{U}, \varphi_0(\boldsymbol{w} - \Pi_w \boldsymbol{w}))\!) + b(\!(\boldsymbol{w} - \Pi_w \boldsymbol{w}, \mathcal{W})\!) \\
&\quad + a(\!(\mathcal{W} - \dot{\mathcal{U}}, \varphi_0(\boldsymbol{u} - \Pi_u \boldsymbol{u}))\!) - \sum_{q=1}^{N_\varphi} a(\!(\beta_q \mathcal{Z}_q, \boldsymbol{w} - \Pi_w \boldsymbol{w})\!) \\
&\quad - \sum_{n=1}^{N-1} ([\![\mathcal{W}]\!]_n, \varrho(\boldsymbol{w} - \Pi_w \boldsymbol{w})_n^-) + (\mathcal{W}_N^-, \varrho(\boldsymbol{w} - \Pi_w \boldsymbol{w})_N^-) \\
&\quad - \sum_{n=1}^{N-1} a([\![\mathcal{U}]\!]_n, \varphi_0(\boldsymbol{u} - \Pi_u \boldsymbol{u})_n^-) + a(\mathcal{U}_N^-, \varphi_0(\boldsymbol{u} - \Pi_u \boldsymbol{u})_N^-) \\
&\quad + \sum_{q=1}^{N_\varphi} a(\!(\mathcal{Z}_q + \beta_q \mathcal{W}, \boldsymbol{z}_q - \Pi_q \boldsymbol{z}_q)\!) - \sum_{q=1}^{N_\varphi} a(\!(\tau_q \dot{\mathcal{Z}}_q, \boldsymbol{z}_q - \Pi_q \boldsymbol{z}_q)\!) \\
&\quad - \sum_{n=1}^{N-1} \sum_{q=1}^{N_\varphi} a([\![\mathcal{Z}_q]\!]_n, \tau_q(\boldsymbol{z}_q - \Pi_q \boldsymbol{z}_q)_n^-) + \sum_{q=1}^{N_\varphi} a(\mathcal{Z}_{q,N}^-, \tau_q(\boldsymbol{z}_q - \Pi_q \boldsymbol{z}_q)_N^-) \\
&= \sum_{j=1}^{13} \mathscr{E}_j
\end{aligned}
$$

with obvious notation. Recalling $P_{\boldsymbol{X}}$ in (19) and P_I in (30), we choose

$$
\begin{aligned}
\boldsymbol{u} - \Pi_u \boldsymbol{u} &= (\boldsymbol{u} - P_{\boldsymbol{X}} \boldsymbol{u}) + (P_{\boldsymbol{X}} \boldsymbol{u} - P_I P_{\boldsymbol{X}} \boldsymbol{u}), \\
\boldsymbol{w} - \Pi_w \boldsymbol{w} &= (\boldsymbol{w} - P_{\boldsymbol{X}} \boldsymbol{w}) + (P_{\boldsymbol{X}} \boldsymbol{w} - P_I P_{\boldsymbol{X}} \boldsymbol{w}), \\
\boldsymbol{z}_q - \Pi_q \boldsymbol{z}_q &= (\boldsymbol{z}_q - P_{\boldsymbol{X}} \boldsymbol{z}_q) + (P_{\boldsymbol{X}} \boldsymbol{z}_q - P_I P_{\boldsymbol{X}} \boldsymbol{z}_q)
\end{aligned}
$$

[¶[19]]and then, with either $(p, r) = (2, 2)$ or $(p, r) = (\infty, 1)$ in the following Hölder inqualities, we take the error representation term-by-term to get first that,

$$
\begin{aligned}
&\mathscr{E}_1 = -(\!(\varrho \dot{\mathcal{W}}, \boldsymbol{w} - P_{\boldsymbol{X}} \boldsymbol{w})\!) - (\!(\varrho \dot{\mathcal{W}}, P_{\boldsymbol{X}} \boldsymbol{w} - P_I P_{\boldsymbol{X}} \boldsymbol{w})\!) = -(\!(\varrho \dot{\mathcal{W}}, (I - P_{\boldsymbol{X}}) \boldsymbol{w})\!), \\
\implies &|\mathscr{E}_1| \leqslant \|\varrho^{1/2}(I - P_{\boldsymbol{X}}) \dot{\boldsymbol{u}}\|_{L_r(I;\boldsymbol{X})} \|\varrho^{1/2} \dot{\mathcal{W}}\|_{L_p(I;\boldsymbol{X}')},
\end{aligned}
$$

and then second that,

$$
\begin{aligned}
\mathscr{E}_2 &= -a(\!(\varphi_0 \mathcal{U}, \boldsymbol{w} - P_{\boldsymbol{X}} \boldsymbol{w})\!) - a(\!(\varphi_0 \mathcal{U}, P_{\boldsymbol{X}} \boldsymbol{w} - P_I P_{\boldsymbol{X}} \boldsymbol{w})\!), \\
&= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} a(\bar{\mathcal{U}} - \mathcal{U}, \varphi_0(I - P_I) P_{\boldsymbol{X}} \boldsymbol{w}) \, dt,
\end{aligned}
$$

where we introduced the average of $\mathcal{U}$ by virtue of the definition (30) of P_I. Now, from Lemma 4.5, with $t \in I_n$,

$$
\begin{aligned}
a(\bar{\mathcal{U}} - \mathcal{U}(t), \varphi_0(I - P_I) P_{\boldsymbol{X}} \boldsymbol{w}(t))) &= \left(\frac{\varphi_0}{k} \int_{t_{n-1}}^{t_n} \int_t^s \dot{\mathcal{U}}(\eta) \, d\eta ds, (I - P_I) \Delta \boldsymbol{w}(t) \right), \\
&\leqslant \varphi_0^{1/2} k \|\varphi_0^{1/2} \dot{\mathcal{U}}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))} \|(I - P_I) \Delta \boldsymbol{w}(t)\|_0, \\
&\leqslant 2\varphi_0^{1/2} k^3 \|\varphi_0^{1/2} \dot{\mathcal{U}}\|_{L_\infty(I_n;\boldsymbol{L}_2(\Omega))} \|\Delta \ddot{\boldsymbol{w}}\|_{L_1(I_n;\boldsymbol{L}_2(\Omega))}
\end{aligned}
$$

[19]Short Version: $\Pi_u = \Pi_w = \Pi_q = \Pi$ for $I - \Pi = (I - P_{\boldsymbol{X}}) + (I - P_I) P_{\boldsymbol{X}}$

where we recalled that for $\boldsymbol{\vartheta} \in \boldsymbol{X}^h$ we have $a(\boldsymbol{\vartheta}, P_{\boldsymbol{X}}\boldsymbol{w}) = -(\boldsymbol{\vartheta}, \Delta\boldsymbol{w})$ and also noted that

$$
\begin{aligned}
a(P_I P_{\boldsymbol{X}}\boldsymbol{w}, \boldsymbol{\vartheta}) &= a\left(P_{\boldsymbol{X}}\boldsymbol{w}(t_n) - \frac{2(t_n - t)}{k^2}\int_{t_{n-1}}^{t_n}(s - t_{n-1})P_{\boldsymbol{X}}\dot{\boldsymbol{w}}(s)\,ds, \boldsymbol{\vartheta}\right),\\
&= a(P_{\boldsymbol{X}}\boldsymbol{w}(t_n), \boldsymbol{\vartheta}) - \frac{2(t_n - t)}{k^2}\int_{t_{n-1}}^{t_n}(s - t_{n-1})a(P_{\boldsymbol{X}}\dot{\boldsymbol{w}}(s), \boldsymbol{\vartheta})\,ds,\\
&= a(\boldsymbol{w}(t_n), \boldsymbol{\vartheta}) - \frac{2(t_n - t)}{k^2}\int_{t_{n-1}}^{t_n}(s - t_{n-1})a(\dot{\boldsymbol{w}}(s), \boldsymbol{\vartheta})\,ds,\\
&= -(\Delta\boldsymbol{w}(t_n), \boldsymbol{\vartheta}) + \frac{2(t_n - t)}{k^2}\int_{t_{n-1}}^{t_n}(s - t_{n-1})(\Delta\dot{\boldsymbol{w}}(s), \boldsymbol{\vartheta})\,ds,\\
&= -\left(\Delta\boldsymbol{w}(t_n) - \frac{2(t_n - t)}{k^2}\int_{t_{n-1}}^{t_n}(s - t_{n-1})\Delta\dot{\boldsymbol{w}}(s)\,ds, \boldsymbol{\vartheta}\right),\\
&= -(P_I\Delta\boldsymbol{w}, \boldsymbol{\vartheta}).
\end{aligned}
$$

[¶[20]] Therefore

$$
|\mathscr{E}_2| \leqslant 2\varphi_0^{1/2}k^3\|\varphi_0^{1/2}\dot{\mathcal{U}}\|_{L_\infty(I;\boldsymbol{L}_2(\Omega))}\|\Delta\ddot{\boldsymbol{u}}\|_{L_1(I;\boldsymbol{L}_2(\Omega))}.
$$

Next we have

$$
\begin{aligned}
\mathscr{E}_3 &= b(\!(\boldsymbol{w} - P_{\boldsymbol{X}}\boldsymbol{w}, \mathcal{W})\!) + b(\!((I - P_I)P_{\boldsymbol{X}}\boldsymbol{w}, \mathcal{W} - \bar{\mathcal{W}})\!),\\
&= \gamma_M(\!(\boldsymbol{w} - P_{\boldsymbol{X}}\boldsymbol{w}, \varrho\mathcal{W})\!) + \gamma_M(\!(\varrho(I - P_I)P_{\boldsymbol{X}}\boldsymbol{w}, \mathcal{W} - \bar{\mathcal{W}})\!)\\
&\quad + \gamma_E a(\!(\boldsymbol{w} - P_{\boldsymbol{X}}\boldsymbol{w}, \mathcal{W})\!) + \gamma_E a(\!((I - P_I)P_{\boldsymbol{X}}\boldsymbol{w}, \mathcal{W} - \bar{\mathcal{W}})\!),\\
&= \gamma_M(\!(\boldsymbol{w} - P_{\boldsymbol{X}}\boldsymbol{w}, \varrho\mathcal{W})\!) + \gamma_M(\!(\varrho(I - P_I)P_{\boldsymbol{X}}\boldsymbol{w}, \mathcal{W} - \bar{\mathcal{W}})\!)\\
&\quad + \gamma_E a(\!((I - P_I)P_{\boldsymbol{X}}\boldsymbol{w}, \mathcal{W} - \bar{\mathcal{W}})\!),
\end{aligned}
$$

and so [¶[21]]

$$
\begin{aligned}
|\mathscr{E}_3| &= \Big|\gamma_M(\!(\boldsymbol{w} - P_{\boldsymbol{X}}\boldsymbol{w}, \varrho\mathcal{W})\!) + \gamma_M(\!(\varrho(I - P_I)P_{\boldsymbol{X}}\boldsymbol{w}, \mathcal{W} - \bar{\mathcal{W}})\!)\\
&\qquad - \gamma_E(\!((I - P_I)\Delta\boldsymbol{w}, \mathcal{W} - \bar{\mathcal{W}})\!)\Big|,\\
&\leqslant C\|(I - P_{\boldsymbol{X}})\boldsymbol{w}\|_{L_1(I;\boldsymbol{L}_2(\Omega))}\|\varrho^{1/2}\mathcal{W}\|_{L_\infty(I;\boldsymbol{L}_2(\Omega))}\\
&\qquad + C\Big(\gamma_M\|(I - P_I)P_{\boldsymbol{X}}\boldsymbol{w}\|_{L_r(I;\boldsymbol{X})} + \gamma_E\|(I - P_I)\Delta\boldsymbol{w}\|_{L_r(I;\boldsymbol{X})}\Big)\|\mathcal{W} - \bar{\mathcal{W}}\|_{L_p(I;\boldsymbol{X}')}\\
&\leqslant C\|(I - P_{\boldsymbol{X}})\boldsymbol{w}\|_{L_1(I;\boldsymbol{L}_2(\Omega))}\|\varrho^{1/2}\mathcal{W}\|_{L_\infty(I;\boldsymbol{L}_2(\Omega))}\\
&\qquad + Ck^3\Big(\gamma_M\|\ddot{\boldsymbol{u}}\|_{L_r(I;\boldsymbol{X})} + \gamma_E\|\Delta\ddot{\boldsymbol{u}}\|_{L_r(I;\boldsymbol{X})}\Big)\|\varrho^{1/2}\dot{\mathcal{W}}\|_{L_p(I;\boldsymbol{X}')}.
\end{aligned}
$$

Arguing similarly as for $\mathscr{E}_2$ we have,

$$
\begin{aligned}
|\mathscr{E}_4| &= |a(\!(\mathcal{W} - \bar{\mathcal{W}}, \varphi_0(I - P_I)P_{\boldsymbol{X}}\boldsymbol{u})\!)| \leqslant Ck^3\|\varrho^{1/2}\dot{\mathcal{W}}\|_{L_p(I;\boldsymbol{X}')}\|\Delta\ddot{\boldsymbol{u}}\|_{L_r(I;\boldsymbol{X})},\\
|\mathscr{E}_5| &\leqslant \sum_{q=1}^{N_\varphi} Ck^3\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|_{L_\infty(I;\boldsymbol{L}_2(\Omega))}\|\Delta\ddot{\boldsymbol{w}}\|_{L_1(I;\boldsymbol{L}_2(\Omega))},
\end{aligned}
$$

[20] Short Version: $a(P_I P_{\boldsymbol{X}}\boldsymbol{w}, \boldsymbol{\vartheta}) = -(P_I\Delta\boldsymbol{w}, \boldsymbol{\vartheta})$.

[21] Short Version:

(because

$$\mathscr{E}_5 = -\sum_{q=1}^{N_\varphi}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} a(\mathcal{Z}_q - \bar{\mathcal{Z}}_q, \beta_q(I-P_I)P_{\boldsymbol{X}}\boldsymbol{w})\,dt,$$
$$\leqslant \sum_{q=1}^{N_\varphi} 2\varphi_q^{1/2}k^3\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|_{L_\infty(I;\boldsymbol{L}_2(\Omega))}\|\Delta\ddot{\boldsymbol{w}}\|_{L_1(I;\boldsymbol{L}_2(\Omega))}$$

by Hölder's inequality.)[¶[22]] and also,

$$\begin{aligned}|\mathscr{E}_{10}+\mathscr{E}_{11}| &= \Bigg|\sum_{q=1}^{N_\varphi} a\big(\!\!\big(\mathcal{Z}_q+\beta_q\mathcal{W}-\tau_q\dot{\mathcal{Z}}_q, \boldsymbol{z}_q - P_{\boldsymbol{X}}\boldsymbol{z}_q\big)\!\!\big)\\
&\quad + \sum_{q=1}^{N_\varphi} a\big(\!\!\big(\mathcal{Z}_q+\beta_q\mathcal{W}, P_{\boldsymbol{X}}\boldsymbol{z}_q - P_I P_{\boldsymbol{X}}\boldsymbol{z}_q\big)\!\!\big) - \sum_{q=1}^{N_\varphi} a\big(\!\!\big(\tau_q\dot{\mathcal{Z}}_q, P_{\boldsymbol{X}}\boldsymbol{z}_q - P_I P_{\boldsymbol{X}}\boldsymbol{z}_q\big)\!\!\big)\Bigg|,\\
&= \Bigg|\sum_{q=1}^{N_\varphi}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} a(\mathcal{Z}_q+\beta_q\mathcal{W}, (I-P_I)P_{\boldsymbol{X}}\boldsymbol{z}_q)\,dt\Bigg|,\\
&\leqslant \sum_{q=1}^{N_\varphi}\sum_{n=1}^{N}\Bigg(\|\mathcal{Z}_q-\bar{\mathcal{Z}}_q\|_{L_\infty(I_n;\boldsymbol{X})}\|(I-P_I)\boldsymbol{z}_q\|_{L_1(I_n;\boldsymbol{X})}\\
&\qquad\qquad + \|\beta_q(\mathcal{W}-\bar{\mathcal{W}})\|_{L_p(I_n;\boldsymbol{X}')}\|(I-P_I)\Delta\boldsymbol{z}_q\|_{L_r(I_n;\boldsymbol{X})}\Bigg),\\
&\leqslant 2k^3\sum_{q=1}^{N_\varphi}\Bigg(\tau_q^{-1/2}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|_{L_\infty(I;\boldsymbol{X})}\|\ddot{\boldsymbol{z}}_q\|_{L_1(I;\boldsymbol{X})}\\
&\qquad\qquad + \left(\frac{\varphi_q}{\tau_q}\right)^{1/2}\|\dot{\mathcal{W}}\|_{L_p(I;\boldsymbol{X}')}\tau_q\|\Delta\ddot{\boldsymbol{z}}_q\|_{L_r(I;\boldsymbol{X})}\Bigg),\\
&\leqslant 2k^3\left(\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|_{L_\infty(I;\boldsymbol{X})}^2\right)^{1/2}\left(\sum_{q=1}^{N_\varphi}\tau_q^{-1}\|\ddot{\boldsymbol{z}}_q\|_{L_1(I;\boldsymbol{X})}^2\right)^{1/2}\\
&\qquad + 2k^3\left(\frac{|\varphi'(0)|}{\varrho}\right)^{1/2}\|\varrho^{1/2}\dot{\mathcal{W}}\|_{L_p(I;\boldsymbol{X}')}\left(\sum_{q=1}^{N_\varphi}\tau_q^2\|\Delta\ddot{\boldsymbol{z}}_q\|_{L_r(I;\boldsymbol{X})}^2\right)^{1/2}\end{aligned}$$

(because $\varphi'(0) = -\sum_{q=1}^{N_\varphi}\varphi_q/\tau_q$), and so, [¶[23]]

$$\begin{aligned}|\mathscr{E}_{10}+\mathscr{E}_{11}| &\leqslant Ck^3\left(\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|_{L_\infty(I;\boldsymbol{X})}^2\right)^{1/2}\left(\sum_{q=1}^{N_\varphi}\|\ddot{\boldsymbol{z}}_q\|_{L_1(I;\boldsymbol{X})}^2\right)^{1/2}\\
&\qquad + Ck^3\|\varrho^{1/2}\dot{\mathcal{W}}\|_{L_p(I;\boldsymbol{X}')}\left(\sum_{q=1}^{N_\varphi}\|\Delta\ddot{\boldsymbol{z}}_q\|_{L_r(I;\boldsymbol{X})}^2\right)^{1/2}.\end{aligned}$$

[22]Short Version:
[23]Short Version:

Using now (30) and the Cauchy-Schwarz inequality for sums we get

$$
\begin{aligned}
\mathscr{E}_6 + \mathscr{E}_7 &= -\sum_{n=1}^{N-1} (\llbracket \mathcal{W} \rrbracket_n \,, \varrho(\boldsymbol{w} - P_{\boldsymbol{X}}\boldsymbol{w})_n^-) + (\mathcal{W}_N^-, \varrho(\boldsymbol{w} - P_{\boldsymbol{X}}\boldsymbol{w})_N^-) \\
&\quad - \sum_{n=1}^{N-1} (\llbracket \mathcal{W} \rrbracket_n \,, \varrho(P_{\boldsymbol{X}}\boldsymbol{w} - P_I P_{\boldsymbol{X}}\boldsymbol{w})_n^-) + (\mathcal{W}_N^-, \varrho(P_{\boldsymbol{X}}\boldsymbol{w} - P_I P_{\boldsymbol{X}}\boldsymbol{w})_N^-), \\
&= -\sum_{n=1}^{N-1} (\llbracket \mathcal{W} \rrbracket_n \,, \varrho(\boldsymbol{w} - P_{\boldsymbol{X}}\boldsymbol{w})_n^-) + (\mathcal{W}_N^+ - \llbracket \mathcal{W} \rrbracket_N \,, \varrho(\boldsymbol{w} - P_{\boldsymbol{X}}\boldsymbol{w})_N^-), \\
&= -\sum_{n=1}^{N} (\llbracket \mathcal{W} \rrbracket_n \,, \varrho(\boldsymbol{w} - P_{\boldsymbol{X}}\boldsymbol{w})_n^-) + (\mathcal{W}_N^+, \varrho(\boldsymbol{w} - P_{\boldsymbol{X}}\boldsymbol{w})_N^-),
\end{aligned}
$$

and so, [¶[24]]

$$
|\mathscr{E}_6 + \mathscr{E}_7| \leqslant 2 \left(\sum_{n=1}^{N} \| \varrho^{1/2} (I - P_{\boldsymbol{X}}) \dot{\boldsymbol{u}}_n^- \|_0^2 \right)^{1/2} \left(\| \varrho^{1/2} \mathcal{W}_N^+ \|_0^2 + \sum_{n=1}^{N} \| \varrho^{1/2} \llbracket \mathcal{W} \rrbracket_n \|_0^2 \right)^{1/2},
$$

while, again from (30), we have

$$
\begin{aligned}
\mathscr{E}_8 + \mathscr{E}_9 &= -\sum_{n=1}^{N-1} a(\varphi_0 \llbracket \mathcal{U} \rrbracket_n \,, (\boldsymbol{u} - P_{\boldsymbol{X}}\boldsymbol{u})_n^-) + a(\varphi_0 \mathcal{U}_N^-, (\boldsymbol{u} - P_{\boldsymbol{X}}\boldsymbol{u})_N^-) \\
&\quad - \sum_{n=1}^{N-1} a(\varphi_0 \llbracket \mathcal{U} \rrbracket_n \,, (P_{\boldsymbol{X}}\boldsymbol{u} - P_I P_{\boldsymbol{X}}\boldsymbol{u})_n^-) + a(\varphi_0 \mathcal{U}_N^-, (P_{\boldsymbol{X}}\boldsymbol{u} - P_I P_{\boldsymbol{X}}\boldsymbol{u})_N^-), \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\mathscr{E}_{12} + \mathscr{E}_{13} &= -\sum_{q=1}^{N_\varphi} \sum_{n=1}^{N-1} a(\tau_q \llbracket \mathcal{Z}_q \rrbracket_n \,, (\boldsymbol{z}_q - P_{\boldsymbol{X}}\boldsymbol{z}_q)_n^-) + a(\tau_q \mathcal{Z}_{q,N}^-, (\boldsymbol{z}_q - P_{\boldsymbol{X}}\boldsymbol{z}_q)_N^-) \\
&\quad - \sum_{q=1}^{N_\varphi} \sum_{n=1}^{N-1} a(\tau_q \llbracket \mathcal{Z}_q \rrbracket_n \,, ((I - P_I) P_{\boldsymbol{X}}\boldsymbol{z}_q)_n^-) + a(\tau_q \mathcal{Z}_{q,N}^-, ((I - P_I) P_{\boldsymbol{X}}\boldsymbol{z}_q)_N^-), \\
&= 0.
\end{aligned}
$$

