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A Class of C^2 Piecewise Quintic Polynomials

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A B S T R A C T

A new class of C^2 piecewise quintic interpolatory polynomials is defined. It is shown that this new class contains a number of interpolatory functions which present practical advantages, when compared with the conventional cubic spline.

1. Introduction

Given the points

$$a = x \quad < x \quad < \ldots < x_k = b, \tag{1.1}$$

and the corresponding values $y_i = y(x_i)$; $i = 0,1,\ldots,k$, let $H_3(x)$ be the piecewise cubic Hermite polynomial, with knots (1.1), which is such that

$$H_3(x_i) = y_i \quad \text{and} \quad H_3^{(1)}(x_i) = y_i^{(1)}; i = 0,1,\ldots,k.$$

Denote by s the piecewise polynomial obtained from H_3 by replacing the derivatives $y_i^{(1)}$; $i = 0,1,\ldots,k$, respectively by suitable approximations m_i; $i = 0,1,\ldots,k$. Let p_i be the cubic polynomial interpolating the function y at the points x_i, x_{i+1}, x_{i+2}, x_{i+3}, define the quadratic polynomials q_i; $i=0,1,\ldots,k-2$, by

$$\left.\begin{aligned} & q_i = p_i^{(1)}; i = 0,1,\ldots\ldots,k-3, \\ & q_{k-2} = q_{k-3} = p_{k-3}^{(1)}, \end{aligned}\right\} \tag{1.2}$$

and let the approximations m_i satisfy the relations

$$\left.\begin{aligned} & m_0 = y_0^{(1)} \\ & \alpha_i m_{i-1} + m_i + \beta_i m_{i+1} = \alpha_i q_{i-1}(x_{i-1}) + q_{i-1}(x_i) + \beta_i q_{i-1}(x_{i-1}); \\ & \qquad\qquad\qquad\qquad i = 1,2,\ldots,k-1, \\ & m_k = y_k^{(1)} \end{aligned}\right\} \tag{1.3}$$

where the α_i and β_i are real numbers. Then, by the definition of Behforooz, Papamichael and Worsey [3], s is a cubic x-spline with parameters α_i, β_i; i - 1,2,...,k-1. This definition of x-splines is a generalization of an earlier definition due to Clenshaw and Negus [5], and contains the conventional cubic spline s_I as the special case

$$2\alpha_i = h_{i+1} / (h_i + h_{i+1}), \quad 2\beta_i = 1 - 2\alpha_i; \quad i - 1, 2, \ldots, k-1,$$

where $h_i = x_i - x_{i-1}$.

Clearly, a cubic X-spline s is continuous and possesses a continuous first derivative. In general, however, $s^{(2)}$ has a jump discontinuity at each interior knot, and s_I is the only cubic X-spline with C^2 continuity on (a,b).

Regarding the quality of approximation, it is shown in [3] that for any cubic X-spline s

$$\| s-y \| = 0(h^4) \tag{1.4}$$

where $\| \cdot \|$ denotes the uniform norm on [a,b] and $h = \max_i h_i$. Since (1.4) gives the best order of uniform convergence that can be obtained by an interpolatory piecewise cubic polynomial, it follows that no cubic X-spline can achieve substantially higher accuracy than s_I. However, as it is shown in [33], there are cubic X-splines which produce results of comparable accuracy to those obtained by s_I, with much less computational effort.

In the present paper we generalize the results of [3] to the case of piecewise quintic polynomial interpolation. For this we consider the piecewise quintic Hermite polynomial H_5 with knots (1.1) and, by analogy with the definition of cubic X-splines, we define a quintic X-spline as a C^2 interpolant derived from H_5 by replacing the derivatives $y_i^{(1)}$ and $y_i^{(2)}$; $i = 0, 1, \ldots, k$, by approximations which are determined by solving a certain pair of tri-diagonal linear systems. The motivation for this generalization emerges from considering the

problem of constructing C^2 interpolants which lead to $0(h^n)$, $n \geq 5$, convergence and whose construction does not involve excessive computational effort, by comparison with the construction of the conventional cubic spline s_I. The requirements concerning the order of convergence and computational labour are imposed so that the new interpolants may compete, in terms of computational efficiency, with s_I. We show that the class of quintic X-splines, defined in this paper, contains several interpolatory functions which satisfy the above requirements.

2. Interpolatory Piecewise Quintic Polynomials

Given the set of values $y_i = y(x_i)$; $i = 0,1,\dots,k$, where x_i. are the points (1.1), let H_5 be the piecewise quintic Hermite polynomial which is such that

$$H_5(x_i) = y_i,\ H_5^{(1)}(x_i) = y_i^{(1)} \text{ and } H_5^{(2)}(x_i)\ y_i^{(2)};\ i = 0,1,\dots,k.$$

Then if $y \in C^6[a,b]$, the following optimal error bound holds

$$\| H_5 - y \| \leq \frac{1}{46{,}080} h^6 \|y^{(6)}\|, \tag{2.1}$$

where, as before, $\|\cdot\|$ denotes the uniform norm on [a,b], $h_i = x_i - x_{i-1}$; $i = 1,2,\dots,k$, and $h = \max_{1\leq i\leq k} h_i$; see e.g. Birkhoff and Priver [4].

Definition 1. Let Q be the piecewise quintic polynomial derived from H_5 by replacing the derivatives $y_i^{(1)}$ and $y_i^{(2)}$; $i = 0,1,\dots,k$, respectively by suitable approximations m. and M_i; $i=0,1,\dots,k$. Then Q will be called a piecewise quintic polynomial (p.q.p.) with derivatives

m_i and M_i ; $i = 0,1, \ldots k$.

