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Exact formulae for certain integrals arising in potential theory

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ABSTRACT

Exact formulae are derived for certain integrals arising in the solution of potential problems by integral equation methods.

1. Introduction

Let ζ_0, ζ_1 and ζ_2 be three collinear points in the complex z-plane
such that ζ_1 lies between ζ_0 and ζ_2 with $0 \leq |\zeta_1 - \zeta_0| < |\zeta_2 - \zeta_0|$.
Let the equation of the line through these three points be

$$\zeta(s) = ((s_2 - s)\zeta_1 + (s - s_1)\zeta_2)/(s_2 - s_1), \qquad (1.1)$$

where

$$|s| - |\zeta - \zeta_0| , \quad s_1 = |\zeta_1 - \zeta_0| , \quad s_2 - |\zeta_2 - \zeta_0| . \qquad (1.2)$$

Also, denote the open line segment between the points ζ_1 and ζ_2 by

$$L = \{z : z = \zeta(s), \quad s_1 < s < s_2\} \qquad (1.3)$$

and let $\bar{L}$ be the corresponding closed segment. This report is
concerned with the derivation of analytical formulae for the evaluation of the line integral

$$\begin{aligned} S(z) - S(z; \zeta_0, \zeta_1, \zeta_2, \beta) \\ = \int_{\zeta_1}^{\zeta_2} |\zeta - \zeta_0|^{-1+\beta} \operatorname{Log}(z - \zeta) |d\zeta| \\ = \int_{s_1}^{s_2} s^{-1+\beta} \operatorname{Log}(z - \zeta(s)) \, ds \end{aligned} \qquad (1.4)$$

where

$$\operatorname{Log}(z - \zeta| - \log|z - \zeta| + i \operatorname{Arg}(z - \zeta), \qquad (1.5)$$

$\beta > \frac{1}{2}$ is a rational number, $\zeta \varepsilon \bar{L}$ and z is an arbitrary point in the plane. The argument function $\operatorname{Arg}(\zeta - \zeta)$ in (1.5) is defined to be a continuous function for $\zeta \varepsilon \bar{L}$ provided $\zeta \neq z$. This implies that $\operatorname{Arg}(z - \zeta)$ does not necessarily coincide with the principal argument. In this connection we note that the principal argument of any complex number z is defined by

$$-\pi < \arg(z) \leq \pi$$

and that the corresponding principal branch of the complex logarithm is denoted by log z .

Integrals of the form (1.4) arise in the numerical solution of integral equations in potential theory and conformal mapping. For example, the methods of Symm (1966) and Christiansen (1971) require the evaluation of (1.4) with $\beta = 1$, whilst in the method of Hayes, Kahaner and Kellner (1972) 3 is an integer ≤ 3. In such methods the value of ß is determined by the type of approximation used for the replacement of the unknown boundary function. For more accurate approximations than those used in Symm (1966), Christiansen (1971) or Hayes et al (1972) ß might be an integer > 3. Furthermore, in order to overcome the effects of corner singularities, such approximations might introduce fractional values of β; see, e.g., Hough and Papamichael (1980).

The integrand in (1.4) can have the following singularities:

<u>(a) Fractional Power Singularity</u>

This type of singularity occurs when β is not an integer and $\zeta_0 = \zeta_1$. In this case (1·4) gives

$$S(z) = \int_0^{8_2} s^{-1+\beta} \text{ Log } (z - \zeta(s)) \, ds \, ,$$

and the singularity occurs because the kth derivative of $s^{-1+\beta}$, where $k = [\beta]$, becomes unbounded at $s = 0$. (Here [.] denotes the integer part).

(b) Logarithmic Singularity

This type of singularity occurs if $z \in \bar{L}$ because, at $z = \zeta$, $\log | z - \zeta |$ is Unbounded and $\mathrm{Arg}(z - \zeta)$ has a finite jump discontinuity.

For numerical computations the fractional power singularity is not very serious since it can generally be removed by a suitable change of variable; see, e.g., Davis and Rabinowitz (1975 , § 2.12.3). However, the logarithmic singularity is much more difficult to deal with in numerical work; see, e.g., Christiansen (1971) and Hsiao, Kopp and Wendland (1979). For this reason, the ability to evaluate S analytically might be of practical importance.

