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Chapter 18
Rescaling and Trace Operators in Fractional Sobolev Spaces on Bounded Lipschitz Domains with Periodic Structure

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18.1 Introduction

We develop tools, which can be useful for elliptic boundary value problems on domains with a periodic structure with holes involving some linear or non-linear Robin-type conditions on the oscillating interface ([GrPeSh12], [KKG17], [GaNRK16]), or contact problems (see [GaMe18], [GrOr18]).
This paper considers rescaling of functions from the Bessel potential, Riesz potential, and Sobolev-Slobodetskii spaces on the boundary or in the domain and also rescaling of the boundary trace operator.

Denote by Ω a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Let $Y := (0,1)^n$ be the reference cell. We denote by T a hole, that is an open set, which closure is strictly included in Y and let $Y^* := Y \setminus \overline{T}$ (see Figure 1). Let ∂T be the Lipschitz boundary of T and ν be the outward to Y^* unit normal vector on the boundary ∂T. Recall, e.g., from [CDDGZ12] that in the periodic setting, every point $z \in \mathbb{R}^n$ can be written as $z = \left[z\right] + \left\{z\right\}, \quad [z] \in \mathbb{Z}^n, \quad \{z\} \in Y$. Here the integer function $\left[\cdot\right]$ for a vector means the floor function $\lfloor\cdot\rfloor$ for each of its components. Denote $\Xi_\varepsilon = \left\{\xi \in \mathbb{Z}^n \mid \varepsilon\xi + \varepsilon Y \subset \Omega\right\}$, $\widehat{\Omega}_\varepsilon = \text{interior}\left\{\bigcup_{\xi\in\Xi_\varepsilon}\left(\varepsilon\xi + \varepsilon\overline{Y}\right)\right\}$, $\Lambda_\varepsilon = \Omega \setminus \overline{\widehat{\Omega}_\varepsilon}$, i.e., the set Λ_ε contains the parts of the cells intersecting the boundary $\partial\Omega$.

Let us introduce the notations for the unions of all holes in the interior cells, $T_\varepsilon := \left\{x \in \widehat{\Omega}_\varepsilon \;\middle|\; \left\{\frac{x}{\varepsilon}\right\} \in T\right\}$, for the hole boundaries, $\partial T_\varepsilon := \left\{x \in \widehat{\Omega}_\varepsilon \;\middle|\; \left\{\frac{x}{\varepsilon}\right\} \in \partial T\right\}$, in $\widehat{\Omega}_\varepsilon$ and for the remaining part, $\widehat{\Omega}^*_\varepsilon = \widehat{\Omega}_\varepsilon \setminus \overline{T_\varepsilon}$. Let also $\Omega^*_\varepsilon = \Omega \setminus \overline{T_\varepsilon}$.

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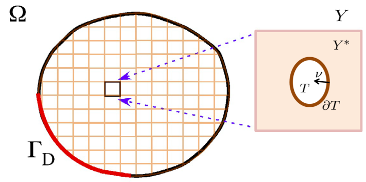

Fig. 18.1 Bounded domain with periodically distributed holes.

18.2 Function spaces

For an arbitrary non-empty open subspace Ω' of $\mathbb{R}^m$, let $W_2^s(\Omega')$, $s \geq 0$ denote the Sobolev-Slobodetskii space, cf. e.g., [McL00]. If s is an integer, the space coincides with the Sobolev space and

$$\|u\|^2_{W_2^s(\Omega')} := \sum_{|\alpha| \leq s} \int_{\Omega'} |\partial^\alpha u(x)|^2 dx.$$

If s is not an integer,

$$\|u\|^2_{W_2^s(\Omega')} := \|u\|^2_{W_2^{[s]}(\Omega')} + \|u\|'^2_{W_2^s(\Omega')},$$

and the Slobodetskii seminorm is defined as

$$\|u\|'^2_{W_2^s(\Omega')} := \sum_{|\alpha|=[s]} \int_{\Omega'} \int_{\Omega'} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{m+2\mu}} dx dy, \quad \mu := s - [s]. \tag{18.1}$$

Let $S(\mathbb{R}^m)$ be the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^m$. Let $S^*(\mathbb{R}^m)$ denote the space of sequentially-continuous linear functionals on $S(\mathbb{R}^m)$ (temperate distributions). Let us denote by

$$\hat{u}(\eta) \equiv \mathscr{F}u(\eta) := \int_{\mathbb{R}^m} u(x) e^{-i2\pi\eta\cdot x} dx, \quad u(x) \equiv \mathscr{F}^{-1}\hat{u}(x) := \int_{\mathbb{R}^m} \hat{u}(\eta) e^{i2\pi\eta\cdot x} d\eta,$$

the direct and inverse Fourier transforms, respectively. These definitions, applicable to functions from $L_1(\mathbb{R}^m)$, are easily extended to $S^*(\mathbb{R}^m)$, see e.g. [McL00, p. 72].

Let us denote $\rho(\eta) := (1+|\eta|^2)^{1/2}$. For $s \in \mathbb{R}$, we define the Bessel potential operator of order s, $J^s : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$,

$$J^s u(x) := \mathscr{F}^{-1}(\rho^s \mathscr{F}u)(x) = \int_{\mathbb{R}^m} (1+|\eta|^2)^{s/2} \hat{u}(\eta) e^{i2\pi\eta\cdot x} d\eta \ \text{ for } x \in \mathbb{R}^m,$$

which is extended, in the distribution sense, to the operator $J^s : S^*(\mathbb{R}^m) \to S^*(\mathbb{R}^m)$.

Let $s \in \mathbb{R}$, $H^s(\mathbb{R}^m) = \{u \in S^*(\mathbb{R}^m) : J^s u(x) \in L_2(\mathbb{R}^m)\}$ denote the Bessel potential space equipped with the norm $\|u\|_{H^s(\mathbb{R}^m)} = \|J^s u\|_{L_2(\mathbb{R}^m)} = \|\rho^s \hat{u}\|_{L_2(\mathbb{R}^m)}$, cf., e.g, [McL00, p. 75-76].

Similarly, let us define the Riesz potential operator

$$\mathscr{J}^s u(x) := \mathscr{F}^{-1}(|\eta|^s \mathscr{F}u)(x) := \int_{\mathbb{R}^m} |\eta|^s \hat{u}(\eta) e^{i2\pi\eta\cdot x} d\eta \quad \text{for } x \in \mathbb{R}^m.$$

Let $h^s(\mathbb{R}^m)$ denotes the Riesz potential space, i.e., the completion in the norm $\|u\|_{h^s(\mathbb{R}^m)} = \|\mathscr{J}^s u\|_{L_2(\mathbb{R}^m)} = \||\eta|^s \hat{u}\|_{L_2(\mathbb{R}^m)}$ of the space of infinitely smooth functions having compact supports in $\mathbb{R}^m$.