It follows at once from the definition that Q can be written as

$$Q(x) = s(x) + Q[x_{i-1}, x_{i-1}, x_{i-1}, x_i, x_i] (x-x_{i-1})^2 (x-x_i)^2$$

$$+ Q[x_{i-1}, x_{i-1}, x_{i-1}, x_i, xi, x_i.] (x-x_{i\_1})^3 (x-x_i)^2,$$

$$x \varepsilon [x_{i\_1}, x_i]; \quad i=1,2,\ldots,k, \qquad (2.2)$$

where s is the piecewise cubic polynomial satisfying $s(x_i) = y_i$, $s^{(1)}(x_i) = m_i$; $i = 0,1,\ldots,k$, and, with the usual notation for divided differences ,

$$\left.\begin{array}{l} Q[x_{i-1}, x_{i-1}, x_i, x_i] = \dfrac{1}{2h_i^4} \ [-6(y_i - y_{i-1}) + 2h_i(m_i = 2m_{i-1}) + h_i^2 m_{i-1}] \\ \text{and} \\ Q[x_{i-1}, x_{i-1}, x_{i-1}, x_i, x_i, x_i] = \dfrac{1}{2h_i^5} \ [12(y_i - y_{i-1}) - 6h_i(m_i + m_{i-1}) + h_i^2(M_i - M_{i-1}) \ ; \\ \qquad\qquad i = 1, 2, \ldots, k. \end{array}\right\} \quad (2.3)$$

The following theorem is a trivial generalization of a result due to Hall [6]. It can be established easily by using (2.1) and the cardinal representations of H_5 and Q.

<u>Theorem 1</u> . Let Q be a p. q.p. with derivatives m_i and M_i; $i = 0,1,\ldots, k$. If $y \varepsilon C^6[a,b]$ then, for $x \varepsilon [x_{i-1}, x_i]$; $i - 1, 2, \ldots, k$,

$$|Q(x) - y(x)| \le \frac{1}{46{,}080} h^6 \left\| y^{(6)} \right\| + \frac{5}{16} h\max\{ \left| m_{i-1} - y_{i-1}^{(1)} \right|, \left| mi - y_{i-1}^{(1)} \right| \}$$

$$+ \frac{1}{32} h^2 \max\{ \left| M_{i-1} - y_{i-1}^{(2)} \right|, \left| Mi - y_i^{(2)} \right| \}. \quad (2.4)$$

The theorem shows that the best order of approximation that can be achieved by an interpolatory p.q.p. Q is

$$\| Q-y \| = 0(h^6) .$$

More specifically, the theorem shows that if the approximations m_i and M_i are such that

$$\left.\begin{aligned} m_i - y_i^{(1)} &= 0(h^r) \\ \text{and} \quad M_i - y_i^{(2)} &= 0(h^s) \end{aligned}\right\} \quad i = 0,1,\ldots, k, \tag{2.5}$$

then,

$$\left.\begin{aligned} \|Q - y\| &= (h^n), \\ \text{where} \quad n &= \min(r+1, S+2, 6. \end{aligned}\right\} \tag{2.6}$$

Since one of our requirements is that the interpolants Q satisfy (2.6) with $n \geq 5$ it follows that, for the purposes of the present paper, the approximations m. and M_i must satisfy (2.5) with $r \geq 4$ and $s \geq 3$ respectively.

Clearly a p.q.p. Q is continuous and possesses continuous first and second derivatives. In general, however, $s^{(3)}$ has a jump discontinuity $d_i^{(3)}$ at each interior knot x.. Using (2.2) and (2.3) it can be shown that

$$\begin{aligned} d_i^3 &= Q^{(3)}(x_i+) - Q^{(3)}(x_i-) \\ &= \frac{3}{h_{i+1}^3 h_i^3}\{20\,[h_i^3 y_{i+1} - (h_{i+1}^3 h_i^3)\, y_i + h_{i+1}^3 y_{i-1}] \\ &\quad - 2h_{i+1}h_i\, 4h_i^2 m_{i+1} + 6(h_i^2 - h_{i+1}^2)m_i - 4h_{i+1}^2 m_{i-1}] \\ &\quad + h_{i+1}^2 h_i^2 [h_i M_{i+1} - 3(h_i + h_{i+1})M_i + h_{i+1}M_{i-1}]\}; \\ &\qquad i = 1, 2, \ldots, k-1. \end{aligned} \tag{2.7}$$

Hence, if $y \varepsilon C^7[a,b]$,

$$\begin{aligned} d_i^{(3)} = \frac{3}{h_{i+1}^2 h_i^2} &\{h_{i+1}h_i[h_i(M_{i+1} - y_{i+1}^{(2)}) - 3(h_i + h_{i+1})(M_i - y_i^{(2)}) + h_{i+1}(M_{i-1} - y_{i-1}^{(2)})] \\ &- 2[4h_i^2(m_{i+1} - y_{i+1}^{(1)}) + 6(h_i^2 - h_{i+1}^2)m_i - y_i^{(1)}) - 4h_{i+1}^2(m_{i-1} - y_i^{(1)})]\} \\ &+ \frac{1}{5!}(h_i^3 + h_{i+1}^3)\, y_i^{(6)} + 0(h^4); \quad i = 1,2,\ldots, k-1. \end{aligned} \tag{2.8}$$

