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THE RESTRICTION MAP FOR COHOMOLOGY AND SYLOW THEORY IN SOLUBLE LOCALLY FINITE GROUPS.

BY ANDREW RAE

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Dedicated to Bertram Huppert on his 75'th birthday.

In extending the Schur Zassenhaus theorem to infinite groups one needs to extend derivations from a subgroups of a soluble locally finite group G; necessary and sufficient conditions are given for this extension to be possible in all cases, provided that the module is a countable *kG* module where the characteristic of k does not divide the order of any element of G.

1. 1 . Introduction

In "Theorems like Sylow's" (1) Phillip Hall distinguishes three types of Sylow Theorems:

(i) E_π theorems which state the existence of at least one Hall π-subgroup.

(ii) D_π theorems which state that every π –subgroup is contained in a Hall π -subgroup.

(iii) C_π theorems which state that any two Hall π -subgroups are conjugate.

Here π denotes a set of primes and a Hall π -subgroup is a π -subgroup whose index is a π' -number

Following Hall we shall refer to E_π , D_π and C_π theory collectively as *Sylow theory.*

For $|\pi| = 1$ these are just the first three parts of Sylow's theorem.

For finite soluble groups Hall deduced E_π and C_π (from which D_π follows) from a result of Schur which states that if *V* is a *kG* module where */G/* is finite and prime to the characteristic of k then $H^2(G,V)$ and $H^1(G,V)$ are trivial.

Note that D_π in effect states that the *maximal* π -subgroups are the Hall π -subgroups; and Hall's theorem states that all *maximal* *roups are con-*

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Jugate and have index coprime to π..

In this paper we are concerned with the extension of these results to infinite, but *locally Finite* groups (a group is *locally finite* if every finite subset generates a finite subgroup). Hartley (2), (3), (4) has successfully characterised the very restricted class of such groups for which the full force of Hall's theorem holds.

A Hall π-subgroup of a locally finite group is one which reduces into some local system consisting of finite subgroups in the sense that it's intersection with every member X of the local system is a Hall π -subgroup of X.

For finite soluble groups both C_π and D_π always hold, but there is a wide class of infinite soluble groups which satisfy D_π without satisfying C_π; all direct products of finite groups for example, and the class L of groups having a subnormal local system which is important in Passman's theory of group rings (see (7)).

Here, then, we consider those groups which satisfy D_π but not necessarily C_π so that maximal π -subgroups are Hall subgroups but are not necessarily conjugate.

This involves a characterisation of those modules for which restriction always gives an epimorphism of cohomology, and we hope that this may have independant interest in homological algebra.

As the groups we are studying are direct limits of finite groups the problem is one of cohomology commuting with direct limits - in general, of course cohomology commutes *with inverse* limits, but not with direct limits.

We now state our principal results.

THEOREM **A.** *Let G be a soluble locally finite n-group, let k be afield whose characteristic is a π′ -number and V be a countable kG module. Then the following conditions are equivalent:*

(i) *res*: $H^1(G,V) \longrightarrow H^1(G_1,V)$ *is an epimorphism for every subgroup* G_1 *of G*

(ii) $H^1(G_1, (V,x]G) = 0$ *for every subgroup* G_1 *of G and element xofG*

Note: that V must be countable is shown by example 2.6 below while example 2.8 shows that solubility is necessary.

We can restate theorem A in group theoretic terms as

THEOREM B. *Let G be a soluble locally finite* π *-group, let k be a field whose characteristic is a* π' *-number and V be a countable kG module. Then the following conditions are equivalent:*

(i) If Γ *denotes the split extension of (V,+) by (G,.) then* Γ *satisfies* D_π.

(ii) For every element x the G closure [V, x]G of the submodule $[V, x] = \{ v(1-x) | v \in V\}$ *satisfies the minimal condition for centralisers of finite subgroups of G.*

(iii) For every element x the group generated by G and [V, x]G satisfies C_π *and so does all its subgroups.*

As subgroup closure plays a vital role in the theory it may be worth mentioning the following corollary.

COROLLARY B1 . *Let G be a soluble locally finite*π*t-group let k be a field whose characteristic is a* π'*-number and V be a countable kG module . Then if* Γ *denotes the split extension of(V,+) by (G,.) the following conditions are equivalent:*

(i) Γ *satisfies* D_π .