[24] Short Version:

[¶[25]] Putting all of these together and taking $\mathfrak{g} = \mathbf{0}$ gives,

$$
\begin{aligned}
&(\varrho \mathcal{W}_N^+, (\boldsymbol{W} - \Pi_w \boldsymbol{w})_N^-) + a(\varphi_0 \mathcal{U}_N^+, (\boldsymbol{U} - \Pi_u \boldsymbol{u})_N^-) \\
&\qquad \leqslant \|\varrho^{1/2}(I - P_{\boldsymbol{X}})\dot{\boldsymbol{u}}\|_{L_r(I;\boldsymbol{X})} \|\varrho^{1/2}\dot{\mathcal{W}}\|_{L_p(I;\boldsymbol{X}')} \\
&\qquad\quad + Ck^3 \|\varphi_0^{1/2}\dot{\mathcal{U}}\|_{L_\infty(I;\boldsymbol{L}_2(\Omega))} \|\Delta \ddot{\boldsymbol{u}}\|_{L_1(I;\boldsymbol{L}_2(\Omega))} \\
&\qquad\quad + C\|(I - P_{\boldsymbol{X}})\boldsymbol{w}\|_{L_1(I;\boldsymbol{L}_2(\Omega))} \|\varrho^{1/2}\mathcal{W}\|_{L_\infty(I;\boldsymbol{L}_2(\Omega))} \\
&\qquad\qquad\quad + Ck^3 \Big(\gamma_M \|\ddot{\boldsymbol{u}}\|_{L_r(I;\boldsymbol{X})} + \gamma_E \|\Delta \ddot{\boldsymbol{u}}\|_{L_r(I;\boldsymbol{X})}\Big) \|\varrho^{1/2}\dot{\mathcal{W}}\|_{L_p(I;\boldsymbol{X}')} \\
&\qquad\quad + Ck^3 \|\varrho^{1/2}\dot{\mathcal{W}}\|_{L_p(I;\boldsymbol{X}')} \|\Delta \ddot{\boldsymbol{u}}\|_{L_r(I;\boldsymbol{X})} \\
&\qquad\quad + \sum_{q=1}^{N_\varphi} Ck^3 \|\tau_q^{1/2} \dot{\mathcal{Z}}_q\|_{L_\infty(I;\boldsymbol{L}_2(\Omega))} \|\Delta \ddot{\boldsymbol{w}}\|_{L_1(I;\boldsymbol{L}_2(\Omega))} \\
&\qquad\quad + Ck^3 \left(\sum_{q=1}^{N_\varphi} \|\tau_q^{1/2} \dot{\mathcal{Z}}_q\|_{L_\infty(I;\boldsymbol{X})}^2\right)^{1/2} \left(\sum_{q=1}^{N_\varphi} \|\ddot{\boldsymbol{z}}_q\|_{L_1(I;\boldsymbol{X})}^2\right)^{1/2} \\
&\qquad\qquad\quad + Ck^3 \|\varrho^{1/2}\dot{\mathcal{W}}\|_{L_p(I;\boldsymbol{X}')} \left(\sum_{q=1}^{N_\varphi} \|\Delta \ddot{\boldsymbol{z}}_q\|_{L_r(I;\boldsymbol{X})}^2\right)^{1/2} \\
&\qquad\quad + C\left(\sum_{n=1}^{N} \|\varrho^{1/2}(I - P_{\boldsymbol{X}})\dot{\boldsymbol{u}}_n^-\|_0^2\right)^{1/2} \left(\|\varrho^{1/2}\mathcal{W}_N^+\|_0^2 + \sum_{n=1}^{N} \|\varrho^{1/2} [\![\mathcal{W}]\!]_n\|_0^2\right)^{1/2}.
\end{aligned}
$$

[25] Short Version: $\mathscr{E}_8 = \mathscr{E}_9 = \mathscr{E}_{12} = \mathscr{E}_{13} = 0$.

Therefore (keeping the places and indents for easy checking),

$$
\begin{aligned}
&(\varrho\mathcal{W}_N^+, (\boldsymbol{W} - \Pi_w\boldsymbol{w})_N^-) + a(\varphi_0\mathcal{U}_N^+, (\boldsymbol{U} - \Pi_u\boldsymbol{u})_N^-) \\
&\qquad \leqslant \Bigg(\|\varrho^{1/2}(I - P_{\boldsymbol{X}})\dot{\boldsymbol{u}}\|^2_{L_r(I;\boldsymbol{X})} \\
&\qquad\qquad + Ck^6\|\Delta\ddot{\boldsymbol{u}}\|^2_{L_1(I;\boldsymbol{L}_2(\Omega))} \\
&\qquad\qquad + C\|(I - P_{\boldsymbol{X}})\boldsymbol{w}\|^2_{L_1(I;\boldsymbol{L}_2(\Omega))} \\
&\qquad\qquad\qquad\qquad + Ck^6\Big(\gamma_M\|\dddot{\boldsymbol{u}}\|_{L_r(I;\boldsymbol{X})} + \gamma_E\|\Delta\dddot{\boldsymbol{u}}\|_{L_r(I;\boldsymbol{X})}\Big)^2 \\
&\qquad\qquad + Ck^6\|\Delta\ddot{\boldsymbol{u}}\|^2_{L_r(I;\boldsymbol{X})} \\
&\qquad\qquad + \sum_{q=1}^{N_\varphi} Ck^6\|\Delta\ddot{\boldsymbol{w}}\|^2_{L_1(I;\boldsymbol{L}_2(\Omega))} \\
&\qquad\qquad + Ck^6\sum_{q=1}^{N_\varphi}\|\dddot{\boldsymbol{z}}_q\|^2_{L_1(I;\boldsymbol{X})} \\
&\qquad\qquad\qquad\qquad + Ck^6\sum_{q=1}^{N_\varphi}\|\Delta\dddot{\boldsymbol{z}}_q\|^2_{L_r(I;\boldsymbol{X})} \\
&\qquad\qquad + \sum_{n=1}^{N}\|\varrho^{1/2}(I - P_{\boldsymbol{X}})\dot{\boldsymbol{u}}_n^-\|_0^2\Bigg)^{1/2} \\
&\times \Bigg(\|\varrho^{1/2}\dot{\mathcal{W}}\|^2_{L_p(I;\boldsymbol{X}')} \\
&\qquad\qquad + \|\varphi_0^{1/2}\dot{\mathcal{U}}\|^2_{L_\infty(I;\boldsymbol{L}_2(\Omega))} \\
&\qquad\qquad + \|\varrho^{1/2}\mathcal{W}\|^2_{L_\infty(I;\boldsymbol{L}_2(\Omega))} \\
&\qquad\qquad\qquad\qquad + \|\varrho^{1/2}\dot{\mathcal{W}}\|^2_{L_p(I;\boldsymbol{X}')} \\
&\qquad\qquad + \|\varrho^{1/2}\dot{\mathcal{W}}\|^2_{L_p(I;\boldsymbol{X}')} \\
&\qquad\qquad + \left(\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|_{L_\infty(I;\boldsymbol{L}_2(\Omega))}\right)^2 \\
&\qquad\qquad + \sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|^2_{L_\infty(I;\boldsymbol{X})} \\
&\qquad\qquad\qquad\qquad + \|\varrho^{1/2}\dot{\mathcal{W}}\|^2_{L_p(I;\boldsymbol{X}')} \\
&\qquad\qquad + \|\varrho^{1/2}\mathcal{W}_N^+\|_0^2 + \sum_{n=1}^{N}\|\varrho^{1/2}\,[\![\mathcal{W}]\!]_n\,\|_0^2\Bigg)^{1/2}.
\end{aligned}
$$

which when better organized is,[¶[26]]

$$
\begin{aligned}
&(\varrho\mathcal{W}_N^+,(\boldsymbol{W}-\Pi_w\boldsymbol{w})_N^-)+a(\varphi_0\mathcal{U}_N^+,(\boldsymbol{U}-\Pi_u\boldsymbol{u})_N^-)\\
&\quad\leqslant C\Bigg(\|(I-P_{\boldsymbol{X}})\dot{\boldsymbol{u}}\|^2_{L_r(I;\boldsymbol{X})}+\|(I-P_{\boldsymbol{X}})\dot{\boldsymbol{u}}\|^2_{L_1(I;\boldsymbol{L}_2(\Omega))}+\sum_{n=1}^{N}\|(I-P_{\boldsymbol{X}})\dot{\boldsymbol{u}}_n^-\|_0^2\\
&\qquad\quad+k^6\|\Delta\ddot{\boldsymbol{u}}\|^2_{L_1(I;\boldsymbol{L}_2(\Omega))}+k^6\|\ddot{\boldsymbol{u}}\|^2_{L_r(I;\boldsymbol{X})}+k^6\|\Delta\ddot{\boldsymbol{u}}\|^2_{W^1_r(I;\boldsymbol{X})}\\
&\qquad\qquad\quad+k^6\sum_{q=1}^{N_\varphi}\|\ddot{\boldsymbol{z}}_q\|^2_{L_1(I;\boldsymbol{X})}+k^6\sum_{q=1}^{N_\varphi}\|\Delta\ddot{\boldsymbol{z}}_q\|^2_{L_r(I;\boldsymbol{X})}\Bigg)^{1/2}\\
&\qquad\quad\times\Bigg(\|\varrho^{1/2}\dot{\mathcal{W}}\|^2_{L_p(I;\boldsymbol{X}')}+\|\varphi_0^{1/2}\dot{\mathcal{U}}\|^2_{L_\infty(I;\boldsymbol{L}_2(\Omega))}+\|\varrho^{1/2}\mathcal{W}\|^2_{L_\infty(I;\boldsymbol{L}_2(\Omega))}\\
&\qquad\qquad\quad+\sum_{q=1}^{N_\varphi}\|\tau_q^{1/2}\dot{\mathcal{Z}}_q\|^2_{L_\infty(I;\boldsymbol{X})}+\|\varrho^{1/2}\mathcal{W}_N^+\|_0^2+\sum_{n=1}^{N}\|\varrho^{1/2}[\![\mathcal{W}]\!]_n\|_0^2\Bigg)^{1/2}
\end{aligned}
$$

and we can obtain $\|(I-P_{\boldsymbol{X}})\dot{\boldsymbol{u}}\|_{L_r(I;\boldsymbol{X})}\leqslant Ch\|\dot{\boldsymbol{u}}\|_{L_r(I;\boldsymbol{H}^2(\Omega))}$, for $r\in[1,\infty]$, for the spatial errors using standard arguments.

Specifically, to deal with the spatial errors we now consider the auxiliary problem of finding $\chi\in\boldsymbol{X}\cap\boldsymbol{H}^2(\Omega)$ such that $-\Delta\boldsymbol{\chi}=\varrho(I-P_{\boldsymbol{X}})\dot{\boldsymbol{u}}$. Then $a(\boldsymbol{v},\boldsymbol{\chi})=(\varrho(I-P_{\boldsymbol{X}})\dot{\boldsymbol{u}},\boldsymbol{v})$ and so by standard arguments $\|(I-P_{\boldsymbol{X}})\dot{\boldsymbol{u}}\|_{\boldsymbol{X}}\leqslant Ch\|\dot{\boldsymbol{u}}\|_{\boldsymbol{H}^2(\Omega)}$ and $\|(I-P_{\boldsymbol{X}})\dot{\boldsymbol{u}}\|_0\leqslant Ch^2\|\dot{\boldsymbol{u}}\|_{\boldsymbol{H}^2(\Omega)}$.

The second of these comes from the following argument:

$$
\begin{aligned}
\|\varrho^{1/2}(I-P_{\boldsymbol{X}})\dot{\boldsymbol{u}}\|_0^2&=a((I-P_{\boldsymbol{X}})\dot{\boldsymbol{u}},\boldsymbol{\chi})=a((I-P_{\boldsymbol{X}})\dot{\boldsymbol{u}},(I-P_{\boldsymbol{X}})\boldsymbol{\chi}),\\
\Rightarrow\|\varrho^{1/2}(I-P_{\boldsymbol{X}})\dot{\boldsymbol{u}}\|_0^2&\leqslant\|(I-P_{\boldsymbol{X}})\dot{\boldsymbol{u}}\|_{\boldsymbol{X}}\|(I-P_{\boldsymbol{X}})\boldsymbol{\chi}\|_{\boldsymbol{X}},\\
&\leqslant Ch^2\|\dot{\boldsymbol{u}}\|_{\boldsymbol{H}^2(\Omega)}\|\boldsymbol{\chi}\|_{\boldsymbol{H}^2(\Omega)},\\
&\leqslant Ch^2\|\dot{\boldsymbol{u}}\|_{\boldsymbol{H}^2(\Omega)}\|\varrho^{1/2}(I-P_{\boldsymbol{X}})\dot{\boldsymbol{u}}\|_0,
\end{aligned}
$$

[¶[27]]Now, from (8) we have $\boldsymbol{z}_q=(\psi*\dot{\boldsymbol{u}})$ for $\psi(t)=(\varphi_q/\tau_q)^{1/2}\exp(-t/\tau_q)$ and so using Hölder's inequality for convolutions, $\|\boldsymbol{z}_q\|_{L_r(I;\cdot)}\leqslant\beta_q\|\dot{\boldsymbol{u}}\|_{L_r(I;\cdot)}$ because $\|\psi\|_{L_1(I)}\leqslant\beta_q$. From (9) we then obtain first that $\|\dot{\boldsymbol{z}}_q\|_{L_r(I;\cdot)}\leqslant 2(\varphi_q/\tau_q)^{1/2}\|\dot{\boldsymbol{u}}\|_{L_r(I;\cdot)}$ and then secondly that $\|\ddot{\boldsymbol{z}}_q\|_{L_r(I;\cdot)}\leqslant C\|\dot{\boldsymbol{u}}\|_{W^1_r(I;\cdot)}$.

Using Theorem 4.6 we now conclude that,

$$
\begin{aligned}
&(\varrho\mathcal{W}_N^+,(\boldsymbol{W}-\Pi_w\boldsymbol{w})_N^-)+a(\varphi_0\mathcal{U}_N^+,(\boldsymbol{U}-\Pi_u\boldsymbol{u})_N^-)\\
&\quad\leqslant CT^{1/p}(1+c_T)\mathscr{R}(\boldsymbol{u})\left(h+k^3+T^{1/2}k^{-1/2}h^2\right)\left(\|\varrho^{1/2}\mathcal{W}_N^+\|_0^2+\|\varphi_0^{1/2}\mathcal{U}_N^+\|^2_{\boldsymbol{X}}\right)^{1/2},
\end{aligned}
$$

[¶[28]]and then choosing $\mathcal{W}_N^+=(\boldsymbol{W}-\Pi_w\boldsymbol{w})_N^-$ and $\mathcal{U}_N^+=(\boldsymbol{U}-\Pi_u\boldsymbol{u})_N^-$ gives

$$
\begin{aligned}
\|\varrho^{1/2}(\boldsymbol{W}-\Pi_w\boldsymbol{w})_N^-\|_0+&\|\varphi_0^{1/2}(\boldsymbol{U}-\Pi_u\boldsymbol{u})_N^-\|_{\boldsymbol{X}}\\
&\leqslant CT^{1/2+1/p}(1+c_T)\mathscr{R}(\boldsymbol{u})\left(h+k^3+k^{-1/2}h^2\right).
\end{aligned}
$$

[26]Short Version:
[27]Short Version:
[28]Short Version:

The proof is then completed by using the triangle inequality and more approximation error bounds for $\|\varrho^{1/2}(\boldsymbol{w}-\Pi_w\boldsymbol{w})_N^-\|_0+\|\varphi_0^{1/2}(\boldsymbol{u}-\Pi_u\boldsymbol{u})_N^-\|_{\boldsymbol{X}}$.

The kinetic plus energy error is estimated by terms of order $O\left(h+k^3+k^{-1/2}h^2\right)$ in Theorem 4.7 which, because $h \leqslant c_T k$, is of size $O(h+k^3)$ and since we can allow $k \sim h^q$ for $q \in (0,1]$ (as $h \to 0$, because $h = h^{1-q}h^q \leqslant h^q \sim k$), we may have errors of size $O(h+h^{3q}) = O(h^\gamma)$ for $\gamma < 1$. We illustrate and discuss this later at the end of Section 5.

The $O(k^3)$ superconvergence in time in the bound $O\left(h+k^3+k^{-1/2}h^2\right)$ is expected for temporally piecewise linear approximations and was reported in [1, Thm. 2.3, Rem. 2.6] for parabolic problems. The $O(h)$ term is very standard and arises from error bounds for the elliptic projection. The $O(k^{-1/2}h^2)$ is more unusual in that it is not seen in error bounds for finite-difference-in-time methods. It arises here because the term $\mathscr{E}_6$ in the proof of Theorem 4.7 contains a sum of squared $\boldsymbol{L}_2(\Omega)$ spatial errors over all N time levels with no compensating weight of the time step k to kill the growth. Hence the sum of $N \sim k^{-1}$ terms of size $O(h^4)$ is controlled by a bound of order $O(k^{-1/2}h^2)$.

The bound in Theorem 4.7 is only optimal if we regard the left hand norms as inseparable. Otherwise, experience tells us that we could expect $\|\varrho^{1/2}(\boldsymbol{w}-\boldsymbol{W})_N^-\|_0 = O\left(h^2+k^3+k^{-1/2}h^2\right)$ and $\|\varphi_0^{1/2}(\boldsymbol{u}-\boldsymbol{U})_N^-\|_{\boldsymbol{X}} = O\left(h+k^3+k^{-1/2}h^2\right)$ — although the first of these these is not proven here.

Furthermore, we can expect that using piecewise polynomials of degree $p>1$ in space would (regularity permitting) result in a bound of size $O\left(h^p+k^3+k^{-1/2}h^{p+1}\right)$ in Theorem 4.7. We can also see that while higher degree temporal DG polynomial approximation would improve the $O(k^3)$ term, it would not affect the factor of $k^{-1/2}$.

5 Implementation and results

The implementation given below is restricted to piecewise linears in time in order to illustrate Theorem 4.7. Unlike Li and Wiberg's method in [17], we do not need an iterative solution but instead eliminate the displacements so the linear block-solve is for just the velocities. Only the main steps are outlined, the full details are in Appendix A. [¶[29]] The formulation includes the case where a traction is imposed on Γ_N but, to remain consistent with Theorem 4.7 we revert to $\Gamma_N = \varnothing$ for the numerical results.