If $s > 0$, then $H^s(\mathbb{R}^m) \subset h^s(\mathbb{R}^m)$ and $\|u\|_{h^s(\mathbb{R}^m)}$ is equivalent to the Sobolev-Slobodetskii semi-norm $\|u\|'_{W_2^s(\mathbb{R}^m)}$, see, e.g., Theorem 4 in [Ma11, Section 10.1.2]. Particularly, if $0 < s < 1$, then

$$\|u\|'^2_{W_2^s(\mathbb{R}^m)} = a_{s,m} \|u\|^2_{h^s(\mathbb{R}^m)}, \tag{18.2}$$

where $a_{s,m}$ is a number depending only on s and m, which is finite and positive for any $s \in (0,1)$, see e.g. [McL00, Lemma 3.15], hence

$$\|u\|^2_{W_2^s(\mathbb{R}^m)} = \|u\|^2_{L_2(\mathbb{R}^m)} + \|u\|'^2_{W_2^s(\mathbb{R}^m)} = \|u\|^2_{L_2(\mathbb{R}^m)} + a_{s,m} \|u\|^2_{h^s(\mathbb{R}^m)}. \tag{18.3}$$

On the other hand, from the inequality

$$2^{s-1}(1+\psi^s) \le (1+\psi)^s \le 1+\psi^s, \quad \forall\ \psi \in (0,\infty), \quad s \in [0,1]$$

we obtain the following norm equivalence inequalities for any $s \in [0,1]$,

$$2^{s-1}[\|u\|^2_{L_2(\mathbb{R}^m)} + \|u\|^2_{h^s(\mathbb{R}^m)}] \le \|u\|^2_{H^s(\mathbb{R}^m)} \le \|u\|^2_{L_2(\mathbb{R}^m)} + \|u\|^2_{h^s(\mathbb{R}^m)}. \tag{18.4}$$

For any non-empty open set $\Omega \subset \mathbb{R}^m$, $H^s(\Omega) := \{u = U|_\Omega \text{ for some } U \in H^s(\mathbb{R}^m)\}$. This space is equipped with the norm $\|u\|_{H^s(\Omega)} = \inf_{U|_\Omega = u,\ U \in H^s(\mathbb{R}^m)} \|U\|_{H^s(\mathbb{R}^m)}$. Also for the space $h^s(\Omega)$ we define the norm in the similar way, $\|u\|_{h^s(\Omega)} = \inf_{U|_\Omega = u,\ U \in h^s(\mathbb{R}^m)} \|U\|_{h^s(\mathbb{R}^m)}$. Moreover, one can prove that $h^s(\Omega) = H^s(\Omega)$ if the domain Ω is bounded and $-m/2 < s < m/2$, cf. [Du77, Section 1.3].

We further follow the notations of [McL00, p. 98] for the definition of Bessel potential spaces on Lipschitz manifolds. Let ∂T be a Lipschitz boundary, and for a partition of unity $\{\phi_j\}$, ∂T is locally a Lipschitz hypograph of some function ζ_j up to some rigid motion $\kappa_j \equiv \omega_j(\cdot - a_j) \colon \mathbb{R}^n \to \mathbb{R}^n$, where $a_j \in \mathbb{R}^n$ and ω_j is a rotation. Let $u_j := u\phi_j$. Then $u = \sum_j u_j$ and for $s \in [-1,1]$ we have the following definition of the Bessel potential norm on the boundary,

$$\|u\|^2_{H^s(\partial T)} \equiv \sum_j \|u_j(\kappa_j^{(-1)}(\cdot, \zeta_j(\cdot)))\sqrt{1+|\nabla\zeta_j(\cdot)|^2}\,\|^2_{H^s(\mathbb{R}^{n-1})}, \tag{18.5}$$

We recall that H^s coincide with the Sobolev–Slobodetskii spaces W_2^s for any non-negative s. Replacing H^s with h^s or W_2^s in (18.5), we arrive at definitions of the norms in $h^s(\partial T)$ and $W_2^s(\partial T)$, respectively, in terms of their counterparts on $\mathbb{R}^{n-1}$. The same argument is valid, of course, if we replace ∂T with ∂T_ε. This implies an extension of the equality (18.2) to T and T_ε,

$$\|u\|'^2_{W_2^s(T)} = a_{s,n-1}\|u\|^2_{h^s(T)}, \quad \|u\|'^2_{W_2^s(T_\varepsilon)} = a_{s,n-1}\|u\|^2_{h^s(T_\varepsilon)}, \tag{18.6}$$

where $a_{s,n-1}$ is a number still depending only on s and $n-1$, which is finite and positive for any $s \in (0,1)$.

Let $\widehat{v(\varepsilon^{-1}\cdot)}(\eta) := \int_{\mathbb{R}^m} v(\varepsilon^{-1}x)e^{-i2\pi\eta\cdot x}\,dx$, $\hat{v}(\varepsilon\eta) := \int_{\mathbb{R}^m} v(y)e^{-i2\pi\varepsilon\eta\cdot y}\,dy$. Then, changing the variables, we evidently have,

$$\widehat{v(\varepsilon^{-1}\cdot)}(\eta) = \varepsilon^m \hat{v}(\varepsilon\eta), \quad \widehat{v(\varepsilon\cdot)}(\eta) = \varepsilon^{-m}\hat{v}(\varepsilon^{-1}\eta), \ \forall \eta \in \mathbb{R}^m. \tag{18.7}$$

We will employ these relations for $m = n$ and $m = n-1$.

Let $\varepsilon \in (0,\infty)$. If $\alpha \in \mathbb{R}$, and $v \in h^\alpha(\mathbb{R}^m)$, then the substitution $\bar{\eta} = \varepsilon\eta$ gives

$$\|v(\varepsilon^{-1}\cdot)\|^2_{h^\alpha(\mathbb{R}^m)} = \int_{\mathbb{R}^m} |\widehat{v(\varepsilon^{-1}\cdot)}(\eta)|^2|\eta|^{2\alpha}\,d\eta = \varepsilon^{2m}\int_{\mathbb{R}^m} |\hat{v}(\varepsilon\eta)|^2|\eta|^{2\alpha}\,d\eta$$
$$= \varepsilon^{m-2\alpha}\int_{\mathbb{R}^m} |\hat{v}(\bar{\eta})|^2|\bar{\eta}|^{2\alpha}\,d\bar{\eta} = \varepsilon^{m-2\alpha}\|v\|^2_{h^\alpha(\mathbb{R}^m)}. \tag{18.8}$$

Replacing ε with $1/\varepsilon$ we obtain $\|v(\varepsilon\cdot)\|^2_{h^\alpha(\mathbb{R}^m)} = \varepsilon^{-m+2\alpha}\|v\|^2_{h^\alpha(\mathbb{R}^m)}$.