(ii) *every subgroup of* Γ *satisfies* D_π .

In a forthcoming paper we will use this to develope a D_π theory for soluble locally finite groups.

1.2 The connection between Sylow theory and cohomology.

We start with a word of warning concerning difficulties inherent in the Sylow theory of infinite groups

It is an elementary conseqence of Zom's Lemma that maximal π-subgroups always exist in locally finite groups (see (6) page 160), but it is not obvious how to define a Hall subgroup for an infinite group; for locally finite groups these are generally taken to be those known as S_π subgroups in (7) which have the property of covering any quotient

HIK where *H* and *K* are normal subgroups of *G* and *HIK* is a π-subgroup. They are defined by the property that they "reduce" into some local system of *G*.

Unfortunately it is easy to construct groups all of whose Hall π-subgroups in this sense are conjugate but such that "almost all" maximal π-subgroups fail to be Hall subgroups (see Example 2.7; this is a countable metabelian group of exponent 6 made by adjoining one element to the direct power of $\sum_3$).

To get an interesting theory one needs to consider properties which hold for all subgroups of *G* - as Hartley does in his Cπ theory; our results suggest that the class of countable soluble groups satisfying D_π is subgroup closed.

For the rest of this paper we assume, unless otherwise stated, that *G,V,* Γ and *k* satisfy the following:

> *G is a soluble locally finite π-group, k a field whose characteristic is a π'-number, V is a countable faithfull kG module and Γ is the split extension of (V,+) by (G,.)*

The connection between the Sylow theory of Γ and the cohomology of *G* on *V* is well known - see (6) chapter XII for example - the main link being that the set:

$$\{gd_g / g \varepsilon G\} \qquad d_g \varepsilon V$$

is a subgroup of Γ iff

$$g \to d_g$$

satisfies $$d_g{}^{g'} d_{g'} = d_{gg'}$$

so that it is a derivation; moreover this subgrup is conjugate to G in Γ iff this derivation is a coboundary. Thus

$$H^1\ (G.V) = 0$$

iff all complements to V in Γ are conjugate, and

$$\Gamma \textit{ satisfies } C_\Pi \textit{ iff } H^1(G,V) = 0.$$

It is of course necessarily the case that all complements to V in Γ are conjugate if G is finite and V is finite dimensional, but taking Γ to be the direct power of countably many copies of Σ_3 shows that this need not happen on general (for this is a countable group having uncountably many Hall 2 subgroups).

To get at the D_π theory of Γ we observe that an arbitrary π-subgroup of Γ corresponds to a derivation from a subgroup of G and that this subgroup is contained in a Hall π-subgroup iff the derivation extends to G.

Thus G

$$\Gamma \textit{ satisfies } D_\Pi \textit{ iff } res : z^1(G.V) \to z^1(G_1, V) \textit{ is an epimorphism for every}$$
$$\textit{Subgroup } G_1 \textit{ of } G$$

Now since restriction of 1-coboundaries is necessarily surjective - for they are the coboundaries of constant functions it follows that

$$\Gamma \textit{ satisfies } D_\Pi \textit{ iff } res{:}H^1(G_1,V) \to H^1(G_1,V) \textit{ is an epimorphism for every}$$
$$\textit{Subgroup } G_1 \textit{of } G$$

This completes the proof of the equivalence of theorems A and B.

1.3 Discussion of results.

Hartley, in (2), (3), (4), developes a theory of modules V having

$$H^1(G_1,V) = 0$$

for all subgroups G_1 of G.

he shows that these are just the modules satisfying the minimal condition for centralisers of finite subgroups of G (the *min-c* condition) and completely characterises the very retricted class $\mathfrak{R}$ of groups which can act faithfully on a *min-c* module; these results lead to a reasonably complete C_{Π} theory for locally soluble groups.

If we combine these results with Theorems A and B we see that if Γ satisfies D_{Π} this has dramatic consequences for the structure of G

.

Thus if Γ satisfies D_{Π} Theorem B shows that $[x,V]G$ satisfies the minimal condition for centralisers of finite subgroups of G. This implies that the quotient

$$G/C_G([x,V]G)$$

is a *min-c* head in Hartley's terminology and consequently has a very restricted structure. It must belong to the class $\mathfrak{R}$ (see (4) Theorem 6.2 and Lemma 3.4)

DEFINITION G belongs to $\mathfrak{R}$ if it is a soluble group of finite (special) rank which is the finite extension of a subdirect product of finitely many "pinched" groups.
Here a group is "pinched" if
(a) it has a normal locally cyclic subgroup containing its derived group and all elements of square free order and
(b) all its 2-subgroups are Abelian.