2. Integration Formulae

Let

$$\beta = p/q \, , \tag{2.1}$$

where p,q are relatively prime integers. The integration formulae derived in this section are all given in terms of a function $G(z,\zeta_0,\zeta)$ which, for distinct arguments, is defined by

$$G(z,\zeta_0,\zeta) = \frac{|\varsigma - \varsigma_0|^{\beta}}{\beta} \{\mathrm{Log}\,(z-\zeta) + f\,[\frac{\varsigma - \varsigma_0}{z - \varsigma_0}]\} \tag{2.2}$$

where

$$f(\xi) = -\sum_{k=0}^{0} \frac{\xi^{-\kappa}}{\beta - \kappa} - \xi^{-\beta} \sum_{k=0}^{q-1} \omega_k^{R+1} \log(1 - \xi^{1/q}\, \omega_k^{-1}), \tag{2.3}$$

$$\omega_K = \cos(2k\pi/q) + i \sin(2k\pi/q) \, , \tag{2.4}$$

$$Q = [(p-1)/q] \, , \tag{2.5}$$

$$R - p- 1 -Qq ,\qquad (2.6)$$

and, as before, [.] denotes the integer part. Although G remains bounded at $z = \zeta$, in general $G(z,\zeta_0,z)$ is not uniquely difined. For this reason we also define

$$G(z,\zeta_0,z+) = \lim_{\varepsilon\to 0+} G(z,\zeta_0,z+\varepsilon(z-\zeta_0)),\qquad (2.7)$$

$$G(z,\zeta_0,z-) = \lim_{\varepsilon\to 0+} G(z,\zeta_0,z-\varepsilon(z-\zeta_0)),\qquad (2.8)$$

where ε is real and $\varepsilon\to 0+$ means that $\in$ tends to zero through positive values. Using (2.2) -(2.8) it is possible to extend the definition of G to the case of coincident arguments. This is done by means of elementary limiting processes. The results are contained in the following lemma.

Lemma

Let G be the function defined by (2.2) - (2.8). Then:

(i) For any value of z, (2.9)

$$G(z,\zeta_0,\zeta_0) = 0 .$$

(ii) For $\zeta \neq \zeta_0$

$$G(\zeta_0, \zeta_0, \zeta) = \frac{|\varsigma-\varsigma_0|^\beta}{\beta} \{\log (\zeta_0- \zeta) - \frac{I}{\beta} \} .\qquad (2.10)$$

(iii) For $z \in L \cup \zeta 2$

$$G(z, \zeta_0, z-) = \frac{|z-\varsigma_0|^\beta}{\beta} \{\log |z- \zeta_0|+c+i \operatorname{Arg} (z - \zeta 1)\} .\qquad (2.11)$$

(iv) For $Z\in U\zeta_1$ and $z \neq \zeta_0$

$$G(z, \zeta 0, z+) = \frac{|\zeta-\varsigma_0|^\beta}{\beta} \{\log|z-\zeta_0|+c+i(\operatorname{Arg}(z-\xi_2)-\pi)\}.\qquad (2.12)$$

In (2.11) and (2.12) c is a real constant which depends only on β and is given by

$$C = \log q - \sum_{k=0}^{Q} \frac{1}{\beta - k} - \sum_{k=1}^{q-1} \omega_k^{R+1} \log(1 - \omega_k^{-1}). \tag{2.13}$$

We now derive the formulae for S(z) by considering separately the following four cases.

Case I $z \notin \bar{L}$

In this case the integrand in (1.4) does not have a logarithmic singularity.

We first assume that $z \neq \zeta_0$. Then integration by parts followed by the change of variable

$$\xi = \frac{s(\varsigma_2 - \varsigma_0)}{s2(z - \varsigma_0)} = \frac{\varsigma - \varsigma_0}{z - \varsigma_0}, \tag{2.14}$$

gives

$$S(z) = \{s_2^{\beta} \,\mathrm{Log}\,(z - \zeta_2) - s_1^{\beta} \,\mathrm{Log}(z - \zeta_1) + \mu^{\beta} T\}/\beta, \tag{2.15}$$

where

$$T = \int_{\xi_1}^{\xi_2} \frac{\xi^{\beta}}{1-\xi} \, d\xi, \tag{2.16}$$

$$\xi 2 = \frac{\varsigma_2 - \varsigma_0}{z - \varsigma_0}, \quad \xi_1 = \frac{\varsigma_1 - \varsigma_0}{z - \varsigma_0} = \frac{s_1 \xi_2}{s_2}, \tag{2.17}$$

$$\mu = \frac{s(z - \varsigma_0)}{\varsigma_2 - \varsigma_0} = \frac{s}{\xi}. \tag{2.18}$$

We observe that the restriction $z \notin L$ implies that $\xi \neq 1$.