On a given time interval, I_n, we choose a piecewise linear temporal basis $\theta_1, \theta_2 \colon I_n \to \mathbb{R}$ and, in (21), write $\boldsymbol{U}(t) = \boldsymbol{U}_1\theta_1(t) + \boldsymbol{U}_2\theta_2(t)$, $\boldsymbol{W}(t) = \boldsymbol{W}_1\theta_1(t)+\boldsymbol{W}_2\theta_2(t)$ and, for each q, $\boldsymbol{Z}_q(t) = \boldsymbol{Z}_{q,1}\theta_1(t)+\boldsymbol{Z}_{q,2}\theta_2(t)$ where $\boldsymbol{U}_j, \boldsymbol{W}_j, \boldsymbol{Z}_{q,j} \in \boldsymbol{X}^h$ for each q and for $j=1,2$. Then, defining,

$$\mathsf{M} = \int_{t_{n-1}}^{t_n} \begin{pmatrix} \theta_1(t)\theta_1(t) & \theta_2(t)\theta_1(t) \\ \theta_1(t)\theta_2(t) & \theta_2(t)\theta_2(t) \end{pmatrix} dt$$

and

$$\mathsf{A} = \int_{t_{n-1}}^{t_n} \begin{pmatrix} \dot\theta_1(t)\theta_1(t) & \dot\theta_2(t)\theta_1(t) \\ \dot\theta_1(t)\theta_2(t) & \dot\theta_2(t)\theta_2(t) \end{pmatrix} dt + \begin{pmatrix} \theta_1\theta_1 & \theta_2\theta_1 \\ \theta_1\theta_2 & \theta_2\theta_2 \end{pmatrix}\Bigg|_{t_{n-1}}$$

[29] Short Version: Only the main steps are outlined.

we can choose $\boldsymbol{\vartheta} = \theta_i(t)\boldsymbol{v}$ in (21) and extract the discrete momentum equation,

$$\mathsf{A}\begin{pmatrix} (\varrho \boldsymbol{W}_1, \boldsymbol{v}) \\ (\varrho \boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \mathsf{M}\begin{pmatrix} a(\varphi_0 \boldsymbol{U}_1, \boldsymbol{v}) \\ a(\varphi_0 \boldsymbol{U}_2, \boldsymbol{v}) \end{pmatrix} + \mathsf{M}\begin{pmatrix} b(\boldsymbol{W}_1, \boldsymbol{v}) \\ b(\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \sum_{q=1}^{N_\varphi} \mathsf{M}\begin{pmatrix} a(\beta_q \boldsymbol{Z}_{q,1}, \boldsymbol{v}) \\ a(\beta_q \boldsymbol{Z}_{q,2}, \boldsymbol{v}) \end{pmatrix}$$
$$= \begin{pmatrix} \theta_1(t_{n-1}) \\ \theta_2(t_{n-1}) \end{pmatrix} (\varrho \boldsymbol{W}_{n-1}^-, \boldsymbol{v}) + \int_{t_{n-1}}^{t_n} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} \Big((\boldsymbol{f}(t), \boldsymbol{v}) + (\boldsymbol{g}(t), \boldsymbol{v})_{\Gamma_N} \Big)\, dt$$

for all $\boldsymbol{v} \in \boldsymbol{X}^h$. Next, choosing $\boldsymbol{\zeta} = \theta_i(t)\boldsymbol{v}$ in (21) gives the following discrete enforcement of $\dot{\boldsymbol{u}} = \boldsymbol{w}$ as,

$$\begin{pmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{pmatrix} = \mathsf{A}^{-1}\mathsf{M}\begin{pmatrix} \boldsymbol{W}_1 \\ \boldsymbol{W}_2 \end{pmatrix} + \mathsf{A}^{-1}\begin{pmatrix} \theta_1(t_{n-1})\boldsymbol{U}_{n-1}^- \\ \theta_2(t_{n-1})\boldsymbol{U}_{n-1}^- \end{pmatrix}$$

and, with this, the momentum equations simplify to,

$$\mathsf{A}\begin{pmatrix} (\varrho \boldsymbol{W}_1, \boldsymbol{v}) \\ (\varrho \boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \mathsf{M}\mathsf{A}^{-1}\mathsf{M}\begin{pmatrix} a(\varphi_0 \boldsymbol{W}_1, \boldsymbol{v}) \\ a(\varphi_0 \boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \mathsf{M}\begin{pmatrix} b(\boldsymbol{W}_1, \boldsymbol{v}) \\ b(\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix}$$
$$+ \sum_{q=1}^{N_\varphi} \mathsf{M}\begin{pmatrix} a(\beta_q \boldsymbol{Z}_{q,1}, \boldsymbol{v}) \\ a(\beta_q \boldsymbol{Z}_{q,2}, \boldsymbol{v}) \end{pmatrix} = \int_{t_{n-1}}^{t_n} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} \Big((\boldsymbol{f}(t), \boldsymbol{v}) + (\boldsymbol{g}(t), \boldsymbol{v})_{\Gamma_N} \Big)\, dt$$
$$+ \begin{pmatrix} \theta_1(t_{n-1}) \\ \theta_2(t_{n-1}) \end{pmatrix} (\varrho \boldsymbol{W}_{n-1}^-, \boldsymbol{v}) - \mathsf{M}\mathsf{A}^{-1}\begin{pmatrix} \theta_1(t_{n-1}) \\ \theta_2(t_{n-1}) \end{pmatrix} a(\varphi_0 \boldsymbol{U}_{n-1}^-, \boldsymbol{v}).$$

In a similar way, by choosing $\boldsymbol{\xi}_q = \boldsymbol{v}\theta_i(t)$ in (21) we obtain,

$$(\tau_q \mathsf{A} + \mathsf{M})\begin{pmatrix} \boldsymbol{Z}_{q,1} \\ \boldsymbol{Z}_{q,2} \end{pmatrix} = \beta_q \mathsf{M}\begin{pmatrix} \boldsymbol{W}_1 \\ \boldsymbol{W}_2 \end{pmatrix} + \tau_q \boldsymbol{Z}_{q,n-1}^- \begin{pmatrix} \theta_1(t_{n-1}) \\ \theta_2(t_{n-1}) \end{pmatrix},$$

which can be substituted into the momentum equations to result in a two-by-two block system for $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$.

To make progress we choose the specific forms $\theta_1(t) = 1$ and $\theta_2(t) = (t_n - t)/k$ and then obtain easily that $\mathsf{A} = \frac{1}{2}\left(\begin{smallmatrix} 2 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and $\mathsf{M} = \frac{k}{6}\left(\begin{smallmatrix} 6 & 3 \\ 3 & 2 \end{smallmatrix}\right)$. Moreover $\mathsf{A}^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -2 & 2 \end{smallmatrix}\right)$ and $\mathsf{M}^{-1} = \frac{1}{k}\left(\begin{smallmatrix} 4 & -6 \\ -6 & 12 \end{smallmatrix}\right)$ and, further, $\mathsf{M}^{-1}\mathsf{A} = \frac{1}{k}\left(\begin{smallmatrix} -2 & -3 \\ 6 & 6 \end{smallmatrix}\right)$, $\mathsf{M}\mathsf{A}^{-1} = \frac{k}{6}\left(\begin{smallmatrix} 0 & 6 \\ -1 & 4 \end{smallmatrix}\right)$, $\mathsf{A}^{-1}\mathsf{M} = \frac{k}{6}\left(\begin{smallmatrix} 6 & 3 \\ -6 & -2 \end{smallmatrix}\right)$ and $\mathsf{M}\mathsf{A}^{-1}\mathsf{M} = \frac{k^2}{36}\left(\begin{smallmatrix} 18 & 12 \\ 6 & 5 \end{smallmatrix}\right)$. After a significant amount of routine calculation we arrive at a specific form of the momentum equations as,

$$k^2\left[\begin{pmatrix} 3\varphi_0 + 6\gamma_E k^{-1} & 2\varphi_0 + 3\gamma_E k^{-1} \\ \varphi_0 + 3\gamma_E k^{-1} & 5\varphi_0/6 + 2\gamma_E k^{-1} \end{pmatrix} + \sum_{q=1}^{N_\varphi} d_q \beta_q^2 \begin{pmatrix} 6(3\tau_q + k) & 3(4\tau_q + k) \\ 3(2\tau_q + k) & (5\tau_q + 2k) \end{pmatrix}\right]$$
$$\times \begin{pmatrix} a(\boldsymbol{W}_1, \boldsymbol{v}) \\ a(\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \begin{pmatrix} 6 + 6\gamma_M k & 0 + 3\gamma_M k \\ 6 + 3\gamma_M k & 3 + 2\gamma_M k \end{pmatrix}\begin{pmatrix} (\varrho \boldsymbol{W}_1, \boldsymbol{v}) \\ (\varrho \boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix}$$
$$= \int_{t_{n-1}}^{t_n} \begin{pmatrix} 6 \\ 6(t_n - t)/k \end{pmatrix} \Big((\boldsymbol{f}(t), \boldsymbol{v}) + (\boldsymbol{g}(t), \boldsymbol{v})_{\Gamma_N} \Big)\, dt + \begin{pmatrix} 6 \\ 6 \end{pmatrix} (\varrho \boldsymbol{W}_{n-1}^-, \boldsymbol{v})$$
$$- k\begin{pmatrix} 6 \\ 3 \end{pmatrix} a(\varphi_0 \boldsymbol{U}_{n-1}^-, \boldsymbol{v}) - \sum_{q=1}^{N_\varphi} 6k d_q \beta_q \tau_q \begin{pmatrix} 6\tau_q + k \\ 3\tau_q + k \end{pmatrix} a(\boldsymbol{Z}_{q,n-1}^-, \boldsymbol{v}). \quad (36)$$

Once $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are obtained from this we update $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ with

$$\boldsymbol{U}_1 = k\boldsymbol{W}_1 + \frac{k}{2}\boldsymbol{W}_2 + \boldsymbol{U}_{n-1}^- \qquad \text{and} \qquad \boldsymbol{U}_2 = -k\boldsymbol{W}_1 - \frac{k}{3}\boldsymbol{W}_2,$$

and then obtain $\boldsymbol{Z}_{q,1}$ and $\boldsymbol{Z}_{q,2}$ from

$$\begin{pmatrix} \boldsymbol{Z}_{q,1} \\ \boldsymbol{Z}_{q,2} \end{pmatrix} = kd_q\beta_q \begin{pmatrix} 6\tau_q + k & 3\tau_q \\ -6\tau_q & k - 2\tau_q \end{pmatrix} \begin{pmatrix} \boldsymbol{W}_1 \\ \boldsymbol{W}_2 \end{pmatrix} + 2d_q\tau_q \boldsymbol{Z}_{q,n-1}^{-} \begin{pmatrix} 3\tau_q - k \\ 3k \end{pmatrix}.$$

We now give the results of some computations designed specifically to illustrate the convergence rates of the algorithm derived above.

To verify that the observed convergence rates agree with those stated in Theorem 4.7 we manufacture an exact solution and choose the data consistent with that solution. For this we take $\Omega := (0,1)^2$, the unit square, with $T = 12\pi$ and we consider an exact solution in the form,

$$\boldsymbol{u} = \bar{\boldsymbol{u}}(\boldsymbol{x})\mathcal{T}(t) \qquad \text{for} \qquad \bar{\boldsymbol{u}}(\boldsymbol{x}) := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 16(x^2 - x)(y^2 - y)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where $\mathcal{T}(t) = t + B\cos(t)$ for a constant B (taken as $B = 0$ or $B = 1$ below). Then $\boldsymbol{w} = \bar{\boldsymbol{u}}\mathcal{T}'(t)$ and we see that $\boldsymbol{u}$ satisfies the requirements of Theorem 4.7. As mentioned earlier, we consider the material to be isotropic, homogeneous and synchronous and then, on using (2) with (4) and the assumption of Rayleigh damping as in (14) we can obtain the loads once the coefficients are defined. For these we take $\varrho = 1$, $\lambda = 1$ and $\mu = 1$, with Rayleigh damping given by $\gamma_M = 2$, $\gamma_E = 1$, and three-term, $N_\varphi = 2$, viscoelasticity given by $(\varphi_1, \tau_1) = (0.35, 0.1)$ and $(\varphi_2, \tau_2) = (0.15, 0.05)$ for all but the first example below. In the discrete scheme we used an $N_{xy} \times N_{xy}$ mesh of isosceles triangles with piecewise linear elements and a uniform time step of $k = T/N_t$, for $N_t \in \mathbb{N}$. We set $h = N_{xy}^{-1}$

In Examples I, II, III and IV below (based on Examples 5, 10, 11 and 12 in Appendix B)[¶[30]] the errors, '$\boldsymbol{e}$', are reported in the kinetic energy norm, KEe, $\|\varrho^{1/2}\boldsymbol{e}_w(T)\|_0$ for $\boldsymbol{e}_w(T) := \boldsymbol{w}(T) - \boldsymbol{W}_N^-$, the elastic strain energy norm, ESe, $\|\varphi_0^{1/2}\boldsymbol{e}_u(T)\|_{\boldsymbol{X}}$ for $\boldsymbol{e}_u(T) := \boldsymbol{u}(T) - \boldsymbol{U}_N^-$, the total energy norm, TEe, $(\|\varrho^{1/2}\boldsymbol{e}_w(T)\|_0^2 + \|\varphi_0^{1/2}\boldsymbol{e}_u(T)\|_{\boldsymbol{X}}^2)^{1/2}$, as well as the $\boldsymbol{H}^1(\Omega)$ norm for both $\boldsymbol{e}_u(T)$ and $\boldsymbol{e}_w(T)$.

These results were first computed in a 64 bit bare metal Mint 18.1 ('Serena') FEniCS installation (see Logg *et al.* in [19] and fenicsproject.org), with dolfin version 2016.2.0, on a Dell xps15z laptop with 2 x 4096MB 1333MHz DDR dual channel RAM and 2nd Gen Intel Core i7-2620M (2.7GHz, 4threads, 4MB cache). They were then repeated on a multi-core and larger memory machine in order to get to larger values of N_{xy}.

This collection of results was a bit patchwork. In the end they were all recomputed as described later in the 'updated results section'. In particular, there exists a custom image: pull first,

`docker pull variationalform/fem:dgcgwave`,

(see `https://hub.docker.com/r/variationalform/fem`) and then run with

`docker run -ti variationalform/fem:dgcgwave`.

The command `cd fenics` followed by `./bigrun.sh -J 3 | tee runmeout.txt` will, for a suite of twelve test cases, produce the error results up to $N_{xy} = \text{int}(2^{3/2})$ in the `results` directory. Examples 5,10,11 and 12 (resp.) of those correspond to examples

[30] Short Version:

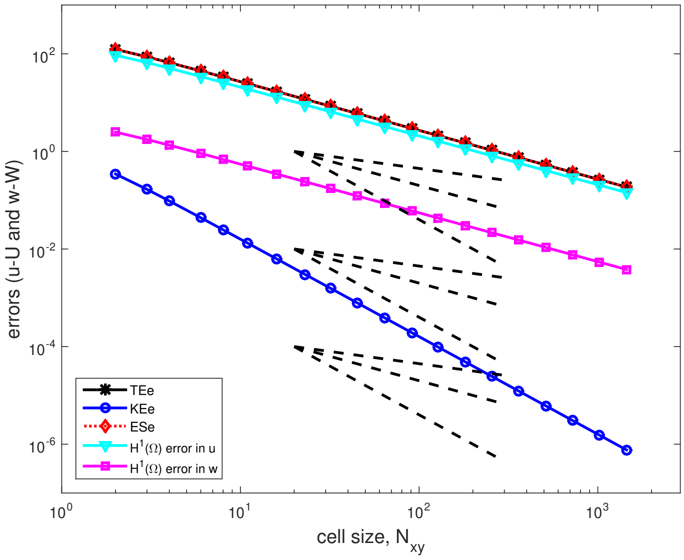

Figure 1: Errors for Example I, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1/2, 1, 2$. In this example $B = 0$ and there are no viscoelastic terms — the Galerkin errors are due only to the spatial approximation.

I, II, III and IV (resp.) below. Use `-J 7` to go up to $N_{xy} = \text{int}(2^{7/2})$ and so on (but it will take longer). [¶[31]]

In Example I we set $B = 0$ and $N_t = 4$ and switch the viscoelasticity off by setting $\varphi_0 = 1$, the Galerkin errors are therefore due only to the spatial approximation and are shown in Figure 1. In each of Examples II, III and IV we choose $B = 1$ with the coefficients given earlier. For these examples we choose the time step $k \sim h^q$, specifically

$$k = \frac{T}{\max\left\{1, \text{int}\left(\frac{T}{h^q}\right)\right\}},$$

for $q = 2/3$, $1/3$ and $1/6$, and show the results in Figure 2 for Example II, Figure 3 for Example III and Figure 4 for Example IV. In each case $h \sim h^{1-q}k \leqslant c_T k$ and so the conditions of Theorem 4.7 are satisfied. Indeed, the order of convergence as predicted by the Theorem becomes $h + k^3 + k^{-1/2}h^2 = h + h^{3q} + h^{2-q/2}$ which is $O(h)$ for $q = 2/3$ and $q = 1/3$, but $O(h^{1/2})$ for $q = 1/6$. In Figures 1 and 2 we can see clearly that the spatial error in the $\boldsymbol{H}^{1-p}(\Omega)$ norm is $O(h^{1+p})$ for $p = 0$ and $p = 1$. This is expected (although the $p = 1$ case is not proven here) but we also see from Figure 3 that when $q = 1/3$ the $O(h)$ term stems from the $k^3 = h^{3q}$ part of the estimate and

[31] Short Version: These results were computed using a 65Gb Intel Xeon E5-2640 v4 CPU (2.40GHz). We used the 2017.1.0 FEniCS (see Logg *et al.* in [19] and fenicsproject.org) docker image started with

`docker run -ti ... quay.io/fenicsproject/stable:2017.1.0`

(... indicates that superfluous details are omitted) on 20 December 2018. A custom image built for this paper can be pulled in docker with

`docker pull variationalform/fem:dgcgwave,`

(see `https://hub.docker.com/r/variationalform/fem`) and then run with

`docker run -ti variationalform/fem:dgcgwave.`

The command `cd fenics` followed by `./bigrun.sh -J 3 | tee runmeout.txt` will, for a suite of twelve test cases, produce the error results up to $N_{xy} = \text{int}(2^{3/2})$ in the `results` directory. Examples 5,10,11 and 12 (resp.) of those correspond to examples I, II, III and IV (resp.) below. Use `-J 7` to go up to $N_{xy} = \text{int}(2^{7/2})$ and so on (but it will take longer).

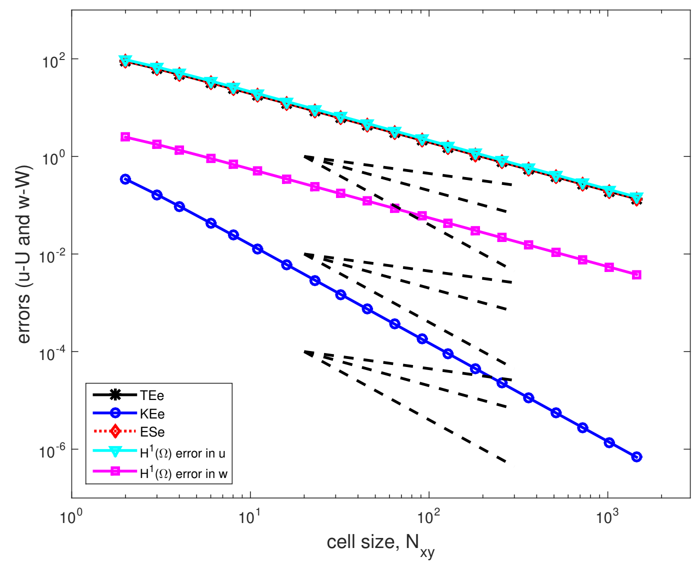

Figure 2: Errors for Example II, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1/2, 1, 2$ and $k \sim h^{2/3}$.

so the expected $O(h^2)$ error associated with the kinetic energy error (the $\boldsymbol{L}_2(\Omega)$ error in $\dot{\boldsymbol{u}}$) does not appear. Furthermore, the $O(h^{1/2})$ error for $q = 1/6$ is beginning to asymptotically show in Figure 4 for all except the dominant elastic strain energy error in $\boldsymbol{u}$. The curves for these $\boldsymbol{H}^1(\Omega)$ type errors in displacement appear indistinguishable in each of Figures 1, 2 and 3. This indicates that those errors are dominated by the $O(h)$ spatial error component for these values of N_{xy} and not by the $O(h^{3q})$ associated with the $O(k^3)$ term.

6 Conclusions

We have extended the formulation and *a priori* error analysis given in [12] from the acoustic wave equation to a viscodynamic system incorporating Rayleigh damping. The elastic term in the Rayleigh damping introduces a multiplicative $T^{1/2}$ growth in the constant but otherwise the error bound is consistent with that obtained in [12], with a constant that grows *a priori* with $T^{1/2}$ and also with the norms in $\mathscr{R}(u)$ (which could of course be simplified at the expense of introducing more powers of T). However, Gronwall's inequality is not used and so we can expect that this bound is of high enough quality to afford confidence in long-time integration.

The results of some numerical experiments are given in Figure 1 for Example I, Figure 2 for Example II, Figure 3 for Example III and Figure 4 for Example IV and these demonstrate that the *a priori* estimate given in Theorem 4.7 is optimal. They also demonstrate that the $\boldsymbol{L}_2(\Omega)$ kinetic energy errors alone can converge at a rate faster than that predicted by the theorem.

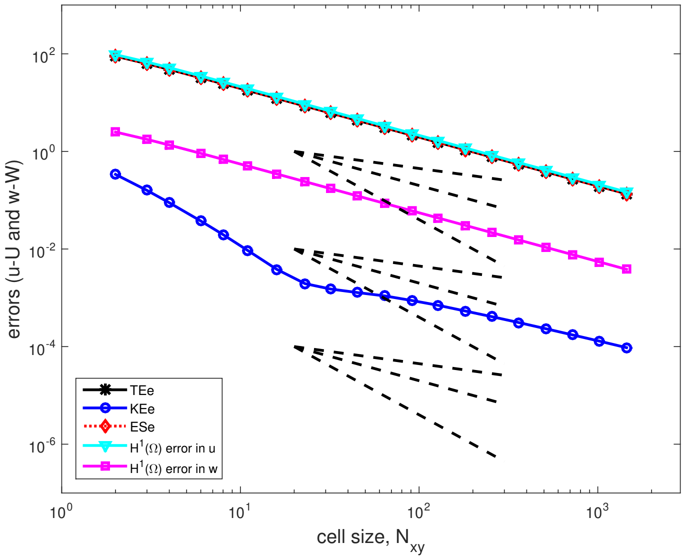

Figure 3: Errors for Example III, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1/2, 1, 2$ and $k \sim h^{1/3}$.