A group has special rank "n" if every finitely generated subgroup can be generated by n elements.

If G belongs to the class $\mathfrak{R}$ then it satisfies min p (the minimal condition for p subgroups) for every prime p and its p subgroups are Cemikov groups of uniformly bounded rank.

Thus $\mathfrak{R}$ groups are, in a sense, not far from being finite.

In addition Theorem B implies (see section 2 for a proof):

COROLLARY B2 . *Let G be a soluble locally finite n-group, k be afield whose charac-- teristic is a* π' *-number, V be a countable, faithfull kG module and Γ denote the split extension of(V,+) by (G, .) . Then if Γsatisfles* D_Π *the normal closure of each element x of G in G must be an* $\Re$ *group.*

Thus *G* may be regarded as a generalisation of a an f.c. group (a group is f.c. if every element has only finitely many conjugates). This is an important point as even finite extensions of f.c. groups can fail to have a D_Π theory (see example 2.7).

On the other hand the same example shows that Γ may be the union of normal C_Π subgroups without satisfying D_Π .

2. Preliminary results.

In this section we recall some facts about *min-c* modules and reduce the proof of Theorems A and B to a key result, Theorem C, which will be proved in section 3. We will also deduce the two corollaries from Theorem B.

Recall our standard hypotheses:

G is a soluble locally finite π-group, k a field whose characteristic is a π'*-number, V is a countable faithfull kG module and Γ is the split extension of (V,+) by (G,.)*

The first result colects up some usefull (and well known) results about min-c modules.

PROPOSITION 2.1.

(a) If V is the extension of a *min-c* module V_1 by another *min-c* module V/V_1 then V is itself *min-c.*

(b) Any submodule or quotient module of a *min-c* module is *min-c.*

(c) The sum of finitely many *min-c* modules is *min-c.*

(d) Let *K* be a subgroup of finite index in *G*. Then if *V* satisfies *min-c* as a *K* module, it follows that V satisfies min-c as a G module.

(e) If *[V,G]* satisfies *min-c* then $G \in \Re$ and, in particular *G* satisfies min p for all primes p.

Proof.

For (a) to (d) we use the fact that *V* satisfies min-c if and only if every subgroup *H* of *G* has a finite subgroup *F* with

$$C_v(F) = \; C_vH)$$

and the following simple, well known, result.

LEMMA 2.2. *V* is completely reducible if *G* is finite; in particular if $x \in G$ then

$$[V,x,x] = [V,x].$$

Proof. Suppose that *G* is finite. Then *V* is the direct limit of finite dimensional G submodules which are competely reducible by Maschke's theorem. It follows that *V* is completely reduce-ble.

In particular if x ∈ G we can apply this to *V* as an $< x >$ module to get

$$V = [V, x] \oplus C_V(x)$$

from which it follows that [V, x, x] = [V, x].

We now return to the proof of Proposition 2.1.

(a) Let V_1 and V_2 = V $_1$ V_1 be *min-c* modules. Let *H* be any subgroup of *G*. Then there are finite subgroups F_1 and F_2 of *G* with the centraliser in V_i of *H* equal to that of F_1 for each i It follows that if F is any finite subgroup containing both F_1 and F_2 then

$$C_V(H) = C_V(F)$$

so that *V* satisfies *min-c* as required.

(b) This is Lemma 2.4 (i) of (4)

(c) This follows from (a) and (b)

(d) See Lemma2.7 of (4).

(e) If *[V, G]* is *min-c* then by (a) *V* is also, since *V*| *[V, G]* is trivially *min-c.*

Thus *G* acts faithfully on a *min-c* module and so is an $\Re$ group (see subsectionl.3) As men-tioned in 1.3 an $\Re$ group satisfies min p for all primes p.

We now prove the sufficiency of the conditions in Theorems A and B. We do not need to assume that *G* is soluble or that *V* is countable.