In order to evaluate the integral T we introduce the change of variable

$$\eta - \xi^{1/q}, \tag{2.19}$$

where the principal branch is always chosen. Then (2.16) gives

$$T = q\int_{\eta_1}^{\eta_2} \frac{\eta^{p+q-1}}{1+\eta^q}\, d\eta$$

$$= -q\sum_{k=O}^{O} \int_{\eta_1}^{\eta_2} \eta^{R+kq}\, d\eta + \sum_{k=O}^{q-1} \int_{\eta_1}^{\eta_2} \frac{\omega_k^{R+1}}{\omega_k - \eta}\, d\eta \quad , \tag{2.20}$$

where

$$\eta_2 = \xi_2^{1/q}\ , \qquad \eta_1 = \xi_1^{1/q} = (s_1/s_2)^{1/q}\ \eta_2\ , \tag{2.21}$$

and w_k, Q and R are given by (2.4) - (2.6). The result (2.20) is obtained by using the standard technique of long division and separation into partial fractions.

The evaluation of the integrals in the first summation of (2.20) is, of course, straight forward. However, some care must be exercised in evaluating the integrals in the second summation, in order to ensure that the correct branch of each integral is chosen. This is done as follows.

By definition, the principal branch of the complex logarithm is

$$\text{Log } z = \int_1^z \frac{d\gamma}{\gamma} \tag{2.22}$$

where the integration path is the line segment from 1 to z, not containing the origin; see, e.g., Hardy (1963, § 231). Therefore

$$\int_{\eta 1}^{\eta 2} \frac{d\eta}{\omega_k - \eta} = -\int_1^{(\omega\omega - \eta 2)/(\omega k - \mu 1)} \frac{d\gamma}{\gamma}$$

$$= -\log\left[\frac{1-\eta_2\omega_k^{-1}}{1-\eta_1\omega_k^{-1}}\right]. \tag{2.23}$$

Also, using the fact that η_1 is a real positive multiple of η_2 it may be shown, that

$$\log\left[\frac{1-\eta_2\omega_k^{-1}}{1-\eta_1\omega_k^{-1}}\right] = \log(1-\eta_2\omega_k^{-1}) - \log(1-\eta_1\omega_k^{-1}). \tag{2.24}$$

Finally it follows from (1.2), (2.2) - (2.6) and (2.15) - (2.24) that, if $z \notin \bar{L}$ and $z \neq \zeta_0$,

$$\begin{aligned} S(z) &= S(z;\zeta_0,\zeta_1,\zeta_2,\beta) \\ &= G(z,\zeta_0,\zeta_2) - G(z,\zeta_0,\zeta_1) \end{aligned} \quad . \tag{2.25}$$

Since $G(z,\zeta_0,\zeta)$ is continuous at $z = \zeta_0$ it follows that the results (2.25) also applies when $z = \zeta_0$; i.e. for all $z \notin \bar{L}$.

Case II $z \in L$.

In this case z is an interior point of the line segment from ζ_1 to ζ_2 and the integrand in (1.4) has a logarithmic singularity.

To evaluate the integral we take the Cauchy principal value of S(z); i.e.

$$S(z) = \lim_{\varepsilon \to 0+} \{S(z;\zeta_0,\zeta_1,z-\epsilon(z-\zeta_0),\beta) + s(z;\zeta_0,z+\epsilon(z-\zeta_0),\zeta_2,\beta)\} . \tag{2.26}$$

Then it follows from (2.7), (2.8) and (2.25) that, for $z \in L$,

$$S(z) \ -G(z,\zeta_0,\zeta_2) \ -G(z,\zeta_0,\zeta_1) + G(z,\zeta_0,z-) \ -G(z,\zeta_0,z+), \tag{2-27}$$

where, from (2.11) and (2.12),

$$G(z,\zeta_0,z-) \ -G(z,\zeta_0,z+) = i(\mathrm{Arg}(z-\zeta_1) \sim \mathrm{Arg}(z-\zeta_2)+\pi). \tag{2.28}$$

The right hand side of (2.28) is either 0 or $2\pi i$, depending on the definition or $\mathrm{Arg}(z-\zeta)$.