Definition 1. For $s \in \mathbb{R}$, let us introduce the following ε-dependent norm in the Bessel potential space $H^s(\mathbb{R}^m)$,

$$\|\phi\|^2_{H^s_\varepsilon(\mathbb{R}^m)} := \int_{\mathbb{R}^m} [\rho(\varepsilon\eta)]^{2s}|\hat{\phi}(\eta)|^2\,d\eta,$$

where $\rho(\varepsilon\eta) = (1+|\varepsilon\eta|^2)^{1/2}$, $\varepsilon \neq 0$.

For a domain $\Omega \subset \mathbb{R}^m$, this norm generates the corresponding ε-dependent norm in the space $H^s(\Omega)$, $\|\phi\|^2_{H^s_\varepsilon(\Omega)} := \inf_{\Phi\in H^s(\mathbb{R}^m): r_\Omega\Phi=\phi} \|\Phi\|^2_{H^s_\varepsilon(\mathbb{R}^m)}$, $s \in \mathbb{R}$.

It is easy to verify that due to the first relation in (18.7),

$$\|\phi(\varepsilon^{-1}\cdot)\|^2_{H^s_\varepsilon(\mathbb{R}^m)} = \varepsilon^m\|\phi\|^2_{H^s(\mathbb{R}^m)}. \tag{18.9}$$

Recall that the volume of the unite cell Y is $|Y| = 1$. Let us provide the following two definitions and two propositions from [CDDGZ12].

Definition 2. Let $p \in [1,+\infty]$ and ϕ be Lebesgue-measurable on $\widehat{\Omega}^*_\varepsilon$ and extended by zero in $\Omega^*_\varepsilon \setminus \widehat{\Omega}^*_\varepsilon$. The unfolding operator $\mathscr{T}_\varepsilon$ from $L_p(\widehat{\Omega}^*_\varepsilon)$ into $L_p(\Omega \times Y^*)$ is defined by

$$\begin{cases} \mathscr{T}_\varepsilon(\phi)(x,y) = \phi(\varepsilon[x/\varepsilon]+\varepsilon y) & \text{for a.e. } (x,y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ \mathscr{T}_\varepsilon(\phi)(x,y) = 0 & \text{for a.e. } (x,y) \in \Lambda_\varepsilon \times Y^*. \end{cases}$$

Proposition 1. *Let* $p \in [1,+\infty]$.
(i) If $\phi \in L_p(\Omega^*_\varepsilon)$ *then* $\|\mathscr{T}_\varepsilon(\phi)\|_{L_p(\Omega\times Y^*)} = \|\mathbf{1}_{\widehat{\Omega}^*_\varepsilon}\phi\|_{L_p(\Omega^*_\varepsilon)} \leq \|\phi\|_{L_p(\Omega^*_\varepsilon)}$.
(ii) If $\phi \in W^1_p(\Omega^*_\varepsilon)$ *then* $\nabla_y\mathscr{T}_\varepsilon(\phi)(x,y) = \varepsilon\mathscr{T}_\varepsilon(\nabla\phi)(x,y)$ *for a.e.* $(x,y) \in \Omega \times Y^*$

and $\|\nabla_y \mathscr{T}_\varepsilon(\phi)\|_{L_p(\Omega\times Y^*)} = \varepsilon\|\mathbf{1}_{\widehat{\Omega}_\varepsilon^*}\nabla\phi\|_{L_p(\Omega_\varepsilon^*)}$.
Here $\mathbf{1}_{\widehat{\Omega}_\varepsilon^*}$ *is the characteristic function of the set* $\widehat{\Omega}_\varepsilon^*$.

Definition 3. Let $p \in [1,+\infty]$. The operator $\mathscr{T}_\varepsilon^b$ from $L_p(\partial T_\varepsilon)$ into $L_p(\Omega\times\partial T)$ is defined by

$$\begin{cases} \mathscr{T}_\varepsilon^b(\phi)(x,y) = \phi(\varepsilon[x/\varepsilon]+\varepsilon y) & \text{for a.e. } (x,y)\in\widehat{\Omega}_\varepsilon\times\partial T, \\ \mathscr{T}_\varepsilon^b(\phi)(x,y) = 0 & \text{for a.e. } (x,y)\in\Lambda_\varepsilon\times\partial T. \end{cases}, \qquad \forall\phi\in L_p(\partial T_\varepsilon).$$

Proposition 2. *Let* $p\in[1,+\infty]$ *and* $\phi\in L_p(\partial T_\varepsilon)$. *Then*

$$\int_{\Omega\times\partial T}\mathscr{T}_\varepsilon^b(\phi)(x,y)\,dx d\sigma_y = \varepsilon\int_{\partial T_\varepsilon}\phi(x)\,d\sigma_x,\ \|\mathscr{T}_\varepsilon^b(\phi)\|_{L_p(\Omega\times\partial T)} = \varepsilon^{1/p}\|\phi\|_{L_p(\partial T_\varepsilon)}.$$

18.3 Rescaling norms on oscillating Lipschitz manifold

Definition 4. Similar to (18.5), we will employ the following norms on $\partial T_\varepsilon = \cup_{\xi\in\Xi_\varepsilon}(\varepsilon\xi+\varepsilon\partial T)$,

$$\|u\|^2_{h^\alpha(\partial T_\varepsilon)} := \sum_{\xi\in\Xi_\varepsilon}\sum_j \|u_{\varepsilon,\xi,j}(\kappa^{(-1)}_{\varepsilon,\xi,j}(\cdot,\zeta_{\varepsilon,\xi,j}(\cdot)))\sqrt{1+|\nabla\zeta_{\varepsilon,\xi,j}(\cdot)|^2}\,\|^2_{h^\alpha(\mathbb{R}^{n-1})},$$

$$\|u\|^2_{H^\alpha_\varepsilon(\partial T_\varepsilon)} := \sum_{\xi\in\Xi_\varepsilon}\sum_j \|u_{\varepsilon,\xi,j}(\kappa^{(-1)}_{\varepsilon,\xi,j}(\cdot,\zeta_{\varepsilon,\xi,j}(\cdot)))\sqrt{1+|\nabla\zeta_{\varepsilon,\xi,j}(\cdot)|^2}\,\|^2_{H^\alpha_\varepsilon(\mathbb{R}^{n-1})}. \tag{18.10}$$

Is is evident that the norms $\|\cdot\|_{H^s_\varepsilon(\partial T)}$ and $\|\cdot\|_{H^s(\partial T)}$ are equivalent if $\varepsilon\neq 0$, although with the equivalence inequality constants depending on ε.