NOTATION

We write $V \in \Psi_G$ if a derivation from a subgroup G_1 of *G* to *V* always extends to the whole of *G.*

PROPOSITION 2.3 . Suppose [x, V] G is min-c as a G module for all $x \in G$. Then $V \in \Psi_G$

Proof

Let H be a subgroup of G and

$$\delta : H \rightarrow V$$

be a derivation. We show that if x is any element of G and X denotes its normal closure in G then δ extends to

$$K = HX$$

The conclusion then follows by transfinite induction.

Let $$W = [X, V].$$

We show first that δ extends to K modulo W.

Let $$H_1 = \{h\, \delta(h) / h \in H\}$$

Then X is a normal subgroup of Γ modulo W so that

$$K_1 = H_1 X\, W$$

is a π group modulo W. It follows that

$$K_1 \cap V = W.$$

Now W is *min-c* as a G module and so it is *min-c* as a K_1 module.

Thus $K_1 \mid C(W)$ supports a *min-c* module and so is an $\mathfrak{R}$ group. In particular it is countable so that K_1 splits over W (see for example (5)) and we can apply the methods of subsection 1.2 to K_1

The proposition now follows from:

LEMMA 2.4. If *V* satisfies *min-c* as a *G* module then $V \in \Psi_G$

This follows from Hartley's result, quoted above, that if *V* satisfies *min-c* then

$$H^1(G, V) = 0$$

For *V* is *min-c* for all subgroups of *G* so that all have trivial cohomology group. Thus *res* must be a surjection.

NOTE. In the proof of Proposition 2.3 we do not need to assume that Γ splits over V; in fact the proof shows that if *Γ* satisfies the other hypotheses of 2.3 then it must split over V. We use this in the deduction of Corolary B_1.

We next deduce the corollaries from Theorem B.

COROLLARY B1 . *Let G be a soluble locally finite n-group let k be a field whose characteristic is a π' -number and V be a countable kG module. Then if Γ denotes the split extension of(V,+) by (G,.) the following conditions are equivalent:*

(i) Γsatisfies D_Π .

(ii) every subgroup of Γsatisfies D_π

Proof.

Note first that Theorem A shows that the submodules of countable ψ_G modules are ψ_G modules.

Let satisfy D_π and Γ_1 be a subgroup of Γ.

Suppose first that $\Gamma_1 \supseteq V$.

Then $\Gamma = V G_1$ where $G_1 = G \cap T_1$

Now clearly Γ_1 satisfies condition(ii) of Theorem B and so, by that result, satisfies D_π

Now consider the general case; then if $\Gamma_0 = \Gamma_1 V$ the about shows that Γ_0 satisfies $D\pi$

Thus we may assume that $\Gamma_1 V = \Gamma$

Now $V_1 = V \cap \Gamma_1$ is a Hall π' subgroup of Γ_1 and so, by Theorem B, we have, for every π' element *x* of Γ_1, that *[V, x]G* satisfies *min*- c as a *G* module.

Now it follows from the proof of Proposition 2.3, as we remarked above, that Γ_1 splits over V_1 and Theorem B shows that Γ_1 satisfies D_π as required.

NOTE. Example 2.6 below shows that the hypothesis that V be countable is essential here.

COROLLARY B2 . *Let G be a soluble locally finite π-group, k be a field whose character- istic is a n'-number, V be a countable kG module and Γdenote the split extension of (V,+) by (G,.). Then if Γsatisfies* D_Π *the normal closure of each element x of G in G must be an* $\mathfrak{R}$ *group.*

Proof. This follows once we can show that if *X* denotes the normal closure of x in *G* then *X* acts faithfully on a min-c module.

Now Lemma 2.2 shows that X acts faithfully on [V, X] so the result follows.

We now reduce the proof of theorems A and B to

THEOREM C. *Let G be countable and soluble and suppose* $V \in \Psi_G$ *Then [V,x]G satisfies* *min*-c *for all* $x \in G$.

First we show

LEMMA 2.5. *Theorem C suffices to prove Theorems A and B.*

Proof. Proposition 2.3 proves the sufficiency of the condition and Theorem C the necessity when *G* and *V* are countable.

Now suppose that *G* is soluble, but not necessarily countable, and that $V \in \Psi_G$

Let $x \in G$, and suppose if possible that

$$W = [x ,V]G$$

does not satisfy *min-c* as a G module.

Then there is a chain of finite subgroups

$$I = F_0 \subseteq F_1 .... \quad \subseteq F_i ....$$

of *G* with $C_w(F_i) \neq C_w(F_{i+1})$ for i=0,1,2,3 …….