Equation (2.8) follows from (2.7) by using the result

$$20[h_i^3 y_{i+1} - (h_i^3 + h_{i+1}^3) y_i + h_{i+1}^3 y_{i-1}] - 2h_{i+1}h_i[4h_i^2 y_{i+1}^{(1)} + 6(h_i^2 - h_{i-1}^2) y_i^{(1)} - 4h_{i+1}^2 y_{i-1}^{(1)}]$$

$$+ h_{i+1}^2 \; h_i^2 \; [h_i \, y_{i+1}^{(2)} - 3(h_i \; h_{i+1}) \; y_i^{(2)} + h_{i+1} \; y_{i-1}^{(2)}] = \frac{2}{6!} \; (h_i^3 + h_{i+1}^3) \; y_i^{(6)} + 0\,(h^{10}) \; ;$$

$$i = 1, 2, \ldots, k-1 ,$$

which is established by Taylor series expansions about the point x_i. Thus the magnitude of $d_i^{(3)}$, like the order of convergence of Q, depends only on the quality of the approximations m_i and M_i. More specifically, if the derivatives m_i and M_i of Q satisfy (2.5) then

$$\left. \begin{array}{l} d_i^{(3)} = 0(h^n) , \\ \\ n = \min\{r-2, \; s-1, 3\} \; . \end{array} \right\}$$

where

(2.9)

3. Quintic X-splines

Let q.; i = 1,2,...,k-2, be the quadratic polynomials defined by (1.2). Then, by analogy with the definition of cubic X-splines of Behforooz et al [3], we define the class of quintic X-splines as follows.

<u>Definition 2</u>. Let a., β_i and γ_i, δ_i .; i = 1, 2, . . . , k - 1, be 4k-4 real numbers. Then, a p.q.p. Q whose derivatives m. and M. ; i = 0,1,...,k satisfy respectively the relations

$$\left. \begin{array}{l} m_0 = y_0^{(1)} , \\ \\ \alpha_i m_{i-1} + m_i + \beta_i m_{i+1} = \alpha_i q_{i-1}(x_{i-1}) + q_{i-1}(x_i) + \beta_i q_{i-1}(x_{i-1}); \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots, k-1, \\ \\ m_k = y_k^{(1)} , \end{array} \right\} \qquad (3.1)$$

and

$$\left.\begin{aligned} &M_0 = y_0^{(2)} ,\\ &\gamma_i M_{i-1} + M_i + \delta_i M_{i+1} = \gamma_i \, q_{i-1}^{(1)}(x_{i-1}) + q_{i-1}^{(1)}(x_i) + \delta_i \, q_{i-1}^{(1)}(x_{i-1}); \\ &\qquad\qquad\qquad\qquad\qquad i = 1,2,\ldots,k-1, \\ &M_k = y_k^{(2)} , \end{aligned}\right\} \tag{3.2}$$

will be called a quintic X-spline with parameters α_i, β_i and γ_i, δ_i ; $i = 1,2,\ldots,k-1$.

By Definition 2 the derivatives m_i and M_i ; i - 1,2,...,k-1, of a quintic X—spline Q are determined by solving the two (k-1)x(k-1) tri-diagonal linear systems defined by (3.1) and (3.2). Thus, a sufficient condition for the unique existence of Q is that its parameters α_i., β_i and γ_i , δ_i satisfy respectively the inequalities

$$|\alpha_i| + |\beta_i| < 1 ; \quad i = 1,2,\ldots,k-1 , \tag{3.3}$$

and

$$|\gamma_i| + |\delta_i| < 1 ; I = 1,2,\ldots,k-1 . \tag{3.4}$$

It follows at once from the definition that, in the representation (2.2) of a quintic X-spline Q, s is a cubic X-spline with parameters α_i, β_i, i=1,2,...,k—1 - The convergence properties of such an s are discussed fully in [3]. In particular, it is shown that if $y \, \varepsilon \, C^6[a,b]$ and (3.3) holds then the derivatives m_i of s satisfy

$$m_i - y_i^{(1)} = 0(h^r); \quad i = 1,2,\ldots,k-1 ,$$

where in general $r = 3$. However, there are several choices of α_i, β_i, for which $r = 4$ and one choice for which $r = 5$. Keeping in mind our requirements concerning the order of $\|Q-y\|$ and the amount of labour involved in computing Q, we conclude from [3: Section 4] that there

are two choices of α_i, β_i, which are of particular interest. These are the values,

$$\left.\begin{array}{l} \alpha_i^{(1)} = \dfrac{h_{i+1}(h_{i+1}+h_{i+2})}{(h_i+h_{i+1})(h_i+h_{i+1}+h_{i+2})}, \quad \beta_i^{(1)} = 0, i=1,2,\ldots\ldots,k-2, \\ \\ \alpha_{k-1}^{(1)} = 0, \beta_{k-1}^{(1)} = \dfrac{h_{k-1}(h_{k-1}+h_{k-2})}{(h_{k-1}+h_k)(h_{k-2}+h_{k-1}+h_k)} \end{array}\right\} \qquad (3.5)$$

and

$$\left.\begin{array}{l} \alpha_i^{(2)} = \dfrac{h_{i+1}^2(h_{i+1}+h_{i+2})}{(h_i+h_{i+1}+h_{i+2})(h_i+h_{i+1})^2}, \beta_i^{(2)} = \dfrac{h_i^2(h_{i+1}+h_{i+2})}{h_{i+2}(h_i+h_{i+1})^2}; \\ \qquad\qquad\qquad i = 1,2,\ldots,k-1, \end{array}\right\} \qquad (3.6)$$