In Definition 4, $u_{\varepsilon,j}(x) := u(x)\phi_{\varepsilon,j}(x)$, while $\phi_{\varepsilon,j}$, $\kappa_{\varepsilon,j}$, $\zeta_{\varepsilon,j}$ are some periodic families of partitions of unity, local rigid rotations and local Lipschitz hypographs. To this end, we can exploit the families ϕ_j, κ_j, ζ_j, associated with ∂T, and set

$$\phi_{\varepsilon,j}(x) := \phi_j(\{x/\varepsilon\})\,,\ \ \zeta_{\varepsilon,j}(x) := \varepsilon\zeta_j(\{x/\varepsilon\})\,.$$

where, as before, $\{\cdot\}$ denotes the fractional part of the vector (components). Moreover, if $\kappa_j(x) = \omega_j(x-a_j)$, we also set $\kappa_{\varepsilon,j}(x) := \varepsilon\omega_j(\{x/\varepsilon\}-a_j)$. Note that

$$x = \kappa^{(-1)}_{\varepsilon,\xi,j}(\bar{x}) = \varepsilon\left[\omega_j^{(-1)}(\bar{x}/\varepsilon)+a_j\right]+\varepsilon\xi = \varepsilon\kappa_j^{(-1)}(\bar{x}/\varepsilon)+\varepsilon\xi\,.$$

As a consequence,

$$\begin{aligned}\kappa^{(-1)}_{\varepsilon,\xi,j}\left(\overline{x}',\zeta_{\varepsilon,j}(\overline{x}')\right) &= \varepsilon\left[\kappa_j^{(-1)}\left(\overline{x}'/\varepsilon,\frac{1}{\varepsilon}\varepsilon\zeta_j(\overline{x}'/\varepsilon)\right)+\xi\right] \\ &= \varepsilon\xi+\varepsilon\kappa_j^{(-1)}\left(\overline{x}'/\varepsilon,\zeta_j(\overline{x}'/\varepsilon)\right).\end{aligned}$$

Moreover,

$$\phi_{\varepsilon,j}\left(\kappa^{(-1)}_{\varepsilon,\xi,j}\left(\overline{x}',\zeta_{\varepsilon,j}(\overline{x}')\right)\right)=\phi_{\varepsilon,j}\left(\varepsilon\xi+\varepsilon\kappa_j^{(-1)}\left(\overline{x}'/\varepsilon,\zeta_j(\overline{x}'/\varepsilon)\right)\right)$$
$$=\phi_j\left(\kappa_j^{(-1)}\left(\overline{x}'/\varepsilon,\zeta_j(\overline{x}'/\varepsilon)\right)\right).$$

Finally, $\sqrt{1+|\nabla\zeta_{\varepsilon,j}(\overline{x}')|^2}=\sqrt{1+\varepsilon^2\frac{1}{\varepsilon^2}|\nabla_{\overline{y}'}\zeta_j(\overline{x}'/\varepsilon)|^2}=\sqrt{1+|\nabla_{\overline{y}'}\zeta_j(\overline{x}'/\varepsilon)|^2}$.

Let us return to the geometric setting from Sec. 18.1 and prove the following proposition.

Theorem 1. *Let $u\in H^\alpha(\partial T_\varepsilon)$, $-1\le\alpha\le 1$. Then*

$$\|u\|^2_{h^\alpha(\partial T_\varepsilon)}=\varepsilon^{-1-2\alpha}\|\mathscr{T}^b_\varepsilon(u)\|^2_{L_2(\Omega,h^\alpha(\partial T))}:=\varepsilon^{-1-2\alpha}\int_\Omega\|\mathscr{T}^b_\varepsilon(u)(x,\cdot)\|^2_{h^\alpha(\partial T)}\,dx. \tag{18.11}$$

Proof. By (18.8) and taking into account that $|\varepsilon Y|=\varepsilon^n$, we obtain

$$\begin{aligned}
&\sum_j\|u_{\varepsilon,j}(\kappa^{(-1)}_{\varepsilon,\xi,j}(\cdot,\zeta_{\varepsilon,j}(\cdot)))\sqrt{1+|\nabla\zeta_{\varepsilon,j}(\cdot)|^2}\,\|^2_{h^\alpha(\mathbb{R}^{n-1})}\\
=&\sum_j\|u\Big(\varepsilon\xi+\varepsilon\kappa_j^{(-1)}(\cdot/\varepsilon,\zeta_j(\cdot/\varepsilon))\Big)\phi_j\Big(\kappa_j^{(-1)}(\cdot/\varepsilon,\zeta_j(\cdot/\varepsilon))\Big)\\
&\qquad\times\sqrt{1+|\nabla\zeta_j(\cdot/\varepsilon)|^2}\,\|^2_{h^\alpha(\mathbb{R}^{n-1})}\\
=&\sum_j\varepsilon^{n-1-2\alpha}\|u\Big(\varepsilon\xi+\varepsilon\kappa_j^{(-1)}(\cdot,\zeta_j(\cdot))\Big)\phi_j\Big(\kappa_j^{(-1)}(\cdot,\zeta_j(\cdot))\Big)\sqrt{1+|\nabla\zeta_j(\cdot)|^2}\,\|^2_{h^\alpha(\mathbb{R}^{n-1})}\\
=&\varepsilon^{n-1-2\alpha}\|u(\varepsilon\xi+\varepsilon\cdot)\|^2_{h^\alpha(\partial T)}=\frac{\varepsilon^{n-1-2\alpha}}{|\varepsilon Y|}\int_{\varepsilon(\xi+Y)}\|u(\varepsilon\lfloor x/\varepsilon\rfloor+\varepsilon\cdot)\|^2_{h^\alpha(\partial T)}\,dx\\
=&\varepsilon^{-1-2\alpha}\int_{\varepsilon(\xi+Y)}\|\mathscr{T}^b_\varepsilon(u)(x,\cdot)\|^2_{h^\alpha(\partial T)}\,dx.
\end{aligned}$$

Finally, summing up in $\xi\in\Xi_\varepsilon$, and taking into account that $\mathscr{T}^b_\varepsilon(u)(x,y)=0$ at $x\in\Lambda_\varepsilon=\Omega\setminus\overline{\widehat{\Omega}_\varepsilon}$, we obtain (18.11). □

Theorem 2. *Let $u\in H^\alpha(\partial T_\varepsilon)$, $-1\le\alpha\le 1$. Then*

$$\|u\|^2_{H^\alpha_\varepsilon(\partial T_\varepsilon)}=\varepsilon^{-1}\|\mathscr{T}^b_\varepsilon(u)\|^2_{L_2(\Omega,H^\alpha(\partial T))}. \tag{18.12}$$