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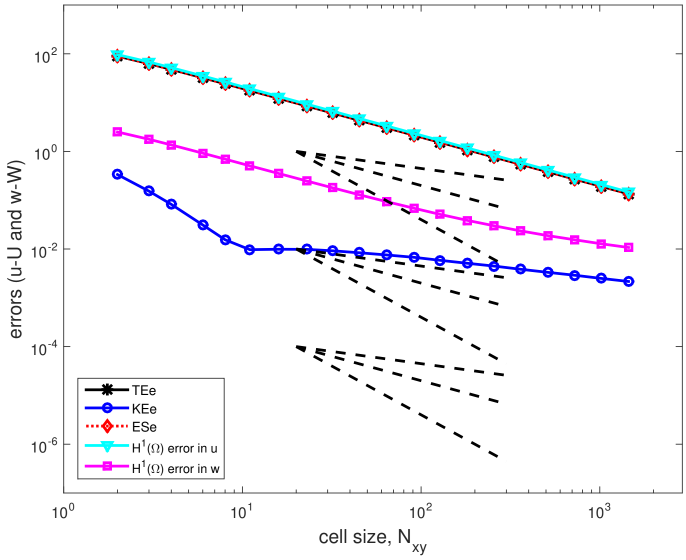

Figure 4: Errors for Example IV, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1/2, 1, 2$ and $k \sim h^{1/6}$.

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A Implementation: the details

In this section we give full details of the scheme described in Section 5.

On a given time interval, I_n, we choose a piecewise linear temporal basis $\theta_1, \theta_2 \colon I_n \to \mathbb{R}$ and, in (21), write $\boldsymbol{U}(t) = \boldsymbol{U}_1\theta_1(t) + \boldsymbol{U}_2\theta_2(t)$, $\boldsymbol{W}(t) = \boldsymbol{W}_1\theta_1(t) + \boldsymbol{W}_2\theta_2(t)$ and, for each q, $\boldsymbol{Z}_q(t) = \boldsymbol{Z}_{q,1}\theta_1(t) + \boldsymbol{Z}_{q,2}\theta_2(t)$ where $\boldsymbol{U}_j, \boldsymbol{W}_j, \boldsymbol{Z}_{q,j} \in \boldsymbol{X}^h$ for each q and for $j = 1, 2$.

We note that it is possible to choose a different temporal basis for each of these but at the moment there seems little point in doing so. Then, defining,

$$\mathsf{M} = \int_{t_{n-1}}^{t_n} \begin{pmatrix} \theta_1(t)\theta_1(t) & \theta_2(t)\theta_1(t) \\ \theta_1(t)\theta_2(t) & \theta_2(t)\theta_2(t) \end{pmatrix} dt$$

and

$$\mathsf{A} = \int_{t_{n-1}}^{t_n} \begin{pmatrix} \dot\theta_1(t)\theta_1(t) & \dot\theta_2(t)\theta_1(t) \\ \dot\theta_1(t)\theta_2(t) & \dot\theta_2(t)\theta_2(t) \end{pmatrix} dt + \begin{pmatrix} \theta_1\theta_1 & \theta_2\theta_1 \\ \theta_1\theta_2 & \theta_2\theta_2 \end{pmatrix}\Bigg|_{t_{n-1}}$$

we can choose $\boldsymbol{\vartheta} = \theta_i(t)\boldsymbol{v}$ in (21) and extract the discrete momentum equation,

$$\mathsf{A}\begin{pmatrix} (\varrho \boldsymbol{W}_1, \boldsymbol{v}) \\ (\varrho \boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \mathsf{M}\begin{pmatrix} a(\varphi_0 \boldsymbol{U}_1, \boldsymbol{v}) \\ a(\varphi_0 \boldsymbol{U}_2, \boldsymbol{v}) \end{pmatrix} + \mathsf{M}\begin{pmatrix} b(\boldsymbol{W}_1, \boldsymbol{v}) \\ b(\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \sum_{q=1}^{N_\varphi} \mathsf{M}\begin{pmatrix} a(\beta_q \boldsymbol{Z}_{q,1}, \boldsymbol{v}) \\ a(\beta_q \boldsymbol{Z}_{q,2}, \boldsymbol{v}) \end{pmatrix}$$
$$= \begin{pmatrix} \theta_1(t_{n-1}) \\ \theta_2(t_{n-1}) \end{pmatrix} (\varrho \boldsymbol{W}_{n-1}^-, \boldsymbol{v}) + \int_{t_{n-1}}^{t_n} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} \Big((\boldsymbol{f}(t), \boldsymbol{v}) + (\boldsymbol{g}(t), \boldsymbol{v})_{\Gamma_N} \Big)\, dt$$

for all $\boldsymbol{v} \in \boldsymbol{X}^h$. Moreover, for use below, we also notice that we can choose differing test functions, $\boldsymbol{v}$, in each row of this block system. Next, choosing $\boldsymbol{\zeta} = \theta_i(t)\boldsymbol{v}$ in (21) gives the following discrete enforcement of $\dot{\boldsymbol{u}} = \boldsymbol{w}$ as,

$$\mathsf{M}\begin{pmatrix} a(\varphi_0 \boldsymbol{W}_1, \boldsymbol{v}) \\ a(\varphi_0 \boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} = \mathsf{A}\begin{pmatrix} a(\varphi_0 \boldsymbol{U}_1, \boldsymbol{v}) \\ a(\varphi_0 \boldsymbol{U}_2, \boldsymbol{v}) \end{pmatrix} - \begin{pmatrix} \theta_1(t_{n-1}) \\ \theta_2(t_{n-1}) \end{pmatrix} a(\varphi_0 \boldsymbol{U}_{n-1}^-, \boldsymbol{v})$$

With these, we get,

$$\begin{pmatrix} a(\varphi_0 \boldsymbol{U}_1, \boldsymbol{v}) \\ a(\varphi_0 \boldsymbol{U}_2, \boldsymbol{v}) \end{pmatrix} = \mathsf{A}^{-1}\mathsf{M}\begin{pmatrix} a(\varphi_0 \boldsymbol{W}_1, \boldsymbol{v}) \\ a(\varphi_0 \boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \mathsf{A}^{-1}\begin{pmatrix} \theta_1(t_{n-1}) \\ \theta_2(t_{n-1}) \end{pmatrix} a(\varphi_0 \boldsymbol{U}_{n-1}^-, \boldsymbol{v}),$$

and so,

$$\mathsf{A}\begin{pmatrix} (\varrho \boldsymbol{W}_1, \boldsymbol{v}) \\ (\varrho \boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \mathsf{M}\begin{pmatrix} a(\varphi_0 \boldsymbol{U}_1, \boldsymbol{v}) \\ a(\varphi_0 \boldsymbol{U}_2, \boldsymbol{v}) \end{pmatrix} + \mathsf{M}\begin{pmatrix} b(\boldsymbol{W}_1, \boldsymbol{v}) \\ b(\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \sum_{q=1}^{N_\varphi} \mathsf{M}\begin{pmatrix} a(\beta_q \boldsymbol{Z}_{q,1}, \boldsymbol{v}) \\ a(\beta_q \boldsymbol{Z}_{q,2}, \boldsymbol{v}) \end{pmatrix}$$
$$= \begin{pmatrix} \theta_1(t_{n-1}) \\ \theta_2(t_{n-1}) \end{pmatrix} (\varrho \boldsymbol{W}_{n-1}^-, \boldsymbol{v}) + \int_{t_{n-1}}^{t_n} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} \Big((\boldsymbol{f}(t), \boldsymbol{v}) + (\boldsymbol{g}(t), \boldsymbol{v})_{\Gamma_N} \Big)\, dt$$

becomes

$$\mathsf{A}\begin{pmatrix} (\varrho \boldsymbol{W}_1, \boldsymbol{v}) \\ (\varrho \boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \mathsf{M}\mathsf{A}^{-1}\mathsf{M}\begin{pmatrix} a(\varphi_0 \boldsymbol{W}_1, \boldsymbol{v}) \\ a(\varphi_0 \boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \mathsf{M}\begin{pmatrix} b(\boldsymbol{W}_1, \boldsymbol{v}) \\ b(\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix}$$
$$+ \sum_{q=1}^{N_\varphi} \mathsf{M}\begin{pmatrix} a(\beta_q \boldsymbol{Z}_{q,1}, \boldsymbol{v}) \\ a(\beta_q \boldsymbol{Z}_{q,2}, \boldsymbol{v}) \end{pmatrix} = \int_{t_{n-1}}^{t_n} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} \Big((\boldsymbol{f}(t), \boldsymbol{v}) + (\boldsymbol{g}(t), \boldsymbol{v})_{\Gamma_N} \Big)\, dt$$
$$+ \begin{pmatrix} \theta_1(t_{n-1}) \\ \theta_2(t_{n-1}) \end{pmatrix} (\varrho \boldsymbol{W}_{n-1}^-, \boldsymbol{v}) - \mathsf{M}\mathsf{A}^{-1}\begin{pmatrix} \theta_1(t_{n-1}) \\ \theta_2(t_{n-1}) \end{pmatrix} a(\varphi_0 \boldsymbol{U}_{n-1}^-, \boldsymbol{v}).$$

Furthermore, by coercivity, we can simplify

$$\begin{pmatrix} a(\varphi_0 \boldsymbol{U}_1, \boldsymbol{v}) \\ a(\varphi_0 \boldsymbol{U}_2, \boldsymbol{v}) \end{pmatrix} = \mathsf{A}^{-1}\mathsf{M}\begin{pmatrix} a(\varphi_0 \boldsymbol{W}_1, \boldsymbol{v}) \\ a(\varphi_0 \boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \mathsf{A}^{-1}\begin{pmatrix} \theta_1(t_{n-1}) \\ \theta_2(t_{n-1}) \end{pmatrix} a(\varphi_0 \boldsymbol{U}_{n-1}^-, \boldsymbol{v}),$$

to

$$\begin{pmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{pmatrix} = \mathsf{A}^{-1}\mathsf{M}\begin{pmatrix} \boldsymbol{W}_1 \\ \boldsymbol{W}_2 \end{pmatrix} + \mathsf{A}^{-1}\begin{pmatrix} \theta_1(t_{n-1})\boldsymbol{U}_{n-1}^- \\ \theta_2(t_{n-1})\boldsymbol{U}_{n-1}^- \end{pmatrix}.$$

In a similar way, by choosing $\boldsymbol{\xi}_q = \boldsymbol{v}\theta_i(t)$ in (21) we obtain,

$$(\tau_q\mathsf{A} + \mathsf{M})\begin{pmatrix} a(\boldsymbol{Z}_{q,1}, \boldsymbol{v}) \\ a(\boldsymbol{Z}_{q,2}, \boldsymbol{v}) \end{pmatrix} = \beta_q\mathsf{M}\begin{pmatrix} a(\boldsymbol{W}_1, \boldsymbol{v}) \\ a(\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \tau_q a(\boldsymbol{Z}_{q,n-1}^-, \boldsymbol{v})\begin{pmatrix} \theta_1(t_{n-1}) \\ \theta_2(t_{n-1}) \end{pmatrix}$$

which, again by virtue of the coercivity, simplifies to,

$$(\tau_q\mathsf{A} + \mathsf{M})\begin{pmatrix} \boldsymbol{Z}_{q,1} \\ \boldsymbol{Z}_{q,2} \end{pmatrix} = \beta_q\mathsf{M}\begin{pmatrix} \boldsymbol{W}_1 \\ \boldsymbol{W}_2 \end{pmatrix} + \tau_q\boldsymbol{Z}_{q,n-1}^-\begin{pmatrix} \theta_1(t_{n-1}) \\ \theta_2(t_{n-1}) \end{pmatrix}.$$

To make progress we choose the specific forms $\theta_1(t) = 1$ and $\theta_2(t) = (t_n - t)/k$ and then obtain easily that $\mathsf{A} = \frac{1}{2}\left(\begin{smallmatrix} 2 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and $\mathsf{M} = \frac{k}{6}\left(\begin{smallmatrix} 6 & 3 \\ 3 & 2 \end{smallmatrix}\right)$. Moreover $\mathsf{A}^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -2 & 2 \end{smallmatrix}\right)$ and $\mathsf{M}^{-1} = \frac{1}{k}\left(\begin{smallmatrix} 4 & -6 \\ -6 & 12 \end{smallmatrix}\right)$ and, further, $\mathsf{M}^{-1}\mathsf{A} = \frac{1}{k}\left(\begin{smallmatrix} -2 & -3 \\ 6 & 6 \end{smallmatrix}\right)$, $\mathsf{M}\mathsf{A}^{-1} = \frac{k}{6}\left(\begin{smallmatrix} 0 & 6 \\ -1 & 4 \end{smallmatrix}\right)$, $\mathsf{A}^{-1}\mathsf{M} = \frac{k}{6}\left(\begin{smallmatrix} 6 & 3 \\ -6 & -2 \end{smallmatrix}\right)$ and $\mathsf{M}\mathsf{A}^{-1}\mathsf{M} = \frac{k^2}{36}\left(\begin{smallmatrix} 18 & 12 \\ 6 & 5 \end{smallmatrix}\right)$. For the boundary conditions notice that $\boldsymbol{W}_1 = \boldsymbol{W}(t_n)$ and $\boldsymbol{W}_2 = \boldsymbol{W}(t_{n-1}) - \boldsymbol{W}(t_n)$. Compare it to [17], which is different.

Therefore,

$$\begin{aligned} \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} (\varrho\boldsymbol{W}_1, \boldsymbol{v}) \\ (\varrho\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} &+ \frac{k^2}{36}\begin{pmatrix} 18 & 12 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} a(\varphi_0\boldsymbol{W}_1, \boldsymbol{v}) \\ a(\varphi_0\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} \\ &+ \frac{\gamma_M k}{6}\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} (\varrho\boldsymbol{W}_1, \boldsymbol{v}) \\ (\varrho\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \frac{\gamma_E k}{6}\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} a(\boldsymbol{W}_1, \boldsymbol{v}) \\ a(\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} \\ + \sum_{q=1}^{N_\varphi}\frac{\beta_q k}{6}\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} a(\boldsymbol{Z}_{q,1}, \boldsymbol{v}) \\ a(\boldsymbol{Z}_{q,2}, \boldsymbol{v}) \end{pmatrix} &= \int_{t_{n-1}}^{t_n}\begin{pmatrix} 1 \\ (t_n - t)/k \end{pmatrix}\Big((\boldsymbol{f}(t), \boldsymbol{v}) + (\boldsymbol{g}(t), \boldsymbol{v})_{\Gamma_N}\Big)\,dt \\ &\quad + \begin{pmatrix} 1 \\ 1 \end{pmatrix}(\varrho\boldsymbol{W}_{n-1}^-, \boldsymbol{v}) - k\begin{pmatrix} 1 \\ 1/2 \end{pmatrix}a(\varphi_0\boldsymbol{U}_{n-1}^-, \boldsymbol{v}) \end{aligned}$$

or, slightly simpler,

$$\begin{aligned} \begin{pmatrix} 6 & 0 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} (\varrho\boldsymbol{W}_1, \boldsymbol{v}) \\ (\varrho\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} &+ k^2\begin{pmatrix} 3 & 2 \\ 1 & 5/6 \end{pmatrix}\begin{pmatrix} a(\varphi_0\boldsymbol{W}_1, \boldsymbol{v}) \\ a(\varphi_0\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} \\ &+ \gamma_M k\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} (\varrho\boldsymbol{W}_1, \boldsymbol{v}) \\ (\varrho\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \gamma_E k\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} a(\boldsymbol{W}_1, \boldsymbol{v}) \\ a(\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} \\ + \sum_{q=1}^{N_\varphi}\beta_q k\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} a(\boldsymbol{Z}_{q,1}, \boldsymbol{v}) \\ a(\boldsymbol{Z}_{q,2}, \boldsymbol{v}) \end{pmatrix} &= \int_{t_{n-1}}^{t_n}\begin{pmatrix} 6 \\ 6(t_n - t)/k \end{pmatrix}\Big((\boldsymbol{f}(t), \boldsymbol{v}) + (\boldsymbol{g}(t), \boldsymbol{v})_{\Gamma_N}\Big)\,dt \\ &\quad + \begin{pmatrix} 6 \\ 6 \end{pmatrix}(\varrho\boldsymbol{W}_{n-1}^-, \boldsymbol{v}) - k\begin{pmatrix} 6 \\ 3 \end{pmatrix}a(\varphi_0\boldsymbol{U}_{n-1}^-, \boldsymbol{v}). \end{aligned}$$

Also,

$$\begin{pmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{pmatrix} = \mathsf{A}^{-1}\mathsf{M}\begin{pmatrix} \boldsymbol{W}_1 \\ \boldsymbol{W}_2 \end{pmatrix} + \mathsf{A}^{-1}\begin{pmatrix} \theta_1(t_{n-1})\boldsymbol{U}_{n-1}^- \\ \theta_2(t_{n-1})\boldsymbol{U}_{n-1}^- \end{pmatrix}$$

becomes

$$\begin{pmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{pmatrix} = \frac{k}{6}\begin{pmatrix} 6 & 3 \\ -6 & -2 \end{pmatrix}\begin{pmatrix} \boldsymbol{W}_1 \\ \boldsymbol{W}_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} \boldsymbol{U}_{n-1}^- \\ \boldsymbol{U}_{n-1}^- \end{pmatrix}$$

which is,

$$\boldsymbol{U}_1 = k\boldsymbol{W}_1 + \frac{k}{2}\boldsymbol{W}_2 + \boldsymbol{U}^-_{n-1} \qquad \text{and} \qquad \boldsymbol{U}_2 = -k\boldsymbol{W}_1 - \frac{k}{3}\boldsymbol{W}_2. \tag{37}$$

Recalling from earlier that,

$$(\tau_q \mathsf{A} + \mathsf{M}) \begin{pmatrix} \boldsymbol{Z}_{q,1} \\ \boldsymbol{Z}_{q,2} \end{pmatrix} = \beta_q \mathsf{M} \begin{pmatrix} \boldsymbol{W}_1 \\ \boldsymbol{W}_2 \end{pmatrix} + \tau_q \boldsymbol{Z}^-_{q,n-1} \begin{pmatrix} \theta_1(t_{n-1}) \\ \theta_2(t_{n-1}) \end{pmatrix}$$

we can now calculate,

$$(\tau_q \mathsf{A} + \mathsf{M})^{-1} = \frac{2}{6\tau_q^2 + 4k\tau_q + k^2} \begin{pmatrix} 3\tau_q + 2k & -3k \\ -6\tau_q - 3k & 6\tau_q + 6k \end{pmatrix},$$

and conclude that

$$\begin{pmatrix} \boldsymbol{Z}_{q,1} \\ \boldsymbol{Z}_{q,2} \end{pmatrix} = kd_q\beta_q \begin{pmatrix} 6\tau_q + k & 3\tau_q \\ -6\tau_q & k - 2\tau_q \end{pmatrix} \begin{pmatrix} \boldsymbol{W}_1 \\ \boldsymbol{W}_2 \end{pmatrix} + 2d_q\tau_q \boldsymbol{Z}^-_{q,n-1} \begin{pmatrix} 3\tau_q - k \\ 3k \end{pmatrix} \tag{38}$$

for $d_q = (6\tau_q^2 + 4k\tau_q + k^2)^{-1}$. The adjustment to the momentum equation follows by observing that,

$$\begin{aligned} \beta_q k \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} a(\boldsymbol{Z}_{q,1}, \boldsymbol{v}) \\ a(\boldsymbol{Z}_{q,2}, \boldsymbol{v}) \end{pmatrix} &= 2kd_q\beta_q\tau_q \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 3\tau_q - k \\ 3k \end{pmatrix} a(\boldsymbol{Z}^-_{q,n-1}, \boldsymbol{v}) \\ &\quad + d_q k^2 \beta_q^2 \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} a(3\tau_q \boldsymbol{W}_2 + (6\tau_q + k)\boldsymbol{W}_1, \boldsymbol{v}) \\ a((k - 2\tau_q)\boldsymbol{W}_2 - 6\tau_q \boldsymbol{W}_1, \boldsymbol{v}) \end{pmatrix} \end{aligned}$$

or, on simplifying,

$$\begin{aligned} \beta_q k \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} a(\boldsymbol{Z}_{q,1}, \boldsymbol{v}) \\ a(\boldsymbol{Z}_{q,2}, \boldsymbol{v}) \end{pmatrix} &= 6kd_q\beta_q\tau_q \begin{pmatrix} 6\tau_q + k \\ 3\tau_q + k \end{pmatrix} a(\boldsymbol{Z}^-_{q,n-1}, \boldsymbol{v}) \\ &\quad + d_q k^2 \beta_q^2 \begin{pmatrix} a(6(3\tau_q + k)\boldsymbol{W}_1 + 3(4\tau_q + k)\boldsymbol{W}_2, \boldsymbol{v}) \\ a(3(2\tau_q + k)\boldsymbol{W}_1 + (5\tau_q + 2k)\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix}. \end{aligned}$$

Finally then we collect results and summarise the solution algorithm. The momentum equation with these substitutions is,

$$\begin{aligned} &\begin{pmatrix} 6 & 0 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} (\varrho \boldsymbol{W}_1, \boldsymbol{v}) \\ (\varrho \boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + k^2 \begin{pmatrix} 3 & 2 \\ 1 & 5/6 \end{pmatrix} \begin{pmatrix} a(\varphi_0 \boldsymbol{W}_1, \boldsymbol{v}) \\ a(\varphi_0 \boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} \\ &\qquad + \gamma_M k \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} (\varrho \boldsymbol{W}_1, \boldsymbol{v}) \\ (\varrho \boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} + \gamma_E k \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} a(\boldsymbol{W}_1, \boldsymbol{v}) \\ a(\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} \\ &\qquad + \sum_{q=1}^{N_\varphi} d_q k^2 \beta_q^2 \begin{pmatrix} a(6(3\tau_q + k)\boldsymbol{W}_1 + 3(4\tau_q + k)\boldsymbol{W}_2, \boldsymbol{v}) \\ a(3(2\tau_q + k)\boldsymbol{W}_1 + (5\tau_q + 2k)\boldsymbol{W}_2, \boldsymbol{v}) \end{pmatrix} \\ &= \int_{t_{n-1}}^{t_n} \begin{pmatrix} 6 \\ 6(t_n - t)/k \end{pmatrix} \Big((\boldsymbol{f}(t), \boldsymbol{v}) + (\boldsymbol{g}(t), \boldsymbol{v})_{\Gamma_N} \Big)\, dt + \begin{pmatrix} 6 \\ 6 \end{pmatrix} (\varrho \boldsymbol{W}^-_{n-1}, \boldsymbol{v}) \\ &\qquad - k \begin{pmatrix} 6 \\ 3 \end{pmatrix} a(\varphi_0 \boldsymbol{U}^-_{n-1}, \boldsymbol{v}) - \sum_{q=1}^{N_\varphi} 6kd_q\beta_q\tau_q \begin{pmatrix} 6\tau_q + k \\ 3\tau_q + k \end{pmatrix} a(\boldsymbol{Z}^-_{q,n-1}, \boldsymbol{v}). \end{aligned}$$

We then update with (37) and (38).