$$h_{k+1} = -(h_{k-2}+h_{k-1}+h_k).$$

The values $\alpha_i^{(1)}, \beta_i^{(1)}$ reduce the three-term recurrence relation in (3.1) to a two-term relation. Thus, in this case, the derivatives m_i are determined from (3.1) by forward substitution. It is shown in [3: Section 4.5] that these m_i satisfy

$$m_i - y_i^{(1)} = 0(h^4); \quad i = 1, 2, \ldots, k\text{-}1, \qquad (3.7)$$

In particular, when the knots are equally spaced then

$$\left.\begin{array}{l} \alpha_i^{(1)} = 1/3, \ \beta_i^{(1)} = 0; \ i = 1,2,\ldots\ldots \ k-2, \\ \alpha_{k-1}^{(1)} = 0, \qquad \beta_{k-1}^{(1)} = 1/3, \end{array}\right\}$$

and, if $y \in C^7[a,b]$,

$$|m_i - y_i^{(1)}| \le \frac{1}{40} h^4 \|y^{(5)}\| + \frac{1}{240} h^5 \|y^{(6)}\| + 0(h^6); \quad i = 1,2,\ldots,k-1. \qquad (3.8)$$

The values (3.6) are the only choice of parameters α_i, β_i for which the derivatives m_i satisfy

$$m_i - y_i^{(1)} = 0(h^5);\ i = 1, 2, \ldots, k-1\ ; \tag{3.9}$$

see [3: Section 4.6]. It should be observed that, in this case, the conditions (3*3) which ensure that the tri-diagonal linear system (3.1) has a unique solution are satisfied only if

$$(h_i+h_{i+1}))\ \ (h_i-h_{i+2}) < 2h_{i+2}(h_{i+1}+h_{i+2});$$
$$i = 1,2, \ldots , k-1\ .$$

When the knots are equally spaced then

$$\alpha_i^{(2)} = 1/6, \quad \beta_i^{(2)} = 1/2; \quad i = 1,2, \ldots, k-2$$

$$\alpha_{k-1}^{(2)} = 1/2, \quad \beta_{k-1}^{(2)} = 1/6,$$

and, if $y \in C^7[a,b]$,

$$\left|m_i - y_i^{(1)}\right| \le \frac{1}{120} h^5 \left\|y^{(6)}\right\| + 0(h^6); \quad i = 1,2,\ldots, k\text{-}1 \tag{3.10}$$

We consider now the effect that the parameters γ_i, δ_i have on the quality of the second derivatives of a quintic X-spline Q. For this we assume that the parameters satisfy (3.4), let

$$\varepsilon_i = \gamma_i \{q_{i-1}^{(1)}(x_{i-1}) - y_{i-1}^{(2)}\} + \{q_{i-1}^{(1)}(x_i) - y_{i-1}^{(2)}\} + \delta_i \{q_{i-1}^{(1)}(x_{i-1}) - y_{i+1}^{(2)}\};$$
$$i = 1, 2, \ldots, k-1, \tag{3.11}$$

and denote by A the matrix of the (k-1) x (k-1) linear system defined by (3.2). Then, using a result of Lucas [7, p.5763,

$$\| A^{-1}\|_\infty \le \nu, \tag{3.12}$$

where $\nu \geq 1$ is such that

$$\left|\gamma_i\right| + \left|\delta_i\right| + 1/\upsilon \leq 1; \quad i = 1, 2, \ldots, k-1.$$

Hence, from (3.2),

$$\left|M_i - y_i^{(2)}\right| \leq \upsilon \max_i \left|\varepsilon_i\right| ; \quad i = 1,2, \ldots, k-1 \qquad (3.13)$$

Also, by Taylor series expansion about the point x_i we find that if $y \in C^6[a,b]$ then,

$$\varepsilon_i = \frac{1}{12} F_i y_i^{(4)} + \frac{1}{60} G_i y_i^{(5)} + 0(h^4) \quad i = 1, 2, \ldots, k-1, \qquad (3.14)$$

where

$$\left.\begin{aligned}
F_i = {} & \gamma_i \{-h_{i+1}(h_{i+1} + 4h_i + h_{i+2}) - h_i(2h_{i+2} + 3h_i)\} \\
& + \{h_{i+1}(2h_i - h_{i+2} - h_{i+1}) + h_i h_{i+2}\} \\
& + \delta_i \{hi + 1(2h_{i+2} - h_{i+1} - h_i) h_i h_{i+2}\}, \\
G_i = {} & \gamma_i \{-(h_{i+1} + 2h_i)(h_{i+1} + h_{i+2} - h_i)(2h_{i+1} + h_{i+2}) \\
& + h_i (7h_i^2 - 3h_{i+1}^2)\} \\
& - (h_{i+1} - h_i)(h_{i+1} + h_{i+2} - h_i)(2h_{i+1} + h_{i+2}) \\
& + \delta_i \{(2h_{i+1} + h_i)(h_{i+1} + h_{i+2} - h_i)(2h_{i+1} + h_{i+2}) \\
& + h_{i+1}(3h_i^2 - 7h_{i+1}^2)\}; \\
& \qquad i = 1, 2, \ldots, k-1,
\end{aligned}\right\} \qquad (3.15)$$

and, as in (3.6),

$$h_{k+1} = -(h_{k-2} + h_{k-1} + h_k) \qquad (3.16)$$

When the knots are equally spaced then (3.14) simplifies considerably and, if $y \; \varepsilon \; C^7[a,b]$, it gives