Proof. We will follow the same pattern as in the proof of Theorem 1. By (18.9), (18.10) and taking into account that $|\varepsilon Y|=\varepsilon^n$, we obtain

$$\begin{aligned}
&\sum_j\|u_{\varepsilon,j}(\kappa^{(-1)}_{\varepsilon,\xi,j}(\cdot,\zeta_{\varepsilon,j}(\cdot)))\sqrt{1+|\nabla\zeta_{\varepsilon,j}(\cdot)|^2}\,\|^2_{H^\alpha_\varepsilon(\mathbb{R}^{n-1})}\\
=&\sum_j\|u\Big(\varepsilon\xi+\varepsilon\kappa_j^{(-1)}(\cdot/\varepsilon,\zeta_j(\cdot/\varepsilon))\Big)\phi_j\Big(\kappa_j^{(-1)}(\cdot/\varepsilon,\zeta_j(\cdot/\varepsilon))\Big)
\end{aligned}$$

$$\times \sqrt{1+|\nabla\zeta_j(\cdot/\varepsilon)|^2}\,\|^2_{H^\alpha_\varepsilon(\mathbb{R}^{n-1})}$$

$$=\sum_j \varepsilon^{n-1}\|u\Big(\varepsilon\xi+\varepsilon\kappa_j^{(-1)}(\cdot,\zeta_j(\cdot))\Big)\phi_j\Big(\kappa_j^{(-1)}(\cdot,\zeta_j(\cdot))\Big)\sqrt{1+|\nabla\zeta_j(\cdot)|^2}\,\|^2_{H^\alpha(\mathbb{R}^{n-1})}$$

$$=\varepsilon^{n-1}\|u(\varepsilon\xi+\varepsilon\cdot)\|^2_{H^\alpha(\partial T)}=\frac{\varepsilon^{n-1}}{|\varepsilon Y|}\int_{\varepsilon(\xi+Y)}\|u(\varepsilon\lfloor x/\varepsilon\rfloor+\varepsilon\cdot)\|^2_{H^\alpha(\partial T)}\,dx$$

$$=\varepsilon^{-1}\int_{\varepsilon(\xi+Y)}\|\mathscr{T}^b_\varepsilon(u)(x,\cdot)\|^2_{H^\alpha(\partial T)}\,dx.$$

Finally, summing up in $\xi\in\Xi_\varepsilon$, and again taking into account that $\mathscr{T}^b_\varepsilon(u)(x,y)=0$ at $x\in\Lambda_\varepsilon=\Omega\setminus\overline{\widehat{\Omega}_\varepsilon}$, we obtain (18.12). □

Remark 1. Rescaling (18.12) coincides with (18.9), i.e., passing from the hyperplane to the Lipschitz boundaries and rescaling the parametrization and its Jacobian does not influence the order of the norm rescaling.

Let us now obtain some inequalities for standard norms.

Theorem 3. *Let $u\in H^\alpha(\partial T_\varepsilon)$. For $\alpha\in[0,1]$, the following norm equivalence holds*

$$2^{\alpha-1}\varepsilon^{-1}\left[\|\mathscr{T}^b_\varepsilon(u)\|^2_{L_2(\Omega\times\partial T)}+\varepsilon^{-2\alpha}\|\mathscr{T}^b_\varepsilon(u)(x,\cdot)\|^2_{L_2(\Omega,h^\alpha(\partial T))}\right]\le$$
$$\|u\|^2_{H^\alpha(\partial T_\varepsilon)}\le\varepsilon^{-1}\left[\|\mathscr{T}^b_\varepsilon(u)\|^2_{L_2(\Omega\times\partial T)}+\varepsilon^{-2\alpha}\|\mathscr{T}^b_\varepsilon(u)(x,\cdot)\|^2_{L_2(\Omega,h^\alpha(\partial T))}\right].$$

Proof. Owing to (18.4), for $\alpha\in[0,1]$,

$$2^{\alpha-1}[\|u\|^2_{L_2(\partial T_\varepsilon)}+\|u\|^2_{h^\alpha(\partial T_\varepsilon)}]\le\|u\|^2_{H^\alpha(\partial T_\varepsilon)}\le\|u\|^2_{L_2(\partial T_\varepsilon)}+\|u\|^2_{h^\alpha(\partial T_\varepsilon)}.$$

This gives the equivalence of the norms. It suffices to note that by Proposition 2,

$$\|u\|^2_{L_2(\partial T_\varepsilon)}=\varepsilon^{-1}\int_\Omega\|\mathscr{T}^b_\varepsilon(u)(x,\cdot)\|^2_{L_2(\partial T)}\,dx=\varepsilon^{-1}\|\mathscr{T}^b_\varepsilon(u)\|^2_{L_2(\Omega\times\partial T)},$$

and by Theorem 1,

$$\|u\|^2_{h^\alpha(\partial T_\varepsilon)}=\varepsilon^{-1-2\alpha}\int_\Omega\|\mathscr{T}^b_\varepsilon(u)(x,\cdot)\|^2_{h^\alpha(\partial T)}\,dx=\varepsilon^{-1-2\alpha}\|\mathscr{T}^b_\varepsilon(u)\|^2_{L_2(\Omega,h^\alpha(\partial T))}.$$

□

Definition 5. On pair with Definition 1, we also define the following ε-dependent norms equivalent to the standard ones for the Sobolev-Slobodetskii spaces $W^s_2(\widehat{\Omega}^*_\varepsilon)$, $s\ge 0$. If s is integer, then let

$$\|v\|^2_{W^s_{2,\varepsilon}(\widehat{\Omega}^*_\varepsilon)}:=\sum_{|\alpha|\le s}\varepsilon^{2|\alpha|}\|\partial^\alpha v\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)}.$$

If s is not integer, then let

$$\|v\|^2_{W^s_{2,\varepsilon}(\widehat{\Omega}^*_\varepsilon)} := \sum_{|\alpha|\le[s]} \varepsilon^{2|\alpha|}\|\partial^\alpha v\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)} + \varepsilon^{2s}\|v\|'^2_{W^s_2(\widehat{\Omega}^*_\varepsilon)}$$

Similarly, we define the equivalent ε-dependent norm in the space $W^s_2(\partial T_\varepsilon)$, $s \in (0,1)$, cf. [GrOr18],

$$\|v\|^2_{W^s_{2,\varepsilon}(\partial T_\varepsilon)} := \|v\|^2_{L_2(\partial T_\varepsilon)} + \varepsilon^{2s}\|v\|'^2_{W^s_2(\partial T_\varepsilon)}. \tag{18.13}$$

Note that the semi-norm in (18.13) can be expressed as $\|v\|'^2_{W^\alpha_2(\partial T_\varepsilon)} := a_\alpha \|v\|^2_{h^\alpha(\partial T_\varepsilon)}$ with the constant a_α depending on α but not on ε, cf. (18.3).

The following assertion is an immediate consequence of Theorem 1, Proposition 2 and relations (18.6).

Corollary 1. *Let $u \in H^\alpha(\partial T_\varepsilon)$, $\alpha \in (0,1)$. Then*

$$\|u\|^2_{W^\alpha_2(\partial T_\varepsilon)} = \varepsilon^{-1}\left[\int_\Omega \|\mathscr{T}^b_\varepsilon(u)(x,\cdot)\|^2_{L_2(\partial T)}\,dx + \varepsilon^{-2\alpha}\int_\Omega \|\mathscr{T}^b_\varepsilon(u)(x,\cdot)\|'^2_{W^\alpha_2(\partial T)}\,dx\right],$$

and in terms of the ε-dependent norm,

$$||u||^2_{W^\alpha_{2,\varepsilon}(\partial T_\varepsilon)} = \varepsilon^{-1}\|\mathscr{T}^b_\varepsilon u\|^2_{L_2(\Omega, W^\alpha_2(\partial T))}. \tag{18.14}$$

Remark 2. By [GrOr18, Lem. 4.1], equality (18.14) is also valid for negative α in the sense of the dual to the Sobolev-Slobodetskii spaces.

