TR/13 August 1972.

MELTING ICE BY THE ISOTHERM MIGRATION METHOD

by

J. Crank and R. D. Phahle

Stefan Problems

A common-place physical phenomenon is the melting of a block of ice by raising its surface to a temperature above 0^0C. The two phases, water and ice, are separated by a boundary on which melting occurs at 0^0C and which moves further into the ice as time progresses. In the mathematical treatment, the motion of the boundary has to be determined and the usual equations of heat flow solved in the water and the ice. The solutions and the boundary movement are dependent on each other. The melting of ice is just one example of a whole class of problems commonly referred to as Stefan Problems. They include the propagation of phase changes in metals diffusion with absorption and processes controlled by discontinuous diffusion coefficients.

Methods of Solution

In some cases analytical solutions can be obtained. (1) Several numerical methods, all based on finite-difference replacements of the original partial differential equation, differ in the way they cope with the movement of the boundary. As usual, in a one dimensional problem the region is covered by a grid of equally spaced lines. The various numerical methods have really explored all possible ways of using the grid. Special finite-difference formulae based on Lagrangian interpolation formulae for unequal intervals have been used in the neighbourhood of the boundary when it falls between two grid lines. (2) Unequal time intervals have been used, calculated so that the boundary moves always from one grid time to the next in one time step. (3) The grid has been deformed so that the number of space intervals between the outer surface and the moving boundary remains constant, with suitable transformation of the basis equation. (4)

Another method employs an apparent specific heat modified to include the latent heat in the appropriate region.[5] Finally, the whole grid has been moved with the velocity of the moving boundary in a method incorporating interpolating splines.[6] Recently a novel way of handling heat flow problems has been proposed which is especially useful for tracking a moving boundary that occurs at a fixed temperature. In the usual heat flow equation in one dimension the temperature is expressed as a function of the independent space variable x, and time t i.e. $u = u(x,t)$. An alternative, however, is to seek a solution in which x is expressed as a function of u and t i.e. $x = x(u,t)$ so that x becomes the dependent variable. We calculate the positions of a given temperature, that is of an isotherm at known times. Hence, the method is known as the Isotherm Migration Method[7] (IMM). Philip[8] dealt with a problem in concentration dependent diffusion by making concentration an independent variable but he did not transform the diffusion equation in the same way as Dix and Cizek. Rose[9] derived a related transformed equation but did not develop a numerical method. The idea of tracking the moving isotherms in media with phase changes was published by Chernoua'ko[10] in 1969, though the English version appeared only in 1970.[11]

Melting Ice

Suppose a plane sheet of ice initially occupies the region $0 \leq x \leq a$ and is being melted by the application of a constant temperature, u_0, on the surface $x = 0$. At any time, t, let the moving boundary separating the water from the ice be at $x_o(t)$. The region $0 \le x \le x_o(t)$ then consists of water with specific heat, density and thermal conductivity denoted by p, C and K respectively. The temperature, u, of the water satisfies the heat flow equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $k = K/c\rho$, the heat diffusivity.

We take the ice to be initially at 0°c throughout. Otherwise we should have an equation similar to (1) for the temperature in the ice phase and containing appropriate heat parameters.

At the melting boundary, $x_o(t)$, the heat flowing per unit area from the water into the ice in a short time, δt, is $-(K\partial u/\partial x)\delta t$. If the boundary moves a distance δx_o in time δt, the heat required to melt the mass $\rho\delta x_o$ of ice per unit area is $L\rho\,\delta x_o$ where L is the latent heat of fusion for ice. Equating these two amounts of heat and proceeding to the limit $\delta t = 0$, we see that the first condition to be satisfied on the boundary is

$$L\rho \frac{dx_o}{dt} = -K \frac{\partial u}{\partial x} \quad \bullet \tag{2}$$