$$\varepsilon_i = \frac{1}{12}\{-\tilde{\gamma}_i + 1 + \tilde{\delta}_i\} h^2 y_i^{(4)} + \frac{1}{12}\{\delta_i - \gamma_i\} h^3 y_i^{(5)}$$

$$+ \frac{1}{360}\{16 \tilde{\delta}_i + 1 - 44 \tilde{\gamma}_i\} h^4 y_i^{(6)} + 0(h^5);$$

$$i = 1, 2, ..., k-1 , \tag{3.17}$$

where

$$\left.\begin{array}{l} \tilde{\gamma}_i = \gamma_i, \; \tilde{\delta}_i = \delta_i; \quad i = 1, 2, \dots, k-2 \\ \text{and} \\ \tilde{\gamma}_{k-1} = \delta_{k-1}, \; \tilde{\delta}_{k-1} = \gamma_{k-1}. \end{array}\right\} \tag{3.18}$$

The results (3.13) and (3.14)- (3.15) show that if the parameters γ_i, δ_i ; i= 1,2 ,..., k-1 of a quintic X-spline satisfy (3.4) then

$$M_i - y_i^{(2)} = 0(h^s); \qquad i = 1, 2,..., k-1,$$

where, in general $s = 2$. However, if the γ_i , d_i , are such that $F_i = 0$; i - 1,2,...,k-1 then $s = 3$, and if $F. = G_i = 0$; i = 1 ,2,. . . ,k-1 then $s = 4$.

Corresponding to the two choices (3.5) and (3.6) of the parameters α_i, βi , there are two choices of the γ_i , d_i which are of particular interest. These are the values

$$\left.\begin{array}{l} \gamma_i^{(1)} = \dfrac{h_{i+1}(2h_i - h_{i+1} - h_{i+2}) + h_i h_{i+2}}{h_{i+1}(4h_i + h_{i+1} + h_{i+2}) + h_i(3h_i + 2h_{i+2})} , \quad \delta_i^{(1)} = 0 ; \\ \qquad\qquad i = 1,2,...,k-2 , \\ \gamma_{k-1}^{(1)} = 0, \; \delta_{k-1}^{(1)} = \dfrac{h_{k-1}(2h_k - h_{k-1} - h_{k-2}) + h_k h_{k-2}}{h_{k-1}(4h_k - h_{k-1} + h_{k-2}) + h_k(3h_k + 2h_{k-2})} , \end{array}\right\} \tag{3.19}$$

and

$$\gamma_i^{(2)} = \frac{h_{i+1}A_i}{D_i(h_i + h_{i+1})} \ , \ \delta_i^{(2)} = \frac{h_i B_i}{D_i(h_i + h_{i+1})} \ ;$$

$$i = 1, 2, \ldots, k-1 \ , \qquad (3.20)$$

where

$$\left.\begin{aligned} A_i &= (h_{i+1} + h_{i+2})\ (h_{i+1} - h_i)\ \{-3h_{i+2}\ (h_i + h_{i+1}) + h\, h_{i+1}^2\} \\ &\quad + h_i h_{i+1}\ \{3\ (h_{i+1} + h_{i+2})^2 - h_{i+1}(3h_i + 4h_{i+1})\} \ . \\ B_i &= (h_{i+1} + h_{i+2})\ (h_{i+1} - h_i)\ \{h_i + h_{i+1})\ (h_i + h_{i+1} + h_{i+2}) + h_i^2\} \\ &\quad + h_i h_{i+1}\ \{3(h_{i+1} + h_{i+2})^2 - h_i(4h_i + 3h_{i+1})\}, \\ D_i &= (h_{i+1} + h_{i+2})^2 \{(h_i + 2\text{-}h_{i+1})\ (4h_i + h_{i+1}) + h_{i+2}(3h_{i+2} + h_{i+1})\} \\ &\quad + 3h_i h_{i+1}\ h_{i+2}\ (h_i + h_{i+1} + h_{i+2}) \ , \end{aligned}\right\} \qquad (3.21)$$

and h_{k+1} is given by (3.16).

The parameters $\gamma_i^{(1)}$, $\delta_i^{(1)}$ are such that, in (3.14) , $F_i = 0$, $i = 1, 2, \ldots, k-1$. Therefore, in this case,

$$M_i - y_i^{(2)} = 0(h^3) \ ; \ i = 1, 2, \ldots, k-1 \ . \qquad (3.22)$$

Also, since the values (3.19) reduce the three-term recurrence relation in (3.2) to a two-term relation, the M_i's are determined from (3.2) by forward substitution.

When the knots are equally spaced then

$$\gamma_i^{(1)} = 1/11 \ , \ \delta_i^{(1)} = 0 \ ; \quad i = 1, 2, \ldots, k-2 \ ,$$

$$\gamma_{k-1}^{(1)} = 0 \ , \qquad \delta_{k-1}^{(1)} = 1/11 \ ,$$

and, since V= 1 1/10, (3.13) and (3.17) give

$$| M_i - y_i^{(2)} | \le \frac{1}{120} h^3 \| y^{(5)} \| + \frac{11}{1200} h^4 \| y^{(6)} \| + 0(h^5) ;$$
$$i = 1,2....,k-1. \qquad (3.23)$$

The values $\gamma_i^{(2)}$, $\delta_i^{(2)}$ are the only values of γ_i ,δ_i for which $F_i = G_i = 0$; $i = 1,2,...,k-1$. This implies that (3.20) is the only choice of parameters γ_i , d_i , for which

$$M_i - y_i^{(2)} = 0(h^4) ; \; i = 1,2,..., k-1 . \qquad (3.24)$$

Clearly, $\gamma_i^{(2)}$, $\delta_i^{(2)}$ are defined only if, in (3.20), $D_i \neq 0$; $i = 1,2,...,k-1$. A sufficient condition for this to hold is that

$$h_{i+2} (3h_{i+2} + h_{i+1}) > (h_{i+1} - h_{i+2}) (4h_i + h_{i+1}) ;$$
$$i = 1,2,..., k-1 . \qquad (3.25)$$

It should be observed however that (3.25) does not imply the conditions (3.4) which ensure that the linear system (3.2) has a unique solution.