18.4 Unfolding in Sobolev-Slobodetskii spaces in perforated domains

Theorem 4.
(i) If $\phi \in W^s_2(\widehat{\Omega}^*_\varepsilon)$, $0 < s < 1$, then

$$\|\mathscr{T}_\varepsilon(\phi)\|'^2_{L_2(\Omega, W^s_2(Y^*))} \le \varepsilon^{2s}\|\phi\|'^2_{W^s_2(\widehat{\Omega}^*_\varepsilon)}, \tag{18.15}$$

$$\|\mathscr{T}_\varepsilon(\phi)\|^2_{L_2(\Omega, W^s_2(Y^*))} \le \|\phi\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)} + \varepsilon^{2s}\|\phi\|'^2_{W^s_2(\widehat{\Omega}^*_\varepsilon)} = \|\phi\|^2_{W^s_{2,\varepsilon}(\widehat{\Omega}^*_\varepsilon)}, \tag{18.16}$$

where $\|\mathscr{T}_\varepsilon(\phi)\|'^2_{L_2(\Omega, W^s_2(Y^*))} := \int_\Omega \|\mathscr{T}_\varepsilon(\phi)(x,\cdot)\|'^2_{W^s_2(Y^*)}\,dx$.
(ii) If $\phi \in W^1_2(\widehat{\Omega}^*_\varepsilon)$, i.e., $s = 1$, then

$$\|\nabla\mathscr{T}_\varepsilon(\phi)\|^2_{L_2(\Omega, L_2(Y^*))} = \varepsilon^2\|\nabla\phi\|^2_{L^s_2(\widehat{\Omega}^*_\varepsilon)}, \tag{18.17}$$

$$\|\mathscr{T}_\varepsilon(\phi)\|^2_{L_2(\Omega, W^1_2(Y^*))} = \|\phi\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)} + \varepsilon^2\|\nabla\phi\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)} = \|\phi\|^2_{W^1_{2,\varepsilon}(\widehat{\Omega}^*_\varepsilon)}. \tag{18.18}$$

(iii) If $\phi \in W^s_2(\widehat{\Omega}^*_\varepsilon)$, $1 < s < 2$, then (18.15) *still holds and*

$$\|\mathscr{T}_\varepsilon(\phi)\|^2_{L_2(\Omega,W_2^s(Y^*))} \le \|\phi\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)} + \varepsilon^2\|\nabla\phi\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)} + \varepsilon^{2s}\|\phi\|'^2_{W_2^s(\widehat{\Omega}^*_\varepsilon)}$$
$$= \|\phi\|^2_{W^s_{2,\varepsilon}(\widehat{\Omega}^*_\varepsilon)}. \quad (18.19)$$

(iv) If $\phi \in W_2^s(\widehat{\Omega}^*_\varepsilon)$, $0 < s < 1/2$, *then*

$$\varepsilon^{2s}\|\phi\|'^2_{W_2^s(\widehat{\Omega}^*_\varepsilon)} \le C_1\|\mathscr{T}_\varepsilon(\phi)\|^2_{L_2(\Omega,W_2^s(Y^*))}, \quad (18.20)$$
$$\|\phi\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)} + \varepsilon^{2s}\|\phi\|'^2_{W_2^s(\widehat{\Omega}^*_\varepsilon)} = \|\phi\|^2_{W^s_{2,\varepsilon}(\widehat{\Omega}^*_\varepsilon)} \le C_2\|\mathscr{T}_\varepsilon(\phi)\|^2_{L_2(\Omega,W_2^s(Y^*))}, \quad (18.21)$$

where C_1 *and* C_2 *are independent on* ε *and* ϕ.

(v) If $\phi \in W_2^s(\widehat{\Omega}^*_\varepsilon)$, $1 < s < 3/2$, *then* (18.20) *still holds and*

$$\|\phi\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)} + \varepsilon^2\|\nabla\phi\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)} + \varepsilon^{2s}\|\phi\|'^2_{W_2^s(\widehat{\Omega}^*_\varepsilon)}$$
$$= \|\phi\|^2_{W^s_{2,\varepsilon}(\widehat{\Omega}^*_\varepsilon)} \le C_3\|\mathscr{T}_\varepsilon(\phi)\|^2_{L_2(\Omega,W_2^s(Y^*))}, \quad (18.22)$$

where C_3 *is independent on* ε *and* ϕ.

Proof. (i) Let $s \in (0,1)$. Then

$$\|\phi\|'^2_{W_2^s(\widehat{\Omega}^*_\varepsilon)} = \int_{\widehat{\Omega}^*_\varepsilon}\int_{\widehat{\Omega}^*_\varepsilon} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy$$
$$= \sum_{\xi_1\in\Xi_\varepsilon}\sum_{\xi_2\in\Xi_\varepsilon}\int_{\varepsilon(\xi_1+Y^*)}\int_{\varepsilon(\xi_2+Y^*)} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy$$
$$\ge \sum_{\xi\in\Xi_\varepsilon}\int_{\varepsilon(\xi+Y^*)}\int_{\varepsilon(\xi+Y^*)} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \sum_{\xi\in\Xi_\varepsilon}\|\phi\|'^2_{W_2^s(\varepsilon\xi+\varepsilon Y^*)}$$
$$= \sum_{\xi\in\Xi_\varepsilon}\varepsilon^{n-2s}\|\phi(\varepsilon\xi+\varepsilon\cdot)\|'^2_{W_2^s(Y^*)} = \sum_{\xi\in\Xi_\varepsilon}\frac{1}{|Y|}\varepsilon^{-2s}\|\phi(\varepsilon\xi+\varepsilon\cdot)\|'^2_{W_2^s(Y^*)}\int_{\varepsilon\xi+\varepsilon Y}dx$$
$$= \sum_{\xi\in\Xi_\varepsilon}\frac{1}{|Y|}\varepsilon^{-2s}\int_{\varepsilon\xi+\varepsilon Y}\|\phi(\varepsilon[x/\varepsilon]+\varepsilon\cdot)\|'^2_{W_2^s(Y^*)}\,dx$$
$$= \frac{1}{|Y|}\varepsilon^{-2s}\int_\Omega \|\mathscr{T}_\varepsilon(\phi)(x,\cdot)\|'^2_{W_2^s(Y^*)}\,dx.$$

Since $|\mathbf{Y}| = 1$, we obtain (18.15).

Taking into account that $\|\phi\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)} = \|\mathscr{T}_\varepsilon(\phi)\|^2_{L_2(\Omega\times Y^*)}$ by item (i) of Proposition 1, the definition $\|u\|^2_{W_2^s(\Omega')} := \|u\|^2_{L_2(\Omega')} + \|u\|'^2_{W_2^s(\Omega')}$, employed with $\Omega' = \widehat{\Omega}^*_\varepsilon$ and $\Omega' = Y^*$, implies (18.16).