Case III $z = \zeta_2$.

As in case II the integrand has a logarithmic singularity and we consider

$$S(\zeta_2) = \lim_{\varepsilon \to 0+} S(\zeta_2;\zeta_0,\zeta_1,\zeta_2-\varepsilon(\zeta_2-\zeta_0),\beta) \quad . \tag{2.29}$$

Then it follows from (2.8) and (2.25) that

$$S(\zeta_2) = G(\zeta_2,\zeta_0,\zeta_2-) - G(\zeta_2,\zeta_0,\zeta_1) . \tag{2.30}$$

<u>Case IV</u> $z = \zeta_1$.

Again the integrand has a logarithmic singularity and we consider

$$S(\zeta_1) = \lim_{\varepsilon \to 0+} S(\zeta_1;\zeta_0,\zeta_1+\varepsilon(\zeta_1-\zeta_0),\zeta_2,\beta) . \tag{2.31}$$

Then it follows from (2.7) and (2.25) that

$$S(\zeta_1) = G(\zeta_1,\zeta_0,\zeta_2) - G(\zeta_1,\zeta_0,\zeta_1+) \tag{2.32}$$

3. Discussion

The formulae of section 2 have been used successfully by Hough and Pspamichael (1980) for the evaluation of all the integrals that arise in the numerical solution of an integral equation method for conformal mapping. In this method spline functions of various degrees and singular functions involving fractional powers of $|\zeta-\zeta_0|$ are used to approximate the unknown source density of the integral equation.

We point out that if $|\xi|$ is small then, for computational work, care must be taken in evaluating $f(\xi_.)$ from (2.3). The reason for this is as follows. Although $f(\xi_.) \to 0$ as $\xi_. \to 0$, the evaluation of $f(\xi_.)$ when $|\xi|$ is small involves the cancellation of terms of large magnitude and opposite sign. This clearly can lead to a loss of significance, which becomes more pronounced as ß increases- For practical applications this source of numerical inaccuracy might not be very serious. For example, no difficulties were encountered in performing the integrations needed in the applications of Hough and Papamichael (1980). However, in other applications it might be necessary to use an alternative formula for the evaluation of $f(\xi)$ when $|\xi|$ is small. For $|\xi| << |$ such an alternative is provided by the rapidly converging series representation

$$f(\xi) = \sum_{k=1}^{\infty} \frac{\xi^k}{k+\beta} ; \ |\xi| < 1$$

which does not lead to loss of accuracy.

Finally we point out that the exact formula for Re $\{S(z;\zeta_1,\zeta_2,1)\}$ is derived in Jaswon and Symm (1977, p. 149). Also the exact formula for Im $\{S(z;\zeta_1,\zeta_1,\zeta_2,1)\}$ was derived by Meek (1976).

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References

1. Christiansen, S.: Numerical Solution of an Integral Equation with a Logarithmic Kernel. BIT- 11, 276-287 (1971).

2, Davis, P.J. and Rabinowitz, P.: Methods of Numerical Integration. New York.: Academic Press 1975.

3. Hardy, G.H.; A Course of Pure Mathematics. Cambridge University Press 1963.

4. Hayes, J.K. , Kahaner, D.K. and Kellner, R.G.: An Improved Method for Numerical Conformal Mapping. Math. Comp 26, 327-334 (1972).

5. Hough, D.M. and Papamichael, N. : The Use of Splines and Singular Functions in an Integral Equation Method for Conformal Mapping. Technical Report TR/99, Dept, of Maths, Brunei University 1980.

6. Jaswon, M.A. and Symm, G.T.: Integral Equation Methods in Potential Theory and Elastostatics. London: Academic Press 1977.

7. Meek, D.: Private Communication (1976).

8. Symm, G.T. : An Integral Equation Method in Conformal Mapping. Numer. Math. 9, 250-258 (1966).

9. Hsiao, G.C, Kopp, P. and Wendland, W.L.: A Galerkin Collocation Method for some Integral Equations of the First Kind. Preprint - Nr. 514, Maths Department, Technische Hochschule Darmstadt, 1979.