When the knots are equally spaced then

$$\gamma_i^{(2)} = \delta_i^{(2)} = 1/10 ; \; i = 1,2, . . . , k-1 ,$$

and, since $v = 5/4$, (3.13) and (3.17) give

$$| M_i - y_i^{(2)} | \le \frac{1}{160} h^4 \| y^{(6)} \| + 0(h^5) ; \; i = 1, 2, . . ., k-1 . \qquad (3.26)$$

The remainder of this paper is concerned with examining the quality of the four quintic X-splines with parameters taken from the four Possible combinations of the values $\alpha_i^{(r)}$, $\beta_i^{(r)}$ and $\gamma_i^{(s)} = \delta_i^{(s)}$; $r,s = 1,2$.

4. Quintic X-splines of special interest

We let

$$E - \| Q-y \| \quad \text{and} \quad (4.1) \quad D^{(3)} = \max_i \; | d_i^{(3)} | , \qquad (4.1)$$

Where, as in Section 2, $d_i^{(3)}$ denotes the jump discontinuity of $Q^{(3)}$ at an interior knot x_i. We also let $Q_{r,s}$ denote the quintic X-spline with parameters $\alpha_i^{(r)}, \beta_i^{(r)}$ and $\gamma_i^{(s)} = \delta_i^{(s)}$; $r, s = 1, 2$. Then, with this notation, the derivatives m_i and M_i of each of the four $Q_{r,s}$; $r,s=1,2$, are determined as follows:

(i) The m_i of $Q_{1,1}$ and $Q_{1,2}$, by forward substitution, from the lower triangular system defined by (3.1) with $\alpha_i = \alpha_i^{(1)}$, $\beta_i = \beta_i^{(1)}$, where $\alpha_i^{(1)}$, $\beta_i^{(1)}$ are the values (3.5).

(ii) The m_i of $Q_{2,1}$ and $Q_{2,2}$, by solving the tri-diagonal system defined by (3.1)with $\alpha_i = \alpha_i^{(2)}$, $\beta_i = \beta_i^{(2)}$, where $\alpha_i^{(2)}$ $\beta_i^{(2)}$ are the values (3.6).

(iii) The M_i of $Q_{1,1}$ and $Q_{2,1}$, by forward substitution, from the lower triangular system defined by (3.2) with $\gamma_i = \gamma_i^{(1)}$, $\delta_i = \delta_i^{(1)}$, where $\gamma_i^{(1)}$, $\delta_i^{(1)}$ are the values (3.19).

(iv) The M_i of $Q_{1,2}$ and $Q_{2,2}$, by solving the tri-diagonal system defined by (3.2) with $\gamma_i = \gamma_i^{(2)}$, $\delta_i = \delta_i^{(2)}$, where $\gamma_i^{(2)}$, $\delta_i^{(2)}$ are the values (3.20).

The results of the previous section in conjunction with (2.5) - (2.6) and (2.9) show that for each of the X-splines $Q_{1,1}$, $Q_{1,2}$ and $Q_{2,1}$,

$$E = 0(h^5) \qquad \text{and} \qquad D^{(3)} = 0(h^2) . \qquad (4.2)$$

These results also show that $Q_{2,2}$ is the only quintic X—spline for which

$$E = 0(h^6) \quad \text{and} \quad D^{(3)} = 0(h^3) \ . \tag{4.3}$$

we consider now the case of equally spaced knots and, for each of the four $Q_{r,s}$, we list bounds on E and on . These bounds are derived easily from (2.4), (2.8) and (3.8), (3.10), (3.23) and (3.26).

(i) Quintic X-spline $Q_{1,1}$

$$E \le \frac{31}{3,840} h^5 \| y^{(5)} \| + \frac{371}{230,400} h^6 \| y^{(6)} \| + 0(h^7) \ , \tag{4.4}$$

$$D^{(3)} \le \frac{7}{5} h^2 \| y^{(5)} \| + \frac{131}{300} h^3 \| y^{(6)} \| + 0(h^4) \ . \tag{4.5}$$

(ii) Quintic X-spline $Q_{1,2}$

$$E \le \frac{1}{128} h^5 \| y^{(5)} \| + \frac{350}{230,400} h^6 \| y^{(6)} \| + 0(h^7) \ , \tag{4.6}$$

$$D^{(3)} \le \frac{6}{5} h^2 \| y^{(5)} \| + \frac{110}{300} h^3 \| y^{(6)} \| + 0(h^4) \ . \tag{4.7}$$

(iii) Quintic X-spline $Q_{2,1}$

$$E \le \frac{1}{840} h^5 \| y^{(5)} \| + \frac{671}{230,400} h^6 \| y^{(6)} \| + 0(h^7) \ , \tag{4.8}$$

$$D^{(3)} \le \frac{1}{5} h^2 \| y^{(5)} \| + \frac{191}{300} h^3 \| y^{(6)} \| + 0(h^4) \ . \tag{4.9}$$