(ii) Equalities (18.17) and (18.18) for the case $s = 1$ immediately follow from Proposition 1.

(iii) Let now $s \in (1,2)$ and $\mu = s-1$. Then by (18.1), item (i) of Proposition 1 and inequality (18.15) with ϕ replaced by $\nabla\phi$ and s by μ, we obtain

$$\|\mathcal{T}_\varepsilon(\phi)\|'^2_{L_2(\Omega,W_2^s(Y^*))} = \|\nabla\mathcal{T}_\varepsilon(\phi)\|'^2_{L_2(\Omega,W_2^\mu(Y^*))}$$
$$= \|\varepsilon\mathcal{T}_\varepsilon(\nabla\phi)\|'^2_{L_2(\Omega,W_2^\mu(Y^*))} \le \varepsilon^{2\mu}\varepsilon^2\|\nabla\phi\|'^2_{W_2^\mu(\widehat{\Omega}^*_\varepsilon)} = \varepsilon^{2s}\|\phi\|'^2_{W_2^s(\widehat{\Omega}^*_\varepsilon)},$$

which implies inequality (18.15) also for $s \in (1,2)$.

Definition of the Sobolev-Slobodetskii space $\|\mathcal{T}_\varepsilon(\phi)\|^2_{L_2(\Omega,W_2^s(Y^*))}$ for $1 < s < 2$ together with relations (18.18) and (18.15) imply (18.19).

(iv) Let $s \in (0,1/2)$. Then

$$\begin{aligned}
&\|\phi\|'^2_{W_2^s(\widehat{\Omega}^*_\varepsilon)} = \int_{\widehat{\Omega}^*_\varepsilon}\int_{\widehat{\Omega}^*_\varepsilon} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&= \sum_{\xi_1\in\Xi_\varepsilon}\sum_{\xi_2\in\Xi_\varepsilon}\int_{\varepsilon(\xi_1+Y^*)}\int_{\varepsilon(\xi_2+Y^*)} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&= \sum_{\xi_1\in\Xi_\varepsilon}\sum_{\xi_2\in\Xi_\varepsilon}\int_{Y^*}\int_{Y^*} \frac{|\phi(\varepsilon\xi_1+\varepsilon q)-\phi(\varepsilon\xi_2+\varepsilon t)|^2}{\varepsilon^{n+2s}|\xi_1+q-\xi_2-t|^{n+2s}}\varepsilon^{2n}dq\,dt\\
&\le \varepsilon^{n-2s}\sum_{\xi\in\Xi_\varepsilon}\int_{Y^*}\int_{Y^*} \frac{|\phi(\varepsilon\xi+\varepsilon q)-\phi(\varepsilon\xi+\varepsilon t)|^2}{|\xi+q-\xi-t|^{n+2s}}dq\,dt\\
&+2\varepsilon^{n-2s}\sum_{\xi_1\in\Xi_\varepsilon}\int_{Y^*}|\phi(\varepsilon\xi_1+\varepsilon q)|^2 \sum_{\substack{\xi_2\in\Xi_\varepsilon\\ \xi_2\neq\xi_1}}\int_{Y^*}\frac{1}{|\xi_1+q-\xi_2-t|^{n+2s}}\,dt\,dq\\
&+2\varepsilon^{n-2s}\sum_{\xi_2\in\Xi_\varepsilon}\int_{Y^*}|\phi(\varepsilon\xi_2+\varepsilon t)|^2 \sum_{\substack{\xi_1\in\Xi_\varepsilon\\ \xi_1\neq\xi_2}}\int_{Y^*}\frac{1}{|\xi_1+q-\xi_2-t|^{n+2s}}\,dq\,dt\\
&= \varepsilon^{n-2s}\sum_{\xi\in\Xi_\varepsilon}\|\phi(\varepsilon\xi+\varepsilon\cdot)\|'^2_{W_2^s(Y^*)}\\
&+4\varepsilon^{n-2s}\sum_{\xi_1\in\Xi_\varepsilon}\int_{Y^*}|\phi(\varepsilon\xi_1+\varepsilon q)|^2\left[\sum_{\substack{\xi_2\in\Xi_\varepsilon\\ \xi_2\neq\xi_1}}\int_{Y^*}\frac{dt}{|\xi_1+q-\xi_2-t|^{n+2s}}\right]dq\\
&\le \varepsilon^{n-2s}\frac{1}{|\varepsilon Y|}\left(\|\mathcal{T}_\varepsilon(\phi)\|'_{L^2(\Omega,W_2^s(Y^*))}\right)^2\\
&+4C_s\varepsilon^{n-2s}\sum_{\xi_1\in\Xi_\varepsilon}\int_{Y^*}|\phi(\varepsilon\xi_1+\varepsilon q)|^2 dist(\xi_1+q,\partial Y_{\xi_1})^{-2s}dq.
\end{aligned}$$

Here, similar to the proof of Theorem 3.33 in [McL00], we used the estimate

$$\sum_{\substack{\xi_2\in\Xi_\varepsilon\\ \xi_2\neq\xi_1}}\int_{Y^*}\frac{dt}{|\xi_1+q-\xi_2-t|^{n+2s}} \le \int_{\mathbb{R}^n\setminus Y_{\xi_1}}\frac{d\tau}{|\xi_1+q-\tau|^{n+2s}} \le C_s dist(\xi_1+q,\partial Y_{\xi_1})^{-2s},$$

where C_s is a constant and $Y_{\xi_1} := \xi_1 + Y$. Applying now Lemma 3.32 from [McL00], we obtain

$$
\begin{aligned}
&\|\phi\|'^2_{W_2^s(\widehat{\Omega}_\varepsilon^*)} \le \varepsilon^{-2s}\frac{1}{|Y|}\|\mathscr{T}_\varepsilon(\phi)\|'^2_{L^2(\Omega,W_2^s(Y^*))} \\
&+4C_s\varepsilon^{n-2s}\sum_{\xi_1\in\Xi_\varepsilon} C_{Y^*}\|\phi(\varepsilon\xi_1+\varepsilon\cdot)\|^2_{W_2^s(Y^*)} \\
&=\varepsilon^{-2s}\frac{1}{|Y|}\|\mathscr{T}_\varepsilon(\phi)\|'^2_{L^2(\Omega,W_2^s(Y^*))}+4C_sC_{Y^*}\varepsilon^{-2s}\frac{1}{|Y|}\|\mathscr{T}_\varepsilon(\phi)\|^2_{L^2(\Omega,W_2^s(Y^*))} \\
&\le C_1\varepsilon^{-2s}\frac{1}{|Y|}\|\mathscr{T}_\varepsilon(\phi)\|^2_{L^2(\Omega,W_2^s(Y^*))},
\end{aligned}
$$

where C_{Y^*} and hence C_1 do not depend on ε. Since $|\mathbf{Y}| = 1$, we obtain (18.20).