Let's collect like terms together, first step,

$$\begin{aligned}
&\begin{pmatrix} 6+6\gamma_M k & 0+3\gamma_M k \\ 6+3\gamma_M k & 3+2\gamma_M k \end{pmatrix}\begin{pmatrix} (\varrho\boldsymbol{W}_1,\boldsymbol{v}) \\ (\varrho\boldsymbol{W}_2,\boldsymbol{v}) \end{pmatrix} \\
&\qquad + k^2\begin{pmatrix} 3\varphi_0+6\gamma_E k^{-1} & 2\varphi_0+3\gamma_E k^{-1} \\ \varphi_0+3\gamma_E k^{-1} & 5\varphi_0/6+2\gamma_E k^{-1} \end{pmatrix}\begin{pmatrix} a(\boldsymbol{W}_1,\boldsymbol{v}) \\ a(\boldsymbol{W}_2,\boldsymbol{v}) \end{pmatrix} \\
&\qquad + \sum_{q=1}^{N_\varphi} d_q k^2\beta_q^2\begin{pmatrix} 6(3\tau_q+k) & 3(4\tau_q+k) \\ 3(2\tau_q+k) & (5\tau_q+2k) \end{pmatrix}\begin{pmatrix} a(\boldsymbol{W}_1,\boldsymbol{v}) \\ a(\boldsymbol{W}_2,\boldsymbol{v}) \end{pmatrix} \\
&= \int_{t_{n-1}}^{t_n}\begin{pmatrix} 6 \\ 6(t_n-t)/k \end{pmatrix}\Big((\boldsymbol{f}(t),\boldsymbol{v})+(\boldsymbol{g}(t),\boldsymbol{v})_{\Gamma_N}\Big)\,dt + \begin{pmatrix} 6 \\ 6 \end{pmatrix}(\varrho\boldsymbol{W}_{n-1}^-,\boldsymbol{v}) \\
&\qquad - k\begin{pmatrix} 6 \\ 3 \end{pmatrix}a(\varphi_0\boldsymbol{U}_{n-1}^-,\boldsymbol{v}) - \sum_{q=1}^{N_\varphi} 6kd_q\beta_q\tau_q\begin{pmatrix} 6\tau_q+k \\ 3\tau_q+k \end{pmatrix}a(\boldsymbol{Z}_{q,n-1}^-,\boldsymbol{v}).
\end{aligned}$$

Second step,

$$\begin{aligned}
&k^2\left[\begin{pmatrix} 3\varphi_0+6\gamma_E k^{-1} & 2\varphi_0+3\gamma_E k^{-1} \\ \varphi_0+3\gamma_E k^{-1} & 5\varphi_0/6+2\gamma_E k^{-1} \end{pmatrix} + \sum_{q=1}^{N_\varphi} d_q\beta_q^2\begin{pmatrix} 6(3\tau_q+k) & 3(4\tau_q+k) \\ 3(2\tau_q+k) & (5\tau_q+2k) \end{pmatrix}\right] \\
&\qquad \times\begin{pmatrix} a(\boldsymbol{W}_1,\boldsymbol{v}) \\ a(\boldsymbol{W}_2,\boldsymbol{v}) \end{pmatrix} + \begin{pmatrix} 6+6\gamma_M k & 0+3\gamma_M k \\ 6+3\gamma_M k & 3+2\gamma_M k \end{pmatrix}\begin{pmatrix} (\varrho\boldsymbol{W}_1,\boldsymbol{v}) \\ (\varrho\boldsymbol{W}_2,\boldsymbol{v}) \end{pmatrix} \\
&= \int_{t_{n-1}}^{t_n}\begin{pmatrix} 6 \\ 6(t_n-t)/k \end{pmatrix}\Big((\boldsymbol{f}(t),\boldsymbol{v})+(\boldsymbol{g}(t),\boldsymbol{v})_{\Gamma_N}\Big)\,dt + \begin{pmatrix} 6 \\ 6 \end{pmatrix}(\varrho\boldsymbol{W}_{n-1}^-,\boldsymbol{v}) \\
&\qquad - k\begin{pmatrix} 6 \\ 3 \end{pmatrix}a(\varphi_0\boldsymbol{U}_{n-1}^-,\boldsymbol{v}) - \sum_{q=1}^{N_\varphi} 6kd_q\beta_q\tau_q\begin{pmatrix} 6\tau_q+k \\ 3\tau_q+k \end{pmatrix}a(\boldsymbol{Z}_{q,n-1}^-,\boldsymbol{v}).
\end{aligned}$$

B Numerical results

In this section we provide full details of the results quoted in Section 5, as well as of some simpler example computations.

The algorithm has been implemented in the FEniCS environment, see Logg *et al.* in [19] and fenicsproject.org. The codes was developed in a virtual box installation of 64 bit Linux Mint 17.3 using the FEniCS environment with dolfin version 1.6.0.

The actual results were computed in a 64 bit bare metal Mint 18.1 ('Serena') FEniCS installation with dolfin version 2016.2.0, on a Dell xps15z laptop with 2 x 4096MB 1333MHz DDR dual channel RAM and 2nd Gen Intel Core i7-2620M (2.7GHz, 4threads, 4MB cache).

B.1 Convergence tests

To verify that the observed convergence rates agree with those stated in Theorem 4.7 we need to manufacture an exact solution and choose set the data consistent with that solution. We start by defining $\Omega := (0,1)^2$, the unit square, and consider the exact solution to be of the form,

$$\boldsymbol{u} = \bar{\boldsymbol{u}}(\boldsymbol{x})\mathcal{T}(t) \qquad \text{for} \qquad \bar{\boldsymbol{u}}(\boldsymbol{x}) := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 16(x^2-x)(y^2-y)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and where $\mathcal{T}(t) = t + B\cos(t)$ for a constant B. Then $\boldsymbol{w} = \bar{\boldsymbol{u}}\mathcal{T}'(t)$ and we see that $\boldsymbol{u}$ satisfies the requirements of Theorem 4.7.

We consider the material to be isotropic, homogeneous and synchronous (as discussed in the introduction) then, on using (2) with (4) and the assumption of Rayleigh damping as in (14), we obtain (with $\boldsymbol{x}$-dependence suppressed),

$$\begin{aligned}
\sigma_{ij} &= \gamma_E\lambda\nabla\cdot\boldsymbol{w}(t)\delta_{ij} + 2\gamma_E\mu\varepsilon_{ij}(\boldsymbol{u}(t)) + \lambda\nabla\cdot\boldsymbol{u}(t)\delta_{ij} + 2\mu\varepsilon_{ij}(\boldsymbol{u}(t)) \\
&\qquad - \int_0^t \varphi_s(t-s)\Big(\lambda\nabla\cdot\boldsymbol{u}(s)\delta_{ij} + 2\mu\varepsilon_{ij}(\boldsymbol{u}(s))\Big)\,ds, \\
&= \Big(\lambda\nabla\cdot\bar{\boldsymbol{u}}\mathcal{T}(t)\delta_{ij} + 2\mu\varepsilon_{ij}(\bar{\boldsymbol{u}})\mathcal{T}(t) + \gamma_E\lambda\nabla\cdot\bar{\boldsymbol{u}}\mathcal{T}'(t)\delta_{ij} + 2\gamma_E\mu\varepsilon_{ij}(\bar{\boldsymbol{u}})\mathcal{T}'(t)\Big) \\
&\qquad - \int_0^t \varphi_s(t-s)\Big(\lambda\nabla\cdot\bar{\boldsymbol{u}}\mathcal{T}(s)\delta_{ij} + 2\mu\varepsilon_{ij}(\bar{\boldsymbol{u}})\mathcal{T}(s)\Big)\,ds, \\
&= \Big(\lambda\nabla\cdot\bar{\boldsymbol{u}}\delta_{ij} + 2\mu\varepsilon_{ij}(\bar{\boldsymbol{u}})\Big)\Big(\mathcal{T}(t) + \gamma_E\mathcal{T}'(t) - \int_0^t \varphi_s(t-s)\mathcal{T}(s)\,ds\Big).
\end{aligned}$$

From the strong form of the problem we therefore need the body forces to be,

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \Big(\mathcal{T}'' + \gamma_M\mathcal{T}'\Big)\varrho\bar{\boldsymbol{u}} - \nabla\cdot\underline{\boldsymbol{D}}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})\Big(\mathcal{T}(t) + \gamma_E\mathcal{T}'(t) - \int_0^t \varphi_s(t-s)\mathcal{T}(s)\,ds\Big).$$

Noting that

$$\begin{aligned}
\varepsilon_{11}(\bar{\boldsymbol{u}}) &= 16(2x-1)(y^2-y) \\
\varepsilon_{22}(\bar{\boldsymbol{u}}) &= 16(2y-1)(x^2-x) \\
\varepsilon_{12}(\bar{\boldsymbol{u}}) &= \frac{16}{2}\big((x^2-x)(2y-1) + (2x-1)(y^2-y)\big)
\end{aligned}$$

we get the divergence of $\sigma_{ij} = D_{ijkl}\varepsilon_{kl}(\bar{\boldsymbol{u}}) = \lambda\nabla\cdot\bar{\boldsymbol{u}}\delta_{ij} + 2\mu\varepsilon_{ij}(\bar{\boldsymbol{u}})$ to be given by,

$$\nabla\cdot\underline{\boldsymbol{D}}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) = 16\begin{pmatrix} 2(\lambda+2\mu)(y-y^2) - (\lambda+\mu)(1-2x)(1-2y) + 2\mu(x-x^2) \\ 2(\lambda+2\mu)(x-x^2) - (\lambda+\mu)(2x-1)(2y-1) + 2\mu(y-y^2). \end{pmatrix}$$

The data are $T = 12\pi$, $\varrho = 1$, $\lambda = 1$ and $\mu = 1$ for all twelve of the following examples. In examples 1,2,3,4 we set $\gamma_M = \gamma_E = 0$ and switch the viscoelasticity off by setting $\varphi_0 = 1$. These examples are then repeated in examples 5,6,7,8 but with $\gamma_M = 2$ and $\gamma_E = 1$, and these are then repeated in examples 9,10,11,12 but with $N_\varphi = 2$ and $(\varphi_1, \tau_1) = (0.35, 0.1)$ and $(\varphi_2, \tau_2) = (0.15, 0.05)$.

In each example we use an $N_{xy} \times N_{xy}$ mesh of isoceles triangles with piecewise linear elements in space-time. The number of time steps varies according to the example.

Remark B.1 *The original runs used the option* `-W 5` *(bicgstab). Examples 11 (for* `Nxy` *values 362, 512) and 6 (for* `-Nxy` *724) showed some poor numerical results. The command line was changed to* `-W 3` *(gmres with ilu) for Example 11 so that the quoted data are for the runs:*

```
time ./solver.py -v 20 -r 1 -R 4 -X 11 -U 1 -W 3 --Nxy 362 \
      -C '-W 3 not -W 5' | tee -a newresults362.txt
time ./solver.py -v 20 -r 1 -R 4 -X 11 -U 1 -W 3 --Nxy 512 \
      -C '-W 3 not -W 5' | tee -a newresults512.txt
```

(*The* `-C` *option is a new feature on a modified FEniCS code*).

For Example 6 (*with* `-Nxy 724`) *the run was terminated early. It took over 160 hours to get just 25% through the time stepping. Instead Example 6 was run as:*

```
time ./solver.py -v 20 -r 1 -R 4 -X 6 -U 1 -W 5 --Nxy 724 \
      -C '-W 5 with both tols 1e-10' | tee -a \
                  ~/Dropbox/AccessAndShare/output724.txt
```

with (*after 6.5 days*) *the eventual output stored in* `newresultrs724.txt` *but with the following additions to the FEniCS solver:*

```
prm = solver.parameters
prm['linear_solver'] = 'bicgstab' # appears best for memory
# new - testing for bad 724 result: successful so retain for the future
prm['krylov_solver']['absolute_tolerance']  = 1e-10     # default 1e-9 ?
prm['krylov_solver']['relative_tolerance']  = 1e-10     # default 1e-7 ?
```

in the `elif Wmethod == 5:` *clause. This has been retained as a permanent edit. It is possible that this will correct the Example 11 anomaly also.*

The next runs for `-Nxy 1024` *were carried out using docker on heron12. The docker image was pulled on 3 Oct 2017 with*

```
docker pull quay.io/fenicsproject/stable:latest
```

and the run was executed with (*for example 6, as discussed below*):

```
mpirun -np 10 ./solver.py -v 20 -i 100 -r 1 \
                  -R 4 -X 6 -U 1 -W 0 --Nxy 1024|
```

because any other choice for `-W ...` *to compare with those above caused an MPI or PetSc crash (no idea why). The results are in* `extra_runs`*. Here the* `-i 100` *option caused an update to be written every 1% of time stepping progress.*

Example 6, as just described above for `-Nxy 724` *began time stepping around* `03/09/2017` *at* `07:23:04` *and finished around* `08/09/2017` *at* `04:08:42` (*these are taken from the output file* `newresults724.txt`)*. That's just under five days without the Initial Condition calculations.*

On the other hand, Example 6 with `-Nxy 1024` *began time stepping on* `24/10/2017` *at* `13:58:10` *and finished on* `26/10/2017` *at* `20:16:27`*. That's about 2¼ days, or twice as fast as the non-MPI* `-Nxy 724` *run.*

`-Nxy 1448` *was run for examples 1 to 12. It began on Sun Oct 29 15:47:56 UTC 2017 and finished around 20/11/2017 at 22:14:58. It was run with command lines of the form (for* `-X 12` *backwards to* `-X 1`*):*

```
mpirun -np 10 ./solver.py -v 20 -i 100 -r 1 \
                  -R 4 -X 12 -U 1 -W 0 --Nxy 1448
```

This from the email exchange with Asif regarding the Heron cluster.

Hi Simon,

Please find below the detail specification of the servers and their total cost.

```
1 x PowerEdge R730xd               2 x PowerEdge FX2s
2 x E5-2640v4 RAM                   12 x FC430 quarter width server
8 x 16GB                           2 x E5 - 2640 v4
2 x SSD 200GB (RAID 1)               8 x 8GB
12 x 2TB (RAID 10)                   1 x SSD 200GB
RAID 0, 1, 5, 6 supported           Dual Port 1GbE
Dual Port 10GbE + Dual Port 1GbE In FX2 chassis { 2 x 2000Watt PSU Chassis
2 x 750W PSU Chassis               Rackmount chassis with rails
Rackmount chassis with rails       5 year 9x5 NBD
5 year 9x5 NBD                       iDRAC 8 Enterprise on each FC430 leading to CMC controll
IDRAC8 Enterprise OOBs controller
```

ex VAT £39,671.85
inc VAT £47,606.22

If you require any further information please let me know.

Kind Regards,
Asif

Thanks Asif,
Was it one of each? I may be reading it incorrectly ..
Simon

Hi Simon,

R730xd is the data/head node and we purchased one of it.
PowerEdge FX2s are the blade enclosures and we purchased two of those.
Each FX2s can have eight blade servers in it
but we evenly distributed the twelve FC430
blade servers between the two chassis.
Each blade has two E5-2640 v4 processors and 64GB RAM.
If you require any further information please let me know.

Kind Regards,
Asif

This needs to be filed somewhere for grant use. Here is my summary:
inc VAT £47,606.22 worth of gear in the heron cluster.
One PowerEdge R730xd head node with 2 x E5-2640v4 processors, 8 x 16GB RAM
and 2 x SSD 200GB (RAID 1), 12 x 2TB (RAID 10) with RAID 0, 1, 5, 6 supported

N_{xy}	TEe	KEe	ESe	$\|\boldsymbol{e}_u(T)\|_{\boldsymbol{H}^1(\Omega)}$	$\|\boldsymbol{e}_w(T)\|_{\boldsymbol{H}^1(\Omega)}$
Nxy	En	L2w	SEu	H1u	H1w
2	1.236e+02	3.448e-01	1.236e+02	9.359e+01	2.483e+00
3	8.659e+01	1.668e-01	8.659e+01	6.618e+01	1.755e+00
4	6.603e+01	9.653e-02	6.603e+01	5.063e+01	1.343e+00
6	4.454e+01	4.376e-02	4.454e+01	3.423e+01	9.080e-01
8	3.354e+01	2.478e-02	3.354e+01	2.580e+01	6.843e-01
11	2.446e+01	1.316e-02	2.446e+01	1.882e+01	4.991e-01
16	1.684e+01	6.236e-03	1.684e+01	1.296e+01	3.437e-01
23	1.172e+01	3.021e-03	1.172e+01	9.022e+00	2.393e-01
32	8.427e+00	1.562e-03	8.427e+00	6.487e+00	1.721e-01
45	5.994e+00	7.899e-04	5.994e+00	4.614e+00	1.224e-01
64	4.215e+00	3.906e-04	4.215e+00	3.244e+00	8.606e-02
91	2.964e+00	1.932e-04	2.964e+00	2.282e+00	6.053e-02
128	2.107e+00	9.765e-05	2.107e+00	1.622e+00	4.303e-02
181	1.490e+00	4.884e-05	1.490e+00	1.147e+00	3.043e-02
256	1.054e+00	2.441e-05	1.054e+00	8.112e-01	2.152e-02
362	7.452e-01	1.221e-05	7.452e-01	5.736e-01	1.522e-02
512	5.269e-01	6.103e-06	5.269e-01	4.056e-01	1.076e-02
724	3.726e-01	3.052e-06	3.726e-01	2.868e-01	7.608e-03
1024	2.634e-01	1.526e-06	2.634e-01	2.028e-01	5.379e-03
1448	1.863e-01	7.631e-07	1.863e-01	1.434e-01	3.804e-03

Table 1: Results for Example 1.


```
Two PowerEdge FX2s blade enclosures with six FC430 blade servers in each (up
to a max of eight). each blade has 2 x E5 - 2640 v4 processors,
64Gb = 8 x 8GB RAM and 1 x SSD 200GB
```

Example 1 *Here we switch off the Rayleigh damping and viscoelasticity and choose* $B = 0$, *so that* $\varrho\ddot{\boldsymbol{u}} = \boldsymbol{0}$. *For the implementation we need the integrals of the load against the temporal basis functions. For our choice of manufactured solution here, these are given by*

$$6\int_{t_{n-1}}^{t_n} t\,dt = 6kt_{n-1/2}, \qquad \text{and} \qquad 6\int_{t_{n-1}}^{t_n} \frac{(t_n-t)t}{k}\,dt = (3t_n-2k)k.$$

We take $N_t = 4$ *time steps so that* $k = 1/4$ *and show the results are in Figure 5 and Table 1.*

Example 2 *We choose* $B = 1$ *so that the exact solution is*

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 16(x^2-x)(y^2-y) \\ 16(x-x^2)(y-y^2) \end{pmatrix} (t + B\cos(t)).$$

Hence,

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = -B \begin{pmatrix} 16(x^2-x)(y^2-y) \\ 16(x-x^2)(y-y^2) \end{pmatrix} \sin(t)$$