Let G_I be the union of all the subgroups F_i, for i=0,l,2…..

Then G_I is countable; adjoin to G_I a countable number of elements so as to make

$$W = [x,V]G_I$$

which is possible since *V* is countable.

Now G_I and *V* contradict Theorem C.

We now give some examples to show that the restrictions placed on *G* and *V* are necessary.

The first example shows that V must be countable and that a submodule of a ψ_G module need not be a ψ_G module.

EXAMPLE 2.6. We construct Γas a subdirect power of countably many copies of $\sum_3$ the symmetric group of degree 3.
Let A be the complete (unrestricted) power of countably many copies of A_3 the alternating group of degree 3 and let Γ_0 be the direct (restricted) power of countably many copies of $\sum_3$, consisting of those members having finite support. Finally let α be the element of the complete power having every component (12).

Put $\Gamma = < A\ \Gamma_0 .\alpha >$

Then maximal 2 subgroups of Γ all complement A and are even conjugate. Thus if we take V to be *(A,+)* and G to be a Sylow 2 subgroup of Γ_0 containing a we have that $V \in \Psi_G$ but *[α, V]* = V is infinite dimensional and G acts faithfully on it. Since G does not satisfy min p it is clear that V does not satisfy *min-c* so that Theorem B does not hold here.

This example also shows that the class of Ψ_G modules is not submodule closed.

Put $A_0 = \Gamma_0 \cap A$

Then the countable submodule $V_0 = (A_0,+)$

has $[\alpha, V_0] = V_0$

so that it cannot be a member of. Ψ_G.

EXAMPLE 2.7. Let G be the same group as in Example 2.6 and let V be the module denoted by V_0 there. Then we claim that all complements to V in VG are conjugate – they must all contain a conjugate of α - but that almost all maximal 2 subgroups are contained in Γ_0 and so fail to be complements.

Note that this group is the finite extension of the f.c. group Γ_0 and that it is even the union of an ascending chain of normal C_π subgroups - finite extensions of V containing a for example.

EXAMPLE 2.8. Example 6.1 in (7) shows that solubility is necessary if we take the first layer to be V and G to be the second and higher layers.

In this group evey finite subgroup is contained in a two step subnormal subgroup and this implies, as is shown there, that D_π holds.

On the other hand, G acts faithfully on V and the normal closure of any element of G contains an infinite elementary abelian subgroup so that it cannot be satisfy min p. This shows that the conclusion of Theorem B cannot hold.

3. *Proof of the main result.*

We assume throughout this section that *G* is countable and soluble.
Our method is to assume that for some $x \in G$ we have that

$$W = [x, V] G$$

does not satisfy *min-c,* and to use this to construct a π-subgroup *H* of Γ which is not consonant with any *V*-conjugate of *x* (π elements are said to be consonant if they generate a π subgroup).

This then shows that some derivation from

$$G_1 = H V \cap G$$

into *V* does not extend to the subgroup $<x, G_1>$ generatesd by *x* and G_1.

The basic idea is to produce a chain

$$1 = F_0 \subseteq F_1 .... \subseteq F_i ......$$

of finite subgroups of *G* with centralizers $C_i = C_w(F_i)$ such that

$$C_i \neq C_{i+1} \qquad \text{for all } i \geq 0,$$

and elements $y_i \in C_i \setminus C_{i+1}$

such that

$$H = \cup F_i y_1 + y_2 + ...y_i \qquad \text{for } i = 1,2......$$

is not consonant with any *V* conjugate of *x*

The first three lemmas show how to do this in certain special cases; in particular Lemma3.2 proves Theorem C for Abelian groups *G*.

LEMMA 3.1 *Let F be a finite subgroup of G and let* $x \in G$ *Suppose that u and v are elements of V.*

Then if x^u *and* F^v *are consonant we have that*

$$u\text{-}v \in C_v(x) + Cv(F)$$

Proof. Suppose first that

$$< x^u, F^v> = T$$

where *T* is a π-subgroup. Since *T* is finite we have

$$T^w \subseteq G$$

for some w ∈ V. We claim that

$$x^u = x^w \qquad .... \qquad (1)$$

and $$F^V = F^W \qquad .... \qquad (2)$$

For, if $z = u\text{-}W$ then x^z and x are consonant, while

$$[x.z] \in V.$$

Thus [x, z] = 1, and (1) is proved.