A second condition, since ice melts at 0°C is

$$u = 0, \quad x = x_o, \quad t \ge 0 . \tag{3}$$

Commonly used variables

$$X = x/_a, \quad T = kt/_a2, \quad X_o = X_o/a, \quad s = \frac{L}{o} \tag{4}$$

lead to the following system of equations

$$\frac{\partial u}{\partial T} = \frac{\partial^2 u}{\partial X^2} \quad , \; 0 < X < X_o \, , \; T \geq 0 \; , \tag{5}$$

$$S\frac{dx_o}{dT} = -\frac{\partial u}{\partial X} \, , \quad X = X_o \, , \quad T \geq 0 \; , \tag{6}$$

$$u = 0, \; X = X_o \, , \; T \geq 0 \; , \tag{7}$$

$$u = u_o \, , \; X = 0, \quad T \geq 0 \; , \tag{8}$$

$$u = 0, \; 0 < X < 1, \quad T = 0. \tag{9}$$

<u>Isotherm Migration Equations</u>

We wish to re-write the heat flow equation (5) so that X is expressed as a function of u and T. Remembering that

$$\frac{\partial u}{\partial X} = \left(\frac{\partial X}{\partial u} \right)^{-1} \text{ and } \frac{\partial X}{\partial T} = -\left(\frac{\partial u}{\partial T} \right)\left(\frac{\partial X}{\partial u} \right)$$

we readily see that

$$\frac{\partial X}{\partial T} = \left(\frac{\partial X}{\partial u} \right)^{-2} \frac{\partial^2 X}{\partial u^2} \tag{10}$$

The other equations (6) - (8) become

$$s\frac{dx_o}{dT} = -\left(\frac{\partial x}{\partial u}\right)^{-1} \quad , \quad u=0,\ T\geq 0 \tag{11}$$

$$X = X_o,\ u=0,\ \ T\geq 0, \tag{12}$$

$$X=0, \quad U = U_o,\ T\geq 0. \tag{13}$$

We can approximate the derivatives in (10) by finite difference, in; the usual way and obtain an explicit expression for X_i^{n+1}, the value of X at $u=i\,\delta u$, $T = (n+1)\,\delta T$ in terms of values already available at $(i\delta u,\ n\delta T)$.

We find

$$X_i^{n+1} = X_i^n + 4\delta t \left\{\frac{X_{i+1}^n - 2X_i^n + X_{i-1}^n}{\left(X_{i-1}^n - X_{i+1}^n\right)^2}\right\} \quad . \tag{14}$$

The corresponding finite-difference replacement of

(11) is

$$(X_o)^{n+1} = (X_o)^n - \frac{\delta T\ \delta u}{s(X_o^n - X_1^n)} \quad . \tag{15}$$

A rigorous analysis of the stability of the non-linear finite difference scheme has not been attempted. Dix and Cizek [7] consider instead the coefficient of

X_i^n in (14). In order that the isotherms should move with time in the manner expected X_i^{n+1} should increase with X_i^n i.e. the coefficient of X_i^n must be positive. This leads to the criterion

$$\partial T < \frac{1}{8}\,(X_{i-1} - X_{i+1})^2 \quad , \tag{16}$$

which allows ∂t to change as the solution proceeds. The truncation error [7] of the IMM is proportional to Δt and $(Au)^2$.

Example

The method has been applied to the problem of a block of ice, of unit thickness and initially at zero temperature throughout. One face is maintained at 10° C.

There is a well known analytical solution of this problem. [1] It is

$$\left.\begin{aligned} u &= u_0 - u_0 \frac{\operatorname{erf}(x/2T^{\frac{1}{2}})}{\operatorname{erf}\phi} \ , \ 0 < X < X_{0,} T \geq 0 \\ u &= 0.\ \ X_0 < x < 1, \ \ T > 0 \end{aligned}\right\} \tag{17}$$

$$X_0 = 2\phi T^{\frac{1}{2}}, \tag{18}$$

where ϕ is given by

$$\pi^{\frac{1}{2}}\ \phi \operatorname{erf} \phi \exp\ (\phi^2) = u_0/s. \tag{19}$$

The analytical solution provides a means of checking the accuracy of other methods of solution. It can also be used as a starting solution since all finite-difference methods present difficulties at small times when the surface temperature has changed discontinuously at T = 0. Equation (17) is used to start the IMM method at time 0.1 and for isotherms 2^0C apart.

In order to advance the solution from T = 0.1, the requirement (16) is used. A least upper bound of the δT's for the different isotherms is adopted as the next time step. In this way the method selects its own time steps as the solution proceeds. In fact, in this example the time step is found to increase in an arithmetic progression every 0.1 units of time, rising from 0.0005 at 0.1 to 0.019 at 3.8. We note in passing that at approximately this latter time the whole block of ice has melted. In Table 1 isotherm positions at selected times calculated by the IMM are compared with those obtained from the analytical solution.