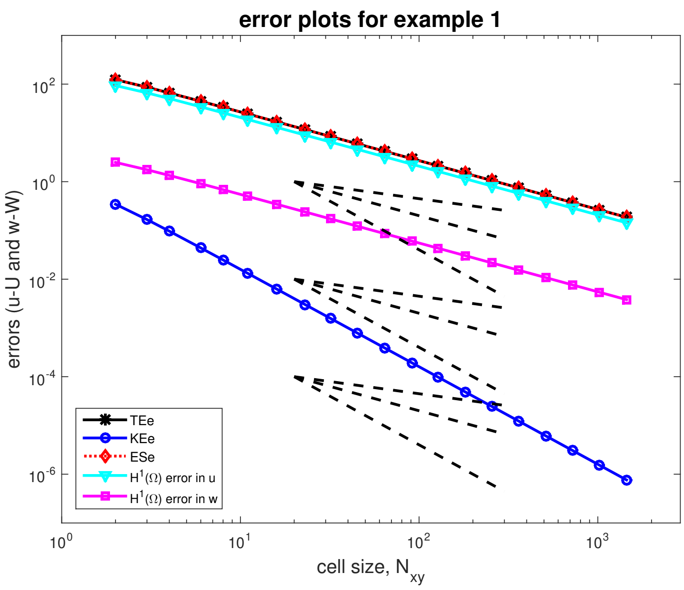

Figure 5: Errors for Example 1, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1, 2, 3$.

and

$$\varrho\ddot{\boldsymbol{u}} = -\varrho B \begin{pmatrix} 16(x^2 - x)(y^2 - y) \\ 16(x - x^2)(y - y^2) \end{pmatrix} \cos(t)$$

the assumption of homogeneous essential boundary conditions is still satisfied and, on using

$$\sigma_{ij} = \lambda \nabla \cdot \boldsymbol{u} \delta_{ij} + 2\mu\varepsilon_{ij}(\boldsymbol{u})$$

with

$$f_i = \varrho \ddot{u}_i - \sigma_{ij,j}$$

we get that

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = -\varrho B \cos(t) \begin{pmatrix} 16(x^2 - x)(y^2 - y) \\ 16(x - x^2)(y - y^2) \end{pmatrix}$$
$$+ 16\Big(t + B\cos(t)\Big) \begin{pmatrix} 2(\lambda + 2\mu)(y - y^2) - (\lambda + \mu)(1 - 2x)(1 - 2y) + 2\mu(x - x^2) \\ 2(\lambda + 2\mu)(x - x^2) - (\lambda + \mu)(2x - 1)(2y - 1) + 2\mu(y - y^2) \end{pmatrix}.$$

So, this time, for the implementation, the integrals of the load against the temporal basis functions are given for the contributions from the $\sigma_{ij,j}$ terms by,

$$6\int_{t_{n-1}}^{t_n} t + \cos(t)\,dt = 6kt_{n-1/2} + 6B\Big(\sin(t_n) - \sin(t_{n-k})\Big),$$

$$6\int_{t_{n-1}}^{t_n} \frac{(t_n - t)(t + B\cos(t))}{k}\,dt = (3t_n - 2k)k + \frac{6B}{k}\Big(\cos(t_{n-1}) - \cos(t_n) - k\sin(t_{n-1})\Big)$$

and for the contributions from $\varrho\ddot{\boldsymbol{u}}$ by

$$6\varrho\int_{t_{n-1}}^{t_n} -B\cos(t)\,dt = 6B\varrho\Big(\sin(t_{n-1}) - \sin(t_n)\Big),$$

$$6\varrho\int_{t_{n-1}}^{t_n} \frac{(t_n - t)(-B\cos(t))}{k}\,dt = \frac{6B\varrho}{k}\Big(k\sin(t_{n-1}) + \cos(t_n) - \cos(t_{n-1})\Big).$$

N_{xy}	TEe	KEe	ESe	$\|\boldsymbol{e}_u(T)\|_{\boldsymbol{H}^1(\Omega)}$	$\|\boldsymbol{e}_w(T)\|_{\boldsymbol{H}^1(\Omega)}$
Nxy	En	L2w	SEu	H1u	H1w
2	1.269e+02	3.147e-01	1.269e+02	9.608e+01	2.561e+00
3	8.888e+01	1.434e-01	8.888e+01	6.794e+01	1.789e+00
4	6.778e+01	8.002e-02	6.778e+01	5.197e+01	1.360e+00
6	4.572e+01	3.510e-02	4.572e+01	3.514e+01	9.140e-01
8	3.443e+01	1.947e-02	3.443e+01	2.648e+01	6.871e-01
11	2.510e+01	1.020e-02	2.510e+01	1.932e+01	5.003e-01
16	1.728e+01	4.826e-03	1.728e+01	1.330e+01	3.442e-01
23	1.203e+01	2.392e-03	1.203e+01	9.261e+00	2.395e-01
32	8.651e+00	1.283e-03	8.651e+00	6.659e+00	1.721e-01
45	6.153e+00	6.704e-04	6.153e+00	4.736e+00	1.224e-01
64	4.326e+00	3.375e-04	4.326e+00	3.330e+00	8.607e-02
91	3.043e+00	1.678e-04	3.043e+00	2.342e+00	6.053e-02
128	2.163e+00	8.526e-05	2.163e+00	1.665e+00	4.303e-02
181	1.530e+00	4.298e-05	1.530e+00	1.178e+00	3.043e-02
256	1.082e+00	2.165e-05	1.082e+00	8.327e-01	2.152e-02
362	7.649e-01	1.091e-05	7.649e-01	5.888e-01	1.522e-02
512	5.408e-01	5.485e-06	5.408e-01	4.163e-01	1.076e-02
724	3.825e-01	2.755e-06	3.825e-01	2.944e-01	7.608e-03
1024	2.704e-01	1.382e-06	2.704e-01	2.082e-01	5.379e-03
1448	1.912e-01	6.933e-07	1.912e-01	1.472e-01	3.804e-03

Table 2: Results for Example 2.

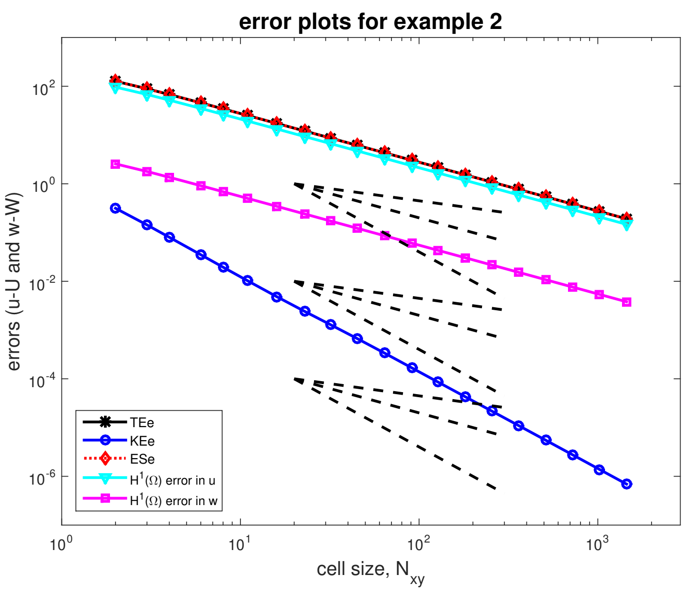

Figure 6: Errors for Example 2, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1, 2, 3$.

N_{xy}	TEe	KEe	ESe	$\|\boldsymbol{e}_u(T)\|_{\boldsymbol{H}^1(\Omega)}$	$\|\boldsymbol{e}_w(T)\|_{\boldsymbol{H}^1(\Omega)}$
Nxy	En	L2w	SEu	H1u	H1w
2	1.269e+02	2.997e-01	1.269e+02	9.608e+01	2.633e+00
3	8.888e+01	1.210e-01	8.888e+01	6.794e+01	1.867e+00
4	6.778e+01	6.166e-02	6.778e+01	5.198e+01	1.429e+00
6	4.572e+01	3.158e-02	4.572e+01	3.514e+01	9.652e-01
8	3.443e+01	2.614e-02	3.443e+01	2.648e+01	7.281e-01
11	2.510e+01	2.318e-02	2.510e+01	1.932e+01	5.334e-01
16	1.728e+01	1.900e-02	1.728e+01	1.330e+01	3.695e-01
23	1.203e+01	1.454e-02	1.203e+01	9.261e+00	2.583e-01
32	8.651e+00	1.148e-02	8.651e+00	6.659e+00	1.874e-01
45	6.153e+00	8.570e-03	6.153e+00	4.736e+00	1.342e-01
64	4.326e+00	6.424e-03	4.326e+00	3.330e+00	9.553e-02
91	3.043e+00	4.673e-03	3.043e+00	2.342e+00	6.792e-02
128	2.163e+00	3.459e-03	2.163e+00	1.665e+00	4.896e-02
181	1.530e+00	2.522e-03	1.530e+00	1.178e+00	3.508e-02
256	1.082e+00	1.881e-03	1.082e+00	8.327e-01	2.525e-02
362	7.649e-01	1.415e-03	7.649e-01	5.888e-01	1.824e-02
512	5.408e-01	1.062e-03	5.408e-01	4.163e-01	1.318e-02
724	3.825e-01	7.942e-04	3.825e-01	2.944e-01	9.545e-03
1024	2.704e-01	5.926e-04	2.704e-01	2.082e-01	6.927e-03
1448	1.912e-01	4.359e-04	1.912e-01	1.472e-01	5.019e-03

Table 3: Results for Example 3.

We take $k \sim h^{2/3}$ as described earlier for Example II.

The results are in Figure 6 and Table 2.

Example 3 *This is precisely the same as in Example 2 except here we take $k \sim h^{1/3}$ as described earlier for Example III. The results are in Figure 7 and Table 3.*

Example 4 *Repeat of Example 3 but with we take $k \sim h^{1/6}$ as described earlier for Example IV. The results are in Figure 8 and Table 4.*

Example 5 *This is a repeat of Example 1 but with $\gamma_M = 2$ and $\gamma_E = 1$. The results are in Figure 9 and Table 5.*

Example 6 *This is a repeat of Example 2 but with $\gamma_M = 2$ and $\gamma_E = 1$. The results are in Figure 10 and Table 6.*

Example 7 *This is a repeat of Example 3 but with $\gamma_M = 2$ and $\gamma_E = 1$. The results are in Figure 11 and Table 7.*

Example 8 *This is a repeat of Example 4 but with $\gamma_M = 2$ and $\gamma_E = 1$. The results are in Figure 12 and Table 8.*

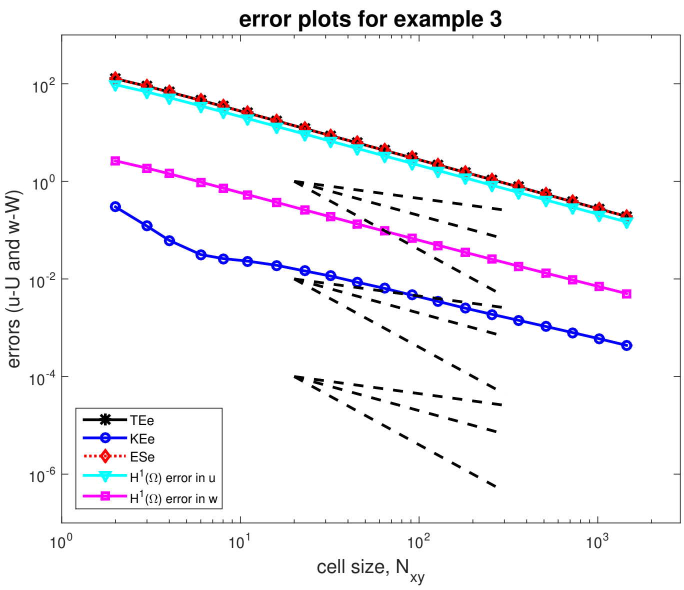

Figure 7: Errors for Example 3, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1, 2, 3$.

N_{xy}	TEe	KEe	ESe	$\|\boldsymbol{e}_u(T)\|_{\boldsymbol{H}^1(\Omega)}$	$\|\boldsymbol{e}_w(T)\|_{\boldsymbol{H}^1(\Omega)}$
Nxy	En	L2w	SEu	H1u	H1w
2	1.269e+02	2.924e-01	1.269e+02	9.609e+01	2.686e+00
3	8.888e+01	1.167e-01	8.888e+01	6.795e+01	1.952e+00
4	6.778e+01	7.896e-02	6.778e+01	5.198e+01	1.530e+00
6	4.572e+01	8.017e-02	4.572e+01	3.514e+01	1.087e+00
8	3.443e+01	7.940e-02	3.443e+01	2.648e+01	8.533e-01
11	2.510e+01	7.665e-02	2.510e+01	1.932e+01	6.655e-01
16	1.728e+01	7.227e-02	1.728e+01	1.330e+01	5.158e-01
23	1.203e+01	6.373e-02	1.203e+01	9.262e+00	4.065e-01
32	8.651e+00	5.575e-02	8.651e+00	6.659e+00	3.323e-01
45	6.153e+00	4.879e-02	6.153e+00	4.736e+00	2.766e-01
64	4.327e+00	4.278e-02	4.326e+00	3.331e+00	2.344e-01
91	3.043e+00	3.763e-02	3.043e+00	2.342e+00	2.023e-01
128	2.164e+00	3.221e-02	2.163e+00	1.665e+00	1.724e-01
181	1.530e+00	2.775e-02	1.530e+00	1.178e+00	1.487e-01
256	1.082e+00	2.406e-02	1.082e+00	8.327e-01	1.296e-01
362	7.653e-01	2.044e-02	7.650e-01	5.889e-01	1.112e-01
512	5.412e-01	1.750e-02	5.409e-01	4.164e-01	9.636e-02
724	3.828e-01	1.509e-02	3.825e-01	2.945e-01	8.417e-02
1024	2.708e-01	1.280e-02	2.705e-01	2.082e-01	7.256e-02
1448	1.916e-01	1.095e-02	1.913e-01	1.472e-01	6.307e-02

Table 4: Results for Example 4.

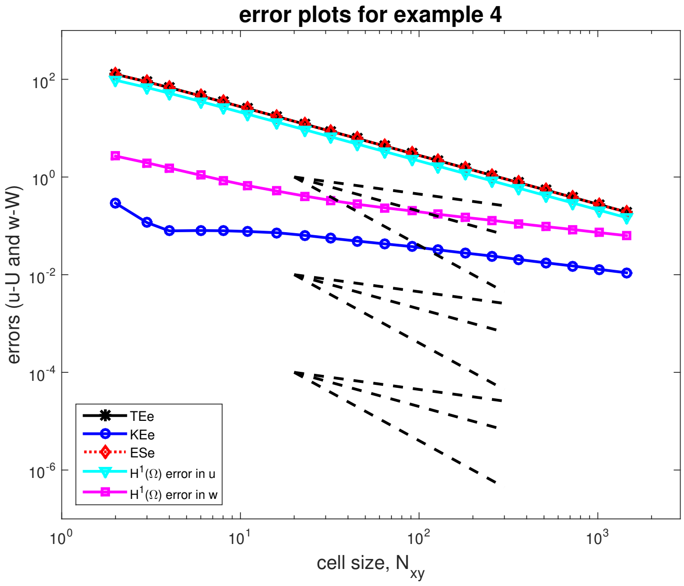

Figure 8: Errors for Example 4, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1, 2, 3$.

N_{xy}	TEe	KEe	ESe	$\|\boldsymbol{e}_u(T)\|_{\boldsymbol{H}^1(\Omega)}$	$\|\boldsymbol{e}_w(T)\|_{\boldsymbol{H}^1(\Omega)}$
Nxy	En	L2w	SEu	H1u	H1w
2	1.236e+02	3.448e-01	1.236e+02	9.360e+01	2.483e+00
3	8.659e+01	1.668e-01	8.659e+01	6.618e+01	1.755e+00
4	6.603e+01	9.653e-02	6.603e+01	5.063e+01	1.343e+00
6	4.454e+01	4.376e-02	4.454e+01	3.423e+01	9.080e-01
8	3.354e+01	2.478e-02	3.354e+01	2.580e+01	6.843e-01
11	2.446e+01	1.316e-02	2.446e+01	1.882e+01	4.991e-01
16	1.684e+01	6.236e-03	1.684e+01	1.296e+01	3.437e-01
23	1.172e+01	3.021e-03	1.172e+01	9.022e+00	2.393e-01
32	8.427e+00	1.562e-03	8.427e+00	6.487e+00	1.721e-01
45	5.994e+00	7.899e-04	5.994e+00	4.614e+00	1.224e-01
64	4.215e+00	3.906e-04	4.215e+00	3.244e+00	8.606e-02
91	2.964e+00	1.932e-04	2.964e+00	2.282e+00	6.053e-02
128	2.107e+00	9.765e-05	2.107e+00	1.622e+00	4.303e-02
181	1.490e+00	4.884e-05	1.490e+00	1.147e+00	3.043e-02
256	1.054e+00	2.441e-05	1.054e+00	8.112e-01	2.152e-02
362	7.452e-01	1.221e-05	7.452e-01	5.736e-01	1.522e-02
512	5.269e-01	6.104e-06	5.269e-01	4.056e-01	1.076e-02
724	3.726e-01	3.052e-06	3.726e-01	2.868e-01	7.608e-03
1024	2.634e-01	1.526e-06	2.634e-01	2.028e-01	5.379e-03
1448	1.863e-01	7.632e-07	1.863e-01	1.434e-01	3.804e-03

Table 5: Results for Example 5.

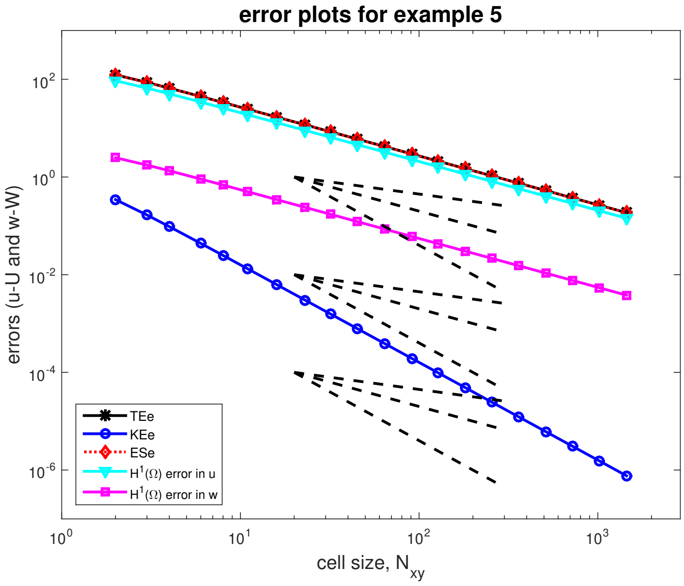

Figure 9: Errors for Example 5, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1, 2, 3$.

N_{xy}	TEe	KEe	ESe	$\|\boldsymbol{e}_u(T)\|_{\boldsymbol{H}^1(\Omega)}$	$\|\boldsymbol{e}_w(T)\|_{\boldsymbol{H}^1(\Omega)}$
Nxy	En	L2w	SEu	H1u	H1w
2	1.269e+02	3.390e-01	1.269e+02	9.609e+01	2.493e+00
3	8.888e+01	1.621e-01	8.888e+01	6.794e+01	1.760e+00
4	6.778e+01	9.293e-02	6.778e+01	5.198e+01	1.346e+00
6	4.572e+01	4.162e-02	4.572e+01	3.514e+01	9.090e-01
8	3.443e+01	2.333e-02	3.443e+01	2.648e+01	6.848e-01
11	2.510e+01	1.226e-02	2.510e+01	1.932e+01	4.994e-01
16	1.728e+01	5.727e-03	1.728e+01	1.330e+01	3.438e-01
23	1.203e+01	2.734e-03	1.203e+01	9.261e+00	2.393e-01
32	8.651e+00	1.395e-03	8.651e+00	6.659e+00	1.721e-01
45	6.153e+00	6.961e-04	6.153e+00	4.736e+00	1.224e-01
64	4.326e+00	3.397e-04	4.326e+00	3.330e+00	8.606e-02
91	3.043e+00	1.658e-04	3.043e+00	2.342e+00	6.053e-02
128	2.163e+00	8.276e-05	2.163e+00	1.665e+00	4.303e-02
181	1.530e+00	4.090e-05	1.530e+00	1.178e+00	3.043e-02
256	1.082e+00	2.022e-05	1.082e+00	8.327e-01	2.152e-02
362	7.649e-01	1.001e-05	7.649e-01	5.888e-01	1.522e-02
512	5.408e-01	4.956e-06	5.408e-01	4.163e-01	1.076e-02
724	3.825e-01	2.457e-06	3.825e-01	2.944e-01	7.608e-03
1024	2.704e-01	1.219e-06	2.704e-01	2.082e-01	5.379e-03
1448	1.912e-01	6.052e-07	1.912e-01	1.472e-01	3.804e-03

Table 6: Results for Example 6.

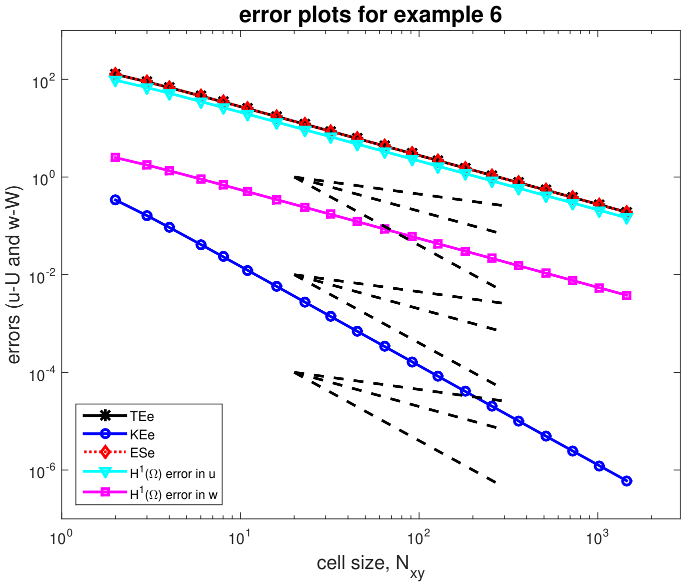

Figure 10: Errors for Example 6, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1, 2, 3$.