The proof of (2) is similar.

But now

$$u\text{-}w \in C_v(x)$$

and $$V\text{-}W \in C_V(F).$$

Subtracting these, the result follows.

LEMMA 3.2. *Suppose that the normal closure* $< x^G >$ *of* x *in G is finite and that*

$$W = [x,V]G$$

does not satisfy min-c. Then there exists a π-subgroup H of Γ *which is not consonant with any V-conjugate of x.*

Proof. Suppose first that *x* is in the centre of *G*. Then

$$W = [V,x].$$

Since *W* does not satisfy *min-c* there is a chain

$$1 = F_0 \subseteq F_1 .... \qquad \subseteq F_1 ....$$

of finite subgroups of *G* with centralizers $C_i = C_w(F_i)$ such that

$$C_i \neq C_{i+1} \qquad \text{for all } i \geq 0.$$

We construct *H* as a local conjugate (see (6)) of $\cup\ F_i$

Let

$$\Delta_i \; [C_{i-1}, F_i] \qquad \text{for all } i \geq 1.$$

Then $\Delta_i \neq 0$ so

$$|\Delta_i| \neq 1 \qquad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Now since V is completely reducible for fininite subgroups of G (Lemma 2.1) we have for i $\geq$ 1 that

$$W = \Delta_1 \oplus \Delta_2 \oplus \ldots\ldots \oplus \Delta_i \oplus C_i. \qquad \ldots\ldots\ldots\ldots \quad (2)$$

and, since $W = [V, x]$

$$C_w(x) = 0 \qquad \ldots\ldots\ldots\ldots \quad (3)$$

Now since W is countable we can enumerate the elements of W as $\{u_i \mid i=1,2,3,\ldots\}$.

Pick $y_i \in \Delta_i$ such that y_i. is not congruent to u_i modulo $\Delta_1 \oplus \Delta_2 \oplus \ldots\ldots\ldots \oplus \Delta_{i-1} \oplus C$. This is possible by (1) above.

Now, for i ≥ 1, put $\qquad z_i = y_1 + y_2 + \ldots + y_i$.

Since z_i is congruent to z_{i+1} modulo C_1 we have that

$$H = \cup F_i^{z} i$$

is a π subgroup of Γ.

We claim that $x^u i$ is not consonant with $F_i^{z} i$. For, by choice of z_i we have

$$u_i - z_i = u_i - y_1 - y_2 - \ldots - y_i$$

$$= u_i - y_i - (y_1 + y_2 + \ldots\ldots + y_{i-1})$$

$$\notin C_i \qquad \ldots\ldots\ldots\ldots\ldots \; (4)$$

since $\qquad u_i - y_i \notin \Delta_1 \oplus \Delta_2 \oplus \ldots\ldots \oplus \Delta_{i-1} \; . \oplus C_i$

and $\qquad y_1 + y_2 + \ldots + y_{i-1} \in \; \Delta_1 \oplus \Delta_2 \oplus \ldots\ldots \oplus \Delta_{i-1}$.

But by (3) above $\quad C_w(x) = 0 \quad$ so (4) shows that

$$u_i - z_i \notin C_w(x) + C_w(F_i)$$

Now Lemma 3.1 applies to the group *W G* to show that

$$x^{u}i \text{ and } F_i^{z} i \text{ are not consonant.}$$

This establishes Lemma 3.2 for the case that *x* is central.

Now in general we have that the centralizer K of $< x^G >$ in *G* has finite index in *G*. Let *T* be a transversal to *K* in *G.*

Then

$$W = [x,V]G = \sum_{r \in T} [x,V]t$$

Since T is finite, Proposition 2.1 (e) and (d) show that *[x, V] t* does not satisfy min-c as a *G* module for some t $\in$ T.

Now x^t is in the centre of K^t

so the above argument shows that there is a π-subgroup H_1 of Γ which is not consonant with any V-conjugate of x'.

Putting $s=t^{-1}$ and $H = H_1^s$

now gives the required result.

Lemma 3.2 now establishes Theorem C for Abelian groups. Foe the soluble case we use an inductive argument based on a variant of Lemma 3.2.

We need another simple Lemma to handle the case of a *min-c* module which is not completely reducible (see (3) for an example of such a non completely reducible module).