Taking into account that $\|\phi\|^2_{L_2(\widehat{\Omega}_\varepsilon^*)} = \|\mathscr{T}_\varepsilon(\phi)\|^2_{L_2(\Omega\times Y^*)}$ by item (i) of Proposition 1, the definition $\|u\|^2_{W_2^s(\Omega')} := \|u\|^2_{L_2(\Omega')} + \|u\|'^2_{W_2^s(\Omega')}$, employed with $\Omega' = \widehat{\Omega}_\varepsilon^*$ and $\Omega' = Y^*$, implies (18.21).

(v) Let now $s \in (1,3/2)$ and, similar to the proof of item (iii), $\mu = s-1$. Then by (18.1), item (i) of Proposition 1 and inequality (18.20) with ϕ replaced by $\nabla\phi$ and s by μ, we obtain

$$
\begin{aligned}
\varepsilon^{2s}\|\phi\|'^2_{W_2^s(\widehat{\Omega}_\varepsilon^*)} = \varepsilon^{2\mu+2}\|\nabla\phi\|'^2_{W_2^\mu(\widehat{\Omega}_\varepsilon^*)} &\le C_1\varepsilon^2\|\mathscr{T}_\varepsilon(\nabla\phi)\|^2_{L_2(\Omega,W_2^\mu(Y^*))} \\
&= C_1\|\nabla\mathscr{T}_\varepsilon(\phi)\|^2_{L_2(\Omega,W_2^\mu(Y^*))} \le C_1\|\mathscr{T}_\varepsilon(\phi)\|^2_{L_2(\Omega,W_2^s(Y^*))}
\end{aligned}
$$

which implies inequality (18.20) also for $s \in (1,3/2)$.

Definition of the Sobolev-Slobodetskii space $\|\mathscr{T}_\varepsilon(\phi)\|^2_{L_2(\Omega,W_2^s(Y^*))}$ for $1 < s < 3/2$ together with relations (18.18) and (18.20) imply (18.22). □

18.5 Rescaling of the Trace Theorem in W_2^s

For $u \in W_2^s(\Omega_\varepsilon^*)$, $s \in (1/2,3/2)$, the trace operator (in the Gagliardo sense) $\gamma : W_2^s(\Omega_\varepsilon^*) \to W_2^{s-1/2}(\partial\Omega_\varepsilon^*)$, is continuous, see e.g. [McL00].

Now, we can rewrite the trace theorem using the scaling estimates from Theorems 3 and 4.

Theorem 5. *Let $u \in W_2^s(\widehat{\Omega}_\varepsilon^*)$, $s \in (1/2,3/2)$, $\varepsilon > 0$.*

(i) If $s \in (1/2,1)$, then

$$
\begin{aligned}
&\varepsilon\left(\|\gamma_{\partial T_\varepsilon}u\|^2_{L_2(\partial T_\varepsilon)} + \varepsilon^{2s-1}\|\gamma_{\partial T_\varepsilon}u\|'^2_{W_2^{s-1/2}(\partial T_\varepsilon)}\right) \\
&\qquad\qquad \le C\left(\|u\|^2_{L_2(\widehat{\Omega}_\varepsilon^*)} + \varepsilon^{2s}\|u\|'^2_{W_2^s(\widehat{\Omega}_\varepsilon^*)}\right). \quad (18.23)
\end{aligned}
$$

(ii) If $s = 1$, then

$$\varepsilon\left(\|\gamma_{\partial T_\varepsilon} u\|^2_{L_2(\partial T_\varepsilon)} + \varepsilon\|\gamma_{\partial T_\varepsilon} u\|'^2_{W_2^{1/2}(\partial T_\varepsilon)}\right) \le C\Big(\|u\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)}\Big). \tag{18.24}$$

(iii) If $s \in (1, 3/2)$, then

$$\varepsilon\left(\|\gamma_{\partial T_\varepsilon} u\|^2_{L_2(\partial T_\varepsilon)} + \varepsilon^{2s-1}\|\gamma_{\partial T_\varepsilon} u\|'^2_{W_2^{s-1/2}(\partial T_\varepsilon)}\right) \le C\Big(\|u\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L_2(\widehat{\Omega}^*_\varepsilon)} + \varepsilon^{2s}\|u\|'^2_{W_2^s(\widehat{\Omega}^*_\varepsilon)}\Big). \tag{18.25}$$

In all three cases the constant C is independent of u and ε and, using the ε-dependent norms, they can be written in the same form,

$$\varepsilon\|\gamma_{\partial T_\varepsilon} u\|^2_{W_{2,\varepsilon}^{s-1/2}(\partial T_\varepsilon)} \le C\|u\|^2_{W_{2,\varepsilon}^s(\widehat{\Omega}^*_\varepsilon)}, \quad 1/2 < s < 3/2.$$

Proof. If $1/2 < s < 3/2$, then by the trace theorem in Y^*, there exists a constant C independent of u and ε, such that

$$\|\gamma_{\partial T}\mathscr{T}_\varepsilon(u)(x,\cdot)\|^2_{L_2(\partial T)} + \|\gamma_{\partial T}\mathscr{T}_\varepsilon(u)(x,\cdot)\|'^2{}_{W_2^{s-1/2}(\partial T)} \le C\Big(\|\mathscr{T}_\varepsilon(u)(x,\cdot)\|^2_{L_2(Y^*)} + \|\mathscr{T}_\varepsilon(u)(x,\cdot)\|'^2{}_{W_2^s(Y^*)}\Big).$$

Integrating in x, we have

$$\|\gamma_{\partial T}\mathscr{T}_\varepsilon(u)\|^2_{L_2(\Omega, W_2^{s-1/2}(\partial T))} \le C\|\mathscr{T}_\varepsilon(u)\|^2_{L_2(\Omega, W_2^s(Y^*))}. \tag{18.26}$$

Let first $1/2 < s < 1$. Employing inequality (18.16) in the right hand hand side of (18.26) and, cf. (18.14), the relation

$$\|\gamma_{\partial T}\mathscr{T}_\varepsilon(u)\|^2_{L_2(\Omega, W_2^{s-1/2}(\partial T))} = \|\mathscr{T}^b_\varepsilon(\gamma_{\partial T_\varepsilon} u)\|^2_{L_2(\Omega, W_2^{s-1/2}(\partial T))} = \varepsilon\left(\|\gamma_{\partial T_\varepsilon} u\|^2_{L_2(\partial T_\varepsilon)} + \varepsilon^{2s-1}\|\gamma_{\partial T_\varepsilon} u\|'^2_{W_2^{s-1/2}(\partial T_\varepsilon)}\right)$$

in the left hand side, we arrive at (18.23).

Similar reasoning with relations (18.18) and (18.19) instead of (18.16) lead to (18.24) and (18.25), respectively. □

Note that the inequality similar to (18.24), for $s = 1$, was first given in [GaKNR14, Lem.3.1(iv)], and appears as an auxiliary result in [GrOr18].

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