N_{xy}	TEe	KEe	ESe	$\|\boldsymbol{e}_u(T)\|_{\boldsymbol{H}^1(\Omega)}$	$\|\boldsymbol{e}_w(T)\|_{\boldsymbol{H}^1(\Omega)}$
Nxy	En	L2w	SEu	H1u	H1w
2	1.269e+02	3.329e-01	1.269e+02	9.609e+01	2.507e+00
3	8.888e+01	1.528e-01	8.888e+01	6.794e+01	1.773e+00
4	6.778e+01	8.314e-02	6.778e+01	5.198e+01	1.356e+00
6	4.572e+01	3.338e-02	4.572e+01	3.514e+01	9.160e-01
8	3.443e+01	1.681e-02	3.443e+01	2.648e+01	6.900e-01
11	2.510e+01	8.028e-03	2.510e+01	1.932e+01	5.033e-01
16	1.728e+01	4.660e-03	1.728e+01	1.330e+01	3.466e-01
23	1.203e+01	3.778e-03	1.203e+01	9.261e+00	2.413e-01
32	8.651e+00	3.354e-03	8.651e+00	6.659e+00	1.736e-01
45	6.153e+00	2.780e-03	6.153e+00	4.736e+00	1.235e-01
64	4.326e+00	2.272e-03	4.326e+00	3.330e+00	8.697e-02
91	3.043e+00	1.776e-03	3.043e+00	2.342e+00	6.123e-02
128	2.163e+00	1.388e-03	2.163e+00	1.665e+00	4.360e-02
181	1.530e+00	1.051e-03	1.530e+00	1.178e+00	3.087e-02
256	1.082e+00	7.979e-04	1.082e+00	8.327e-01	2.186e-02
362	7.649e-01	6.026e-04	7.649e-01	5.888e-01	1.549e-02
512	5.408e-01	4.509e-04	5.408e-01	4.163e-01	1.097e-02
724	3.825e-01	3.360e-04	3.825e-01	2.944e-01	7.778e-03
1024	2.704e-01	2.503e-04	2.704e-01	2.082e-01	5.513e-03
1448	1.912e-01	1.847e-04	1.912e-01	1.472e-01	3.907e-03

Table 7: Results for Example 7.

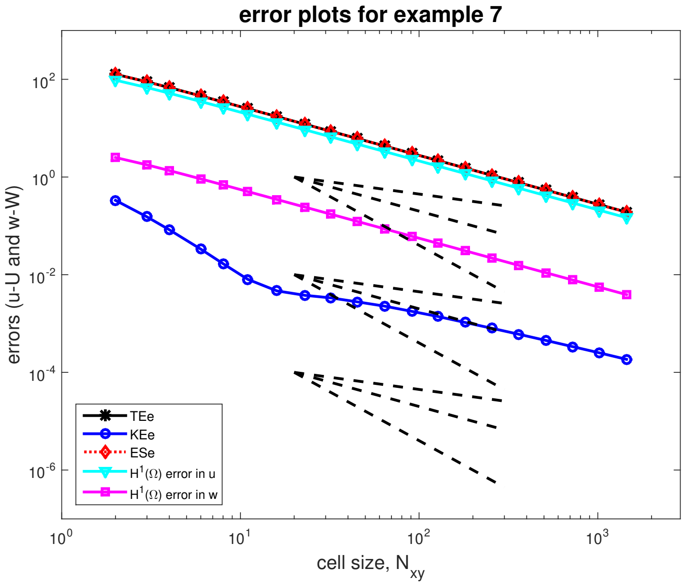

Figure 11: Errors for Example 7, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1, 2, 3$.

N_{xy}	TEe	KEe	ESe	$\|\boldsymbol{e}_u(T)\|_{\boldsymbol{H}^1(\Omega)}$	$\|\boldsymbol{e}_w(T)\|_{\boldsymbol{H}^1(\Omega)}$
Nxy	En	L2w	SEu	H1u	H1w
2	1.269e+02	3.288e-01	1.269e+02	9.609e+01	2.517e+00
3	8.888e+01	1.450e-01	8.888e+01	6.794e+01	1.787e+00
4	6.778e+01	7.458e-02	6.778e+01	5.198e+01	1.371e+00
6	4.572e+01	2.774e-02	4.572e+01	3.514e+01	9.315e-01
8	3.443e+01	1.811e-02	3.443e+01	2.648e+01	7.049e-01
11	2.510e+01	1.759e-02	2.510e+01	1.932e+01	5.182e-01
16	1.728e+01	1.878e-02	1.728e+01	1.330e+01	3.631e-01
23	1.203e+01	1.785e-02	1.203e+01	9.261e+00	2.584e-01
32	8.651e+00	1.630e-02	8.651e+00	6.659e+00	1.911e-01
45	6.153e+00	1.470e-02	6.153e+00	4.736e+00	1.419e-01
64	4.326e+00	1.318e-02	4.326e+00	3.330e+00	1.066e-01
91	3.043e+00	1.180e-02	3.043e+00	2.342e+00	8.208e-02
128	2.163e+00	1.027e-02	2.163e+00	1.665e+00	6.441e-02
181	1.530e+00	8.990e-03	1.530e+00	1.178e+00	5.169e-02
256	1.082e+00	7.915e-03	1.082e+00	8.327e-01	4.255e-02
362	7.650e-01	6.842e-03	7.649e-01	5.889e-01	3.517e-02
512	5.409e-01	5.960e-03	5.408e-01	4.163e-01	2.964e-02
724	3.825e-01	5.228e-03	3.825e-01	2.944e-01	2.539e-02
1024	2.705e-01	4.523e-03	2.704e-01	2.082e-01	2.165e-02
1448	1.913e-01	3.943e-03	1.912e-01	1.472e-01	1.868e-02

Table 8: Results for Example 8.

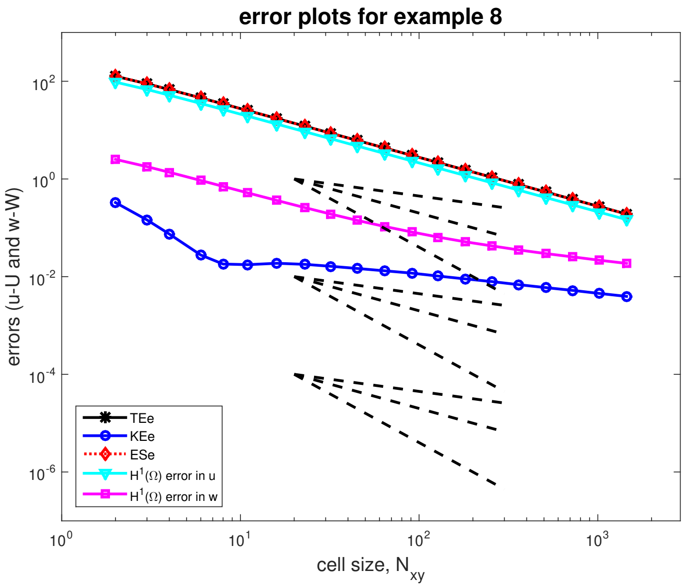

Figure 12: Errors for Example 8, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1, 2, 3$.

B.2 Including viscoelasticity

Sage Math (89 at, p...)

```
t,s,tau = var('t s tau'); assume(tau>0);
II = integrate(exp(-((t-s)/tau))/tau*(A*s+B*cos(s)),s,0,t);
II; latex(II)

(((tau^4 + tau^2)*A - B*tau)*e^(-t/tau)/(tau^2 + 1) ...
+ ((t*tau^3 - tau^4 + t*tau - tau^2)*A + (tau^2*sin(t) ...
+ tau*cos(t))*B)/(tau^2 + 1))/tau ...
\frac{
  \frac{{\left({\left(\tau^{4} + \tau^{2}\right)} A ...
          - B \tau\right)} e^{\left(-\frac{t}{\tau}\right)}}
  {\tau^{2} + 1} ...
  + \frac{{\left(t \tau^{3} - \tau^{4} + t \tau - \tau^{2}\right)} A ...
           + {\left(\tau^{2} \sin\left(t\right) + \tau \cos\left(t\right)\right)} B}
    {\tau^{2} + 1}
}{\tau}
```

In LaTeX 2_ε this is,

$$\frac{\frac{\left(\left(\tau^{4} + \tau^{2}\right) A - B \tau\right) e^{\left(-\frac{t}{\tau}\right)}}{\tau^{2} + 1} + \frac{\left(t \tau^{3} - \tau^{4} + t \tau - \tau^{2}\right) A + \left(\tau^{2} \sin\left(t\right) + \tau \cos\left(t\right)\right) B}{\tau^{2} + 1}}{\tau}$$

Simplifying

$$\begin{aligned}
&\frac{(\tau^2(\tau^2+1)A - B\tau)e^{-t/\tau} + A\tau^3(t-\tau) + A\tau(t-\tau) + B\tau(\tau\sin(t)+\cos(t))}{(\tau^2+1)\tau} \\
&\quad = \frac{(\tau(\tau^2+1)A - B)e^{-t/\tau} + A\tau^2(t-\tau) + A(t-\tau) + B(\tau\sin(t)+\cos(t))}{(\tau^2+1)} \\
&\quad = \frac{\tau(\tau^2+1)Ae^{-t/\tau} - Be^{-t/\tau} + A(\tau^2+1)(t-\tau) + B(\tau\sin(t)+\cos(t))}{(\tau^2+1)} \\
&\quad = \frac{\tau(\tau^2+1)Ae^{-t/\tau} + A(\tau^2+1)(t-\tau) + B(\tau\sin(t)+\cos(t) - e^{-t/\tau})}{(\tau^2+1)} \\
&\quad = A(t-\tau+\tau e^{-t/\tau}) + \frac{B(\tau\sin(t)+\cos(t)-e^{-t/\tau})}{(\tau^2+1)}
\end{aligned}$$

Next step is F_1 and F_2. For the 'A' part of F_1

```
t,s,tau,t1,t2 = var('t s tau t1 t2'); assume(tau>0);
III = integrate(A*(t-tau+tau*exp(-t/tau)),t,t1,t2);
III; latex(III)

-1/2*(2*tau^2*e^(t1/tau)
      - (2*tau^2 - (t1^2 - t2^2 - 2*(t1 - t2)*tau)*e^(t1/tau)
        )*e^(t2/tau)
     )*A*e^(-t1/tau - t2/tau)

-\frac{1}{2} \, {\left(
  2 \, \tau^{2} e^{\frac{t_{1}}{\tau}} - {\left(
      2 \, \tau^{2} - {\left(t_{1}^{2} - t_{2}^{2}
    - 2 \, {\left(t_{1} - t_{2}\right)} \tau\right)} e^{\frac{t_{1}}{\tau}}
  \right)} e^{\frac{t_{2}}{\tau}}
\right)} A e^{\left(-\frac{t_{1}}{\tau} - \frac{t_{2}}{\tau}\right)}
```

This is,

$$\begin{aligned}
&-\frac{1}{2}\left(2\,\tau^2 e^{\frac{t_1}{\tau}} - \left(2\,\tau^2 - \left(t_1^2 - t_2^2 - 2\,(t_1-t_2)\tau\right)e^{\frac{t_1}{\tau}}\right)e^{\frac{t_2}{\tau}}\right)Ae^{\left(-\frac{t_1}{\tau}-\frac{t_2}{\tau}\right)} \\
&\quad = -\frac{A}{2}\left(2\tau^2 e^{\frac{t_1}{\tau}} - \left(2\,\tau^2 - \left(t_1^2 - t_2^2 - 2\,(t_1-t_2)\tau\right)e^{\frac{t_1}{\tau}}\right)e^{\frac{t_2}{\tau}}\right)e^{\left(-\frac{t_1}{\tau}-\frac{t_2}{\tau}\right)} \\
&\quad = -\frac{A}{2}\left(2\tau^2 e^{\frac{t_1}{\tau}}e^{\left(-\frac{t_1}{\tau}-\frac{t_2}{\tau}\right)} - \left(2\,\tau^2 - \left(t_1^2 - t_2^2 - 2\,(t_1-t_2)\tau\right)e^{\frac{t_1}{\tau}}\right)e^{\frac{t_2}{\tau}}e^{\left(-\frac{t_1}{\tau}-\frac{t_2}{\tau}\right)}\right) \\
&\quad = -\frac{A}{2}\left(2\tau^2 e^{\frac{-t_2}{\tau}} - \left(2\,\tau^2 e^{\frac{-t_1}{\tau}} - \left(t_1^2 - t_2^2 - 2\,(t_1-t_2)\tau\right)\right)\right) \\
&\quad = -\frac{A}{2}\left(2\tau^2 e^{\frac{-t_2}{\tau}} - \left(2\,\tau^2 e^{\frac{-t_1}{\tau}} + \left(t_2^2 - t_1^2 - 2\,(t_2-t_1)\tau\right)\right)\right) \\
&\quad = -\frac{A}{2}\left(2\tau^2 e^{\frac{-t_2}{\tau}} - 2\,\tau^2 e^{\frac{-t_1}{\tau}} - ((t_2-t_1)(t_2+t_1) - 2\,(t_2-t_1)\tau)\right) \\
&\quad = -A\tau^2(e^{\frac{-t_2}{\tau}} - e^{\frac{-t_1}{\tau}}) + A(t_2-t_1)\left(\frac{1}{2}(t_2+t_1) - \tau\right)
\end{aligned}$$

And for the 'B' part of F_1

```
t,s,tau,t1,t2 = var('t s tau t1 t2'); assume(tau>0);
IV = integrate(B*(tau*sin(t)+cos(t)-exp(-t/tau) )/(tau^2+1),t,t1,t2);
IV; latex(IV)

(tau*e^(t1/tau) + ((tau*(cos(t1) - cos(t2)) - sin(t1) ...
  + sin(t2))*e^(t1/tau) - tau)*e^(t2/tau))*B*e^(-t1/tau - t2/tau)/(tau^2 + 1)

\frac{{\left(
          \tau e^{\frac{t_{1}}{\tau}} + {\left(
            {\left(\tau {\left(\cos\left(t_{1}\right)
            - \cos\left(t_{2}\right)\right)} - \sin\left(t_{1}\right)
            + \sin\left(t_{2}\right)\right)} e^{\frac{t_{1}}{\tau}} - \tau
          \right)} e^{\frac{t_{2}}{\tau}}
        \right)} B e^{\left(-\frac{t_{1}}{\tau} - \frac{t_{2}}{\tau}\right)}}
     {\tau^{2} + 1}
```

This is

$$
\begin{aligned}
&\frac{\left(\tau e^{\frac{t_1}{\tau}} + \left((\tau(\cos(t_1) - \cos(t_2)) - \sin(t_1) + \sin(t_2))e^{\frac{t_1}{\tau}} - \tau\right)e^{\frac{t_2}{\tau}}\right)Be^{\left(-\frac{t_1}{\tau}-\frac{t_2}{\tau}\right)}}{\tau^2+1} \\
&= \frac{Be^{\left(-\frac{t_1}{\tau}-\frac{t_2}{\tau}\right)}\left(\tau e^{\frac{t_1}{\tau}} + \left((\tau(\cos(t_1) - \cos(t_2)) - \sin(t_1) + \sin(t_2))e^{\frac{t_1}{\tau}} - \tau\right)e^{\frac{t_2}{\tau}}\right)}{\tau^2+1} \\
&= \frac{B\left(\tau e^{-\frac{t_2}{\tau}} + \left((\tau(\cos(t_1) - \cos(t_2)) - \sin(t_1) + \sin(t_2))e^{\frac{t_1}{\tau}} - \tau\right)e^{-\frac{t_1}{\tau}}\right)}{\tau^2+1} \\
&= \frac{B\left(\tau e^{-\frac{t_2}{\tau}} - \tau e^{-\frac{t_1}{\tau}} + \tau\cos(t_1) - \tau\cos(t_2) - \sin(t_1) + \sin(t_2)\right)}{\tau^2+1} \\
&= \frac{B\left(\tau(e^{-\frac{t_2}{\tau}} - e^{-\frac{t_1}{\tau}}) + \tau(\cos(t_1) - \cos(t_2)) - \sin(t_1) + \sin(t_2)\right)}{\tau^2+1}
\end{aligned}
$$

Now for the 'A' part of F_2,

```
t,s,tau,t1,t2 = var('t s tau t1 t2'); assume(tau>0); assume(t2>t1)
V = integrate(A*(t2-t)/(t2-t1)*(t-tau+tau*exp(-t/tau)),t,t1,t2);
V; latex(V)

1/6*((6*(t1 - t2)*tau^2 + 6*tau^3 - (2*t1^3 - 3*t1^2*t2
  - 3*(t1^2 - 2*t1*t2)*tau)*e^(t1/tau))*e^(-t1/tau)
  - (6*tau^3 + (t2^3 - 3*t2^2*tau)*e^(t2/tau))*e^(-t2/tau))*A/(t1 - t2)

\frac{{\left(
  {\left(6 \, {\left(t_{1} - t_{2}\right)} \tau^{2} + 6 \, \tau^{3}
  - {\left(2 \, t_{1}^{3} - 3 \, t_{1}^{2} t_{2} - 3 \, {\left(t_{1}^{2}
  - 2 \, t_{1} t_{2}\right)} \tau\right)} e^{\frac{t_{1}}{\tau}}
  \right)} e^{\left(-\frac{t_{1}}{\tau}\right)}
  - {\left(6 \, \tau^{3} + {\left(t_{2}^{3} - 3 \, t_{2}^{2} \tau
  \right)} e^{\frac{t_{2}}{\tau}}
  \right)} e^{\left(-\frac{t_{2}}{\tau}\right)}\right)} A}
{6 \, {\left(t_{1} - t_{2}\right)}}
```

In $\LaTeX\,2_\varepsilon$ this is,

$$\frac{\left(\left(6\,(t_1 - t_2)\tau^2 + 6\,\tau^3 - \left(2\,t_1^3 - 3\,t_1^2 t_2 - 3\left(t_1^2 - 2\,t_1 t_2\right)\tau\right)e^{\frac{t_1}{\tau}}\right)e^{\left(-\frac{t_1}{\tau}\right)} - \left(6\,\tau^3 + \left(t_2^3 - 3\,t_2^2\tau\right)e^{\frac{t_2}{\tau}}\right)e^{\left(-\frac{t_2}{\tau}\right)}\right)A}{6\,(t_1 - t_2)}$$

Lastly, for the 'B' part of F_2,

```
t,s,tau,t1,t2 = var('t s tau t1 t2'); assume(tau>0); assume(t2>t1)
VI = integrate(B*(t2-t)/(t2-t1)*(tau*sin(t)+cos(t)-exp(-t/tau) )/(tau^2+1),t,t1,t2);
VI; latex(VI)
```

```
-(((t1 - t2)*tau + tau^2 - ((t1*cos(t1) - t2*cos(t1) - sin(t1))*tau
  - t1*sin(t1) + t2*sin(t1) - cos(t1))*e^(t1/tau))*e^(-t1/tau)
  - (tau^2 + (tau*sin(t2) + cos(t2))*e^(t2/tau))*e^(-t2/tau))*B
  /((tau^2 + 1)*(t1 - t2))
```

```
-\frac{
  {\left({\left({\left(t_{1} - t_{2}\right)} \tau + \tau^{2}
    - {\left({
            \left(t_{1} \cos\left(t_{1}\right)
            - t_{2} \cos\left(t_{1}\right) - \sin\left(t_{1}\right)
       \right)} \tau
       - t_{1} \sin\left(t_{1}\right) + t_{2} \sin\left(t_{1}\right)
       - \cos\left(t_{1}\right)\right)} e^{\frac{t_{1}}{\tau}}
       \right)} e^{\left(-\frac{t_{1}}{\tau}\right)}
    - {\left(
         \tau^{2} + {\left(\tau \sin\left(t_{2}\right)
         + \cos\left(t_{2}\right)\right)} e^{\frac{t_{2}}{\tau}}
      \right)} e^{\left(-\frac{t_{2}}{\tau}\right)}\right)} B}
{{\left(\tau^{2} + 1\right)}{\left(t_{1} - t_{2}\right)}}
```

and in $\LaTeX\,2_\varepsilon$ this is,

$$\begin{aligned}
&-\frac{\left((t_1 - t_2)\tau + \tau^2 - ((t_1\cos(t_1) - t_2\cos(t_1) - \sin(t_1))\tau - t_1\sin(t_1) + t_2\sin(t_1) - \cos(t_1))e^{\frac{t_1}{\tau}}\right)e^{\left(-\frac{t_1}{\tau}\right)}}{B^{-1}(\tau^2+1)(t_1 - t_2)} \\
&\qquad + \frac{\left(\tau^2 + (\tau\sin(t_2) + \cos(t_2))e^{\frac{t_2}{\tau}}\right)e^{\left(-\frac{t_2}{\tau}\right)}}{B^{-1}(\tau^2+1)(t_1 - t_2)} \\
&\quad = \frac{\tau^2\left(e^{\left(-\frac{t_1}{\tau}\right)} - e^{\left(-\frac{t_2}{\tau}\right)}\right) + \tau\sin(t_1) + \cos(t_1) - \tau\sin(t_2) - \cos(t_2)}{B^{-1}(\tau^2+1)(t_2 - t_1)} \\
&\qquad + \frac{\tau\cos(t_1) - \sin(t_1) - \tau e^{\left(-\frac{t_1}{\tau}\right)}}{B^{-1}(\tau^2+1)}
\end{aligned}$$

or, bigger,

$$\begin{aligned}
&\frac{\tau^2\left(e^{\left(-\frac{t_1}{\tau}\right)} - e^{\left(-\frac{t_2}{\tau}\right)}\right) + \tau\sin(t_1) + \cos(t_1) - \tau\sin(t_2) - \cos(t_2)}{B^{-1}(\tau^2+1)(t_2 - t_1)} \\
&\qquad + \frac{\tau\cos(t_1) - \sin(t_1) - \tau e^{\left(-\frac{t_1}{\tau}\right)}}{B^{-1}(\tau^2+1)}
\end{aligned}$$