LEMMA 3.3. If *[V, G] satisfies min-c then* $V = [V,G] \oplus C_{v(}G).$

Proof. Since *[V, G]* satisfies *min-c* there is a finite subgroup *F* of *G* with

$$C_{[V,G]}(F) = C_{[V,G]}(G) = 0.$$

Since *V* is completely reducible for *F* by Lemma 2.2, we have that

$$V = [V,F] \oplus C_v(F).$$

Thus $[V, F] = [V, G]$.

But this holds for all finite subgroups containg F, so clearly

$$C_V(F) = C_VG)$$

and $V = [V, G] \oplus C_v (G)$ as required.

We now come to the required extension of Lemma 3.2

LEMMA 3.4. *Let K be generated by normal subgroups F of G such that [V, F] satisfies min-c. Let* $x \in G$ *and put*

$$L = < x^G > \cap K$$

and $W = [V, L]$.

Then if W fails to satisfy min-c as a G module there is a local conjugate of L which is not consonant with any conjugate of x; in particular $V \notin \Psi_G$

Proof. Assume that W does not satisfy *min -c.*
Now L is a subgroup of K and so is generated by normal subgroups F such that $[V, F]$ satisfies *min-c.* By Proposition 2.1(c) we can assume that

$$L = \cup F_i$$

where $1=F_0 \subseteq F_1 \ldots\ldots \subseteq F_i \ldots\ldots$,
F_i is G invariant and $[V, F_i]$ satisfy min-c for all $i \geq 0$.

Now, by Lemma 3.3 applied to W as an F_i- module,

$$W = [W, F_i] \oplus C_w (F_i) \qquad \text{for all} \quad i = 1,2,3...$$

Let

$$C_i = C_W(F_i)$$

and $\Delta_i = [C_{i-1} , F_i]$ for all $i \geq 1$.

Then as in Lemma 3.2,

$$W = \Delta_1 \oplus \Delta_2 \oplus \ldots\ldots \oplus \Delta_i \oplus C_i \qquad \ldots\ldots\ldots\ldots(1)$$

and, since $W = [V, L]$

$$C_W(L) = 0 \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

Since W does not satisfy *min -c* we have that

$$[V,F_i] \neq W$$

Since $$W = \cup [V.F_i]$$

we. may refine the chain F_i if necessary so that

$$\Delta_i \neq 0 \qquad \text{all i}$$

Now, as in Lemma 3.2 we enumerate the elements of W as $[u_i | i=1,2,3,....\}$.

We next choose elements $y_i \in \Delta_i$ allowing for the fact that in general it may happen that $C_W(x)$ is non zero.

We may, however, assume that

$$[x,\Delta_i] \neq 0$$

for Δ_i is G invariant and if centralized by x is also centralized by $<x^G>$. But this would imply that it commutes with L which contradicts (2) above.

Let B_i be the centralizer of x in Δ_i and pick $y_i \notin \Delta_i$ such that

$$y_i - u_i \notin \Delta_1 \oplus \Delta_2 \oplus \ldots\ldots\ldots \oplus \Delta_{i-1} \oplus C_i \oplus B_i = \Delta_1 \oplus \Delta_2 \oplus \ldots\ldots \oplus \Delta_{i-1} \oplus C_i \oplus C_W(x)$$

We now claim that if

$$z_i = y_i + y_2 + \ldots.. + yi$$

and

$$H_I = \cup F_i^z i$$

then H_I is a π subgroup which is not consonant with any V-conjugate of x. For x^ui is not consonant with $F_i^z i$ since

$$u_i - z_i = u_i - y_i - (y_1 + y_2 + \ldots. + y_{i-1})$$

$$\notin C_i + C_W(x)$$

and we can apply Lemma3.1 to the group *W G.* This establishes Lemma3.4.

We now come to a technical result which is needed for the induction step in the proof of Theorem C.

From now on we assume that derivations from subgroups of *G* into *V* always extend to the whole of *G;* in the notation of section2, $V \in \Psi_G$.

LEMMA 3.5. *Suppose* $V \in \Psi_G$ *and G has a normal subgroup A with G/A finite and Abelian. Then if [V, x]A satisfies min-c as an A module for every* $x \in A$ *it follows that [V, x]G satisfies min-c as an G module for every* $x \in G$*; moreover* $G^{(i)}$ *is covered by finite characteristic subgroups modulo* $A^{(i)}$ *for each i = 1,2,3....*

Proof. If x $\in$ A then Proposition 2.1 (c) show that *[V, x]G* satisfies min-c as an A module; for *G/A* is finite so that *[V, x]G* is the join of finitely many translates of the *minc*-c module *[V, x]A* and Proposition 2.1 (d) shows that *[V, x]G* satisfies *min-c* as a G module.