Table 2 compares the positions of the moving boundary at several times given by the analytic solution with those calculated. by the IMM and by the methods of Crank[(2)] and Goodman[(12)] The latter two solutions are tabulated by Coldrey.[(13)] Crank[(2)] uses Lagrangian interpolation formulae to track the moving boundary. Goodman's method[(12)] is an approximate analytic method in which the temperature profile is assumed in this example to take the form of a second degree polynomial satisfying the boundary conditions at any time. This profile is substituted into an integrated form of the heat flow equation to give an ordinary differential equation expressing the position of the moving boundary as a function of time.

Table 1. Selected Isotherm Positions.

For each time the upper entry shows the isotherm position (10^4 X) obtained from the IMM and the lower Talues from the analytical solution. The 10°C isotherm remains at x= 0 throughout.

Time	8^0C	6^0C	4^0C	2^0C	0^0C
0.1005	317	636	956	1283	1617
	317	635	957	1281	1617
0.1570	397	796	1199	1608	2026
	396	794	1197	1601	2021
0.3500	594	1191	1794	2406	2032
	592	1186	1787	2390	3017
0.6260	795	1594	2401	3221	4058
	791	1586	2389	3197	4035
0.9655	983	1970	2698	3982	5017
	983	1969	2967	3970	5011
1.5720	1257	2520	3796	5092	6416
	1254	2513	3787	5065	6394
2.0225	1427	2860	4306	5780	7283
	1422	2850	4295	5746	7253
2.6030	1619	3247	4891	6561	8267
	1613	3233	4872	6518	8228
3.2020	1797	3602	5427	7280	9173
	1789	3586	5404	7229	9126
3.8411	1968	3947	5945	7976	10050
	1690	3928	5919	7918	9995

Table 2. Comparison of positions $10^3\ \mathbf{x}_0$ (T) of the moving boundary.

Time	IMM	Goodman	Crank	Analytic
0.5	362	367	392	361
1.0	510	520	513	510
1.5	627	636	628	625
2.0	725	735	725	722
2.5	810	822	810	807
3.0	888	900	887	884
3.5	959	972	958	955
3.8	1000	1000	998	995

Advantages of IMM

The relative ease with which the IMM copes with a moving boundary occurring at a prescribed temperature is obvious. The boundary is always on a grid line in the (u,T) plane (in this example the line u = 0). It's position $X_o(\tau)$ is evolved by the solution with no special treatment other than the use of (15).

Another strong advantage arises if the heat parameters are temperature dependent. These need only be known for the temperatures corresponding to the grid lines chosen. The evaluation of new values at each time step, which can be time consuming when working in the (x,t) plane, is avoided.

Other applications of the IMM are being explored.

References

1. Carslaw, H.S., and Jaeger,J.C. Conduction of Heat in Solids, O.U.P., 1959, Chap.XI .
2. Crank J. Quart.J.Mech App.Math. 10 (1957) 220.
3. Douglas, J., and Gallie T.M. Duke Math. J.,22 (1955) 557.
4. Murray, W.D., and Landis, F. Trana.ASME 81 (1959) 106.
5. Albasiny, E.L.: Proc.I.E.E. 103 (1956) Series B, Parts 1-3, 158.
6. Crank,J., and Gupta,R.S., JIMA (in the press)
7. Bix, R.C., and Cizek, J. Heat Transfer 1970, Vol.1, 4th International Heat Transfer Conference, Paris Versailles, Elsevier, 1970.
8. Philip, J.R. Trans.Faraday Soo 51 (1955) 885.
9. Rose, M.E. SIAM J.of Applied Maths.15 (1967) 495.
10. Chernous'ko, F.I, Zh.Prikl.Mekh.i,Tekhn. Fis.No.2 (1969) 6.
11. Chernoua'ko, F.L. International ChenuEng.10, No.1, (l970) 42.
12. Goodman, T.R Adavances-5 in Heat Transfer, Vol.1 Academic Press, New York, 1964.
13. Coldrey, T.J. M.Tech Dissertation, Brunel University, 1966.