Example 9 *This is a repeat of Example 5, retaining* $\gamma_M = 2$ *and* $\gamma_E = 1$*, but now also with* $N_\varphi = 2$ *with* $(\varphi_1, \tau_1) = (0.35, 0.1)$ *and* $(\varphi_2, \tau_2) = (0.15, 0.05)$*. The results are in Figure 13 and Table 9.*

N_{xy}	TEe	KEe	ESe	$\\|\boldsymbol{e}_u(T)\\|_{\boldsymbol{H}^1(\Omega)}$	$\\|\boldsymbol{e}_w(T)\\|_{\boldsymbol{H}^1(\Omega)}$
Nxy	En	L2w	SEu	H1u	H1w
2	8.743e+01	3.448e-01	8.742e+01	9.361e+01	2.483e+00
3	6.123e+01	1.668e-01	6.123e+01	6.619e+01	1.755e+00
4	4.669e+01	9.653e-02	4.669e+01	5.064e+01	1.343e+00
6	3.150e+01	4.376e-02	3.150e+01	3.423e+01	9.080e-01
8	2.372e+01	2.478e-02	2.372e+01	2.580e+01	6.843e-01
11	1.729e+01	1.316e-02	1.729e+01	1.882e+01	4.991e-01
16	1.191e+01	6.236e-03	1.191e+01	1.296e+01	3.437e-01
23	8.288e+00	3.021e-03	8.288e+00	9.022e+00	2.393e-01
32	5.959e+00	1.562e-03	5.959e+00	6.487e+00	1.721e-01
45	4.238e+00	7.899e-04	4.238e+00	4.614e+00	1.224e-01
64	2.980e+00	3.906e-04	2.980e+00	3.244e+00	8.606e-02
91	2.096e+00	1.932e-04	2.096e+00	2.282e+00	6.053e-02
128	1.490e+00	9.765e-05	1.490e+00	1.622e+00	4.303e-02
181	1.054e+00	4.884e-05	1.054e+00	1.147e+00	3.043e-02
256	7.451e-01	2.441e-05	7.451e-01	8.112e-01	2.152e-02
362	5.269e-01	1.221e-05	5.269e-01	5.736e-01	1.522e-02
512	3.725e-01	6.102e-06	3.725e-01	4.056e-01	1.076e-02
724	2.635e-01	3.051e-06	2.635e-01	2.868e-01	7.608e-03
1024	1.863e-01	1.525e-06	1.863e-01	2.028e-01	5.379e-03
1448	1.317e-01	7.618e-07	1.317e-01	1.434e-01	3.804e-03

Table 9: Results for Example 9.

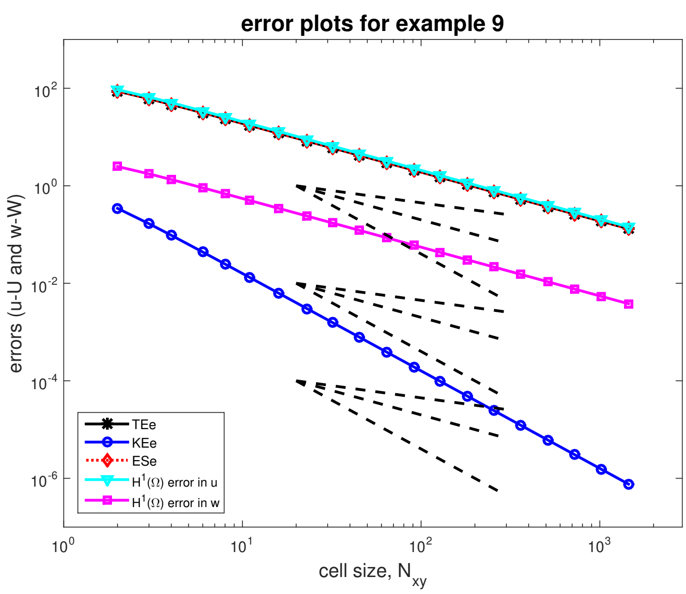

Figure 13: Errors for Example 9, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1, 2, 3$.

N_{xy}	TEe	KEe	ESe	$\lVert \boldsymbol{e}_u(T) \rVert_{\boldsymbol{H}^1(\Omega)}$	$\lVert \boldsymbol{e}_w(T) \rVert_{\boldsymbol{H}^1(\Omega)}$
Nxy	En	L2w	SEu	H1u	H1w
2	8.974e+01	3.426e-01	8.974e+01	9.610e+01	2.486e+00
3	6.285e+01	1.652e-01	6.285e+01	6.794e+01	1.757e+00
4	4.793e+01	9.525e-02	4.793e+01	5.198e+01	1.344e+00
6	3.233e+01	4.297e-02	3.233e+01	3.514e+01	9.083e-01
8	2.435e+01	2.423e-02	2.435e+01	2.648e+01	6.845e-01
11	1.775e+01	1.281e-02	1.775e+01	1.932e+01	4.992e-01
16	1.222e+01	6.029e-03	1.222e+01	1.330e+01	3.438e-01
23	8.508e+00	2.901e-03	8.508e+00	9.261e+00	2.393e-01
32	6.117e+00	1.490e-03	6.117e+00	6.659e+00	1.721e-01
45	4.350e+00	7.489e-04	4.350e+00	4.736e+00	1.224e-01
64	3.059e+00	3.680e-04	3.059e+00	3.330e+00	8.606e-02
91	2.152e+00	1.808e-04	2.152e+00	2.342e+00	6.053e-02
128	1.530e+00	9.087e-05	1.530e+00	1.665e+00	4.303e-02
181	1.082e+00	4.519e-05	1.082e+00	1.178e+00	3.043e-02
256	7.649e-01	2.247e-05	7.649e-01	8.327e-01	2.152e-02
362	5.409e-01	1.118e-05	5.409e-01	5.888e-01	1.522e-02
512	3.824e-01	5.564e-06	3.824e-01	4.163e-01	1.076e-02
724	2.704e-01	2.774e-06	2.704e-01	2.944e-01	7.608e-03
1024	1.912e-01	1.380e-06	1.912e-01	2.082e-01	5.379e-03
1448	1.352e-01	6.865e-07	1.352e-01	1.472e-01	3.804e-03

Table 10: Results for Example 10.

Example 10 *This is a repeat of Example 6, retaining* $\gamma_M = 2$ *and* $\gamma_E = 1$*, but now also with* $N_\varphi = 2$ *with* $(\varphi_1, \tau_1) = (0.35, 0.1)$ *and* $(\varphi_2, \tau_2) = (0.15, 0.05)$*. The results are in Figure 14 and Table 10.*

Example 11 *This is a repeat of Example 7, retaining* $\gamma_M = 2$ *and* $\gamma_E = 1$*, but now also with* $N_\varphi = 2$ *with* $(\varphi_1, \tau_1) = (0.35, 0.1)$ *and* $(\varphi_2, \tau_2) = (0.15, 0.05)$*. The results are in Figure 15 and Table 11.*

Example 12 *This is a repeat of Example 8, retaining* $\gamma_M = 2$ *and* $\gamma_E = 1$*, but now also with* $N_\varphi = 2$ *with* $(\varphi_1, \tau_1) = (0.35, 0.1)$ *and* $(\varphi_2, \tau_2) = (0.15, 0.05)$*. The results are in Figure 16 and Table 12.*

B.3 Updated numerical results

To harmonise all results the entire run was re-executed. These are the preliminary notes.

Note: 9 Jan 2019. Run terminated during example 6 (100 minutes for 1%) in order to get 5 for the paper.

Make sure the notes in the earlier section for the report version are also updated.

On the heron11 machine, 20 Dec 2018,

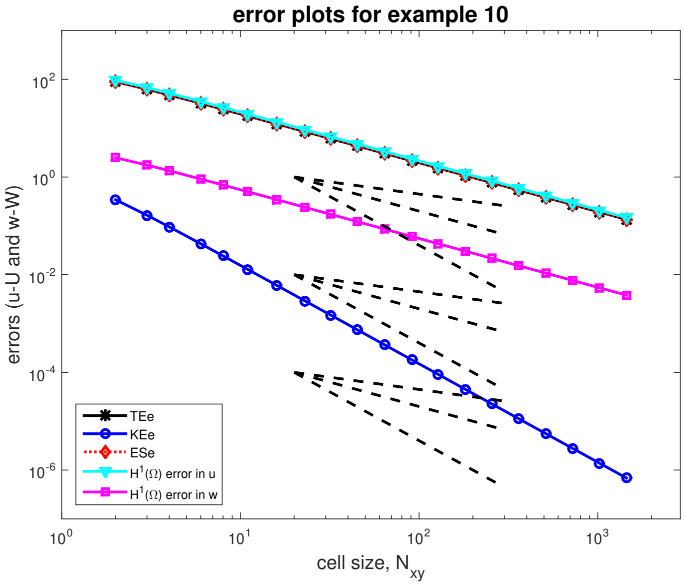

Figure 14: Errors for Example 10, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1, 2, 3$.

N_{xy}	TEe	KEe	ESe	$\|\boldsymbol{e}_u(T)\|_{\boldsymbol{H}^1(\Omega)}$	$\|\boldsymbol{e}_w(T)\|_{\boldsymbol{H}^1(\Omega)}$
Nxy	En	L2w	SEu	H1u	H1w
2	8.974e+01	3.384e-01	8.974e+01	9.610e+01	2.495e+00
3	6.285e+01	1.589e-01	6.285e+01	6.794e+01	1.764e+00
4	4.793e+01	8.879e-02	4.793e+01	5.198e+01	1.350e+00
6	3.233e+01	3.752e-02	3.233e+01	3.514e+01	9.118e-01
8	2.435e+01	1.973e-02	2.435e+01	2.648e+01	6.869e-01
11	1.775e+01	9.313e-03	1.775e+01	1.932e+01	5.009e-01
16	1.222e+01	3.818e-03	1.222e+01	1.330e+01	3.449e-01
23	8.508e+00	1.952e-03	8.508e+00	9.261e+00	2.400e-01
32	6.117e+00	1.512e-03	6.117e+00	6.659e+00	1.726e-01
45	4.350e+00	1.283e-03	4.350e+00	4.736e+00	1.228e-01
64	3.059e+00	1.096e-03	3.059e+00	3.330e+00	8.634e-02
91	2.152e+00	8.811e-04	2.152e+00	2.342e+00	6.074e-02
128	1.530e+00	7.014e-04	1.530e+00	1.665e+00	4.320e-02
181	1.082e+00	5.365e-04	1.082e+00	1.178e+00	3.056e-02
256	7.649e-01	4.098e-04	7.649e-01	8.327e-01	2.161e-02
362	5.409e-01	3.105e-04	5.409e-01	5.888e-01	1.529e-02
512	3.824e-01	2.325e-04	3.824e-01	4.163e-01	1.082e-02
724	2.704e-01	1.733e-04	2.704e-01	2.944e-01	7.654e-03
1024	1.912e-01	1.291e-04	1.912e-01	2.082e-01	5.415e-03
1448	1.352e-01	9.520e-05	1.352e-01	1.472e-01	3.832e-03

Table 11: Results for Example 11.

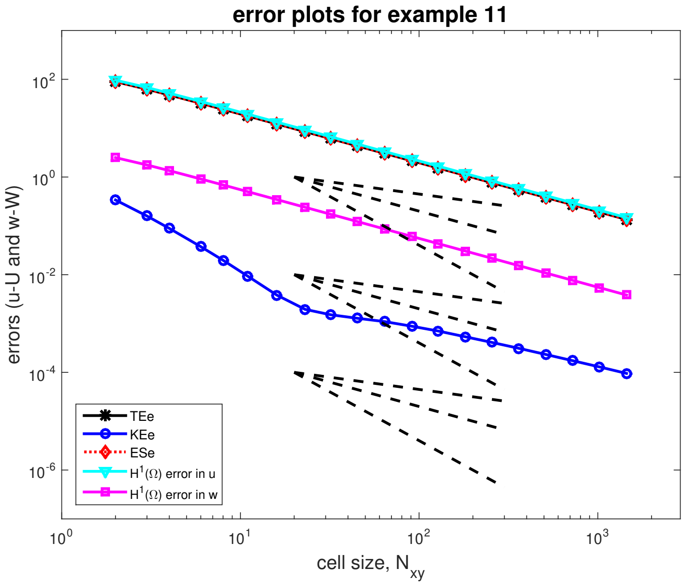

Figure 15: Errors for Example 11, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1, 2, 3$.

N_{xy}	TEe	KEe	ESe	$\|\boldsymbol{e}_u(T)\|_{\boldsymbol{H}^1(\Omega)}$	$\|\boldsymbol{e}_w(T)\|_{\boldsymbol{H}^1(\Omega)}$
Nxy	En	L2w	SEu	H1u	H1w
2	8.974e+01	3.353e-01	8.974e+01	9.610e+01	2.501e+00
3	6.285e+01	1.532e-01	6.285e+01	6.794e+01	1.772e+00
4	4.793e+01	8.221e-02	4.793e+01	5.198e+01	1.357e+00
6	3.233e+01	3.111e-02	3.233e+01	3.514e+01	9.194e-01
8	2.435e+01	1.548e-02	2.435e+01	2.648e+01	6.938e-01
11	1.775e+01	9.697e-03	1.775e+01	1.932e+01	5.074e-01
16	1.222e+01	9.888e-03	1.222e+01	1.330e+01	3.516e-01
23	8.508e+00	9.820e-03	8.508e+00	9.261e+00	2.468e-01
32	6.117e+00	9.182e-03	6.117e+00	6.659e+00	1.793e-01
45	4.350e+00	8.386e-03	4.350e+00	4.736e+00	1.296e-01
64	3.059e+00	7.556e-03	3.059e+00	3.330e+00	9.364e-02
91	2.152e+00	6.762e-03	2.152e+00	2.342e+00	6.858e-02
128	1.530e+00	5.859e-03	1.530e+00	1.665e+00	5.112e-02
181	1.082e+00	5.103e-03	1.082e+00	1.178e+00	3.866e-02
256	7.649e-01	4.469e-03	7.649e-01	8.327e-01	2.993e-02
362	5.409e-01	3.838e-03	5.409e-01	5.888e-01	2.344e-02
512	3.824e-01	3.323e-03	3.824e-01	4.163e-01	1.880e-02
724	2.705e-01	2.898e-03	2.704e-01	2.944e-01	1.545e-02
1024	1.912e-01	2.492e-03	1.912e-01	2.082e-01	1.276e-02
1448	1.352e-01	2.161e-03	1.352e-01	1.472e-01	1.074e-02

Table 12: Results for Example 12.

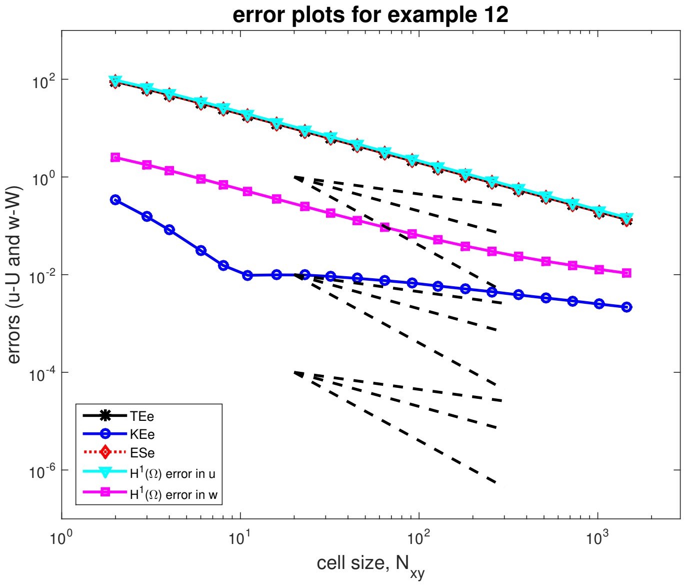

Figure 16: Errors for Example 12, the dashed lines indicate rates of N_{xy}^{-p} for $p = 1, 2, 3$.

```
screen -S dgcgwave
C-AC
```

created the screen session and then from

```
https://quay.io/repository/fenicsproject/stable?tag=latest&tab=tags
```

the 2017.1.0 FEniCS container was pulled with (you can substitute in your own UID GID if needed)

```
heron11 code_v02/fenics% docker run -ti --env HOST_UID=xyz --env HOST_GID=uvw --name dgcgwave \
?     -v /home/icsrsss/fenics_docker/shared:/home/fenics/shared quay.io/fenicsproject/stable:2017.1.0
```

It wasnt already there so after this, and about 10 minutes,

```
Unable to find image 'quay.io/fenicsproject/stable:2017.1.0' locally
2017.1.0: Pulling from fenicsproject/stable
```

the container was running. This pull was at 15:22 GMT 20 Dec 2018. The following command will therefore recreate this container.

```
docker run -ti --env HOST_UID=xyz --env HOST_GID=uvw \
    --name dgcgwave \
    -v /home/icsrsss/fenics_docker/shared:/home/fenics/shared \
    quay.io/fenicsproject/stable:2017.1.0
```

A few errors were corrected in the `solver.py` script. First `print` commands without brackets had them added (to be compatible with python3). These were:

```
 - line 108 to print('Command Line: using: ')
 - lines 196-200 to print(...)
```

Also,

```
%d changed to %f for T = in usage()
several other small bug fixes related to the command line args
```

The re-run of all twelve examples then began at

```
20 Dec 2018 at 16:30

with the run command:

  ./bigrun_np.sh -J 21 | tee runmeout_np.txt
```

The `bigrun_np.sh` script was also altered so as not to use `mpi` for 'small' problems, because it results in a crash. Each example takes just over 8 hours — should be finished by the end of Dec 24th.

In the above we have used the non-latest version of FEniCS. The latest appears to require python3. This was one reason to alter the brackets above, but that wasn't enough.
This didn't work: we obtained the latest version with

```
docker run -ti --env HOST_UID=xyz --env HOST_GID=uvw --name fenics_dg \
   -v /home/icsrsss/fenics_docker/shared:/home/fenics/shared quay.io/fenicsproject/stable
```

which then pulls everything over:

```
Unable to find image 'quay.io/fenicsproject/stable:latest' locally
latest: Pulling from fenicsproject/stable
```

This pull took place at 12:43 GMT 20 Dec 2018. The code wouldn't run and these quick fixes were implemented:

```
 - #!/bin/python3 in solver.py
 - line 108 to print('Command Line: using: ')
 - lines 196-200 to print(...)
 - line 37,38 comment: #set_log_level(0) #set_log_active(False)
```

These were not enough. There were some difficulties with the mesh, but not time to investigate further. Hence the pull of the 2017.1.0 container which we know works on our architectures.

This is how the docker hub image was created and used.

```
# run the required version of FEniCS in docker with a shared folder
docker run -ti --name dgcgwave \
 -v /..absolute_path.../dgcgwave/code_v02/fenics:/home/fenics/shared \
 quay.io/fenicsproject/stable:2017.1.0
```

```
# copy source codes from shared folder to another inside the container.
fenics@7f85088e2a09:~$ cp -r shared fenics
fenics@7f85088e2a09:~$ ls
WELCOME  demo  fenics  fenics.env.conf  local  shared

# make sure the permissions are set on the 'executables'
chmod u+x bigrun* solver.py
# update and install bc
sudo apt-get update
sudo apt-get install bc
# check that the code runs, and then exit the container
./bigrun_np.sh -J 2 | tee runmeout_np.txt
exit

# from above we see the unique hash in the prompt...
fenics@7f85088e2a09:~$ ls
# now commit the container to freeze its state
docker commit 7f85088
# and obtain a new hash
sha256:628f9736f945a411180be3fa016a1c5f10abb4de4ed24801ec4d64683bc7fca2

# There are now two methods to distribute the container.
# Method #1: create a docker hub account and then use the new hash:
docker login
docker tag 628f9736 variationalform/fem:dgcgwave
docker push variationalform/fem:dgcgwave

# A downloader can now write:
docker pull variationalform/fem:dgcgwave
docker run -ti variationalform/fem:dgcgwave
cd fenics
./bigrun_np.sh -J 4 | tee runmeout_np.txt

# Method #2: distribute a tar archive (I haven't tried this)
docker save 628f9736 > dgcgwave_fenics_docker.tar
docker load < dgcgwave_fenics_docker.tar
docker run -ti 628f9736
```