(iv) Quintic X-spline $Q_{2,2}$

$$E \le \frac{650}{230,400} h^6 \| y^{(6)} \| + 0(h^7) \ , \tag{4.10}$$

$$D^{(3)} \le \frac{170}{300} h^3 \| y^{(6)} \| + 0(h^4) \ . \tag{4.11}$$

5. Numerical results and discussion

In Tables 1 and 2 we present numerical results obtained by taking $y(x) = \exp(x)$,

$$x. = i/20 \; ; \qquad i = 0,1, \ldots ,20 \; , \tag{5.1}$$

and constructing each of the four quintic X-splines considered in section 4. The results listed are values of the absolute error $[Q(x) - y(x)|$, computed at various points between the knots, and the maximum values $D^{(3)}$ of the jump discontinuities in the third derivative at interior knots. The results of Tables 3 and 4 are obtained, in a similar manner, by using the same y and the unequally spaced knots

$$x_i = i^2/8^2 \; ; \qquad i - 0,1 ,. . .,8 \; . \tag{5.2}$$

For comparison purposes, we also include in the first column of each table the corresponding results obtained in [3] by using the conventional cubic spline s_I,

The theoretical results of the previous sections indicate that $Q_{2,2}$ is the most 'accurate' X-spline. These results also show that in approximating a smooth function y by a quintic X-spline, the quality of the first derivatives m. is more critical that that of the second derivatives M_i. This follows from the observation that in (2.4), the magnitudes of the coefficients associated with the terms $(m_j - y_i^{(1)})$ are larger than those associated with $(M_j - y_i^{(2)})$. For this reason we expect $Q_{2,1}$ to produce more accurate approximations that $Q_{1,2}$.

The numerical results in Tables 1 and 3 show that the X-splines $Q_{2,2}$ and $Q_{2,1}$ produce the most accurate results. They also show that there

is no significant overall difference in accuracy between the approximations due to $Q_{2,2}$ and $Q_{2,1}$ and between those of $Q_{1,2}$ and $Q_{1,1}$ This is in accordance with the theory, since when $y(x) = \exp(x)$ and the equally spaced knots (5.1) are used, then the error bounds (4.4), (4.6), (4.8) and (4.10), with the $O(h^7)$ term ignored, give $E \leq n \times 10^{-9}$ where n takes the values 7.0, 6.8, .35 and .13 respectively for each of the X-splines $Q_{1,1}$, $Q_{1,2}$, $Q_{2,1}$ and $Q_{2,2}$, A similar argument would of course explain the results corresponding to the unequally spaced knots (5,2), for which $h = \max_i h_i = 15/64$. Naturally, as h decreases the difference in accuracy between the results due to $Q_{2,}$ and $Q_{2,1}$ becomes more pronounced. However, $Q_{2,2}$ leads to a marked improvement in accuracy only if h is very small.

Of the four X-splines considered here the construction of $Q_{1,1}$ involves the least computational effort. The derivatives of this X-spline are determined by forward substitution from two lower triangular systems and this involves less computational effort than the determination of the parameters of the conventional cubic spline s_I. Also, $Q_{1,1}$ is the only X-spline in Section 4 whose unique existence is guaranteed for any distribution of the knots. For this reason, we consider $Q_{1,1}$ to be of greater practical interest that the other three X—splines considered in Section 4.

By Definition 2, the construction of a quintic X-spline requires knowledge of $y^{(1)}$ and $y^{(2)}$ at the two endpoints x_Q, x_k and, in an interpolation problem, this information is not usually available. However, by using techniques similar to those of Behforooz and Papamichael [1 and 2], the end conditions

$$m_0 = y_0^{(1)} , m_k = y_k^{(1)} ; \tag{5.3}$$

$$M_0 = y_0^{(2)} , \quad M_k = y_k^{(2)} , \tag{5.4}$$

can be replaced by conditions which use only the available function values of y at the knots whilst retaining the order of the X-spline approximation. For example, if $\pi_i(X)$ is the quartic polynomial interpolating y at the points x. , x_{i+1} , x_{i+2}, x_{i+3} and x_{i+4} ; j = 0,k-4, then the following end conditions can be used, instead of (5.3), for the construction of $Q_{1,1}$ and $Q_{1,2}$

$$m_0 = \pi_0^{(1)}(x_0), \quad m_k = \pi_{k-4}^{(1)}(x_k) . \tag{5.5}$$

Similarly, the end conditions

$$\left.\begin{aligned} m_0 + \alpha_0 m_1 &= \pi_0^{(1)}(x_0) + = \alpha_0 \pi_0^{(1)}(x_1) , \\ \alpha_k m_{k-1} + m_k &= \alpha_k \pi_{k-4}^{(1)}(x_{k-1}) + \pi_{k-4}^{(1)}(x_k) , \end{aligned}\right\} \tag{5.6}$$

can be used for the construction of $Q_{2,1}$ and $Q_{2,2}$, where, in (5.6)

$$\alpha_j = (1 + u_j + v_j + w_j + u_j v_j + v_j w_j + w_j u_j + u_j v_j w_j) / u_j v_j w_j \; ; \quad j = 0,$$

with

$$u_0 = h_2 / h_1, \quad v_0 = (h_1 u_0 + h_3) / h_1, \quad w_0 = (h_1 v_0 + h_4 / h_1 ,$$

and

$$u_k = h_{k-1}, \quad v_k = (h_k u_k + h_{k-2}) / h_k, \quad w_k = (h_k v_k + h_{k-3}) / h_k.$$