Now $$[V, x]G = [V, < x^G >]$$

so that *A* is generated by normal subgroups *F* of *G* such that *[V, F]* satisfies *min-c,* and so A satisfies the conditions on *K* in Lemma 3.4.

Now let y $\notin$ A.

Then, since *G/A* is Abelian we have that

$$Y=[y,G] \subseteq A \cap < y^G >$$

Next we apply Lemma 3.4 to *Y* to deduce that *[V, Y]* satisfies *min-c* as a *G* module.

Factoring out *[V,Y]*, as we may since quotients of Ψ_G modules clearly belong to the class Ψ_G, we have that y is central in *G* so that Lemma 3.2 applies to show that

[V, y] satisfies *min-c* modulo *[V,Y]*

Now, using Proposition 2.1 (a) we have that

[V, y]G satisfies *min-c* as a *G* module.

Finally, we must show that $G^{(i)}$ is covered by finite characteristic subgroups modulo $A^{(i)}$ for each i= 1,2,3....

We do this by showing that $G^{(i)} / A^{(i)}$ is an Abelian section of an $\mathfrak{R}$ group and so satisfies min p for all primes p. Such a group is the product of Cemikov p groups(for different primes p) - see (6) for example, and so is covered by finite characteristic subgroups (those consisting of all elements of exponent n for any natural number n).

Let *T* be a transversal to *A* in *G.* Then

$$G^{(i)} \subseteq < T^G > A^{(i)}$$

But, by Proposition2.1(c) we have

[T, V]G satisfies *min-c* as a *G* module.

Thus $< T^G >$ is an $\mathfrak{R}$ group ; since $G^{(i)}/A^{(i)}$ is Abelian and is a section of an $\mathfrak{R}$ group it is an Abelian $\mathfrak{R}$ group and the result follows from Proposition 2.1 (f).

Proof of Theorem C.

Suppose that *G* has soluble length n and that $V \in \Psi_G$; we show by induction on n that

[V, x]G satisfies *min- c* for all $x \in G$.

If n=l this follows from Lemma 3.2 as we have already remarked. Suppose then that it holds for *G* having soluble length at most n-1.

We divide the induction step into two parts: we first show that

(1) $G^{(i)}$ is covered by finite normal subgroups modulo $G^{(i+1)}$ for all $i \geq 0$. This is obvious for i=0; for i >0 we use Lemma 3.5. Let *F* be any finite subgroup of G. Then

$D = F\,G'$ is a normal subgroup of *G.*

By induction applied to G' we have that, for all $x \in G'$

[V, x]G' satisfies *min-c* as a G' module.

Thus we can apply Lemma 3.5 to *D* to deduce that for $i \geq 1$

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$$D^{(i)}|G^{(i+1)}$$

is generated by finite characteristic subgroups; since *D* is normal in *G* these are normalised by *G*. But *G* is generated modulo $G^{(i+1)}$ by subgroups $D^{(i)}$ as *F* ranges over the finite subgroups of *G*.

This establishes (1).

(2) We now use induction on *i* to prove that if $x \in G^{(n-i)}$ then

[V, x]G satisfies *min-c* as a *G* module.

First let $i = 1$. Then by (1) above $< x^G >$ is finite, and so Lemma 3.2 applies.

Now assume as inductive hypothesis that

for all $x \in G^{(n-i+1)}$ we have that *[V, x]G* satisfies *min-c* ……… (α_i)

Let y x $\in G^{(n-i)}$.

Then, by Lemma 3.4 we have that

$[< y^G > \cap G^{(n-i+1)}, V]$ satisfies *min-c* as a *G* module.

Thus, as in the proof of Lemma 3.5, we may factor out

$(< y^G > \cap G^{(n-i+1)}) [< y^G > \cap G^{(n-i+1)}, V]$

We now find, using (1), that $< y^G >$ is finite; thus Lemma 3.2 shows that

[V,y]G satisfies *min-c*

as required.

This proves (α_{i+1}) and so completes the proof of Theorem C.