By analogy, the second derivative end conditions (5.4) can be replaced by

$$M_0 = \pi_0^{(2)}(x_0), \quad M_k = \pi_{k-4}^{(2)}(x_k) , \tag{5.7}$$

for the construction of $Q_{1,1}$ and $Q_{2,1}$ and

$$\left.\begin{aligned} M_0 + \gamma_0 M_1 &= \pi_0^{(2)}(x_0) + \gamma_0 \pi_0^{(2)}(x_1) , \\ \gamma_k M_{k-1} + M_k &= \gamma_k \pi_{k-4}^{(2)}(x_{k-1}) + \pi_{k-4}^{(2)}(x_k) , \end{aligned}\right\} \tag{5.8}$$

for $Q_{1,2}$ for $Q_{2,2}$, where in (5.8)

$$\gamma_0 = \frac{[h_1(7h_1^2 - 3h_2^2) + (h_2 + 2h_1)(h_3 + h_2 - h_1)(h_4 + h_3 + h_2 - h_1) + h_1(h_4 + 2h_3 + 3h_2 - h_1)(5h_1 + 3h_2)]}{[(h_1 - h_2)(h_2 + h_3)(h_4 + h_3 + h_2) + h_1 h_2 (h_4 + 2h_3 + 2h_2)]}$$

and γ_k is obtained from γ_0 by replacing h_j by h_{k+1-j} ; $j = 1,2,3,4$, throughout.

Table 1

Values of |Q(x)—y(x)|. (Knots as in 5.1)

X	S_I	$Q_{1,1}$	$Q_{1,2}$	$Q_{2,1}$	$Q_{2,2}$
0.01	$.674x10^{-8}$	$.114x10^{-9}$	$.120x10^{-9}$	$.733x10^{-11}$	$.803x10^{-12}$
0.02	$.151x10^{-7}$	$.564x10^{-9}$	$.593x10^{-9}$	$.334x10^{-10}$	$.402x10^{-11}$
0.09	$.705x10^{-8}$	$.497x10^{-9}$	$.529x10^{-9}$	$.364x10^{-10}$	$.472x10^{-11}$
0.22	$.189x10^{-7}$	$.446x10^{-9}$	$.366x10^{-9}$	$.797x10^{-10}$	$.519x10^{-12}$
0.36	$.990x10^{-8}$	$.840x10^{-9}$	$.799x10^{-9}$	$.369x10^{-10}$	$.412x10^{-11}$
0.62	$.281x10^{-7}$	$.683x10^{-9}$	$.563x10^{-9}$	$.117x10^{-9}$	$.245x10^{-11}$
0.93	$.374x10^{-7}$	$.152x10^{-8}$	$.148x10^{-8}$	$.102x10^{-9}$	$.621x10^{-10}$
0.96	$.184x10^{-7}$	$.213x10^{-8}$	$.219x10^{-8}$	$.230x10^{-10}$	$.381x10^{-10}$
0.99	$.179x10^{-7}$	$.276x10^{-9}$	$.291x10^{-9}$	$.102x10^{-10}$	$.510x10^{-11}$

Table 2

Values of $D^{(3)}$. (Knots as in 5.1)

	S_I	$Q_{1,1}$	$Q_{1,2}$	$Q_{2.1}$	$Q_{2,2}$
$D^{(3)}$	. 130	$.285x10^{-2}$	$.186x10^{-2}$	$.921x10^{-3}$	$.714x10^{-4}$

Table 3

Values of $|Q(x)-y(x)|$. (Knots as in 5.2)

X	S_I	$Q_{1,1}$	$Q_{1,2}$	$Q_{2,1}$	$Q_{2,2}$
0.01	$.512\times10^{-9}$	$.252\times10^{-10}$	$.380\times10^{-10}$	$.227\times10^{-11}$	$.105\times10^{-10}$
0.05	$.287\times10^{-8}$	$.200\times10^{-8}$	$.253\times10^{-8}$	$.842\times10^{-9}$	$.315\times10^{-9}$
0.1	$.804\times10^{-7}$	$.858\times10^{-8}$	$.139\times10^{-7}$	$.341\times10^{-8}$	$.194\times10^{-8}$
0.17	$.297\times10^{-6}$	$.182\times10^{-7}$	$.577\times10^{-8}$	$.172\times10^{-7}$	$.484\times10^{-8}$
0.35	$.589\times10^{-6}$	$.293\times10^{-6}$	$.352\times10^{-6}$	$.314\times10^{-7}$	$.277\times10^{-7}$
0.5	$.272\times10^{-5}$	$.758\times10^{-6}$	$.960\times10^{-6}$	$.325\times10^{-6}$	$.122\times10^{-6}$
0.6	$.325\times10^{-5}$	$.964\times10^{-6}$	$.836\times10^{-6}$	$.413\times10^{-8}$	$.123\times10^{-6}$
0.8	$.721\times10^{-5}$	$.233\times10^{-5}$	$.229\times10^{-5}$	$.194\times10^{-6}$	$.154\times10^{-6}$
0.9	$.207\times10^{-4}$	$.220\times10^{-5}$	$.212\times10^{-5}$	$.227\times10^{-6}$	$.150\times10^{-6}$

Table 4

Values of $D^{(3)}$. (Knots as in 5.2)

	S_I	$Q_{1,1}$	$Q_{1,2}$	$Q_{2,1}$	$Q_{2,2}$
$D^{(3)}$	.484	$.433\times10^{-1}$	$.324\times10^{-1}$	$.272\times10^{-1}$	$.423\times10^{-2}$

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